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Characterizations of Kurzweil–Henstock–Pettis integrable functions

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Abstract. We prove that several results of Talagrand proved for the Pettis integral also hold for the Kurzweil–Henstock–Pettis integral. In particular the Kurzweil–Henstock–Pettis integrability can be characterized by cores of the functions and by properties of suitable operators defined by integrands.

1. Introduction. Our intention is to continue the study of the Kurzweil–Henstock–Pettis integral, started in [9] in the case of multifunctions. The integral is the generalization of the Pettis integral of a function, obtained by replacing the Lebesgue integrability of scalar functions by the Kurzweil–Henstock integrability. It integrates essentially more functions than the Pettis integrable ones (cf. [9, Example 1]). We refer to [19] and [20] for information about Pettis integrability.

Here we find some conditions guaranteeing the Kurzweil–Henstock–Pettis integrability of a single function. To this end we associate to each scalarly Kurzweil–Henstock integrable function $f : [0,1] \to X$ an operator $T_f : X^* \to \mathrm{KH}[0,1]$, defined on the dual of the range space of f and taking its values in the space of real-valued Kurzweil–Henstock integrable functions. In Theorem 2, generalizing a classical result for the Pettis integral, we prove that a function f is Kurzweil–Henstock–Pettis integrable if and only if the operator T_f is weak*-weakly continuous on the unit ball. In Theorem 3 we prove that Kurzweil–Henstock–Pettis integrable functions are determined by weakly compactly generated subspaces. In Theorem 4 we characterize the Kurzweil–Henstock–Pettis integrability in terms of the notion of core, introduced by Geitz [13]. Theorems 3 and 4 are generalizations of the celebrated results of Talagrand [24] characterizing Pettis integrable functions.

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Theorems 1–4 also hold for the Denjoy–Pettis integral (cf. [14, 12]), which is a generalization of the Kurzweil–Henstock–Pettis integral obtained by replacing the Kurzweil–Henstock integrability of scalar functions by the Denjoy–Khinchin integrability.

2. Basic facts. Let $[0,1]$ be the unit interval equipped with the usual topology and the Lebesgue measure λ; $\mathcal{L}$ denotes the family of all Lebesgue measurable subsets of $[0,1]$ and if $E \in \mathcal{L}$, then $|E|$ is its Lebesgue measure. $\mathcal{L}^+$ denotes the family of all sets of positive Lebesgue measure, and $\mathcal{I}$ the family of all closed subintervals of $[0,1]$. A *partition in* $[0,1]$ is a collection $P = \{(I_1,t_1),\dots,(I_p,t_p)\}$, where $I_1,\dots,I_p$ are non-overlapping subintervals of $[0,1]$ and $t_i \in I_i$, $i = 1,\dots,p$. If $\bigcup_{i=1}^p I_i = [0,1]$, we say that P is a *partition of* $[0,1]$. Given a subset E of $[0,1]$, we say that the partition P is *anchored on* E if $t_i \in E$ for each $i = 1,\dots,p$. A *gauge* on $[0,1]$ is a positive function on $[0,1]$. For a given gauge δ on $[0,1]$, we say that a partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ is *δ-fine* if $I_i \subset (t_i - \delta(x_i), t_i + \delta(x_i))$, $i = 1,\dots,p$.

Functions $f, g : [0,1] \to \mathbb{R}$ are called *equivalent* if $f = g$ almost everywhere, in the sense of the Lebesgue measure. Identifying in this way real functions of bounded variation on $[0,1]$, we obtain the space $\mathrm{BV}[0,1]$. We always consider this space with the quotient variation norm $\|\cdot\|_{\mathrm{BV}}$.

Throughout, X is an arbitrary Banach space with dual X^*. We do not assume separability of X. The closed unit ball of X is denoted by $B(X)$. If $Y \subset X$, then $Y^\perp$ denotes the annihilator of Y in X^*. The weak topology of X is often denoted by $\sigma(X, X^*)$. The topology generated on X by a Banach space Y is denoted by $\sigma(X,Y)$.

Functions $f, g : [0,1] \to X$ are said to be *scalarly equivalent* if for each $x^* \in X^*$ the set $N_{x^*} := \{t \in [0,1] : x^*f(t) \neq x^*g(t)\}$ is of Lebesgue measure zero.

DEFINITION 1. A function $g : [0,1] \to \mathbb{R}$ is said to be *Kurzweil–Henstock integrable* (or simply *KH-integrable*) on $[0,1]$ if there exists a real number z with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that

$$\Big|\sum_{i=1}^{p} g(t_i)|I_i| - z\Big| < \varepsilon$$

for each δ-fine partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ of $[0,1]$.

We set $z := (\mathrm{KH})\int_0^1 g(t)\,dt$.

We denote by $\mathcal{KH}[0,1]$ the set of all real-valued Kurzweil–Henstock integrable functions on $[0,1]$. When the equivalent functions are identified we denote the quotient space by $\mathrm{KH}[0,1]$. Throughout the paper also the elements of $\mathrm{KH}[0,1]$ are called KH-integrable functions. So, as is common in measure theory, quite often we do not distinguish functions from their equivalence classes, especially in proofs. In particular, stating that $x_n^* f \to h$ weakly in KH[0, 1] means that the equivalence classes of the functions $x_n^* f$ converge to the equivalence class of h.

The space KH[0, 1] is endowed with the *Alexiewicz norm* (cf. [1])

$$\|g\|_{\mathrm{A}} = \sup_{0<\alpha\le 1} \Big|(\mathrm{KH}) \int_0^\alpha g(t)\,dt\Big|.$$

The completion $\widehat{\mathrm{KH}}[0,1]$ of KH[0, 1] is isomorphic to the space of distributions which are the distributional derivatives of continuous functions (cf. [2]). As is known, KH-integrability coincides with Denjoy–Perron integrability (cf. [15]). The Denjoy–Perron integral is also called the Denjoy integral in the restricted sense (see [21]). If $\mathcal{D}[0,1]$ is the (quotient) space of Denjoy–Perron integrable functions endowed with the Alexiewicz norm, then its conjugate space is linearly isometric to the space BV[0, 1] (see [1]). Consequently, the conjugate space $\mathrm{KH}^*[0,1]$ is linearly isometric to BV[0, 1].

When we write that on a set $L \subset \mathrm{KH}[0,1]$, weak sequential convergence coincides with convergence in measure, we mean that any sequence in L converges (or not) in both senses to the same limit.

A family $\mathcal{A} \subset \mathrm{KH}[0,1]$ is said to be *Kurzweil–Henstock equiintegrable*, or simply *KH-equiintegrable*, on [0, 1] if in Definition 1, for every $\varepsilon > 0$ there exists a gauge δ on [0, 1] which works for all functions in $\mathcal{A}$.

