

## Computation of the regime configuration of a meandering stream

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**Abstract** Regime channel formation is a delicate adaptation to the imposed environmental conditions compatible with flow and sediment transport mechanisms. The present paper concerns the computation of the regime configuration of a meandering stream. It is supposed that the channel develops until it reaches the state of “final thermodynamic equilibrium”, where the ratio of the kinetic energy of the flow to its cross-sectional potential energy is minimum. An optimization procedure that allows the analysis of the regime channel formation is presented and it is checked using a real case.

**Key words** predictive model; regime configuration; river's plane-form evolution

### INTRODUCTION

Natural rivers tend to adjust their plane shape and longitudinal profile in order to assume a configuration compatible with the changing hydraulic and man-made constraints. The final configuration, called “regime configuration”, represents the equilibrium geometry self-formed by the alluvial stream. The prediction of the regime channel configuration has generated a lot of interest amongst scientists and engineers. In technical terms the approach consists of the identification of the independent variables (often called controlling variables) and of the dependent variables (channel width, water depth, channel slope, meandering or braiding pattern). The independent variables, which are imposed upon the river by nature or by man, maintain their constant values through the duration of the regime development and they can be considered as the cause of the evolution process; the dependent variables vary during the channel evolution in order to obtain a value compatible with the imposed conditions. In selecting the independent variables for the prediction analysis, it is necessary to take into account the time scale of the process under study. In long-term processes, as considered in the present work, the mechanisms controlling the planimetric channel evolution become significant (Chang, 1988; Di Silvio, 1994): the water discharge and the sediment discharge can be considered as independent variables and the channel characteristics (width, depth and slope) as dependent variables. The planimetric evolution of the channel follows as effect of the regime-formation process that could take place in different ways: in some cases the stream is meandering, in others it is braiding (Yalin & da Silva, 2001).

There is still no unanimity on the question of which process is mainly responsible for the initiation and development of meandering stream. In particular, two major theories can be considered: the stability theories (Engelund & Skovgaard, 1973; Callander, 1978; Parker *et al.*, 1982; Zolezzi & Seminara, 2001) which consider the meandering pattern as a

consequence of an unstable response of the river to some perturbations; the rational theories or extreme methods (Yang & Song, 1979; White *et al.*, 1982; Davies & Sutherland, 1983; Yang, 1984), which are motivated by the conviction that the regime configuration develops because an energy related physical quantity tends to acquire its minimum value. In the latter case the regime configuration is revealed by solving a system of equations determining the dependent variables (width, depth and slope). Within the rational theories various physical quantities reaching the minimum value have been considered: the rate of energy dissipation (Yang & Song, 1979), the transport rate (White *et al.*, 1982), the unit stream power (Yang, 1984), the stream power (Chang, 1988), etc.

Recently Yalin & da Silva (2000, 2001) demonstrated, on the basis of thermodynamics considerations, that the channel develops until it reaches the state of “final thermodynamic equilibrium”, where the ratio of the kinetic energy of the flow to its cross-sectional potential energy (as implied by the flow Froude number) is minimum. In the present work, this last approach is applied and a computational procedure to determine the regime channel formation is presented.

## THE SYSTEM OF EQUATIONS

In accordance with the rational theories, the regime characteristics (channel width, water depth, channel bed slope) of an alluvial stream of constant flow rate,  $Q$ , can be estimated by solving a system of equations formed by:

—the resistance equation, that can be written as:

$$Q = Bhc\sqrt{gSh} \quad (1)$$

where  $c$  is the friction factor;  $g$  is the acceleration due to gravity;  $B$  is channel width;  $h$  is the water depth,  $S$  is the bed slope;

—the transport equation, written in the following general form:

$$Q_s = D_{50}u_*\phi_{Q_s}(X, Y, Z) \quad (2)$$

where  $Q_s$  is the sediment transport rate;  $u_*$  is the shear flow velocity;  $D_{50}$  is the representative sediment diameter;  $\phi_{Q_s}$  is in general a function of the non-dimensional parameters

$X = \frac{u_*D_{50}}{\nu}$ ,  $Y = \frac{\rho u_*^2}{\gamma_s D_{50}}$ ,  $Z = \frac{h}{D_{50}}$ , where  $\rho$  is water density;  $\gamma_s$  is specific weight of water;  $\nu$

is kinematic viscosity, and it has to be specified on the basis of the relationship used to express the transport rate (Einstein, 1942; Meyer-Peter, 1949; Yalin, 1963).

