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Proper identities, Lie identities and exponential codimension growth ☆

Antonio Giambruno [a,*], Mikhail Zaicev [b]

[a] *Dipartimento di Matematica e Applicazioni, Università di Palermo, Via Archirafi 34, 90123 Palermo, Italy*
[b] *Department of Algebra, Faculty of Mathematics and Mechanics, Moscow State University, Moscow 119992, Russia*

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Abstract

The exponent $\exp(A)$ of a PI-algebra A in characteristic zero is an integer and measures the exponential rate of growth of the sequence of codimensions of A [A. Giambruno, M. Zaicev, On codimension growth of finitely generated associative algebras, Adv. Math. 140 (1998) 145–155; A. Giambruno, M. Zaicev, Exponential codimension growth of P.I. algebras: An exact estimate, Adv. Math. 142 (1999) 221–243]. In this paper we study the exponential rate of growth of the sequences of proper codimensions and Lie codimensions of an associative PI-algebra. We prove that the corresponding proper exponent exists for all PI-algebras, except for some algebras of exponent two strictly related to the Grassmann algebra. We also prove that the Lie exponent exists for any finitely generated PI-algebra. The value of both exponents is always equal to $\exp(A)$ or $\exp(A) - 1$.

Keywords: Polynomial identities; Codimension growth

1. Introduction

Let A be an associative PI-algebra over a field F of characteristic zero, $F\langle X\rangle$ the free algebra on a countable set $X = \{x_1, x_2, \ldots\}$ and $\mathrm{Id}(A)$ the T-ideal of $F\langle X\rangle$ of polynomial identities

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* Corresponding author.
E-mail addresses: agiambr@unipa.it (A. Giambruno), zaicev@mech.math.msu.su (M. Zaicev).

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of A. Since in characteristic zero all identities of A are consequences of the multilinear ones, an effective way of measuring the identities satisfied by A is provided by the sequence of codimensions $c_n(A)$, $n = 1, 2, \ldots$. Recall that if P_n is the space of multilinear polynomials in $x_1, \ldots, x_n$ then $c_n(A) = \dim_F P_n/(P_n \cap \mathrm{Id}(A))$. It is well known that since A satisfies a non-trivial identity, this sequence is exponentially bounded [18], i.e., $c_n(A) \leqslant d^n$, for some real number d. Moreover it was recently shown that

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exists and is an integer called the exponent of the PI-algebra A [8,9].

Two more numerical sequences can be naturally associated to a T-ideal, the sequence of proper codimensions and the sequence of Lie codimensions. They are defined as follows.

If we consider A as a Lie algebra under the Lie bracket $[a, b] = ab - ba$, we can study its Lie polynomial identities; if V_n denotes the space of multilinear Lie polynomials in the first n variables, then $c_n^L(A) = \dim V_n/(V_n \cap \mathrm{Id}(A))$, $n = 1, 2, \ldots$, is the sequence of Lie codimensions of A.

On the other hand, recall that a polynomial is proper if it is a linear combination of products of (long) Lie commutators; the relevance of these polynomials is strengthened by the property that any T-ideal of identities of an algebra with 1 can be generated by its proper polynomials [4, Proposition 4.3.3]. If Γ_n denotes the space of multilinear proper polynomials in the first n variables, then $c_n^p(A) = \dim \Gamma_n/(\Gamma_n \cap \mathrm{Id}(A))$, $n = 1, 2, \ldots$, is the sequence of proper codimensions of A.

It is well known (see for instance [7]) that if A is any algebra, the above three sequences are related by the inequalities

$$c_n^L(A) \leqslant c_n^p(A) \leqslant c_n(A) \tag{1}$$

for all $n \geqslant 1$. Now, for any exponentially bounded non-negative sequence α_n, $n = 1, 2, \ldots$, one can construct the bounded sequence $\sqrt[n]{\alpha_n}$, $n = 1, 2, \ldots$, and compute its upper and lower limit. Hence, since for any PI-algebra A, $c_n(A)$, $n = 1, 2, \ldots$, is exponentially bounded, we define the following real numbers:

$$\underline{\exp}^L(A) = \liminf_{n\to\infty} \sqrt[n]{c_n^L(A)}, \qquad \overline{\exp}^L(A) = \limsup_{n\to\infty} \sqrt[n]{c_n^L(A)},$$

$$\underline{\exp}^p(A) = \liminf_{n\to\infty} \sqrt[n]{c_n^p(A)}, \qquad \overline{\exp}^p(A) = \limsup_{n\to\infty} \sqrt[n]{c_n^p(A)},$$

called the lower and upper Lie exponent and the lower and upper proper exponent of the algebra A, respectively. In case of equality, $\underline{\exp}^L(A) = \overline{\exp}^L(A) = \exp^L(A)$ will be called the Lie exponent of A. Similarly $\underline{\exp}^p(A) = \overline{\exp}^p(A) = \exp^p(A)$ will be the proper exponent of A.

It is an open question if the proper exponent and the Lie exponent exist for an arbitrary PI-algebra. About positive results, in [20] it was proved that for any finite dimensional Lie algebra $\exp^L(A)$ exists and is an integer. On the other hand it is known that in case of PI-Lie algebras, the sequence of codimensions can have an overexponential growth (see [17]). Even if the sequence of Lie codimensions is exponentially bounded, the exponential rate of growth can be non-integer for infinite dimensional finitely generated algebras [16].

In this paper we shall prove that if A is a finitely generated PI-algebra, $\exp^L(A)$ and $\exp^p(A)$ both exist and coincide with $\exp(A)$ or $\exp(A)-1$.

For infinitely generated PI-algebras the same result will be proved for a wide class of algebras and it will be shown that it does not hold in general. In fact, it can be shown that if G is the infinite dimensional Grassmann algebra, B a finite dimensional algebra with Jacobson radical of codimension 1 and N a nilpotent algebra, then any algebra of the type $A = G \oplus B \oplus N$ is such that $\exp^p(A)$ does not exist.

Here we shall prove that if A is a PI-algebra with $\exp(A) = d$, then $\exp^p(A)$ exists, is an integer and $\exp^p(A) = d$ or $d-1$, unless $\exp(A) = 2$ and A is an algebra of the type above.

2. Finitely generated algebras

Throughout F will be a field of characteristic zero and A an F-algebra satisfying a non-trivial polynomial identity.

Let $\bar F$ be the algebraic closure of F. If we regard $\bar A = A \otimes_F \bar F$ as an algebra over $\bar F$, it can be easily shown that $c_n(A) = c_n(\bar A)$ (see [8] or [12]). The same conclusion holds for $c_n^L(A)$ and $c_n^p(A)$. Therefore for our purpose we may assume that F is algebraically closed.

We start by observing that since $\exp(A)$ exists, then from (1) we obtain

$$\overline{\exp}^L(A) \leqslant \overline{\exp}^p(A) \leqslant \exp(A). \tag{2}$$

Also, in [1] it was shown that if A is any PI-algebra with 1, then

$$\overline{\exp}^p(A) = \exp(A) - 1. \tag{3}$$

In this section we shall prove the existence of $\exp^L(A)$ and $\exp^p(A)$ for any finitely generated PI-algebra A. We shall also find their precise value.

We start by remarking that if $\exp(A) \leqslant 1$ then $c_n(A)$ is polynomially bounded. In this case the proper exponent and the Lie exponent always exist. In fact we have the following

Proposition 1. *Let A be a PI-algebra whose codimensions are polynomially bounded. Then the Lie and the proper exponent of A exist and are integers. If $\exp(A) = 0$, i.e., A is nilpotent then $\exp^L(A) = \exp^p(A) = 0$. If $\exp(A) = 1$ then $\exp^L(A) = \exp^p(A) = 0$ or 1.*

Proof. For a nilpotent algebra the statement is obvious. Let $\exp(A) = 1$. Since $c_n^p(A) \leqslant c_n(A)$ then

$$\limsup_{n\to\infty} \sqrt[n]{c_n^p(A)} \leqslant 1.$$

Hence it is enough to check that for n large enough, either $c_n^p(A) = 0$ always holds or $c_n^p(A), c_n^L(A) > 0$ always holds.

Recall that by [12, Theorem 7.2.12], A has the same identities as a direct sum $B_1 \oplus \cdots \oplus B_m$ where the B_i's are finite dimensional algebras with Jacobson radical of codimension at most one.

If $B_1, \ldots, B_m$ are all nilpotent, then $\exp(B_1 \oplus \cdots \oplus B_m) = 0$, a contradiction. Let B be a finite dimensional algebra with Jacobson radical J and $\dim B/J = 1$. Suppose that B is an algebra with 1. Since any Lie monomial vanishes if we replace at least one variable with 1, then $c_n^L(B) =$

$c_n^p(B) = 0$ for all $n \geqslant N$ where $J^N = 0$. Let now B be a non-unitary algebra. Then $B = F + J$ and we consider the left and right multiplication by the unit $1 \in F$ on J. As a vector space J can be decomposed into the sum

$$J = J_{11} \oplus J_{10} \oplus J_{01} \oplus J_{00},$$

where $1 \cdot a = 0$ if $a \in J_{01} + J_{00}$ and $1 \cdot a = a$ if $a \in J_{10} + J_{11}$. Similarly $a \cdot 1 = 0$ if $a \in J_{10} + J_{00}$ and $a \cdot 1 = a$ if $a \in J_{01} + J_{11}$.

If $0 \neq a \in J_{01}$ then the left-normed commutator $[a, 1, \ldots, 1] = a$ is non-zero, hence $[x_1, \ldots, x_n]$ is not an identity of B. Similarly, if $0 \neq a \in J_{10}$ then $[a, 1, \ldots, 1] = \pm a \neq 0$, and $[x_1, \ldots, x_n]$ is again a non-identity of B. In other words, if $J_{10} \oplus J_{01} \neq 0$ then $c_n^p(B) \geqslant c_n^L(B) \geqslant 1$ for all $n \geqslant 2$. But in case $J_{10} \oplus J_{01} = 0$ we have the decomposition $B = C \oplus J_{00}$, the sum of two two-sided ideals where J_{00} is nilpotent and $C = F + J_{11}$ is a unitary algebra with 1-dimensional maximal semisimple subalgebra. In this case as before $c_n^L(B) = c_n^p(B) = 0$ for all $n \geqslant N$ where $J^N = 0$.

In conclusion, we have that either $c_n^L(B_1 \oplus \cdots \oplus B_m) = c_n^p(B_1 \oplus \cdots \oplus B_m) = 0$ for all n large enough or $c_n^p(B_j) \geqslant c_n^L(B_j) \geqslant 1$ for some subalgebra B_j, and in the latter case $\exp^L(B_1 \oplus \cdots \oplus B_m) = \exp^p(B_1 \oplus \cdots \oplus B_m) = 1$. □

At the light of the previous proposition we may assume that the PI-algebra A is such that $\exp(A) > 1$. Recall that by a result of Kemer [14], if A is finitely generated, there exists a finite dimensional algebra D such that $\mathrm{Id}(A) = \mathrm{Id}(D)$. In particular A and D have the same proper identities and the same Lie identities. Therefore without lost of generality, we may assume that A is a finite dimensional algebra.