DEFINITION 2. A function $f : [0,1] \to X$ is *scalarly measurable* (resp. *scalarly integrable*) if, for each $x^* \in X^*$, the function $x^* f$ is Lebesgue measurable (integrable). A scalarly integrable function $f : [0,1] \to X$ is *Dunford integrable* if for each nonempty set $A \in \mathcal{L}$ there exists a vector $w_A \in X^{**}$ such that for every $x^* \in X^*$,

$$\langle x^*, w_A\rangle = \int_A x^* f(t)\,dt.$$

If $w_A \in X$ for each $A \in \mathcal{L}$, then f is said to be *Pettis integrable* on [0, 1]. We call w_A the *Pettis integral of* f over A and we write $w_A := (\mathrm{P})\int_A f(t)\,dt$.

DEFINITION 3. A function $f : [0,1] \to X$ is *scalarly Kurzweil–Henstock integrable* if, for each $x^* \in X^*$, the function $x^* f$ is Kurzweil–Henstock integrable. A scalarly Kurzweil–Henstock integrable function $f : [0,1] \to X$ is *Kurzweil–Henstock–Dunford integrable* (or simply *KHD-integrable*) if, for each interval $[a,b] \in \mathcal{I}$, there exists a vector $w_{ab} \in X^{**}$ such that for every $x^* \in X^*$,

$$\langle x^*, w_{ab}\rangle = (\mathrm{KH}) \int_a^b x^* f(t)\,dt.$$

If $w_{ab} \in X$ for each $[a,b] \in \mathcal{I}$, then f is said to be *Kurzweil–Henstock–Pettis integrable* (or simply *KHP-integrable*) on $[0,1]$. We call w_{ab} the *Kurzweil–Henstock–Pettis integral* of f over $[a,b]$ and we write $w_{ab} := (\mathrm{KHP})\int_a^b f(t)\,dt$.

We denote by $\mathrm{KHP}([0,1],X)$ the set of all X-valued Kurzweil–Henstock–Pettis integrable functions on $[0,1]$ (functions that are scalarly equivalent are identified).

It is a classical result that each scalarly integrable function is Dunford integrable. It follows from Theorem 3 of Gámez and Mendoza [12] that a function $f : [0,1] \to X$ is KHD-integrable if and only if f is scalarly Kurzweil–Henstock integrable (they consider the Denjoy–Khinchin integral).

If $f : [0,1] \to X$ is a Kurzweil–Henstock scalarly integrable function, then an operator $T_f : X^* \to \mathrm{KH}[0,1]$, associated with f, is defined by $T_f(x^*) := x^*f$. If f is scalarly integrable, $T_f : X^* \to L_1[0,1]$ is defined in the same way.

For every $f : [0,1] \to X$ we define

$$\mathcal{Z}_f := \{x^*f : x^* \in B(X^*)\} \subset \mathbb{R}^{[0,1]}.$$

Identifying equivalent functions we obtain the set Z_f.

τ_{m} will denote the topology of convergence in measure. Throughout we will frequently apply the fact that, due to Fremlin's subsequence theorem [11], if f is scalarly measurable, then the set Z_f is τ_{m}-compact.

We will be concerned with generalizations of the following three fundamental results concerning the Pettis integral to the case of the Kurzweil–Henstock–Pettis integral:

THEOREM A. *Let $f : [0,1] \to X$ be a scalarly integrable function. Then f is Pettis integrable if and only if $T_f : X^* \to L_1[0,1]$ is weak*-weakly continuous.* ∎

THEOREM B ([24]). *Let $f : [0,1] \to X$ be a scalarly integrable function. Then f is Pettis integrable if and only if the following conditions are satisfied:*

(i) *$T_f : X^* \to L_1[0,1]$ is weakly compact;*
(ii) *there exists a weakly compactly generated subspace $Y \subset X$ such that $x^*f = 0$ a.e. for each $x^* \in Y^\perp$ (the exceptional sets $N_{x^*} := \{t \in [0,1] : x^*f(t) \neq 0\}$ depend on x^*).* ∎

THEOREM C ([24]). *A scalarly integrable function $f : [0,1] \to X$ is Pettis integrable if and only if it satisfies the following conditions:*

(i) *$T_f : X^* \to L_1[0,1]$ is weakly compact;*
(iii) *$\mathrm{cor}_f(E) \neq \emptyset$ for every $E \in \mathcal{L}^+$.* ∎

In the theorem above

$$\mathrm{cor}_f(E) := \bigcap\{\overline{\mathrm{conv}}\, f(E \setminus N) : \lambda(N) = 0\}$$

(see Geitz [13]).

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DEFINITION 4. Let $g : [0,1] \to \mathbb{R}$ be of bounded variation. A function $F : [0,1] \to X$ is *weakly Riemann–Stieltjes integrable on* $[0,1]$ *with respect to* g if for each $x^* \in X^*$ the function x^*F is Riemann–Stieltjes integrable on $[0,1]$ with respect to g, and there is a point $x \in X$ such that

$$x^*(x) = (\mathrm{RS})\int_0^1 x^*F(t)\,dg(t) \quad \text{for every } x^* \in X^*,$$

where the integral is the classical Riemann–Stieltjes integral.

We set $x := (\mathrm{wRS})\int_0^1 F(t)\,dg(t)$. The following result is well known. We present it just for completeness.

PROPOSITION 1. *If* $F : [0,1] \to X$ *is weakly continuous* (*i.e.* $x^*F \in C[0,1]$ *for every* $x^* \in X^*$) *and* $g \in \mathrm{BV}[0,1]$, *then* F *is weakly Riemann–Stieltjes integrable with respect to* g *on* $[0,1]$.

Proof. Since each BV function can be written as a difference of two non-decreasing functions, and each non-decreasing function can be written as a difference of two increasing functions, we may assume that g is increasing. If $\psi_g : [g(0), g(1)] \to [0,1]$ is defined by $\psi_g(t) := \sup\{s : g(s) \le t\}$, then ψ_g is increasing, $\psi_g(g(t)) = t$ and it is continuous. Moreover,

$$(*) \qquad \int_0^1 \alpha(t)\,dg(t) = \int_{g(0)}^{g(1)} \alpha(\psi_g(s))\,ds$$

for every continuous $\alpha : [0,1] \to \mathbb{R}$.

Notice now that $F \circ \psi_g$ is separably valued. Indeed, if Q is a countable dense subset of $[g(0), g(1)]$ then, due to the weak continuity of $F \circ \psi_g$, the set $F \circ \psi_g(Q)$ is weakly dense in the range of $F \circ \psi_g$. As $F \circ \psi_g$ is weakly continuous and weak separability in a Banach space coincides with norm separability, the function $F \circ \psi_g$ is strongly measurable. Since F is bounded, so is $F \circ \psi_g$. Consequently, $F \circ \psi_g$ is Bochner integrable on $[g(0), g(1)]$. Let

$$x_0 := (\mathrm{Bochner}) \int_{g(0)}^{g(1)} F(\psi_g(t))\,dt.$$

If $x^* \in X^*$, then applying $(*)$, we have

$$x^*(x_0) = \int_{g(0)}^{g(1)} x^*(F(\psi_g(t)))\,dt = \int_0^1 x^*(F(t))\,dg(t).$$

This proves the existence of the weak Riemann–Stieltjes integral of F with respect to g. ∎

Now we are in a position to prove the main result of this section.