—the optimization condition of a quantity related to the stream energy. As a rule, it is written as :

$$\text{Minimize } O.F. = \varphi(V, V_c)$$

$$V=(B, h, S, \dots); \quad V_c=(Q, Q_s, \gamma_s, \gamma, \rho, D_{50}, \dots) \quad (3)$$

where  $V$  is the decision variables vector,  $V_c$  is the controlling variables vector. The optimization condition (3) expresses the condition to which the river system tends spontaneously.

The equations (1) and (2) can be treated as constraint equations of the optimization problem (3). The equations (1)–(3) system is solved and the regime values of the variables  $B$ ,  $h$  and  $S$  are estimated. Because it is not easy to evaluate the controlling variable  $Q_s$ , Yalin & da Silva (2001) proposed to replace equation (2) by the following equation:

$$B = f_1 f_2 \sqrt{\frac{Q}{u_{*,cr}}} \tag{4}$$

$$f_1 = 0.639 \left[ \frac{\gamma_s D_{50}^3}{\rho v^2} \right]^{0.3} \tag{5}$$

if the term in square brackets is  $<15$ ;

$$f_1 = 1.42 \text{ otherwise } f_2 = \left[ 0.2 b \left( 1 - e^{-0.35|\bar{c}-27.5|^{1.2}} \right) + 1.2 \right] \frac{c}{\bar{c}} \tag{6}$$

with  $b = 1$  if  $\bar{c} \geq 27.5$ ;  $b = -1$  if  $\bar{c} < 27.5$

where  $u_{*,cr}$  is the critical shear flow velocity,  $\bar{c}$  is the friction factor with flat bed.

In accordance with previous experimental observations (Leopold & Wolman, 1957; Ackers, 1964), equations (4)–(6) were derived by assuming that the variation of channel width essentially takes place during a very short time period (say  $tb$ ) in comparison with the entire duration of the regime channel formation ( $tR$ ); thus, no substantial adjustment in channel width occurs during the time interval ( $tR-tb$ ).

### THERMODYNAMIC CONSIDERATIONS

The second law of thermodynamic establishes that the entropy,  $\zeta$ , of an isolated system tends to increase with the passage of time. If a river system is considered as a closed sub-system (Yalin & da Silva, 2001), it tends to evolve in order to increase its entropy. Thus, it is:

$$\frac{d\zeta}{d\theta} \geq 0 \tag{7}$$

where  $\theta = t/t_R$  represents the no-dimensional time.

On the other hand, in accordance with Gibb's equation, the increment of the system entropy is related to the increment of the system internal energy  $U$ :

$$T \frac{d\zeta}{d\theta} = \frac{dU}{d\theta} \tag{8}$$

where  $T$  is the absolute temperature.

In general, a body of mass  $M$  and volume  $W$  (control volume of the river system) receives both mechanical energy by the work of forces to which it is subjected and thermal energy through the contour surface. The contribution to the system of these forms of energy is equal to the variation in time both of the kinetic energy and of the internal energy of the system. If a river system is considered, as aforementioned, a closed sub-system, it is supposed that the net heat and the work time rates exchanged between the stream and the

boundaries is null. Thus, the variation in time (i.e. during the regime development) of the sum of the kinetic and internal energy has to be equal to zero:

$$\frac{d}{d\theta} \left( U + \frac{\rho}{2} W u^2 \right) = 0 \quad (9)$$

where  $u$  is the average flow velocity.