But then, since $\exp(A) > 1$, by [10] (see also [12]) there exists a subalgebra B of A such that $B \cong UT(t_1, \ldots, t_k)$, an upper block matrix algebra. Recall that

$$UT(t_1, \ldots, t_k) = \begin{pmatrix} A_1 & Q_{12} & \cdots & Q_{1k} \\ & A_2 & \ddots & \vdots \\ & & \ddots & Q_{k-1,k} \\ 0 & & & A_k \end{pmatrix}, \tag{4}$$

where, for every $1 \leqslant i \leqslant k$, $A_i = M_{t_i}(F)$ is the algebra of $t_i \times t_i$ matrices over F, and the Q_{ij} are block matrices over F of corresponding size. Moreover $\exp(A) = \exp(B) = t = t_1^2 + \cdots + t_k^2$.

As we mentioned in the introduction, in [20] it was proved that for any finite dimensional Lie algebra C, $\exp^L(C)$ exists and is an integer. We next describe how the Lie exponent can be computed.

By the Ado–Ivasawa theorem the Lie algebra C decomposes as

$$C = L + R,$$

where L is a maximal semisimple subalgebra of C and R is the solvable radical. Let $I_1, J_1, \ldots, I_q, J_q$ be Lie ideals of C. We say that $I_1, J_1, \ldots, I_q, J_q$ satisfy the condition $(*)$ if

- $J_r \subseteq I_r$ and I_r/J_r is a non-zero irreducible C-module in the adjoint representation, for any $r = 1, \dots, q$;
- for any irreducible L-submodules $P_1, \dots, P_q$ such that $I_r = J_r \oplus P_r$, $r = 1, \dots, q$, there exist $p_1, \dots, p_q \geqslant 0$ such that

$$\big[[P_1, \underbrace{L, \dots, L}_{p_1}], \dots, [P_q, \underbrace{L, \dots, L}_{p_q}]\big] \neq 0.$$

Then according to [20], the Lie exponent of C is equal to

$$\max\left\{\dim \frac{C}{\mathrm{Ann}_C(\frac{I_1}{J_1}) \cap \dots \cap \mathrm{Ann}_C(\frac{I_q}{J_q})}\right\},$$

where $\mathrm{Ann}_C(\frac{I}{J}) = \{a \in C \mid [a, I] \subseteq J\}$ and we consider only ideals $I_1, J_1, \dots, I_q, J_q$ satisfying the condition $(*)$.

We now apply the above to the algebra B, considered as a Lie algebra.

If $k = 1$, then B is isomorphic to the algebra $M_{t_1}(F)$ and $\exp^L(B) = t_1^2 - 1 = t - 1$ (see [7]). In fact, as a Lie algebra, we can decompose $M_{t_1}(F) = sl_n(F) \oplus Z$, where Z is the 1-dimensional center of B. Hence

$$\exp^L(A) \geqslant \exp^L(B) = t - 1. \tag{5}$$

Now let $k \geqslant 2$ and identify the algebra B with the algebra $UT(t_1, \dots, t_k)$ given in (4). For $r = 1, \dots, k-1$, set

$$I_r = \bigoplus_{\substack{r+1 \leqslant \beta \leqslant k \\ 1 \leqslant \alpha \leqslant r}} Q_{\alpha\beta}, \qquad J_r = \bigoplus_{\substack{r+1 \leqslant \beta \leqslant k \\ (\alpha,\beta) \neq (r,r+1) \\ 1 \leqslant \alpha \leqslant r}} Q_{\alpha\beta}.$$

Then $I_1, J_1, \dots, I_{k-1}, J_{k-1}$ are Lie ideals of B and for $r = 1, \dots, k-1$, we have the decomposition $I_r = J_r \oplus Q_{r,r+1}$ with $Q_{r,r+1}$ an irreducible B-module.

If we set $Q = \sum_{1 \leqslant i < j \leqslant k} Q_{ij}$, then clearly $[Q, I_r] \subseteq J_r$ and this says that Q annihilates I_r/J_r, for all $r = 1, \dots, k-1$. It follows that

$$\mathrm{Ann}_B\left(\frac{I_1}{J_1}\right) \cap \dots \cap \mathrm{Ann}_B\left(\frac{I_{k-1}}{J_{k-1}}\right) = Q \oplus \mathrm{Ann}_{A_1 \oplus \dots \oplus A_k}(Q_{12} \oplus \dots \oplus Q_{k-1,k}). \tag{6}$$

The annihilator $\mathrm{Ann}_{A_1 \oplus \dots \oplus A_k}(Q_{12} \oplus \dots \oplus Q_{k-1,k})$ is the centralizer of $Q_{12} \oplus \dots \oplus Q_{k-1,k}$ in $A_1 \oplus \dots \oplus A_k$, i.e., the space of scalar matrices. Therefore the codimension of $\mathrm{Ann}_B(\frac{I_1}{J_1}) \cap \dots \cap \mathrm{Ann}_B(\frac{I_{k-1}}{J_{k-1}})$ in B equals $\dim(A_1 \oplus \dots \oplus A_k) - 1 = t - 1$ and we obtain that

$$\exp^L(B) \geqslant t - 1. \tag{7}$$

If we now assume that A is an algebra with 1 then, by combining (3), (5) and (7) we get

$$t-1 \leqslant \exp^L(A) = \underline{\exp}^L(A) \leqslant \underline{\exp}^p(A) \leqslant \overline{\exp}^p(A) \leqslant \exp(A) - 1 = t-1.$$

Hence $\exp^p(A) = \exp^L(A) = t-1$ and this proves the following

Proposition 2. *Let A be a finite dimensional algebra with 1. Then $\exp^p(A)$ and $\exp^L(A)$ exist and are integers. Moreover*

$$\exp^L(A) = \exp^p(A) = \exp(A) - 1.$$

Given an algebra B without a unit element, we denote by $B^\sharp$ the algebra obtained from B by adjoining a unit element. We have the following

Lemma 1. *For a finite dimensional algebra B we have $\exp(B^\sharp) = \exp(B) + 1$ or $\exp(B)$.*

Proof. Let $B = B_1 \oplus \cdots \oplus B_m + J$ be the Wedderburn–Malcev decomposition of the algebra B, where $B_1 \oplus \cdots \oplus B_m$ is a maximal semisimple subalgebra with simple summands $B_1, \ldots, B_m$, and J is the Jacobson radical of B. Then by [8],

$$\exp(B) = \max \dim(B_{r_1} \oplus \cdots \oplus B_{r_k}),$$

where $B_{r_1}, \ldots, B_{r_k} \in \{B_1, \ldots, B_m\}$ are distinct and satisfy $B_{r_1} J B_{r_2} J \cdots J B_{r_k} \neq 0$.

It is easily seen that $B_1 \oplus \cdots \oplus B_m \oplus F$ is a maximal semisimple subalgebra of $B^\sharp$. Hence it follows that $\exp(B^\sharp)$ equals either $\exp(B)$ or $\exp(B) + 1$. $\square$

We can now prove the existence of the Lie exponent and of the proper exponent for any finitely generated PI-algebra.

Theorem 1. *Let A be a finitely generated PI-algebra. Then*

1) $\exp^L(A)$ *and* $\exp^p(A)$ *exist and are integers*;
2) $\exp^L(A) = \exp^p(A) = \exp(A)$ *or* $\exp(A) - 1$;
3) *if A is a unitary algebra, then* $\exp^L(A) = \exp^p(A) = \exp(A) - 1$.

Proof. As we mentioned at the beginning of the section we may assume that F is algebraically closed. Moreover there exists a finite dimensional algebra B such that $\mathrm{Id}(A) = \mathrm{Id}(B)$. If B is a unitary algebra, then the conclusion of the theorem follows from the previous proposition.

Suppose now that B is not a unitary algebra and consider $B^\sharp$, the algebra obtained by adjoining a unit element to B. Clearly B and $B^\sharp$ have the same Lie and proper identities. Hence $c_n^L(B) = c_n^L(B^\sharp)$ and $c_n^p(B) = c_n^p(B^\sharp)$. But then by Proposition 2, $\exp^L(B)$ and $\exp^p(B)$ exist and also $\exp^L(B) = \exp^p(B) = \exp(B^\sharp) - 1$. By Lemma 1 we then obtain that

$$\exp^L(B) = \exp^p(B) = \exp(B) \text{ or } \exp(B) - 1.$$

In case A is a finitely generated PI-algebra with 1, then the conclusion 3) of the theorem follows from (3). $\square$

We close this section with some examples showing that the equalities $\exp^p(A) = \exp(A)$ and $\exp^p(A) = \exp(A) - 1$ can actually occur.

Consider first the algebra $A = M_k(F)$ of $k \times k$ matrices over F. Then by [12], $\exp(A) = k^2$ and $\exp^L(A) = k^2 - 1$ [7]. Hence $\exp^L(A) = \exp^p(A) = \exp(A) - 1$. This conclusion also follows from Theorem 1.

Example 1. Let G be the infinite dimensional Grassmann algebra over F. Recall that G is the algebra generated by the elements $e_1, e_2, \ldots,$ subject to the conditions $e_i e_j + e_j e_i = 0$ for $i, j \geqslant 1$. If

$$A = \begin{pmatrix} G & G \\ 0 & 0 \end{pmatrix},$$

then by [1], $\exp^L(A) = \exp(A) = 2$. Hence also $\exp^p(A) = 2$.

We next give an example of a finite dimensional algebra A with the property $\exp^p(A) = \exp(A)$.

Example 2. We let A be the algebra of $(k + 1) \times (k + 1)$ matrices with zero last row. Hence

$$A = \left\{ \begin{pmatrix} a_{11} & \ldots & a_{1,k+1} \\ \vdots & & \vdots \\ a_{k1} & \ldots & a_{k,k+1} \\ 0 & \ldots & 0 \end{pmatrix} \,\middle|\, a_{ij} \in F \right\}.$$

By [20], $\exp^L(A) = k^2$ and by [9] $\exp(A) = k^2$. Hence also $\exp^p(A) = k^2$.

3. Hook partitions

In order to study the infinite dimensional case, we shall make use of a standard approach based on the action of the symmetric group on multilinear polynomials (see for instance [12, Chapter 2]). In this section we shall obtain some technical results concerning the dimensions of the irreducible modules for the symmetric group. We refer the reader to [13] for the representation theory of the symmetric group.