THEOREM 1. *If $f : [0,1] \to X$ is KHP-integrable and $g \in \mathrm{BV}[0,1]$, then fg is also KHP-integrable. Conversely, if fg is KHP-integrable for every KHP-integrable $f : [0,1] \to X$, then g is almost everywhere equal to a function in* BV[0, 1].

Proof. Define $F : [0,1] \to X$ by $F(t) := (\mathrm{KHP}) \int_0^t f(s)\,ds$. It is a consequence of the classical theory of Kurzweil–Henstock integrability that F is weakly continuous (cf. [15]). Hence, according to Proposition 1, F is weakly Riemann–Stieltjes integrable with respect to g; let $x_0 = (\mathrm{wRS}) \int_0^1 F(t)\,dg(t)$. If $x^* \in X^*$, then by the classical theorem on integration by parts for the KH-integral (cf. [15]), we have

$$(\mathrm{KH})\int_0^1 x^*(f(t))g(t)\,dt = x^*(F(1))g(1) - (\mathrm{RS})\int_0^1 x^*(F(t))\,dg(t) = x^*[F(1)g(1) - x_0].$$

Thus, fg is KHP-integrable.

The second part of the assertion is well known for real-valued functions $f \in \mathrm{KH}[0,1]$ (see [18, Theorem 12.9]) and this immediately yields the Banach space case. ■

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The theorem below is a KHP-analogue of Theorem A.

THEOREM 2. *Let $f : [0,1] \to X$ be a KH-scalarly integrable function. Then f is KHP-integrable if and only if the operator $T_f : X^* \to \mathrm{KH}[0,1]$ is weak*-weakly continuous.*

Proof. Assume the weak*-weak continuity of T_f. Fix $[a,b] \in \mathcal{I}$ and let $w_{ab} \in X^{**}$ be the KHD-integral of f on $[a,b]$, i.e.

$$\langle x^*, w_{ab}\rangle = (\mathrm{KH})\int_a^b x^* f(t)\,dt = \langle T_f(x^*), \chi_{[a,b]}\rangle.$$

We have to show that $w_{ab} \in X$. But $\chi_{[a,b]} \in \mathrm{BV}[0,1]$ and so, according to the continuity assumption, the right hand side is weak*-continuous. Hence so is the left side. This means however that w_{ab} is a linear weak*-continuous functional on X^*. Consequently, $w_{ab} \in X$.

Assume now that $f : [0,1] \to X$ is KHP-integrable. According to Theorem 1, if $g \in \mathrm{BV}[0,1]$, then fg is KHP-integrable, that is, there is $w \in X$

such that for every $x^* \in X^*$ we have

$$\langle T_f(x^*), g\rangle = (\mathrm{KH})\int_0^1 x^*(f(t))g(t)\,dt = \langle x^*, w\rangle.$$

Hence $\langle T_f(\cdot), g\rangle$ is weak*-continuous, which proves the weak*-weak continuity of T_f. ∎

PROPOSITION 2. *If* $f : [0,1] \to X$ *is KHP-integrable, then* Z_f *is* $\sigma(\mathrm{KH},\mathrm{BV})$*-compact and* $\sigma(\mathrm{KH},\mathrm{BV})$ *coincides with* τ_{m} *on* Z_f.

Proof. According to Theorem 2, the set Z_f is weakly compact. Now take a sequence $(x_n^*) \subset B(X^*)$ such that $x_n^* f \to h$ weakly in KH[0,1]. According to Fremlin's theorem [11, Theorem 2F] each subsequence of $(x_n^* f)$ contains an almost everywhere convergent subsequence. In order to prove that $x_n^* f \to h$ in measure it is enough to show that for each subsequence $(x_{n_k}^*)$ of (x_n^*) such that $x_{n_k}^* f \to g$ almost everywhere in [0,1], we have $h = g$ a.e. So let $(x_{n_k}^*)$ be such a subsequence. If (y_γ^*) is a subnet of $(x_{n_k}^*)$ weak*-converging to x_0^*, then $x_0^* f = g$ almost everywhere in [0,1] and the KHP-integrability of f yields $y_\gamma^* f \to x_0^* f$ weakly in KH[0,1] (cf. Theorem 2). But $(y_\gamma^* f)$ is also a subnet of $(x_{n_k}^* f)$ and so we have $h = x_0^* f = g$ a.e. Consequently, the sequence $(x_{n_k}^* f)$ is almost everywhere pointwise convergent to h and $(x_n^* f)$ converges in measure to h.

Take now a sequence $(x_n^*) \subset B(X^*)$ such that $x_n^* f \to h$ in measure. Applying Fremlin's theorem [11, Theorem 2F] once again, we deduce that $h \in Z_f$. As Z_f is weakly compact, each subsequence of $(x_n^* f)$ contains a weakly convergent subsequence. If $x_{n_k}^* f \to g$ weakly, then we have just proven that $x_{n_k}^* f \to g$ in measure. Hence $g = h$ a.e. Thus, each weakly convergent subsequence of $(x_n^* f)$ converges weakly to h. Consequently, $x_n^* f \to h$ weakly.

It follows that on Z_f, weak-sequential convergence coincides with sequential convergence in measure. Moreover, Fremlin's subsequence theorem implies that Z_f is τ_{m}-compact.

To prove the coincidence of the topologies we will prove that they have the same closed sets. So let $\emptyset \neq W \subset Z_f$ be arbitrary. If $w_0 \in \overline{W}^{\sigma}$ $(= \sigma(\mathrm{KH},\mathrm{BV})$-closure of W), then due to the relative weak compactness of W, there are $x_n^* f \in W$ such that $x_n^* f \to w_0$ weakly, and so by the coincidence of the sequential convergences, also $x_n^* f \to w_0$ in measure. Thus, $\overline{W}^{\sigma} \subseteq \overline{W}^{\tau_{\mathrm{m}}}$.

The proof of the reverse inclusion is similar. Indeed, if $v_0 \in \overline{W}^{\tau_{\mathrm{m}}}$ then, due to metrizability of convergence in measure, there are $\overline{x}_n^* f \in W$ such that $\overline{x}_n^* f \to v_0$, and consequently $x_n^* f \to v_0$ weakly. This proves that $\overline{W}^{\tau_{\mathrm{m}}} \subseteq \overline{W}^{\sigma}$ and completes the whole proof. ∎

For the Lebesgue integral it is easy to deduce that if a sequence of Lebesgue integrable functions is almost everywhere pointwise convergent to

g_1 and weakly convergent to g_2 in $L_1[0,1]$, then $g_1 = g_2$ almost everywhere. In particular, in the case of a scalarly integrable function $f : [0,1] \to X$, due to Fremlin's subsequence theorem, condition (i) in Theorems B and C can be replaced by the following topological condition:

On the set $Z_f \subset L_1[0,1]$ the topology $\sigma(L_1, L_\infty)$ coincides with τ_{m}.

The KH-integral behaves differently. In [7] there is an example of a sequence of real-valued KH-integrable functions with pointwise limit different from its limit in the weak topology of KH[0,1]. We are going to show that for KHP-integrable functions such pathologies cannot take place on Z_f.

Since [7] is not easily accessible, we sketch the construction and deduce the consequence that is important for us.