Consequently, the increment of the system entropy corresponds to a decrement of the kinetic energy and, because  $W$  and  $\rho$  are positive quantities, the decrement of the kinetic energy coincides with the decrement in average flow velocity.

Thus, the kinetic energy (and thus the average flow velocity) represents the physical quantity that tends to reach its minimum value in time, producing the channel regime formation. On the other hand, from the constraint equation (1), for a wide alluvial stream, it is:

$$\frac{u^2}{gh} = c^2 S \quad (10)$$

where the term on right hand side represents the Froude number  $F_r$ . Thus, replacing equation (10) in (9) and taking into account equation (8), after simple passages (Yalin & da Silva, 2001) it can be concluded that, during a time period  $t_b \leq t \leq t_R$ , the increment of the entropy (equation (7)) is necessarily accompanied by a decrement of the stream Froude number.

## SOLUTION OF THE OPTIMIZATION PROBLEM

On the basis of the thermodynamic considerations reported in previous section, it is assumed that the regime channel formation has to be related to the minimization of the stream Froude number, that can be estimated by equation (10). Thus, the objective function of the optimization problem (3) has to express the minimization of the product ( $c^2 S$ ).

For given geometric characteristics of the sediment on the bed, the friction factor  $c$  depends on the hydraulic characteristics of the flow and it can be estimated as (Engelund, 1966; Yalin, 1992):

$$c = \sqrt{1 / \left[ \frac{1}{c^2} + \frac{1}{2h} (\delta_r^2 \Lambda_r + \delta_d^2 \Lambda_d) \right]} \quad \bar{c} = \frac{1}{\kappa} \ln \left( \frac{h}{k_s} \right) + B_s \quad (11)$$

where  $\kappa$  is the Von Karman's constant (assumed equal to 0.4);  $k_s$  is the representative bed roughness (assumed equal to  $2D_{50}$ );  $B_s$  is a function of the Reynolds shear number

$\left( \text{Re}_* = \frac{u_* D_{50}}{\nu} \right)$ ,  $\delta_r$  and  $\delta_d$  represent respectively the ripples and dunes steepness;  $\Lambda_r$  and  $\Lambda_d$

represent respectively the ripples and dunes height. These last four terms ( $\delta_r$ ,  $\delta_d$ ,  $\Lambda_r$ ,  $\Lambda_d$ )

depend on three non-dimensional parameters:  $Z = \frac{h}{D_{50}}$ ,  $\xi = \left( \frac{\gamma_s D_{50}^3}{\rho \nu^2} \right)^{1/3}$  and  $\eta = \frac{\tau}{\tau_{cr}} = \frac{u_*^2}{u_{*,cr}^2}$ .

Thus, it can be synthetically written (Yalin, 1992) that:

$$\delta_r = \delta_r(\eta, Z), \quad \Lambda_r = \Lambda_r(\eta, \xi, Z) \quad \delta_d = \delta_d(\eta, \xi, Z), \quad \Lambda_d = \Lambda_d(\eta, \xi, Z) \quad (12)$$

The parameter  $\xi$  depends on the physical characteristics of water and sediment, that are supposed constant during the regime channel evolution; the parameter  $Z$  depends on the water depth  $h$ , that is a state variable of the considered optimization problem; the parameter  $\eta$  represents the ratio of the average shear stress to the critical shear stress and, thus, it is related to the stream energetic properties; thus,  $\eta$  is considered the decision variable of the optimization problem.