Let S_n be the symmetric group acting on $1, \ldots, n$. Since S_n is finite and $\operatorname{char} F = 0$, any finite dimensional representation of S_n is completely reducible and any irreducible representation corresponds, up to isomorphism, to a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n. Recall that, given a partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, one can consider the corresponding Young diagram D_λ and, if we fill up the boxes of D_λ with the integers $1, \ldots, n$, we get a Young tableau T_λ of shape λ.

Recall also that, given a tableau T_λ, if R_{T_λ} and C_{T_λ} denote the row-stabilizer and column-stabilizer subgroups of S_n, then the element

$$e_{T_\lambda} = \sum_{\substack{\sigma \in R_{T_\lambda} \\ \tau \in C_{T_\lambda}}} (\operatorname{sgn} \tau) \sigma \tau$$

is an essential idempotent of the group algebra FS_n i.e., $e_{T_\lambda}^2 = \alpha e_{T_\lambda}$, for some $0 \neq \alpha \in F$, and $FS_n e_{T_\lambda}$ is an irreducible left FS_n-module. Moreover, if T_λ and T_μ are two Young tableaux, then $FS_n e_{T_\lambda}$ is isomorphic to $FS_n e_{T_\mu}$ if and only if T_λ and T_μ are of the same shape i.e., $\lambda = \mu$.

For a partition $\lambda \vdash n$ we denote by λ' the conjugate partition of λ. Recall that $\lambda' = (\lambda'_1, \ldots, \lambda'_s) \vdash n$, where $\lambda'_1, \ldots, \lambda'_s$ are the lengths of the columns of D_λ. Thus the diagram $D_{\lambda'}$ is obtained from D_λ by flipping D_λ along its main diagonal.

Given a finite dimensional S_n-module M, we denote its character by $\chi(M)$. In case $M \cong FS_n e_{T_\lambda}$ we write $\chi(M) = \chi_\lambda$. Hence, any character $\chi(M)$ has a decomposition of the type

$$\chi(M) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where the integers m_λ are called the multiplicities of χ_λ in $\chi(M)$. In particular, the dimension $\dim M = \deg \chi(M)$ equals

$$\deg \chi(M) = \sum_{\lambda \vdash n} m_\lambda d_\lambda,$$

where $d_\lambda = \deg \chi_\lambda$ is the dimension of the irreducible S_n-module corresponding to $\lambda \vdash n$.

We also recall the hook formula for the dimension d_λ of the irreducible S_n-module with character χ_λ. Let (i, j) be the box of D_λ lying at the intersection of the ith row and the jth column of D_λ. We define the hook number $h_{ij}(\lambda)$ of (i, j) as $h_{ij}(\lambda) = \lambda_i + \lambda'_j - i - j + 1$, where λ' is the conjugate partition of λ. Then

$$d_\lambda = \frac{n!}{\prod_{i,j} h_{ij}(\lambda)}.$$

Below we shall use the notation $H(k, l)$ for the so-called infinite hook with k infinite rows and l infinite columns. Actually $H(k, l)$ is the union of all partitions λ whose $(k + 1)$th row is of length at most l, i.e.,

$$H(k, l) = \bigcup_{n \geqslant 1} \left\{ \lambda = (\lambda_1, \lambda_2, \ldots) \vdash n \;\middle|\; \lambda_{k+1} \leqslant l \right\}.$$

Although we shall consider only infinitely generated PI-algebras, for convenience we allow l to be zero in $H(k, l)$. In this case $H(k, 0)$ is an infinite (horizontal) strip of height $k > 0$.

In the next lemma, in order to compute the degrees d_λ, we shall estimate the product of all hook numbers.

Lemma 2. *Let $n > 100$ and let $\mu = (\mu_1, \ldots, \mu_k) \vdash n$. Then the product $\prod_{i,j} h_{ij}(\mu)$ of the hook numbers of μ satisfies*

$$\frac{{\mu_1}^{\mu_1} \cdots {\mu_k}^{\mu_k}}{e^n} \leqslant \prod_{i,j} h_{ij}(\mu) \leqslant \frac{{\mu_1}^{\mu_1} \cdots {\mu_k}^{\mu_k}}{e^n} n^{k^2+k}.$$

Proof. The product of the hook numbers of the ith row of μ satisfies

$$\mu_i! \leqslant \prod_{j=1}^{\mu_i} h_{ij}(\mu) \leqslant k(k+1)\cdots(k+\mu_i-1) \leqslant (\mu_i+k)! \leqslant n^k \mu_i!.$$

Hence

$$\mu_1!\cdots\mu_k! \leqslant \prod_{i,j} h_{ij}(\mu) \leqslant n^{k^2} \mu_1!\cdots\mu_k!. \tag{8}$$

Recall now that by Stirling formula (see [19])

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{\theta_n}{12n}}$$

for some $0 \leqslant \theta_n \leqslant 1$. It follows that for $n \geqslant 100$,

$$\left(\frac{n}{e}\right)^n \leqslant n! \leqslant n\left(\frac{n}{e}\right)^n.$$

Therefore

$$\left(\frac{\mu_1}{e}\right)^{\mu_1}\cdots\left(\frac{\mu_k}{e}\right)^{\mu_k} \leqslant \mu_1!\cdots\mu_k! \leqslant n^k \left(\frac{\mu_1}{e}\right)^{\mu_1}\cdots\left(\frac{\mu_k}{e}\right)^{\mu_k}.$$

If we write

$$\left(\frac{\mu_1}{e}\right)^{\mu_1}\cdots\left(\frac{\mu_k}{e}\right)^{\mu_k} = \frac{{\mu_1}^{\mu_1}\cdots{\mu_k}^{\mu_k}}{e^n},$$

we obtain that

$$\frac{{\mu_1}^{\mu_1}\cdots{\mu_k}^{\mu_k}}{e^n} \leqslant \mu_1!\cdots\mu_k! \leqslant n^k \frac{{\mu_1}^{\mu_1}\cdots{\mu_k}^{\mu_k}}{e^n}.$$

We now apply the above inequalities to (8) and we obtain the desired result. □

In order to estimate the dimension of an irreducible S_n-module it is convenient to introduce the following function.

Definition 1. Let $m \geqslant 2$ and let

$$\Phi(x_1,\ldots,x_m) = \frac{1}{x_1^{x_1}\cdots x_m^{x_m}}$$

be the function defined for all real numbers $x_1,\ldots,x_m$, $0 < x_1,\ldots,x_m < 1$, such that $x_1 + \cdots + x_m = 1$.

Remark 1. The function Φ is continuous and has only one maximum for $x_1 = \cdots = x_m = \frac{1}{m}$ where $\Phi(\frac{1}{m}, \ldots, \frac{1}{m}) = m$. If $x_m \leqslant x_1, \ldots, x_{m-1}$ and its value $x_m = a$ is fixed then $\Phi(x_1, \ldots, x_m)$ takes the maximal value when $x_1 = \cdots = x_{m-1}$ and takes the minimal value when $x_2 = \cdots = x_m = a$.

Lemma 3. *Let $x_m \leqslant x_1, \ldots, x_{m-1}$. If $x_m \leqslant \frac{\alpha}{m}$ for some $\alpha < 1$ then there exists $\delta > 0$ such that $\Phi(x_1, \ldots, x_m) < m - \delta$. If $x_m \geqslant \frac{1}{2m}$ then $\Phi(x_1, \ldots, x_m) \geqslant \sqrt{m}$.*

Proof. First we prove the upper bound in case $x_m = \frac{\alpha}{m}$. Denote $\varepsilon = \frac{\alpha}{m}$. By Remark 1 we have

$$\Phi(x_1, \ldots, x_m) \leqslant \Phi\left(\frac{1-\varepsilon}{m-1}, \ldots, \frac{1-\varepsilon}{m-1}, \varepsilon\right) = \left(\frac{m-1}{1-\varepsilon}\right)^{1-\varepsilon} \cdot \frac{1}{\varepsilon^{\varepsilon}} = A.$$

Since $1 - \varepsilon = \frac{m-\alpha}{m}$, we obtain

$$A = \left(\frac{m-1}{m-\alpha}\right)^{\frac{m-\alpha}{m}} \cdot m^{\frac{m-\alpha}{m}} \left(\frac{m}{\alpha}\right)^{\frac{\alpha}{m}} = b \cdot m,$$

where

$$b = \left(\frac{m-1}{m-\alpha}\right)^{\frac{m-\alpha}{m}} \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{m}}.$$

Consider the function

$$g(x) = \left(\frac{m-1}{m-x}\right)^{\frac{m-x}{m}} \left(\frac{1}{x}\right)^{\frac{x}{m}}.$$

It is easy to observe that its derivative is positive if $x \leqslant 1$ and $g(1) = 1$. Hence $b = g(\alpha) < 1$. It follows that $bm \leqslant m - \delta$ for some $\delta > 0$ and we are done in case $x_m = \frac{\alpha}{m}$.

Suppose now that $x_m < \frac{\alpha}{m}$. Then $x_m = \frac{\beta}{m}$ where $\beta < \alpha$ and as before $\Phi(x_1, \ldots, x_m) \leqslant cm$ where $c = g(\beta)$. Since $g'(x) > 0$ for $x \leqslant 1$ and $\beta < \alpha$ then

$$\Phi\left(x_1, \ldots, x_{m-1}, \frac{\alpha}{m}\right) \leqslant \Phi\left(x_1, \ldots, x_{m-1}, \frac{\beta}{m}\right) \leqslant m - \delta$$

and we have proved the upper bound.

Let now $x_m = \frac{1}{2m}$. Then by Remark 1

$$\Phi(x_1, \ldots, x_m) \geqslant \Phi\left(\frac{m+1}{2m}, \frac{1}{2m}, \ldots, \frac{1}{2m}\right) = (2m)^{\frac{m-1}{2m}} \left(\frac{2m}{m+1}\right)^{\frac{m+1}{2m}}$$
$$= 2m\left(\frac{1}{m+1}\right)^{\frac{m+1}{2m}} > \frac{2m}{\sqrt{m+1}} > \sqrt{m}$$

and we have proved the second inequality for $x_m = \frac{1}{2m}$.