EXAMPLE 1. Let $\alpha > 3$ be arbitrary and let C_α be the Cantor set in $[0,1]$ with measure $(\alpha-3)/(\alpha-2)$. Denote by $\varrho_k^n = (a_k^n, b_k^n)$, $k = 1, \ldots, 2^{n-1}$, $n = 1, 2, \ldots$, the contiguous intervals of C_α. Then $|\varrho_k^n| = \alpha^{-n}$.

For every $n \in \mathbb{N}$ define a continuous function $F_n : [0,1] \to [0,1]$ by setting

$$F_n(x) = \begin{cases} 2^{1-n} & \text{if } x = 1 \text{ or } x = a_k^s,\ k = 1, \ldots, 2^{s-1},\ s = 1, \ldots, n, \\ 0 & \text{if } x = 0 \text{ or } x = b_k^s,\ k = 1, \ldots, 2^{s-1},\ s = 1, \ldots, n, \end{cases}$$

and extending linearly to the whole interval $[0,1]$, without producing new vertices of the graph. Then let

$$f_n(x) = \begin{cases} F_n'(x) & \text{if } F_n'(x) \text{ exists}, \\ 0 & \text{otherwise}. \end{cases}$$

The sequence (f_n) of KH-integrable functions (they are in fact even Lebesgue integrable) is as desired. Indeed, by easy computations we get

$$f_n(x) = \begin{cases} -2^{1-n}\alpha^i & \text{if } x \in \varrho_s^i,\ i = 1, \ldots, n,\ s = 1, \ldots, 2^{i-1}, \\ 2\left(1 - \frac{1}{\alpha}\sum_{p=0}^{n-1}\left(\frac{2}{\alpha}\right)^p\right)^{-1} & \text{if } x \notin \bigcup_{i=1}^n \bigcup_{s=1}^{2^{i-1}} \overline{\varrho_s^i}, \\ 0 & \text{otherwise}, \end{cases}$$

$\overline{\varrho_s^i}$ being the closure of ϱ_s^i. It can be easily seen that $(f_n(x))$ converges to $2(\alpha-2)(\alpha-3)^{-1}$ on $C_\alpha \setminus \{a_k^n, b_k^n : k = 1, \ldots, 2^{n-1},\ n \in \mathbb{N}\}$ and to zero elsewhere. We denote the limit function by f. Notice that $f \in L_1[0,1] \subset$ KH[0,1].

Now let $h \in \mathrm{BV}[0,1]$ be arbitrary. The sequence (F_n) converges uniformly to zero. Therefore, integrating by parts we get

$$\lim_n \langle h, f_n \rangle = \lim_n (\mathrm{KH})\int_0^1 f_n h\, dx = \lim_n \Big([F_n h]_0^1 - (\mathrm{KH})\int_0^1 F_n\, dh \Big) = 0.$$

So the sequence (f_n) is weakly convergent to zero. Thus, $f - f_n \to 0$ pointwise on $[0,1]$ and $f - f_n \to f$ weakly in KH[0,1].

Define a function $g : [0,1] \to c_0$ by $g(t) := (f(t) - f_n(t))_{n=1}^{\infty}$. If g were KHP-integrable in c_0, then for each $I \in \mathcal{I}$ we would have the equality $(\mathrm{KHP}) \int_I g(t)\,dt = ((\mathrm{KH}) \int_I [f(t) - f_n(t)]\,dt)_{n=1}^{\infty}$. But $(f - f_n)$ is $\sigma(\mathrm{KH}, \mathrm{BV})$-convergent to f that is not negligible and so there is an interval $I \in \mathcal{I}$ such that

$$(\mathrm{KH}) \int_I [f(t) - f_n(t)]\,dt \to (\mathrm{KH}) \int_I f(t)\,dt \neq 0.$$

Consequently, g is not KHP-integrable in c_0.

We now prove that the canonical injection of $\mathcal{Z}_g = \{x^* g : \|x^*\| \leq 1\}$ into $\mathrm{KH}[0,1]$ has relatively weakly compact image. To prove it notice that if $x^* \in l_1$, then $x^* = (\alpha_n)$, where $\sum_{n=1}^{\infty} |\alpha_n| \leq 1$. If $\overline{\mathrm{aco}}\, W$ denotes the closed absolutely convex hall of W, then

$$x^* g = \sum_{n=1}^{\infty} \alpha_n (f - f_n) \in \overline{\mathrm{aco}}\{f - f_n : n \in \mathbb{N}\},$$

and the last set is weakly compact in $\widehat{\mathrm{KH}}[0,1]$ (due to Krein's theorem, cf. [16, p. 162]). Hence, Z_g is relatively weakly compact in $\mathrm{KH}[0,1]$. Equivalently, $T_g : l_1 \to \mathrm{KH}[0,1]$ is weakly compact.

Thus, we have obtained a function $g : [0,1] \to c_0$ such that on Z_g the weak topology $\sigma(\mathrm{KH}, \mathrm{BV})$ and the topology of convergence in measure do not coincide.

As already noted, g is not KHP-integrable but the operator $T_g : l_1 \to \mathrm{KH}[0,1]$ is weakly compact. Since g is determined by a separable space, it is clear that in order to get a generalization of Theorem B for the KHP-integral one cannot simply replace $L_1[0,1]$ by $\mathrm{KH}[0,1]$. An additional assumption is unavoidable. ∎

LEMMA 1. *Let $f \in \mathrm{KH}[0,1]$ be such that, for every $I \in \mathcal{I}$, $(\mathrm{KH}) \int_I f = 0$. Then $f(t) = 0$ almost everywhere in $[0,1]$.*

Proof. By hypothesis $F(t) = (\mathrm{KH}) \int_0^t f \equiv 0$ in $[0,1]$. Since $F'(t) = f(t)$ almost everywhere in $[0,1]$ (cf. for example [15]), the statement follows. ∎

Notice that due to Lemma 1 the topology $\sigma(\mathrm{KH}, \mathrm{BV})$ is Hausdorff on Z_f.

The theorem below is a KHP-analogue of Theorem B.

THEOREM 3. *Let $f : [0,1] \to X$ be a scalarly KH-integrable function. Then f is KHP-integrable if and only if the following two conditions are fulfilled:*

(TC) *on $Z_f \subset \mathrm{KH}[0,1]$ the topology $\sigma(\mathrm{KH}, \mathrm{BV})$ coincides with τ_{m};*

(DC) *there exists a weakly compactly generated subspace $Y \subset X$ such that for each $x^* \in Y^{\perp}$ one has $x^* f = 0$ almost everywhere.*

Proof. KHP $\Rightarrow$ (TC) has been proven in Proposition 2.

KHP $\Rightarrow$ (DC). Let now Y be the closed linear subspace of X generated by the collection $\{(\mathrm{KHP})\int_I f : I \in \mathcal{I}\}$. Since $\mathrm{KH}^*[0,1] = \mathrm{BV}([0,1])$, by the Gantmacher theorem we know that the adjoint operator $T_f^* : \mathrm{BV} \to X^{**}$ is also weakly compact. Let $I \in \mathcal{I}$ be an interval; then its characteristic function χ_I belongs to $\mathrm{BV}([0,1])$.