The bed slope,  $S$ , varies in time and it can also be expressed as a function of the variable  $\eta$ . Taking into account that  $u_*^2 = g h S$ , it is:

$$S(\eta) = \eta \frac{u_{*,cr}^2}{gh} = \eta \left[ \frac{u_{*,cr}^2}{gD_{50}} \right] \frac{D_{50}}{h} \tag{13}$$

The term in square brackets, multiplied by the ratio  $\gamma/\gamma_s$  ( $\gamma$  = specific weight for water), represents the no-dimensional number  $Y_{cr} = \frac{\gamma u_{*,cr}^2}{\gamma_s g D_{50}}$ . Thus (13) becomes:

$$S(\eta) = \eta \frac{\gamma_s}{\gamma Z} Y_{cr} \tag{14}$$

by taking into account equation (1), it is:  $N = \frac{Q}{BD_{50}u_{*,cr}} = \frac{h}{D_{50}} c \frac{\sqrt{gSh}}{u_{*,cr}} = Z c \sqrt{\eta}$  and the equation (14) can be rewritten as:

$$S(\eta) = \frac{\gamma_s}{\gamma} Y_{cr} \frac{c(\eta)\eta^{2/3}}{N} \tag{15}$$

Finally, the optimization problem (3) is expressed as:

$$\text{Minimize FO} = \left[ \frac{\gamma_s}{\gamma N} Y_{cr} \eta^{2/3} c(\eta)^2 \right] \tag{16}$$

subjected to constraints (1) and (4)–(6). The function  $c(\eta)$  is determined by (11).

In gravel-bed channels the evolution process stops when  $\eta = 1$  (i.e.  $\tau = \tau_{cr}$ ). In sand bed channels the regime configuration could be reached for  $\eta > 1$ , because of the presence of bed forms. Thus, the following constraint has also been considered:

$$\eta \geq 1 \tag{17}$$

The critical mobility number  $Y_{cr}$  can be estimated as a function of the parameter  $\xi$  (Yalin & da Silva, 2001), that depends on the physical characteristics of the granular material and of the fluid, both assumed constant during the regime channel formation. Thus, on the plane  $Fr$ - $\eta$ , the objective function (16) is a curve of parameter  $N$ .

## APPLICATION

The optimization problem (1)–(4)–(16)–(17) is nonlinear and it has been solved with the help of the commercial code MINOS (Murtagh & Saunders, 1983). The solution of the

optimization problem allows to estimate the regime values of the channel width, water depth and longitudinal bed slope, for a given bed material and for a given flow rate. The procedure is iterative and it gives as result the values of pairs  $(F_{r,i}-\eta)$  at each  $i$ -th stage of the channel evolution. At  $t = 0$  (initial condition) both the characteristics of the “initial” channel ( $B_o, h_o, S_o$ ) and the physical characteristics of the bed material and of the fluid have to be known. Thus, the parameters  $\xi$  and  $Y_{cr}$  can be estimated and the critical shear velocity is calculated as:

$$u_{*,cr} = \left( \frac{\gamma_s D_{50}}{\rho} \right)^{1/2} Y_{cr}^{1/2} \quad (18)$$

The no-dimensional parameter  $N_i$ , at  $i$ -th evolution stage, is determined and the minimum value of the Froude number ( $F_{r,i}$ ) and the corresponding flow intensity  $\eta_i$  are obtained as solution of the optimization problem (16). Then, the corresponding flow depth is evaluated as  $h_i = \left( \frac{Q^2}{g B_i^2 F_{r,i}} \right)^{1/3}$  and the new channel width is calculated by (4)÷(6). The procedure stops when the values of the channel width estimated for two following iterations are almost the same.

Finally, the regime bed slope is calculated as:

$$S_R = \eta_R \frac{\gamma_s D_{50} Y_{cr}}{\gamma h_R} \quad (19)$$

The flow chart of the proposed procedure is reported in Fig. 1.