Suppose now that $x_m = \beta > \frac{1}{2m}$. Then again by Remark 1

$$\Phi(x_1, \dots, x_m) \geqslant \Phi\bigl(1-(m-1)\beta, \beta, \dots, \beta\bigr) = \frac{1}{(1-(m-1)\beta)^{1-(m-1)\beta}\beta^{(m-1)\beta}}.$$

One can easily check that the function

$$f(x) = \frac{1}{(1-kx)^{1-kx}x^{kx}}$$

has positive derivative if $x < 1-kx$. Since $\beta < 1-(m-1)\beta$ and

$$\frac{1}{2m} < 1-(m-1)\frac{1}{2m} = \frac{m+1}{2m}$$

we obtain

$$\Phi(x_1, \dots, x_{m-1}, \beta) \geqslant \Phi\left(x_1, \dots, x_{m-1}, \frac{1}{2m}\right) \geqslant \sqrt{m}. \qquad \square$$

Let now $\lambda \vdash n$, $\lambda \in H(k,l)$, $k+l \geqslant 2$, and let λ' be the conjugate partition of λ. We assume that $\lambda_1, \dots, \lambda_k > l$ and $\lambda'_1, \dots, \lambda'_l > k$. Define $\bar{n} = n - kl$,

$$\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_k), \quad \text{and} \quad \bar{\lambda}^\circ = \bigl(\bar{\lambda}_1^\circ, \dots, \bar{\lambda}_l^\circ\bigr),$$

where $\bar{\lambda}_i = \lambda_i - l$, $1 \leqslant i \leqslant k$, and $\bar{\lambda}_i^\circ = \lambda'_i - k$, $1 \leqslant i \leqslant l$.

Since $\frac{\bar{\lambda}_1}{\bar{n}} + \dots + \frac{\bar{\lambda}_k}{\bar{n}} + \frac{\bar{\lambda}_1^\circ}{\bar{n}} + \dots + \frac{\bar{\lambda}_l^\circ}{\bar{n}} = 1$, then we make the following

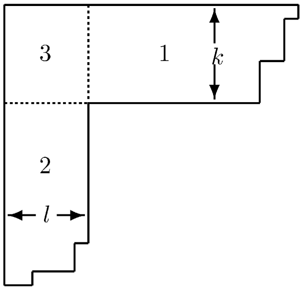
Definition 2. Let $\lambda \vdash n$, $\lambda \in H(k,l)$, $k+l \geqslant 2$. Then we define

$$\Phi(\lambda) = \frac{1}{(\frac{\bar{\lambda}_1}{\bar{n}})^{\frac{\bar{\lambda}_1}{\bar{n}}} \cdots (\frac{\bar{\lambda}_k}{\bar{n}})^{\frac{\bar{\lambda}_k}{\bar{n}}} (\frac{\bar{\lambda}_1^\circ}{\bar{n}})^{\frac{\bar{\lambda}_1^\circ}{\bar{n}}} \cdots (\frac{\bar{\lambda}_l^\circ}{\bar{n}})^{\frac{\bar{\lambda}_l^\circ}{\bar{n}}}}.$$

Lemma 4. *Let $\lambda \in H(k,l)$, $k+l \geqslant 2$, and suppose that $\lambda_k, \lambda'_l \geqslant \frac{2n}{3(k+l)}$. Then, for n large enough,*

$$\frac{1}{(k+l)^{kl}} \frac{1}{n^{(k+l)^3}} \Phi(\lambda)^n \leqslant d_\lambda \leqslant \frac{n^{kl+1}}{(\sqrt{k+l})^{kl}} \Phi(\lambda)^n.$$

Proof. Split the diagram of λ into three areas as shown in the figure below and let $\prod_1, \prod_2, \prod_3$ be the products of the hook numbers of the areas 1, 2 and 3, respectively. In case $l=0$ we set $\prod_2 = \prod_3 = 0$.



Notice that according to our previous definitions, $\prod_1 = \prod h_{ij}(\bar\lambda)$, is the product of the hook numbers of the partition $\bar\lambda$ and $\prod_2 = \prod h_{ij}(\bar\lambda^\circ)$, is the product of the hook numbers of the partition $\bar\lambda^\circ$. Since $\prod_3 \leqslant n^{kl}$, we obtain that

$$\frac{1}{n^{kl}} \leqslant \frac{n(n-1)\cdots(n-kl+1)}{\prod_3} \leqslant n^{kl}. \tag{9}$$

Recalling that $\bar n = n - kl$ and taking into account the above inequality (9) and Lemma 2 we get

$$\frac{e^{\bar n}\bar n!}{n^{k^2+k}n^{l^2+l}n^{kl}\bar\lambda_1^{\bar\lambda_1}\cdots\bar\lambda_k^{\bar\lambda_k}\bar\lambda_1^{\circ\bar\lambda_1^\circ}\cdots\bar\lambda_l^{\circ\bar\lambda_l^\circ}} \leqslant d_\lambda = \frac{n!}{\prod_1\prod_2\prod_3} \leqslant \frac{n^{kl}e^{\bar n}\bar n!}{\bar\lambda_1^{\bar\lambda_1}\cdots\bar\lambda_k^{\bar\lambda_k}\bar\lambda_1^{\circ\bar\lambda_1^\circ}\cdots\bar\lambda_l^{\circ\bar\lambda_l^\circ}}.$$

Since by Stirling formula $(\frac{\bar n}{e})^{\bar n} \leqslant \bar n! \leqslant \bar n(\frac{\bar n}{e})^{\bar n}$, we obtain

$$\frac{\bar n^{\bar n}}{n^{(k+l)^3}\bar\lambda_1^{\bar\lambda_1}\cdots\bar\lambda_k^{\bar\lambda_k}\bar\lambda_1^{\circ\bar\lambda_1^\circ}\cdots\bar\lambda_l^{\circ\bar\lambda_l^\circ}} \leqslant d_\lambda \leqslant \frac{n^{kl+1}\bar n^{\bar n}}{\bar\lambda_1^{\bar\lambda_1}\cdots\bar\lambda_k^{\bar\lambda_k}\bar\lambda_1^{\circ\bar\lambda_1^\circ}\cdots\bar\lambda_l^{\circ\bar\lambda_l^\circ}}.$$

Recalling the definition of $\Phi(\lambda)$, we have that

$$\frac{\bar n^{\bar n}}{\bar\lambda_1^{\bar\lambda_1}\cdots\bar\lambda_k^{\bar\lambda_k}\bar\lambda_1^{\circ\bar\lambda_1^\circ}\cdots\bar\lambda_l^{\circ\bar\lambda_l^\circ}} = \Phi(\lambda)^{\bar n},$$

hence

$$\frac{1}{n^{(k+l)^3}}\Phi(\lambda)^{\bar n} \leqslant d_\lambda \leqslant n^{kl+1}\Phi(\lambda)^{\bar n}. \tag{10}$$

Now, $\bar n = n - kl$ and by Remark 1, $\Phi(\lambda) \leqslant k + l$. Since $\lambda_k, \lambda'_l \geqslant \frac{2n}{3(k+l)}$ then, for n large enough, $\bar\lambda_k, \bar\lambda_l^\circ$ satisfy $\frac{\bar\lambda_k}{\bar n}, \frac{\bar\lambda_l^\circ}{\bar n} \geqslant \frac{1}{2(k+l)}$. Therefore applying Lemma 3 we obtain

$$\frac{1}{(k+l)^{kl}}\frac{1}{n^{(k+l)^3}}\Phi(\lambda)^n \leqslant \frac{1}{n^{(k+l)^3}}\Phi(\lambda)^{-kl}\Phi(\lambda)^n \leqslant d_\lambda \leqslant \frac{n^{kl+1}}{(\sqrt{k+l})^{kl}}\Phi(\lambda)^n,$$

the desired conclusion. □

Lemma 5. *Let* $0 < n_1 < n_2 < \cdots$ *be a sequence of integers and let* $\lambda(1) \vdash n_1, \lambda(2) \vdash n_2, \ldots$ *be partitions. If there exist* k, l *such that* $\lambda(j) \subseteq H(k,l)$ *and* $\lambda(j)_k \leqslant \frac{2n_j}{3(k+l)}$ *or* $\lambda(j)'_l \leqslant \frac{2n_j}{3(k+l)}$, *for all* $j = 1, 2, \ldots$, *then*

$$\limsup_{j\to\infty} \sqrt[n_j]{d_{\lambda(j)}} < k + l.$$

Proof. As in the proof of Lemma 4 (see (10)) we have

$$d_{\lambda(j)} \leqslant n_j^{kl+1} \Phi\big(\lambda(j)\big)^{n_j - kl}.$$

Since $\lambda(j)_k$ or $\lambda(j)'_l$ do not exceed $\frac{2n_j}{3(k+l)}$ one can show that $\frac{\bar{\lambda}(j)_k}{\bar{n}_j}$ or $\frac{\bar{\lambda}(j)^\circ_l}{\bar{n}_j}$ are bounded by $\frac{3}{4(k+l)}$, for n_j large enough. Hence we can write

$$\Phi\big(\lambda(j)\big) = \Phi(x_1, \ldots, x_{k+l})$$

for some $x_1, \ldots, x_{k+l}$ satisfying $x_{k+l} \leqslant x_1, \ldots, x_{k+l-1}$ and $x_{k+l} \leqslant \frac{3}{4(k+l)}$. Then by Lemma 3 there exists $\delta > 0$ not depending on j, such that

$$\Phi\big(\lambda(j)\big) \leqslant k + l - \delta$$

and hence $d_{\lambda(j)} \leqslant n_j^{kl+1}(k + l - \delta)^{n_j - kl}$. Therefore

$$\sqrt[n_j]{d_{\lambda(j)}} \leqslant \sqrt[n_j]{\frac{n_j^{kl+1}}{(k+l-\delta)^{kl}}}(k + l - \delta)$$

and the proof is complete. □

Lemma 6. *Let* $0 < n_1 < n_2 < \cdots$ *be a sequence of integers and, for every* $i \geqslant 1$, *let* $\lambda(i) \vdash n_i$ *be such that* $\lambda(i) \in H(k,l)$. *Then*

$$\lim_{i\to\infty} \Phi\big(\lambda(i)\big) = k + l \quad \textit{if and only if} \quad \lim_{i\to\infty} \frac{\lambda(i)_k}{n_i} = \lim_{i\to\infty} \frac{\lambda(i)'_l}{n_i} = \frac{1}{k+l}.$$

Proof. Let $\lim_{i\to\infty} \Phi(\lambda(i)) = k + l$. Take $\delta > 0$ and consider the interval $(\frac{1}{k+l} - \delta, \frac{1}{k+l} + \delta)$. Suppose that $\frac{\bar{\lambda}(i)_k}{n_i} \notin (\frac{1}{k+l} - \delta, \frac{1}{k+l} + \delta)$ for some i. It is easy to see that in case $x_k \notin (\frac{1}{k+l} - \delta, \frac{1}{k+l} + \delta)$ the function $\Phi(x_1, \ldots, x_{k+l})$ takes a maximal value when

$$x_1 = \cdots = x_{k-1} = x_{k+1} = \cdots = x_{k+l}$$

and $x_k = \frac{1}{k+l} - \delta$ or $\frac{1}{k+l} + \delta$.