For each $x^* \in X^*$ we have

$$\langle T_f^*\chi_I, x^*\rangle = \langle \chi_I, T_f x^*\rangle = (\mathrm{KH})\int_I x^* f = x^*\,(\mathrm{KHP})\int_I f.$$

Thus, $T_f^*\chi_I = (\mathrm{KHP})\int_I f \in X$ and

$$\Big\{(\mathrm{KHP})\int_I f : I \in \mathcal{I}\Big\} \subset 2T_f^* B(\mathrm{BV}[0,1]).$$

As $T_f^* B(\mathrm{BV}[0,1])$ is weakly compact, Y is weakly compactly generated.

Let x^* be in $Y^\perp$. Then for each $I \in \mathcal{I}$ we have

$$0 = x^*\,(\mathrm{KHP})\int_I f = (\mathrm{KH})\int_I x^* f.$$

Therefore, by Lemma 1, $x^* f = 0$ almost everywhere in $[0,1]$ and condition (DC) is satisfied.

(TC) + (DC) $\Rightarrow$ KHP. Here we apply an idea borrowed from Proposition 2.2 of [22]. Assume now that conditions (TC) and (DC) are satisfied and define an operator $Q_Y : Y^* \to \mathrm{KH}[0,1]$ by setting $Q_Y(y^*) = T_f(y^*_{\mathrm{ext}})$, where $y^*_{\mathrm{ext}} \in Y^*$ is an arbitrary norm-preserving extension of y^* to the whole X. A direct calculation shows that Q_Y is well defined, bounded and T_f is weak*-weakly continuous if and only if Q_Y is weak*-weakly continuous if and only if T_f is $\sigma(X^*, Y)$-weakly continuous. Hence, according to Theorem 2, in order to prove the KHP-integrability of f it suffices to show that T_f is $\sigma(X^*, Y)$-weakly continuous.

As Y is weakly compactly generated (hence has the Mazur property, i.e. each element of Y^{**} which is sequentially weak*-continuous on Y^* belongs to Y), it is enough to prove that T_f is sequentially $\sigma(X^*, Y)$-weakly continuous. So let $(x_n^*) \subset B(X^*)$ be a sequence $\sigma(X^*, Y)$-converging to zero. By condition (TC), Z_f is weakly compact. Then there is $h \in \mathrm{KH}[0,1]$ and a subsequence (z_n^*) of (x_n^*) such that $z_n^* f \to h$ weakly in $\mathrm{KH}[0,1]$. By (DC) we have $z_n^* f \to h$ in measure. Let $(z_{n_k}^* : k \in \mathbb{N})$ and $z_0^* \in B(X^*)$ be such that z_0^* is a weak* cluster point of (z_n^*) and $z_{n_k}^* f \to z_0^* f$ a.e. Consequently, $z_0^* \in Y^\perp$ and so $h = z_0^* f = 0$ a.e. It follows that every subsequence of $(x_n^* f)$ contains a subsequence which is weakly convergent to zero in $\mathrm{KH}[0,1]$. Consequently, $T_f x_n^* \to 0$ weakly in $\mathrm{KH}[0,1]$. This gives us the desired sequential $\sigma(X^*, Y)$-weak continuity of T_f. ∎

REMARK 1. As can be easily seen from the proof above, the subspace Y in Theorem 3 may be assumed to have only the Mazur property.

COROLLARY 1. *If $f : [0,1] \to X$ is KHP-integrable, then each set $E \in \mathcal{L}^+$ contains an $F \in \mathcal{L}^+$ such that $f\chi_F$ is Pettis integrable.*

Proof. According to [14, Corollary 32] if $E \in \mathcal{L}^+$, then there is $\mathcal{L}^+ \ni M \subset E$ such that $f|M$ is scalarly integrable (since each KH-integrable real function is Denjoy–Khinchin integrable, we may apply [14]). Then, applying for instance Corollary 3.1 from [19], one finds $M \supset F \in \mathcal{L}^+$ such that $f\chi_F$ is scalarly bounded (i.e. there is $\alpha > 0$ such that for each $x^* \in X^*$ we have $|x^*f|F| \le \alpha\|x^*\|$ a.e.). But this means that $T_{f|F} : X^* \to L_1(\lambda|F)$ is weakly compact. Together with condition (DC) of Theorem 3, this yields the Pettis integrability of $f|F$ (cf. [20, Theorem 4.5]). ∎

Even when $X = \mathbb{R}$ there are simple examples of f such that each $E \in \mathcal{L}^+$ contains an F of positive measure with $f\chi_F$ Lebesgue integrable but f itself not KH-integrable.

LEMMA 2. *Let $f : [0,1] \to X$ be a scalarly integrable function. If T_f is $\sigma(L_1, L_\infty)$-compact, then it is $\sigma(\mathrm{KH}, \mathrm{BV})$-compact and on the set $Z_f \subset \mathrm{KH}[0,1]$ the weak topology $\sigma(\mathrm{KH}, \mathrm{BV})$ coincides with τ_{m}.*

Proof. Let $U : L_1 \to \mathrm{KH}[0,1]$ be the natural injection and let $(x_\alpha^* f)$ be a net of functions in $T_f B(X^*)$. Assume that T_f is $\sigma(L_1, L_\infty)$-compact and the net $(x_\alpha^* f)$ is $\sigma(L_1, L_\infty)$-convergent to a function $g \in L_\infty$. Then $\langle x_\alpha^* f - g, U^* h\rangle = \langle U x_\alpha^* f - Ug, h\rangle$ for every $h \in \mathrm{BV}[0,1]$ and so the net $(U x_\alpha^* f)$ is weakly convergent in $\mathrm{KH}[0,1]$ to Ug. It follows that T_f is $\sigma(\mathrm{KH}, \mathrm{BV})$-compact.

Since T_f is $\sigma(L_1, L_\infty)$-compact, each sequence from $T_f B(X^*)$ contains a $\sigma(L_1, L_\infty)$-convergent subsequence. Let $(x_n^* f)$ with $x_n^* \in B(X^*)$, $n \in \mathbb{N}$, be $\sigma(\mathrm{KH}, \mathrm{BV})$-convergent to $h \in Z_f$. Applying Theorem 2F of [11], we can find a subsequence $(x_{n_k}^* f)$ that is a.e. pointwise convergent to g. The sequence contains a further subsequence $(x_{n_{k_p}})_p$ $\sigma(L_1, L_\infty)$-convergent and a.e. convergent to $h \in L_1[0,1]$. If we now apply the theorem of Mazur, we get convex combinations $h_p \in \operatorname{conv}\{x_{n_{k_q}} : q \ge p\}$ a.e. convergent to h. Clearly $g = h$ a.e. This proves that $\sigma(\mathrm{KH}, \mathrm{BV})$-convergent sequences are convergent in measure to the same limits. The coincidence of the topologies follows as in Proposition 2. ∎

The following result gives a characterization of Pettis integrable functions as a subset of KHP-integrable functions. An example of a KHP-integrable function that is scalarly integrable but not Pettis integrable is presented in [12].