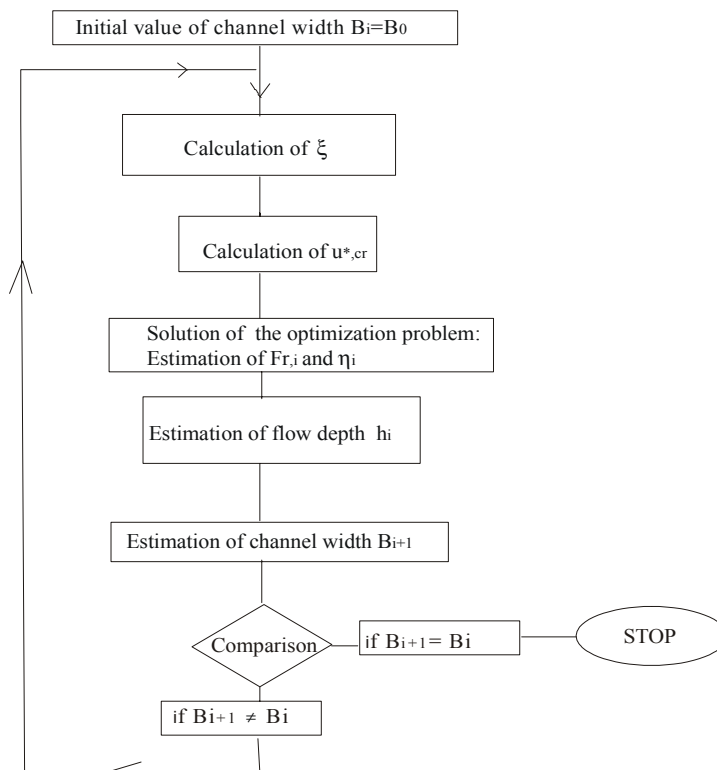


Fig. 1 Flow chart of the proposed procedure.

It is assumed that the channel plane shape follows the sine-generated curve (Langbein & Leopold, 1966). Thus, it is:

$$\vartheta = \vartheta_o \cos\left(2\pi \frac{l}{L_m}\right) \tag{20}$$

where  $l$  is the longitudinal abscissa along the channel centreline;  $L_m$  is the meander length;  $\vartheta_o$  is the deflection angle at  $l = 0$ ;  $\vartheta$  is the deflection angle at the abscissa  $l$ .

Thus, the regime plane configuration of the channel is evaluated by (20), taking into account that  $L_m$  and the meander wave length ( $\Lambda$ ) are interrelated by  $\vartheta_o$  as (Yalin, 1992):

$$\frac{L_m}{\Lambda} = \frac{1}{J(\vartheta_o)} \left( = \frac{S_o}{S_R} \right) \tag{21}$$

where  $J(\vartheta_o)$  is the Bessel function of the first kind and zero-th order.

The procedure has been applied in order to evaluate the regime configuration of Peace River in Alberta (data found in Yalin & da Silva, 2001). The initial channel width ( $B_o$ ) is 619.2 m, the initial water depth ( $h_o$ ) is 9.33 m and the initial bed slope ( $S_o$ ) is 0.0084%; the water discharge is equal to  $9905 \text{ m}^3 \text{ s}^{-1}$  and the median size of bed material ( $D_{50}$ ) is 0.31 mm.

The regime values of channel width, water depth and longitudinal bed slope estimated as solution of problem (1)–(6) and (16)–(17) are:  $B_R = 797.7 \text{ m}$ ;  $h_R = 18.6 \text{ m}$ ;  $S_R = 0.0000339$ . These values have been obtained for  $\eta_R = 29.64$  and  $Fr_{R} = 2.7 \times 10^{-3}$ . The regime (at  $\theta = 1$ ) planimetric channel configuration is characterized by a value of the deflection angle  $\vartheta_o = 96.6^\circ$ , and it has been determined by (20). In Fig. 2 the configurations of the channel axis estimated at three evolution stages are reported.