Let this maximal value be $k + l - \varepsilon$, for some $\varepsilon > 0$. Since $\lim_{i\to\infty} \Phi(\lambda(i)) = k + l$, there exists i_0 such that for all $i \geqslant i_0$, $\Phi(\lambda(i)) > k + l - \varepsilon$. By the above this implies that $\frac{\bar{\lambda}(i)_k}{\bar{n}_i} \in (\frac{1}{k+l} - \delta, \frac{1}{k+l} + \delta)$, for all $i \geqslant i_0$. Hence $|\frac{\bar{\lambda}(i)_k}{\bar{n}_i} - \frac{1}{k+l}| < \delta$, for all $i \geqslant i_0$, and $\lim_{i\to\infty} \frac{\bar{\lambda}(i)_k}{\bar{n}_i} = \frac{1}{k+l}$.

Since $\lambda(i)_k = \bar{\lambda}(i)_k + l$ and $n_i = \bar{n}_i + kl$, we get that also $\lim_{i\to\infty} \frac{\lambda(i)_k}{n_i} = \frac{1}{k+l}$. Similarly one proves that $\lim_{i\to\infty} \frac{\lambda(i)'_l}{n_i} = \frac{1}{k+l}$.

Conversely, suppose that $\lim_{i\to\infty} \frac{\lambda(i)_k}{n_i} = \lim_{i\to\infty} \frac{\lambda(i)'_l}{n_i} = \frac{1}{k+l}$. Then also $\lim_{i\to\infty} \frac{\bar{\lambda}(i)_k}{\bar{n}_i} = \lim_{i\to\infty} \frac{\bar{\lambda}^\circ(i)_l}{\bar{n}_i} = \frac{1}{k+l}$. Since $\bar{\lambda}(i)_1 \geqslant \cdots \geqslant \bar{\lambda}(i)_k$, $\bar{\lambda}^\circ(i)_1 \geqslant \cdots \geqslant \bar{\lambda}^\circ(i)_l$ and

$$\frac{\bar{\lambda}(i)_1}{\bar{n}_i} + \cdots + \frac{\bar{\lambda}(i)_k}{\bar{n}_i} + \frac{\bar{\lambda}^\circ(i)_1}{\bar{n}_i} + \cdots + \frac{\bar{\lambda}^\circ(i)_l}{\bar{n}_i} = 1$$

we also have

$$\lim_{i\to\infty} \frac{\bar{\lambda}(i)_j}{\bar{n}_i} = \lim_{i\to\infty} \frac{\bar{\lambda}^\circ(i)_r}{\bar{n}_i} = \frac{1}{k+l}$$

for $1 \leqslant j \leqslant k$, $1 \leqslant r \leqslant l$. Hence

$$\lim_{i\to\infty} \Phi\big(\lambda(i)\big) = \Phi\left(\frac{1}{k+l}, \ldots, \frac{1}{k+l}\right) = k+l. \qquad \square$$

4. Verbally prime algebras

Recall that P_n is the space of multilinear polynomials in $x_1, \ldots, x_n$ and Γ_n is its subspace of proper polynomials. We consider the permutation action of S_n on P_n given by $\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, with $\sigma \in S_n$, $f(x_1, \ldots, x_n) \in P_n$. Clearly Γ_n is an S_n-submodule of P_n. Moreover, if A is a PI-algebra, then $P_n \cap \mathrm{Id}(A)$ and $\Gamma_n \cap \mathrm{Id}(A)$ are stable under the S_n-action and this allows us to consider

$$P_n(A) = \frac{P_n}{P_n \cap \mathrm{Id}(A)} \quad \text{and} \quad \Gamma_n(A) = \frac{\Gamma_n}{\Gamma_n \cap \mathrm{Id}(A)}$$

as S_n-modules. By complete reducibility we can write

$$P_n = \big(P_n \cap \mathrm{Id}(A)\big) \oplus T_n \quad \text{and} \quad \Gamma_n = \big(\Gamma_n \cap \mathrm{Id}(A)\big) \oplus R_n,$$

where $T_n \cong P_n(A)$ and $R_n \cong \Gamma_n(A)$ as S_n-modules. The S_n-character of $P_n(A)$ is called the nth cocharacter of A and is denoted by $\chi_n(A)$. Similarly the S_n-character of $\Gamma_n(A)$ is the nth proper cocharacter of A and is denoted by $\chi_n^p(A)$. Clearly

$$c_n(A) = \deg \chi_n(A) \quad \text{and} \quad c_n^p(A) = \deg \chi_n^p(A).$$

Since

$$\Gamma_n(A) = \frac{\Gamma_n}{\Gamma_n \cap P_n \cap \mathrm{Id}(A)} \cong \frac{\Gamma_n + P_n \cap \mathrm{Id}(A)}{P_n \cap \mathrm{Id}(A)},$$

it follows that $\Gamma_n(A)$ is isomorphic to a submodule of $P_n(A)$. Hence, if

$$\chi_n(A)=\sum_{\lambda\vdash n} m_\lambda\chi_\lambda \quad\text{and}\quad \chi_n^p(A)=\sum_{\lambda\vdash n} m'_\lambda\chi_\lambda \tag{11}$$

are the decompositions of the nth cocharacter and of the nth proper cocharacter of A into irreducible characters, we obtain that $m'_\lambda\leqslant m_\lambda$, for all $\lambda\vdash n$.

Recall that if $\chi_n(A)$ and $\chi_n^p(A)$ have the decompositions given in (11), then

$$l_n(A)=\sum_{\lambda\vdash n} m_\lambda \quad\text{and}\quad l_n^p(A)=\sum_{\lambda\vdash n} m'_\lambda$$

are the nth colength and the nth proper colength of A. By a result of Berele and Regev [3], if A is a PI-algebra, then the sequence $l_n(A)$, $n=1,2,\dots$ is polynomially bounded. But then, by the above we have

Remark 2. If A is a PI-algebra then, for all $n\geqslant 1$, we have that $l_n^p(A)\leqslant an^t$, for some constants $a,t>0$.

In what follows we shall write $\chi_n(A)\subseteq H(k,l)$, if $m_\lambda=0$ in (11) as soon as $\lambda\notin H(k,l)$. This means that $e_{T_\lambda}f\equiv 0$ is an identity of A, for any $\lambda\notin H(k,l)$ and any multilinear polynomial $f=f(x_1,\dots,x_n)$. Similarly, $\chi_n^p(A)\subseteq H(k,l)$ if and only if $e_{T_\lambda}h\equiv 0$ is an identity of A, for any $\lambda\notin H(k,l)$ and any multilinear proper polynomial $h=h(x_1,\dots,x_n)$. We shall use this approach in order to estimate the asymptotic behavior of $c_n^p(A)$.

Recall that an associative algebra A is a superalgebra or a $\mathbb{Z}_2$-graded algebra if A has a vector space decomposition $A=A^{(0)}\oplus A^{(1)}$ such that $A^{(0)}A^{(0)}+A^{(1)}A^{(1)}\subseteq A^{(0)}$ and $A^{(0)}A^{(1)}+A^{(1)}A^{(0)}\subseteq A^{(1)}$. The subspace $A^{(0)}$ is called the even component while $A^{(1)}$ is the odd component of A. Any element $a\in A^{(0)}\cup A^{(1)}$ is called homogeneous. Moreover, a is even if $a\in A^{(0)}$ and odd if $a\in A^{(1)}$.

If Y and Z are two countable sets of indeterminates, and we consider all $y\in Y$ as even variables and all $z\in Z$ as odd variables, then the free associative algebra $F\langle Y,Z\rangle$ is naturally endowed with a $\mathbb{Z}_2$-grading. We say that a polynomial $f(y_1,\dots,y_r,z_1,\dots,z_t)\equiv 0$ is a graded identity of a superalgebra $A=A^{(0)}\oplus A^{(1)}$ if $f(a_1,\dots,a_r,b_1,\dots,b_t)=0$, for all $a_1,\dots,a_r\in A^{(0)}$, $b_1,\dots,b_t\in A^{(1)}$. Let $P_{r,t}$ be the subspace of $F\langle Y,Z\rangle$ of multilinear polynomials in $y_1,\dots,y_r,z_1,\dots,z_t$. Then $P_{r,t}$ has a natural structure of $S_r\times S_t$-module, if we let S_r act on $y_1,\dots,y_r$ and S_t on $z_1,\dots,z_t$. Recall also that any irreducible $S_r\times S_t$-module Q is isomorphic to $M\otimes N$, where M and N are irreducible modules for S_r and S_t, respectively, i.e., Q corresponds to a pair of partitions (μ,ν) where $\mu\vdash r$, $\nu\vdash t$, and $\chi(M)=\chi_\mu$, $\chi(N)=\chi_\nu$.

An important role in PI-theory is played by the infinite dimensional Grassmann algebra $G=\langle e_1,e_2,\dots \mid e_ie_j+e_je_i=0,\ i,j=1,2,\dots\rangle$. Recall that G has a natural $\mathbb{Z}_2$-grading $G=G^{(0)}\oplus G^{(1)}$ where $G^{(0)}$ and $G^{(1)}$ are the subspaces spanned by the monomials in the e_i's of even and odd length, respectively. A basic tool that we shall use here, is a theorem of Kemer asserting that given a PI-algebra A there exists a finite dimensional superalgebra $B=B^{(0)}\oplus B^{(1)}$ such that $\mathrm{Id}(A)=\mathrm{Id}(G(B))$ where $G(B)=(G^{(0)}\otimes B^{(0)})\oplus(G^{(1)}\otimes B^{(1)})$ is the Grassmann envelope of B (see [14]).

By a result of Berele [2, Theorem 3], for any finite dimensional superalgebra $B=B^{(0)}\oplus B^{(1)}$, the cocharacter of its Grassman envelope $\chi_n(G(B))$ lies in the hook $H(m,l)$, where $m=\dim B^{(0)}$ and $l=\dim B^{(1)}$. In case B is a unitary superalgebra, for the proper cocharacter this result can be improved as follows.

Lemma 7. *Let $B = B^{(0)} \oplus B^{(1)}$ be a finite dimensional superalgebra with 1 and let $A = G(B)$ be its Grassmann envelope. Then $\chi_n^p(A) \subseteq H(k,l)$ for all $n = 1, 2, \ldots,$ where* $\dim B^{(0)} = k+1$, $\dim B^{(1)} = l$.

Proof. Let $\lambda \vdash n$ be a partition such that $m_\lambda \neq 0$ in $\chi_n^p(A)$. Denote by $f = f(x_1, \ldots, x_n)$ a proper polynomial generating an irreducible submodule Q of $R_n \cong \Gamma_n(A)$ with character χ_λ. Since f is not an identity of A, by eventually reordering the variables, there exists an evaluation $x_1 \mapsto a_1 \in A^{(0)}, \ldots, x_p \mapsto a_p \in A^{(0)}, x_{p+1} \mapsto b_1 \in A^{(1)}, \ldots, x_n \mapsto b_{n-p} \in A^{(1)}$ such that $f(a_1, \ldots, a_p, b_1, \ldots, b_{n-p}) \neq 0$ in A. This means that $f(y_1, \ldots, y_p, z_1, \ldots, z_q)$ is not a graded identity of A where $q = n - p$, $y_1, \ldots, y_p$ are even variables and $z_1, \ldots, z_q$ are odd variables.