Proposition 3. *Let $f : [0,1] \to X$ be a scalarly integrable and KHP-integrable function. Then f is Pettis integrable if and only if the operator $T_f : X^* \to L_1[0,1]$ is $\sigma(L_1, L_\infty)$-compact.*

Proof. If f is Pettis integrable, then the weak compactness of T_f follows from Theorem B.

If T_f is $\sigma(L_1, L_\infty)$-compact, then it follows from Lemma 2 that (TC) is fulfilled. ∎

In the following we denote by τ_{p} the topology of pointwise convergence in the space of all real-valued functions. The following lemma is a reformulation of Lemma 5-1-2 from [24].

Lemma 3. *Let $\mathcal{Z} \subset \mathcal{KH}[0,1]$ be a convex and τ_{p}-compact set. Assume that the image of the canonical embedding of $\mathcal{Z}$ into* KH$[0,1]$ *is a weakly compact subset of* KH$[0,1]$ *and the weak topology* $\sigma(\mathrm{KH}, \mathrm{BV})$ *coincides with τ_{m} on that image. If the canonical embedding $\mathcal{Z} \to \mathrm{KH}[0,1]$ is not pointwise weakly continuous at $g \in \mathcal{Z}$, then there is $f \in \mathcal{Z}$ which is not equal a.e. to g, but g is in the τ_{p}-closure of the set*

$$\mathcal{W}_f := \{h \in \mathcal{Z} : h = f \text{ a.e.}\}.$$

Proof. By assumption, we may assume that there are $h' \in \mathrm{BV}$ and $\alpha < \beta$ such that $(\mathrm{KH})\int_0^1 g(t)h'(t)\,dt \le \alpha$ and g is in the τ_{p}-closure of the set $Y = \{h \in \mathcal{Z} : (\mathrm{KH})\int_0^1 h(t)h'(t)\,dt \ge \beta\}$.

For a finite set $F \subset [0,1]$ and $\varepsilon > 0$, let

$$U_{F,\varepsilon} := \{h \in \mathcal{Z} :\ |(h-g)(t)| \le \varepsilon \ \forall t \in F\}.$$

If $C_{F,\varepsilon} := Y \cap U_{F,\varepsilon}$, then $C_{F,\varepsilon} \neq \emptyset$ and the closure $H_{F,\varepsilon}$ of the image $\widetilde{C}_{F,\varepsilon}$ of $C_{F,\varepsilon}$ in KH$[0,1]$ is convex and weakly compact, and $\widetilde{C}_{F,\varepsilon}$ is weakly sequentially dense in $H_{F,\varepsilon}$. The family of sets $H_{F,\varepsilon}$ has the finite intersection property and so there is $f' \in \mathrm{KH}[0,1]$ which is in every $H_{F,\varepsilon}$. In particular, $\langle f', h'\rangle \ge \beta$. Fix (F,ε). Then there is a sequence $\{f_n\} \subset \widetilde{C}_{F,\varepsilon}$ weakly convergent to f'. By Fremlin's theorem [11, Theorem 2F], there is a function f'' and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ converging to f'' almost everywhere. As $\mathcal{Z}$ is τ_{p}-compact, let $f_{F,\varepsilon} \in \mathcal{Z}$ be a τ_{p}-cluster point of $\{f_n\}$. Since $f'' = f_{F,\varepsilon}$ a.e., we get $f'' \in \mathrm{KH}[0,1]$.

By assumption, $f' = f'' \in \mathrm{KH}[0,1]$.

So we have $f_{n_k} \to f'$ a.e. Then $f_{F,\varepsilon} = f'$ a.e. Since $U_{F,\varepsilon}$ is τ_{p}-closed, we have $f_{F,\varepsilon} \in U_{F,\varepsilon}$. It follows that g is a τ_{p}-cluster point of the set $\{f_{F,\varepsilon} : F, \varepsilon\}$. Take for f an arbitrary $f_{F,\varepsilon}$. We then have

$$(\mathrm{KH})\int_0^1 h'f = (\mathrm{KH})\int_0^1 h'f_{F,\varepsilon} = (\mathrm{KH})\int_0^1 h'f' \ge \beta.$$

Since $f = f_{F,\varepsilon}$ a.e. for every (F,ε), we get the result. ∎

LEMMA 4 (cf. [19, Lemma 6.2]). *Assume that* $f:[0,1]\to X$ *is scalarly measurable and* $\operatorname{cor}_f(E)\neq\emptyset$ *for every* $E\in\mathcal{L}^+$. *If* $x^*\in X^*$, *then* $x^*f=0$ *a.e. if and only if* $x^*=0$ *on* $\operatorname{cor}_f[0,1]$.

The theorem below is a KHP-analogue of Theorem C.

THEOREM 4. *A scalarly KH-integrable function* $f:[0,1]\to X$ *is KHP-integrable if and only if the following conditions are satisfied*:

(TC) *on* $Z_f\subset \mathrm{KH}[0,1]$ *the topology* $\sigma(\mathrm{KH},\mathrm{BV})$ *coincides with* τ_{m};
(CC) $\operatorname{cor}_f(E)\neq\emptyset$ *for every* $E\in\mathcal{L}^+$.

Proof. Assume that conditions (TC) and (CC) are fulfilled, but f is not KHP-integrable. By (TC) and Fremlin's subsequence theorem the set Z_f is weakly compact. Let $\mathcal{Z}:=\mathcal{Z}_f$. Applying Theorem 2 and Lemma 3, we get the existence of two functionals $y^*,z^*\in B(X^*)$ such that $|\{t\in[0,1]: y^*f(t)\neq z^*f(t)\}|>0$ and y^*f is in the τ_{p}-closure of the set

$$\mathcal{W}_f:=\{x^*f: x^*\in B(X^*)\ \&\ x^*f=z^*f \text{ a.e.}\}.$$

Let $(x^*_\alpha f)\subset\mathcal{W}_f$ be a net τ_{p}-convergent to y^*f. We may assume that $x^*_\alpha\to y^*$ in $\sigma(X^*,X)$. Since for every α we have $x^*_\alpha f=z^*f$ a.e. on $[0,1]$, it follows from Lemma 4 that $x^*_\alpha|_{\operatorname{cor}_f[0,1]}=z^*|_{\operatorname{cor}_f[0,1]}$. But then $y^*|_{\operatorname{cor}_f[0,1]}=z^*|_{\operatorname{cor}_f[0,1]}$, which yields, again by Lemma 4, $y^*f=z^*f$ a.e. on $[0,1]$. This however contradicts our assumption and so f is KHP-integrable.

Assume now that $f\in\mathrm{KHP}([0,1],X)$. Then (TC) is a consequence of Theorem 3, and (CC) follows from Corollary 1 and Theorem C. ∎

REMARK 2. If $c_0\subseteq X$ isomorphically, then Gámez and Mendoza [12] constructed an example of a KHP-integrable function f which is not Pettis integrable on any portion of some perfect subset F of $[0,1]$. According to [14, Theorem 33] we may assume (taking a suitable subset of F) that f is scalarly integrable on a portion $F\cap I$ of F. Consequently, applying Theorem 4 and then Theorem C, we see that the operator $T_{f|(F\cap I)}:X^*\to L_1(\lambda|(F\cap I))$ is not weakly compact. Equivalently, $T_{f\chi_{F\cap I}}:X^*\to L_1[0,1]$ is not weakly compact.