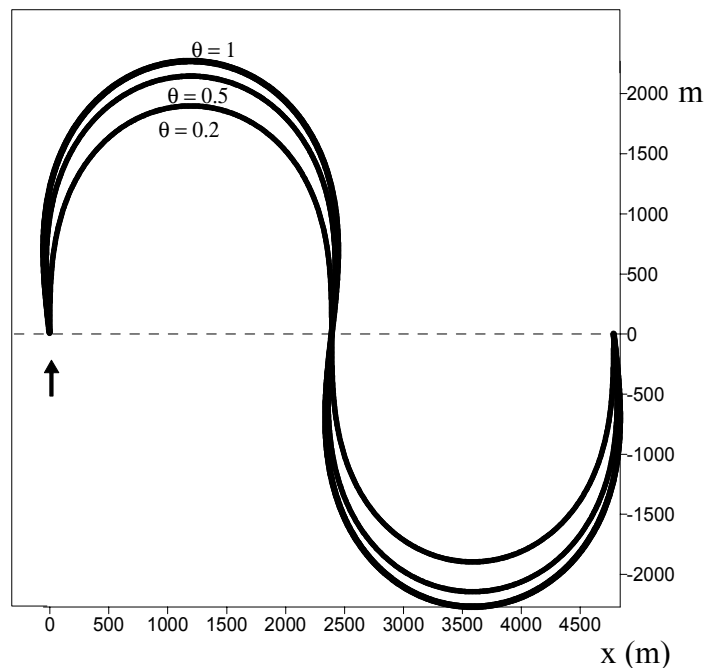
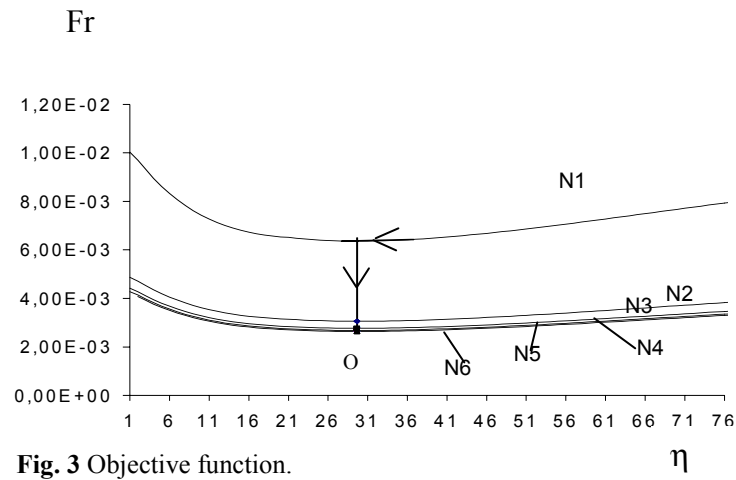
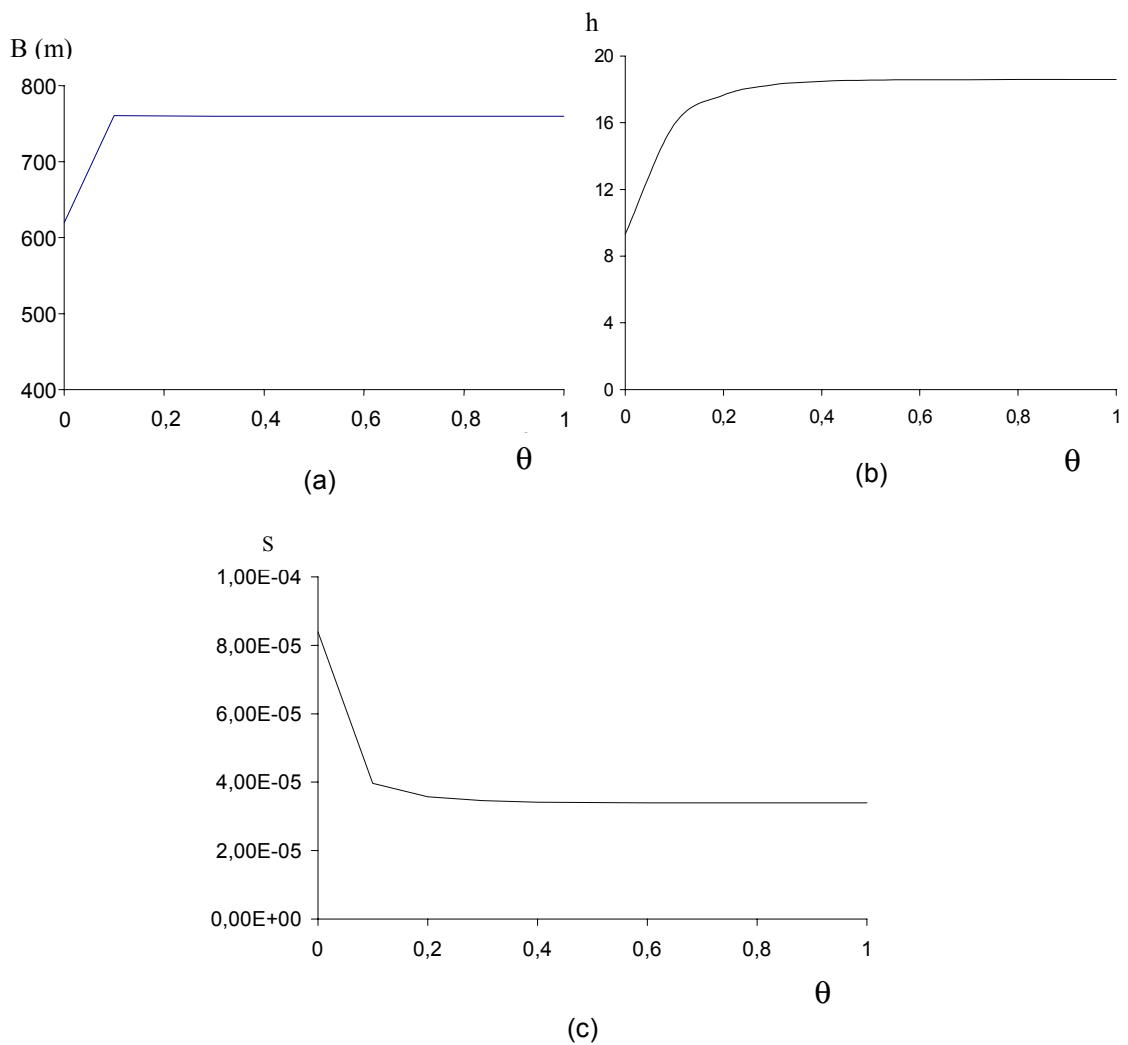