Consider the $S_p \times S_q$-module generated by $f(y_1, \ldots, y_p, z_1, \ldots, z_q)$ and let M be one of its irreducible submodules. It is well known that M corresponds to a pair of partitions (μ, ν), with $\mu \vdash p$, $\nu \vdash q$. There exist Young tableaux T_μ and T_ν and a polynomial $g_0 \in M$ such that $0 \neq e_{T_\mu} e_{T_\nu} g_0 \in M$. It follows that also the polynomial

$$g = \left(\sum_{\sigma \in C_{T_\mu}} (\operatorname{sgn} \sigma) \sigma \right) e_{T_\mu} e_{T_\nu} g_0$$

lies in M and is non-zero, i.e., g is not a graded identity of A. Let $\mu = (\mu_1, \ldots, \mu_r)$ and $\nu = (\nu_1, \ldots, \nu_s)$. Then in particular, g is alternating on r even variables and is symmetric on ν_1 odd variables. Since g is a proper polynomial, in order to get a non-zero value, we cannot evaluate any y_i into $1 \otimes a$, $a \in G^{(0)}$. Since $\dim B^{(0)} = k+1$, it follows that $r \leqslant k$, i.e., μ lies in a horizontal strip of height k. Similarly, since $\dim B^{(1)} = l$, we get that ν lies in a vertical strip of width l.

Now we consider the S_n-action on $x_1 = y_1, \ldots, x_p = y_p, x_{p+1} = z_1, \ldots, x_n = z_q$ and let $FS_n M$ be the S_n-module generated by M. By the Littlewood–Richardson rule (see [12, Theorem 2.3.9]) it follows that the character of $FS_n M$ lies in the hook $H(k,l)$. On the other hand, $FS_n M \subseteq Q$ and Q is an irreducible S_n-module. Hence $FS_n M = Q$ and $\chi(Q) \subseteq H(k,l)$. Since Q is an arbitrary irreducible S_n-submodule of $R_n \cong \Gamma_n(A)$ we get $\chi_n^p(A) \subseteq H(k,l)$. □

Lemma 8. *Let $B = B^{(0)} \oplus B^{(1)}$ be a finite dimensional simple superalgebra over an algebraically closed field F,* $\dim B^{(0)} = k+1$, $\dim B^{(1)} = l$, *with $k + l \geqslant 2$, and let $A = G(B)$. Then for any $\delta > 0$ there exist a natural number N and partitions $\mu(i) \vdash iN$, $i = 1, 2, \ldots,$ such that in $\chi_{iN}^p(A) = \sum_{\lambda(i) \vdash Ni} m_{\lambda(i)} \chi_{\lambda(i)}$ all multiplicities $m_{\mu(i)}$ are non-zero and*

$$\left| \frac{\mu(i)_j}{iN} - \frac{1}{k+l} \right| \leqslant 2(k+l)\delta, \qquad \left| \frac{\mu(i)'_s}{iN} - \frac{1}{k+l} \right| \leqslant 2(k+l)\delta$$

for all $j = 1, \ldots, k$, $s = 1, \ldots, l$. Besides, $|\mu(i+1)_1 - \mu(i)_1| \leqslant N$, $|\mu(i+1)'_1 - \mu(i)'_1| \leqslant N$, for all $i \geqslant 1$.

Proof. Recall that $c_n^p(A) = \dim \Gamma_n(A)$. Since A is a unitary algebra, by [1] we have that $\limsup_{n\to\infty} \sqrt[n]{c_n^p(A)} = \exp(A) - 1$, and by [12]

$$\lim \sup_{n\to\infty} \sqrt[n]{c_n(A)} - 1 = \exp(A) - 1 = k + 1 + l - 1 = k + l.$$

Since by Remark 2 the proper colength of A is polynomially bounded, it follows that there exist a sequence of integers $t_1 < t_2 < \cdots$ and partitions $\lambda(1) \vdash t_1, \lambda(2) \vdash t_2, \ldots$ such that

$$\lim_{j\to\infty} \sqrt[t_j]{d_{\lambda(j)}} = k + l.$$

Note that by Lemma 7, $\lambda(j) \subseteq H(k,l)$, for all $j = 1, 2, \ldots$. Applying Lemma 5 we obtain that $\lambda(j)_k, \lambda(j)'_l \geqslant \frac{2t_j}{3(k+l)}$ for all $j = 1, 2, \ldots$. Then by Lemma 4,

$$\lim_{j\to\infty} \Phi\bigl(\lambda(j)\bigr) = k + l$$

and by Lemma 6,

$$\lim_{j\to\infty} \frac{\lambda(j)_k}{t_j} = \lim_{j\to\infty} \frac{\lambda(j)'_l}{t_j} = \frac{1}{k+l}, \tag{12}$$

where $\lambda(j)'$ is the conjugate partition of $\lambda(j) \vdash t_j$.

From (12) it follows that for any $\delta > 0$ there exist $t_j = N$ and $\lambda = \lambda(j) \vdash N$ such that $\lambda \subseteq H(k,l)$, the multiplicity m_λ is non-zero in $\chi_N^p(A)$ and

$$\left|\frac{\lambda_k}{N} - \frac{1}{k+l}\right| < \delta, \qquad \left|\frac{\lambda'_l}{N} - \frac{1}{k+l}\right| < \delta. \tag{13}$$

As a consequence of (13) we obtain

$$\frac{\lambda_k}{N} \geqslant \frac{1}{k+l} - \delta \quad \text{and} \quad \frac{\lambda'_l}{N} \geqslant \frac{1}{k+l} - \delta. \tag{14}$$

Moreover, we can suppose that N also satisfies

$$\frac{k+l}{N} < \delta. \tag{15}$$

Since $m_\lambda \neq 0$, there exists a multilinear proper polynomial $h = h(x_1, \ldots, x_N)$ generating an irreducible S_N-submodule of Γ_N mod $\Gamma_N \cap \mathrm{Id}(A)$ with character χ_λ.

Given an integer $i > 0$, consider the proper polynomial

$$g_i = h_1 h_2 \cdots h_i,$$

where $h_j = h(x_{(j-1)N+1}, \ldots, x_{jN})$. Since A generates a prime variety, g_i is not an identity of A (see [14] or [12, Theorem 3.7.8]).

Consider the S_{iN}-module generated by g_i and let M be one of its irreducible submodules. Then $\chi(M) = \chi_\mu$ for some partition $\mu \vdash iN$. By Lemma 7, $\chi_n^p(A) \subseteq H(k,l)$. Referring to (13) we can suppose that $\lambda_k > l$ and $\lambda_{k+1} = l$, i.e. $\lambda'_l > k$. Since each h_j generates an S_N-module with character χ_λ and $\chi_\mu \subseteq H(k,l)$, from the Littlewood–Richardson rule it follows that

$$\mu_k \geqslant i\lambda_k - il, \qquad \mu'_l \geqslant i\lambda'_l - ik$$

(see [13, p. 94]). Hence, by (14) and (15),

$$\frac{\mu_k}{iN} \geqslant \frac{\lambda_k}{N} - \frac{l}{N} > \frac{1}{k+l} - 2\delta, \qquad \frac{\mu'_l}{iN} \geqslant \frac{\lambda'_l}{N} - \frac{k}{N} > \frac{1}{k+l} - 2\delta. \tag{16}$$

Denote $\mu = \mu(i)$. Since $\mu(i)_1 \geqslant \cdots \geqslant \mu(i)_k$, $\mu(i)'_1 \geqslant \cdots \geqslant \mu(i)'_l$ and

$$\frac{\mu(i)_1}{iN} + \cdots + \frac{\mu(i)_k}{iN} + \frac{\mu(i)'_1}{iN} + \cdots + \frac{\mu(i)'_l}{iN} = 1,$$

the relation (16) implies that

$$\frac{1}{k+l} - 2\delta \leqslant \frac{\mu(i)_1}{iN} \leqslant \frac{1}{k+l} + 2\delta(k+l)$$

and

$$\frac{1}{k+l} - 2\delta \leqslant \frac{\mu(i)'_1}{iN} \leqslant \frac{1}{k+l} + 2\delta(k+l).$$

This proves the first part of the lemma.

Finally, the inequalities $0 \leqslant |\mu(i+1)_1 - \mu(i)_1| \leqslant N$ and $0 \leqslant |\mu(i+1)'_1 - \mu(i)'_1| \leqslant N$ are obvious. □

5. Minimal superalgebras

Throughout this section we assume that F is an algebraically closed field of characteristic zero. We start by recalling the basic structure theorems of the finite dimensional superalgebras (see for instance [12, Section 3.5]). Let $B = B^{(0)} \oplus B^{(1)}$ be a finite dimensional superalgebra over F and let $J = J(B)$ be its Jacobson radical. Then J is a graded ideal and there exists a maximal semisimple subalgebra B_{ss} such that $B = B_{ss} + J$. Moreover B_{ss} can be chosen to be a superalgebra and we can write $B_{ss} = B_1 \oplus \cdots \oplus B_m$, where $B_1, \ldots, B_m$ are simple superalgebras. Since the field F is algebraically closed, the algebras B_i must be of one of the following three types:

1. $A = M_n(F)$ with trivial grading $A = A^{(0)}$, $A^{(1)} = 0$.
2. $A = M_{k,l}(F) = \{ \left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right) \}$, $k \geqslant l > 0$, where P, Q, R, S are $k \times k$, $k \times l$, $l \times k$ and $l \times l$ matrices, respectively. A is endowed with the grading
$$A^{(0)} = \left\{ \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} \right\}, \qquad A^{(1)} = \left\{ \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \right\}.$$
3. $A = M_n(F \oplus cF)$, where $c^2 = 1$ with grading $A^{(0)} = M_n(F)$, $A^{(1)} = cM_n(F)$.

Next we recall the construction of a minimal superalgebra (see [11], [12, Definition 8.1.3, Lemma 8.1.6]).