When investigating KH-integrals one meets immediately KH-equiintegrable sets of functions. We now formulate consequences of Theorems 3 and 4 in the case when $\mathcal{Z}_f$ satisfies some equiintegrability conditions.

LEMMA 5. *Let* $f:[0,1]\to X$ *be a scalarly KH-integrable function such that each infinite subset of* $\mathcal{Z}_f$ *contains an infinite countable KH-equiintegrable subset. Then on* $Z_f\subset\mathrm{KH}[0,1]$ *the weak topology* $\sigma(\mathrm{KH},\mathrm{BV})$ *coincides with* τ_{m}.

Proof. Let $(x^*_nf)_n\subset Z_f$ be $\sigma(\mathrm{KH},\mathrm{BV})$-convergent to $g\in\mathrm{KH}[0,1]$. According to Fremlin's subsequence theorem there is a subsequence $(x^*_{n_k}f)_k$

almost everywhere converging to a function h. We may assume that $(x^*_{n_k}f)_k$ is KH-equiintegrable. Since it is pointwise bounded, by Theorem 4 of [8] it converges to h in the Alexiewicz norm, and hence weakly. Consequently, $g = h$ a.e. and each subsequence of $(x^*_n f)_n$ contains a further subsequence that is a.e. convergent to g. Thus, $(x^*_n f)_n$ converges in measure to g.

According to Fremlin's subsequence theorem the set Z_f is $\tau_{\rm m}$-compact. Take now a sequence $(x^*_n f)_n \subset Z_f$ converging in measure to $g \in \mathrm{KH}[0,1]$. If $(x^*_{n_k}f)_k$ is almost everywhere converging to g, then clearly $g = x^* f$ for some $x^* \in B(X^*)$ and so $g \in Z_f$. Moreover $(x^*_{n_k}f)_k$ is pointwise bounded. Applying the equiintegrability of a subsequence $(x^*_{n_{k_p}}f)_p$, we see that $(x^*_{n_{k_p}}f)_p$ converges to g weakly in $\mathrm{KH}[0,1]$ (see Theorem 4 of [8]). It follows that each subsequence of $(x^*_n f)_n$ contains a subsequence weakly convergent to g and so $(x^*_n f)_n$ is weakly convergent to g. Consequently, Z_f is $\sigma(\mathrm{KH},\mathrm{BV})$-compact and sequential $\sigma(\mathrm{KH},\mathrm{BV})$-convergence on Z_f coincides with sequential $\tau_{\rm m}$-convergence. The coincidence of the two topologies on Z_f now follows exactly as in Proposition 2. This completes the proof. ∎

REMARK 3. In Lemma 5 one may weaken the assumptions, assuming that for each countable subset $\mathcal{H} \subset \mathcal{Z}_f$ there is a set N of measure zero such that the set $\{h\chi_{N^c} : h \in \mathcal{H}\}$ is KH-equiintegrable. As observed in [15] the concept of KH-equiintegrability, unlike the concept of uniform integrability, does not allow one to ignore sets of measure zero.

COROLLARY 2. *Let $f : [0,1] \to X$ be a scalarly KH-integrable function such that each infinite subset of $\mathcal{Z}_f$ contains a KH-equiintegrable infinite subset. If f satisfies* (DC) *or* (CC), *then $f \in \mathrm{KHP}(X,[0,1])$.*

For a Banach space with the Mazur property we have the following sufficient condition for KHP-integrability:

PROPOSITION 4. *If X has the property of Mazur (in particular when X is separable or weakly compactly generated), then f is KHP-integrable if and only if $f : [0,1] \to X$ is scalarly KH-integrable and each infinite subset of the collection $\{x^* f : \|x^*\| \le 1\}$ contains a KH-equiintegrable sequence.*

Proof. We first show that the linear function $a : X^* \to (-\infty,\infty)$ given by $a(x^*) := (\mathrm{KH})\int_0^1 x^* f(t)\,dt$ is w^*-continuous. So consider a sequence of functionals $x^*_n \in B(X^*)$. Assuming that $x^*_n \to x^*_0$ in the weak* topology, we get the pointwise convergence $x^*_n f(t) \to x^*_0 f(t)$. We may suppose that the sequence $(x^*_n f)$ is KH-equiintegrable; then we have (see [17])

$$a(x^*_0) = (\mathrm{KH})\int_0^1 x^*_0 f(t)\,dt = \lim_n\,(\mathrm{KH})\int_0^1 x^*_n f(t)\,dt = \lim_n a(x^*_n).$$

Consequently, a is w^*-continuous, and so there exists $x_0 \in X$ such that $a(x^*) = x^*(x_0)$. Thus, f is KHP-integrable.

The converse follows from Lemma 5, Theorem 3 and Remark 1. ∎

REMARK 4. If in the definition of the Kurzweil–Henstock–Pettis integral we replace the Kurzweil–Henstock integrability of scalar functions by the Denjoy–Khinchin integrability, we obtain the Denjoy–Pettis integral, introduced in [14] and studied in [14, 12]. Denote by $\mathcal{DK}[0,1]$ the set of all real-valued Denjoy–Khinchin integrable functions on $[0,1]$ and by $\mathrm{DK}[0,1]$ its quotient space. Then endow $\mathrm{DK}[0,1]$ with the Alexiewicz norm

$$\|g\|_A = \sup_{0<\alpha\le 1}\Big|(\mathrm{DK})\int_0^\alpha g(t)\,dt\Big|,$$

where $(\mathrm{DK})\int_0^\alpha$ stands for the Denjoy–Khinchin integral. Then the space $\mathrm{KH}[0,1]$ is dense in $\mathrm{DK}[0,1]$, the completion of $\mathrm{DK}[0,1]$ in the Alexiewicz norm coincides with the completion $\widehat{\mathrm{KH}}[0,1]$ of $\mathrm{KH}[0,1]$, and the conjugate space $\mathrm{DK}^*[0,1]$ is linearly isometric to $\mathrm{BV}[0,1]$ (cf. [4]). Moreover, the classical theorem on integration by parts also holds for the Denjoy–Khinchin integral (cf. [15, Theorem 15.14]). Using the above facts instead of the corresponding ones for the Kurzweil–Henstock integral, it is easy to see that Denjoy–Pettis versions of Theorems 1–4 also hold true. Clearly, we have to take into account that the range space of the operator T_f is, this time, $\mathrm{DK}[0,1]$.

5. Convergence theorems. It is the aim of this section to present a generalization of some convergence theorems proved in [5] and [8].