Fig. 2 Estimated planimetric channel configurations.



**Fig. 3** Objective function.



**Fig. 4** State variables calculated at each time step: (a) channel width; (b) water depth; (c) channel bed slope.



In Fig. 3 the pairs of  $Fr-\eta$  estimated at each evolution stage are reported. The parameter of each curve is the value of  $N$  estimated at the corresponding evolution stage; the point  $O$  identifies the final regime condition.

The channel width, the water depth and the channel bed slope estimated at each evolution stage are reported in Fig. 4(a–c), respectively. Both the channel width and the water depth increase during the channel evolution, while the longitudinal channel bed slope decreases. As Fig. 4(a) shows, the variation of the channel width is significant in the early evolution stages ( $\theta < 0.1$ ); it is almost equal to the final regime value already for  $\theta = 0.1$ .

Furthermore, Fig. 4(b) and (c) show that three variation phases can be identified both for the water depth and for the longitudinal bed slope: a first phase ( $\theta \leq 0.1$ ) where both the water depth and the longitudinal bed slope vary rapidly; a second phase ( $0.1 \leq \theta \leq 0.5$ ) where gradual variations occur, and a third phase ( $\theta > 0.5$ ) where very little variation takes place until the variables reach the corresponding regime values.

## CONCLUSION

In this paper a computational procedure for the prediction of the regime channel formation is presented. The procedure solves an optimization constrained problem where the objective function expresses the minimization of the stream Froude number, as recently suggested by one of the authors. The proposed computational procedure allows the evaluation of the planimetric configuration of the channel at each evolution stage until the regime configuration is reached. The application to a real case highlights the procedure capability.

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