Let $B_1 \oplus \cdots \oplus B_m$ be a maximal semisimple subalgebra of a finite dimensional superalgebra $B = B^{(0)} \oplus B^{(1)}$ where $B_1, \ldots, B_m$ are graded simple and let $J = J^{(0)} \oplus J^{(1)}$ be the Jacobson radical of B. Then B is said to be a minimal superalgebra if there exist homogeneous elements $w_{12}, \dots, w_{m-1,m} \in J^{(0)} \cup J^{(1)}$ and minimal graded idempotents $e_1 \in B_1, \dots, e_m \in B_m$ such that

$$e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1}$$

for all $i = 1, \dots, m-1$, the product $w_{12} w_{23} \cdots w_{m-1,m}$ is non-zero and $w_{12}, \dots, w_{m-1,m}$ generate J as a two-sided ideal of B. As a vector space the minimal superalgebra B has the decomposition

$$B = \bigoplus_{1 \leqslant i \leqslant j \leqslant m} B_{ij},$$

where $B_{11} = B_1, \dots, B_{mm} = B_m$ and

$$B_{ij} = B_{ii} w_{i,i+1} B_{i+1,i+1} \cdots B_{j-1,j-1} w_{j-1,j} B_{jj}$$

for all $i < j$. Moreover, $J = \bigoplus_{i<j} B_{ij}$ and $B_{ij} B_{kl} = \delta_{jk} B_{il}$ where δ_{jk} is the Kronecker delta.

Lemma 9. *Let $B = B_{ss} + J$ be a minimal superalgebra with $B_{ss} = B_1 \oplus \cdots \oplus B_m$ as above. Let $\dim B_{ss}^{(0)} = k+1$, $\dim B_{ss}^{(1)} = l$ and let $A = G(B)$. If $B \neq F \oplus cF$, then for any $\varepsilon > 0$ there exist a constant b and a sequence of integers $N_1 < N_2 < \cdots$ such that*

$$c^p_{N_i+3m-3}(A) \geqslant \frac{1}{(k+l)^{kl}} \frac{1}{N_i^{(k+l)^3}} (k+l-\varepsilon)^{N_i}$$

for all $i = 1, 2, \dots$. Moreover $N_{i+1} - N_i \leqslant b$, for all $i \geqslant 1$.

Proof. The main idea of the proof is to find, by using Lemma 8, a sequence $N_1, N_2, \dots$ and partitions $\rho(i) \vdash N_i$ satisfying $m_{\rho(i)} \neq 0$ and $\Phi(\rho(i)) > k+l-\varepsilon$. We then apply Lemma 4.

Let $B = B_{ss} + J$ where $B_{ss} = B_1 \oplus \cdots \oplus B_m$ and $B_1, \dots, B_m$ are simple superalgebras. Let also $w_{ij} \in J$ be the elements defined above.

First we consider the case $m = 1$, i.e., B is a simple superalgebra. Then by Lemmas 7 and 8, for any $\delta > 0$ one can find N with the following property: for all $N_i = iN$ there exists a partition $\mu(i) \vdash N_i$, $\mu(i) \subseteq H(k,l)$, with multiplicity $m_{\mu(i)} \neq 0$, where $k+1 = \dim B^{(0)}$, $l = \dim B^{(1)}$ and

$$\left| \frac{\mu(i)_j}{iN} - \frac{1}{k+l} \right| < 2(k+l)\delta, \qquad \left| \frac{\mu(i)'_s}{iN} - \frac{1}{k+l} \right| < 2(k+l)\delta \tag{17}$$

for all $j = 1, \dots, k$, $s = 1, \dots, l$. Since $\Phi(\mu(i))$ is a continuous function, by Lemma 6 it follows that for some δ small enough,

$$\Phi\bigl(\mu(i)\bigr) \geqslant k+l-\varepsilon.$$

Hence by applying Lemma 4 we obtain the lower bound

$$c^p_{N_i}(A) = c^p_{iN}(A) \geqslant \deg \chi_{\mu(i)} = d_{\mu(i)} \geqslant \frac{1}{(k+l)^{kl}} \frac{1}{N_i^{(k+l)^3}} (k+l-\varepsilon)^{N_i}.$$

Now let $m \geqslant 2$. Denote $\dim B_j^{(0)} = k_j,\ j = 1, \dots, m-1$, $\dim B_j^{(1)} = l_j,\ j = 1, \dots, m$, and $\dim B_m^{(0)} = k_m + 1$. Let also $k = k_1 + \dots + k_m$, $l = l_1 + \dots + l_m$. Then

$$\dim\bigl(B_1^{(0)} \oplus \dots \oplus B_m^{(0)}\bigr) = k + 1, \qquad \dim\bigl(B_1^{(1)} \oplus \dots \oplus B_m^{(1)}\bigr) = l.$$

Suppose first that $B_m \neq F \oplus cF$, i.e., $G(B_m)$ is not the Grassmann algebra. We start by constructing the sequence $N_1 < N_2 < \cdots$.

By Lemma 8 applied to B_m, for any $\delta > 0$ one can find N such that for all $i = 1, 2, \dots$, there exist partitions $\mu(i) \vdash iN$ with $m_{\mu(i)} \neq 0$ satisfying

$$\left|\frac{\mu(i)_j}{iN} - \frac{1}{k_m + l_m}\right| < 2(k_m + l_m)\delta, \qquad \left|\frac{\mu(i)'_s}{iN} - \frac{1}{k_m + l_m}\right| < 2(k_m + l_m)\delta \tag{18}$$

for all $j = 1, \dots, k_m$, $s = 1, \dots, l_m$. Now we fix i and denote $n_m = iN$, $\lambda(m) = \mu(i) \vdash n_m$. Next we choose positive integers $n_1, \dots, n_{m-1}$ and partitions $\lambda(1) \vdash n_1, \dots, \lambda(m-1) \vdash n_{m-1}$ for $G(B_1), \dots, G(B_{m-1})$ respectively, in the following way.

By [12, Lemma 6.3.3] for any simple superalgebra B_j, $1 \leqslant j \leqslant m-1$, and for any positive integer t_j there exist an integer n_j, a partition $\lambda(j)$ of n_j with

$$h(k_j, l_j, 2t_j - k_j - l_j) \leqslant \lambda(j) \leqslant h(k_j, l_j, 2t_j), \tag{19}$$

and a Young tableau $T_{\lambda(j)}$ such that $G(B_j)$ does not satisfy an identity $f_j \equiv 0$ corresponding to $T_{\lambda(j)}$. Here $h(a, b, t)$ is the finite hook (partition)

$$h(a, b, t) = (\underbrace{b + t, \dots, b + t}_{a}, \underbrace{b, \dots, b}_{t}).$$

We choose the integers $t_{m-1}, \dots, t_2, t_1$ in the following way. We let t_{m-1} be the least integer satisfying $2t_{m-1} - k_{m-1} - l_{m-1} \geqslant \lambda(m)_1, \lambda(m)'_1$. Then

$$\max\bigl\{\lambda(m)_1, \lambda(m)'_1\bigr\} \leqslant 2t_{m-1} - k_{m-1} - l_{m-1} \leqslant \max\bigl\{\lambda(m)_1, \lambda(m)'_1\bigr\} + 1. \tag{20}$$

Let all other t_j, $j = m-2, \dots, 1$, be minimal satisfying $2t_j - k_j - l_j \geqslant 2t_{j+1} + l_{j+1}, 2t_{j+1} + k_{j+1}$. Then

$$2t_{j+1} + \max\{l_{j+1}, k_{j+1}\} \leqslant 2t_j - k_j - l_j \leqslant 2t_{j+1} + \max\{l_{j+1}, k_{j+1}\} + 1 \tag{21}$$

for all $j = 1, \dots, m-2$. Since a hook $h(a, b, t)$ is a partition of $n = ab + (a+b)t$, we obtain that if $\lambda(j) \vdash n_j$, then

$$(2t_j - k_j - l_j)(k_j + l_j) + k_j l_j \leqslant n_j \leqslant 2t_j(k_j + l_j) + k_j l_j \tag{22}$$

for all $j = 1, \dots, m-1$. We then define $N_i = n_1 + \dots + n_m$. We shall prove below that $N_1, N_2, \dots,$ is the desired sequence. Note that N_i, all $n_1, \dots n_{m-1}$, $t_1, \dots, t_{m-1}$, and $\lambda(1), \dots, \lambda(m)$ (but not $k_1, \dots, k_m, l_1, \dots, l_m$) depend on $i = 1, 2, \dots$.

We start by proving that $N_{i+1} - N_i$ is bounded by a constant. Recall that $\lambda(m) = \mu(i) \vdash iN$ and $\mu(i+1)_1 - \mu(i)_1$, $\mu(i+1)'_1 - \mu(i)'_1$ are bounded by N by Lemma 8. Then, if we write $t_j = t_j(i)$, $j = 1, \dots, m-1$, from (20) we get

$$2t_{m-1}(i+1) - 2t_{m-1}(i) \leqslant N + 1.$$

Similarly, by (21) all $t_j(i+1) - 2t_j(i)$ are bounded by constants not depending on i. For instance, $2t_{m-2}(i+1) - 2t_{m-2}(i) \leqslant N + 2$. If we take for convenience a constant b' such that

$$2t_j(i+1) - 2t_j(i) \leqslant b'$$

for all $j = 1, \dots, m-1$, then from (22) we deduce that

$$n_j(i+1) - n_j(i) \leqslant (k_j + l_j)^2 + b'(k_j + l_j).$$

Finally, since $N_i = n_1(i) + \dots + n_{m-1}(i) + iN$ then

$$N_{i+1} - N_i < b,$$

for some constant b not depending on i.

In order to complete the proof of the lemma in case $B_m \neq F \oplus cF$, we need to construct a suitable proper polynomial which is not an identity of A. As we mentioned above, by [12, Lemma 6.3.3], for every j there exists an ordinary non-identity f_j of $G(B_j)$ corresponding to the partition $\lambda(j) \vdash n_j$. The choice of $\lambda(1), \dots, \lambda(m)$ allows us to glue the Young tableaux $T_{\lambda(1)}, \dots, T_{\lambda(m)}$ as in [12, Section 6.4].

More precisely, we start by gluing the Young diagrams $D_{\lambda(j)}$ and $D_{\lambda(j+1)}$ by gluing the first row of $D_{\lambda(j+1)}$ to the (k_j+1)th row of $D_{\lambda(j)}$, the second row of $D_{\lambda(j+1)}$ to the (k_j+2)th row of $D_{\lambda(j)}$, and so on. By the inequalities (21), this procedure gives a new diagram denoted $D_{\lambda(j)} \star D_{\lambda(j+1)}$. We then construct the diagram $D_{\lambda(1)} \star \dots \star D_{\lambda(m)}$ obtained by gluing together the diagrams $D_{\lambda(1)}, \dots, D_{\lambda(m)}$. We can now glue the Young tableaux $T_{\lambda(1)}, \dots, T_{\lambda(m)}$ in a similar way: if α_{uv} is the entry appearing in the (u, v) position of $T_{\lambda(i)}$, we write $T_{\lambda(i)} = D_{\lambda(i)}(\alpha_{uv})$. For $i = 1, \dots, m$, we add $n_1 + \dots + n_{i-1}$ to each entry of $T_{\lambda(i)}$, and we call $T^0_{\lambda(i)}$ the new tableau. Then we define

$$T_\rho = T_{\rho(i)} = T^0_{\lambda(1)} \star \dots \star T^0_{\lambda(m)} \tag{23}$$

to be the diagram $D_{\lambda(1)} \star \dots \star D_{\lambda(m)}$ filled up with the entries of the tableaux $T^0_{\lambda(i)}$. Hence $T_{\rho(i)}$ is a tableau on N_i boxes.