THEOREM 5. *Let $(f_n)_n$ be a sequence of KHP-integrable X-valued functions and let $f:[0,1]\to X$ be a function. Assume that the following conditions are satisfied:*

(a) *$x^*f_n \to x^*f$ a.e. for each $x^*\in X^*$ (the exceptional sets depend on x^*);*
(b) *the sequence $(f_n)_n$ is pointwise bounded;*
(c) *each countable subset of $\{x^*f_n : \|x^*\|\le 1,\ n\in\mathbb{N}\}$ is KH-equiintegrable.*

Then f is KHP-integrable and

$$\lim_n \|x^*f_n - x^*f\|_A = 0 \quad \textit{for each } x^*\in X^*.$$

Proof. We first prove that each sequence $\{x_m^*f : x_m^*\in B(X^*)\}$ is KH-equiintegrable. So let $\{x_m^*f : x_m^*\in B(X^*)\}$ be an arbitrary sequence. Then set $N_m=\{t\in[0,1]: x_m^*f_n(t)\not\to x_m^*f(t)\}$ and $N=\bigcup_{m=1}^\infty N_m$.

Fix an arbitrary $\varepsilon>0$. According to (c) and [15, p. 361], there is a gauge δ' such that if $\{(I_1,t_1),\dots,(I_p,t_p)\}$ is a δ'-fine partition of $[0,1]$, then for every $n, m \in \mathbb{N}$,

$$\Big|\sum_{i=1}^{p} x_m^* f_n(t_i)\chi_{N^c}(t_i)|I_i| - \int_0^1 x_m^* f_n(t)\,dt\Big| < \varepsilon.$$

Moreover, it follows from (a) that for each m there is $n_m \in \mathbb{N}$ such that for every $n \ge n_m$,

$$\Big|\sum_{i=1}^{p} x_m^* f_n(t_i)\chi_{N^c}(t_i)|I_i| - \sum_{i=1}^{p} x_m^* f(t_i)\chi_{N^c}(t_i)|I_i|\Big| < \varepsilon.$$

By (c) the sequence $(x_m^* f_n)_n$ is equiintegrable and by (b) it is pointwise bounded. Then, by Theorem 4 of [8], there is $k_m \ge n_m$ such that for every $n \ge k_m$,

$$\Big|\int_0^1 x_m^* f_n(t)\,dt - \int_0^1 x_m^* f(t)\,dt\Big| < \varepsilon.$$

Consequently, for each m,

$$\begin{aligned}
&\Big|\sum_{i=1}^{p} x_m^* f(t_i)\chi_{N^c}(t_i)|I_i| - \int_0^1 x_{\mathrm{m}}^* f(t)\,dt\Big| \\
&\qquad \le \Big|\sum_{i=1}^{p} x_m^* f(t_i)\chi_{N^c}(t_i)|I_i| - \sum_{i=1}^{p} x_m^* f_{k_m}(t_i)\chi_{N^c}(t_i)|I_i|\Big| \\
&\qquad\quad + \Big|\sum_{i=1}^{p} x_m^* f_{k_m}(t_i)\chi_{N^c}(t_i)|I_i| - \int_0^1 x_m^* f_{k_{\mathrm{m}}}(t)\,dt\Big| \\
&\qquad\quad + \Big|\int_0^1 x_m^* f_{k_m}(t)\,dt - \int_0^1 x_m^* f(t)\,dt\Big| < 3\varepsilon.
\end{aligned}$$

But according to [8] there is a gauge δ'' on N such that if $\{(I_1,t_1),\dots,(I_p,t_p)\}$ is a δ''-fine partition anchored in N, then $\sup_m \sum_{i=1}^{p} x_m^* |f(t_i)|\,|I_i| < \varepsilon$. So if $\delta(t) := \delta'(t)\chi_{N^c}(t) + \delta''(t)\chi_N(t)$, then for each partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ of $[0,1]$ that is δ-fine, we have

$$\Big|\sum_{i=1}^{p} x_m^* f(t_i)|I_i| - \int_0^1 x_m^* f(t)\,dt\Big| < 4\varepsilon,$$

which proves the equiintegrability of the sequence $(x_m^* f)_m$.

It follows from Lemma 5 that on the set Z_f the topologies $\sigma(\mathrm{KH},\mathrm{BV})$ and τ_{m} coincide. Thus condition (TC) of Theorem 3 is satisfied. In order to prove the KHP-integrability of f we need to show yet the existence of a weakly compactly generated space $Y \subset X$ such that $x^* f = 0$ a.e. if $x^* \in Y^{\perp}$. But as each f_n is in $\mathrm{KHP}([0,1],X)$ there is a weakly compact set $W_n \subset B(X)$

such that $x^* f_n = 0$ a.e. if $x^* \in Y_n^\perp$, where Y_n is the Banach space generated by W_n. Consequently, if $W = \bigcup_{n=1}^{\infty} 2^{-n} W_n$, then W is weakly compact and the Banach space Y generated by W has the required property. Thus, f is KHP-integrable.

Since each sequence $(x^* f_n)_n$ is equiintegrable and pointwise bounded, applying Theorem 4 of [8] we get

$$\lim_n \|x^* f_n - x^* f\|_A = 0 \quad \text{for each } x^* \in X^*. \ \blacksquare$$

REMARK 5. A more careful analysis of the above proof shows that one can weaken condition (c) to: each countable set $\{x^*_{m_k} f_{m_k} : k \in \mathbb{N}, \|x^*_{m_k}\| \le 1\}$ contains a subset $\{x^*_{m_{k_l}} f_{m_{k_l}} : l \in \mathbb{N}, \|x^*_{m_{k_l}}\| \le 1\}$ that is KH-equiintegrable.

REMARK 6. We recall that the concept of KH-equiintegrability, unlike uniform integrability, does not allow one to ignore sets of measure zero (see e.g. [15]). As in condition (a) of Theorem 5 the assumption of "pointwise convergence" has been relaxed, we had to assume the pointwise boundedness of the functions. As in Lemma 5 we might have weakened (c), assuming that one can choose countable subsets KH-equiintegrable on a set of full measure.

It is known that a sequence of real-valued pointwise convergent functions is KH-equiintegrable if and only if the sequence of their primitives is uniformly $ACG^*_{(s)}$ (see e.g. [17], [3]). For the definition of the uniform $ACG^*_{(s)}$ condition we refer to [4]. So if we observe that under the assumption of the uniform $ACG^*_{(s)}$ of the primitives the notion of KH-equiintegrability is not altered by possibly changing the values of a function on a null set, then the proof of the next theorem follows, after suitable changes, as for Theorem 1 in [5]. In any case we note that in the proof of Theorem 1 in [5] a few important details have been neglected or overlooked.

THEOREM 6. *Let $(f_n)_n$ be a sequence of KHP-integrable X-valued functions and let $f : [0,1] \to X$ be a function. Assume that:*

(α) *$x^* f_n \to x^* f$ a.e. for each $x^* \in X^*$ (the exceptional sets depend on x^*);*

(β) *each countable subset of $\{x^* F_n : \|x^*\| \le 1, n \in \mathbb{N}\}$ is uniformly $ACG^*_{(s)}$, where F_n's are the KHP-primitives of f_n's.*

Then f is KHP-integrable and

$$\lim_n \|x^* f_n - x^* f\|_A = 0 \quad \textit{for each } x^* \in X^*.$$

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