Next we show how to construct a proper polynomial from the tableau T_ρ. We start with a multilinear polynomial $f_1 = f_1(x_1, \dots, x_{n_1})$ which is not an identity of $G(B_1)$. Then there exist homogeneous elements $a_1, \dots, a_{n_1} \in B_1$, $g_1, \dots, g_{n_1} \in G$ such that

$$0 \neq f_1(a_1 \otimes g_1, \dots, a_{n_1} \otimes g_{n_1}) = \tilde{f}(a_1, \dots, a_{n_1}) \otimes g_1 \cdots g_{n_1} = \bar{a} \otimes g,$$

where $\bar{a} \in B_1$ and $g \in G$. Since B_1 is a matrix algebra or a sum of two matrix algebras, it is easy to find $c_1, c_2 \in B_1$ such that $c_1 \bar{a} c_2 = e_1$ where e_1 is a graded idempotent appearing in the definition of a minimal superalgebra. In particular, $e_1 w_{12} = w_{12}$. Let

$$\bar{f}_1 = y_1 f_1(x_1, \dots, x_{n_1}) y_2 z_1.$$

From the above it follows that under a suitable evaluation of $y_1, y_2, x_1, \dots, x_{n_1}$ in $G(B_1)$ and of z_1 into $w_{12} \otimes g'$, for some $g' \in G$, the polynomial $\bar{f}_1$ takes the non-zero value $w_{12} \otimes h_1$, for some $h_1 \in G$.

On the other hand, since $w_{12} B_1 = 0$, under the same evaluation, any right-normed Lie monomial

$$[y_1, x_{\sigma(1)}, \dots, x_{\sigma(n_1)}, y_2, z_1]$$

takes the same value as the associative monomial $y_1 x_{\sigma(1)} \cdots x_{\sigma(n_1)} y_2 z_1$. This means that after replacing all monomials of $\bar{f}_1$ with the corresponding right-normed Lie monomials we obtain a proper polynomial

$$p_1 = p_1(x_1, \dots, x_{n_1}, y_1, y_2, z_1)$$

which takes the non-zero value $w_{12} \otimes h_1 \in G(B)$.

By applying the above procedure to the algebras $G(B_1), \dots, G(B_{m-1})$, we construct polynomials $p_1, \dots, p_{m-1}$ on disjoint sets of indeterminates such that the product

$$p_1 \cdots p_{m-1}$$

takes the non-zero value $w_{12} \cdots w_{m-1,m} \otimes h_1 \cdots h_{m-1} = w_{1m} \otimes h_1 \cdots h_{m-1} \in G(B)$. Finally, recalling that we are applying Lemma 8 to $G(B_m)$, we let $p_m(x_1, \dots, x_{iN})$ be a proper polynomial corresponding to the partition $\lambda(m)$ and p_m is not an identity of $G(B_m)$. Then by [12, Lemma 8.3.1, part 2], the polynomial p_m can take either the value $\bar{p} = 1 \otimes s_1$ or $\bar{p} = (e_m - e'_m) \otimes s_2$ or $\bar{p} = (e_m + e'_m) \otimes s_3$, for some $s_1, s_2, s_3 \in G$ and for some graded idempotent $e'_m \in B_m$. In any case

$$(w_{1m} \otimes h_1 \cdots h_{m-1}) \bar{p} = w_{1m} \otimes h_1 \cdots h_{m-1} s_j \neq 0.$$

Therefore if we let p be the product of the polynomials $p_1, \dots, p_m$ on disjoint sets of variables

$$p_1 \cdots p_m = p = p(x_1, \dots, x_{N_i}, y_1, \dots, y_{2m-2}, z_1, \dots, z_{m-1}),$$

then p is a proper polynomial and $p \notin \Gamma_{N_i+3m-3} \cap \mathrm{Id}(A)$. Moreover, if we let S_{N_i} act on the variables $x_1, \dots, x_{N_i}$, then $e_{T_\rho} p$ generates an irreducible S_{N_i}-module where T_ρ is defined by (23). Let us verify that $e_{T_\rho} p \notin \Gamma_{N_i+3m-3} \cap \mathrm{Id}(A)$.

Denote by C_0 the subgroup of C_{T_ρ} of permutations preserving the entries inside each one of the tableaux $T^\circ_{\lambda(1)}, \dots, T^\circ_{\lambda(m)}$. Similarly, let R_0 be the subgroup of R_{T_ρ} of permutations preserving the entries inside all tableaux $T^\circ_{\lambda(1)}, \dots, T^\circ_{\lambda(m)}$. Then define

$$e_0 = \sum_{\substack{\sigma \in R_0 \\ \tau \in C_0}} (\operatorname{sgn} \tau) \sigma \tau = e_{T_{\lambda(1)}} \cdots e_{T_{\lambda(m)}}$$

in FS_{N_i}. Let

$$C_{T_\rho} = C_0 \cup C_o\pi_1 \cup \cdots,$$
$$R_{T_\rho} = R_0 \cup \nu_1 R_0 \cup \cdots$$

be the decompositions of C_{T_ρ} and R_{T_ρ} into right and left cosets over C_0 and R_0, respectively. Then, clearly,

$$e_{T_\rho} = e_0 + \sum_{i+j\geqslant 1} \nu_i e_0 \pi_j = e_0 + e'_0,$$

where $\nu_0 = \pi_0$ denotes the unit element of S_{N_i}.

Recall that $\varphi(p) \neq 0$, for some evaluation $\varphi : F\langle X\rangle \mapsto G(B)$ such that $\varphi(x_1), \dots, \varphi(x_{n_1})$, $\varphi(y_1), \varphi(y_2) \in G(B_1)$, $\varphi(x_{n_1+1}), \dots, \varphi(x_{n_1+n_2}), \varphi(y_3), \varphi(y_4) \in G(B_2)$, and so on; also $\varphi(z_1) = w_{12} \otimes h_1, \dots, \varphi(z_{m-1}) = w_{m-1,m} \otimes h_{m-1} \in G(J)$. Note that

$$e_0 p = (e_{T_{\lambda(1)}} p_1) \cdots (e_{T_{\lambda(m)}} p_m) = \gamma p$$

for some non-zero scalar γ. On the other hand, since

$$B_{\sigma(1)} w_{12} B_{\sigma(2)} \cdots B_{\sigma(m-1)} w_{m-1,m} B_{\sigma(m)} = 0$$

in B, for any permutation $\sigma \in S_m$, $\sigma \neq (1)$, we have that

$$\varphi(\pi_j p) = \varphi(\nu_i p) = 0,$$

as soon as $i \geqslant 1$ or $j \geqslant 1$. That is $\varphi(e'_0 p) = 0$. Hence

$$\varphi(e_{T_\rho} p) = \varphi\big(e_0 p + e'_0 p\big) = \gamma\varphi(p) \neq 0,$$

and we get that $c^p_{N_i+3m-3}(A) \geqslant d_{\rho(i)}$.

In order to find a lower bound for $c^p_{N_i+3m-3}(A) = \dim \Gamma_{N_i+3m-3}(A)$ we first recall that $\lambda(m) = \mu(i)$ satisfies (18), where δ can be chosen to be an arbitrary positive real number. From (21) it easily follows that

$$\begin{aligned} 2t_1 &\leqslant 2t_2 + k_1 + l_1 + k_2 + l_2 + 1 \leqslant \cdots \\ &\leqslant 2t_{m-1} + 2(k_1 + \cdots + k_{m-2} + l_1 + \cdots + l_{m-2}) + k_{m-1} + l_{m-1} + m - 2 \end{aligned}$$

and by (20)

$$2t_1 \leqslant \max\big\{\lambda(m)_1, \lambda(m)'_1\big\} + 2(k+l) + m - 1.$$

Hence by (19) we have

$$\lambda(1)_1 \leqslant l_1 + 2t_1 \leqslant \max\big\{\lambda(m)_1, \lambda(m)'_1\big\} + 3(k+l) + m - 1. \tag{24}$$

Similarly,

$$\lambda(1)'_1 \leqslant k_1 + 2t_1 \leqslant \max\{\lambda(m)_1, \lambda(m)'_1\} + 3(k+l) + m - 1. \tag{25}$$

By using (18) we can write

$$\left|\frac{\lambda(m)_{k_m}}{N_i} - \frac{\lambda(m)'_{l_m}}{N_i}\right| \leqslant \frac{iN}{N_i}\left|\frac{\lambda(m)_{k_m}}{iN} - \frac{\lambda(m)'_{l_m}}{iN}\right| \leqslant 4(k_m + l_m)\delta \leqslant 4(k+l)\delta. \tag{26}$$

Also, if N_i is large enough, say $\frac{3(k+l)+m-1}{N_i} < (k+l)\delta$, then by (24) and (18) we obtain

$$\frac{\lambda(1)_1}{N_i} - \frac{\lambda(m)_1}{N_i} \leqslant 5(k+l)\delta.$$

Finally by the above and (18) we get

$$\frac{\lambda(1)_1}{N_i} - \frac{\lambda(m)_{k_m}}{N_i} \leqslant 9(k+l)\delta. \tag{27}$$

Similarly, by (25) and (18) we get

$$\frac{\lambda(1)'_1}{N_i} - \frac{\lambda(m)'_{l_m}}{N_i} \leqslant 9(k+l)\delta. \tag{28}$$

If we now recall the above construction of the partition $\rho(i) \vdash N_i$, we have that $\rho(i)_1 = \lambda(1)_1, \dots, \rho(i)_k = \lambda(m)_{k_m}$, $\rho(i)'_1 = \lambda(1)'_1, \dots, \rho(i)'_l = \lambda(m)'_{l_m}$. Hence from (26), (27) and (28) it follows that any two arguments of the function $\Phi(\rho(i))$ differ at most by $22(k+l)\delta$. Since δ was initially taken arbitrarily small, we get that, given any $\delta' > 0$, we can find δ such that

$$\left|\frac{\rho(i)_j}{N_i} - \frac{1}{k+l}\right| < \delta' \quad \text{and} \quad \left|\frac{\rho(i)'_s}{N_i} - \frac{1}{k+l}\right| < \delta'$$

for all $j = 1, \dots, k$, $s = 1, \dots, l$. But then by Remark 1,

$$\Phi(\rho(i)) > k + l - \varepsilon.$$

Finally, by Lemma 4

$$c^p_{N_i+3m-3}(A) \geqslant d_{\rho(i)} \geqslant \frac{1}{(k+l)^{kl}} \frac{1}{N_i^{(k+l)^3}} (k+l-\varepsilon)^{N_i},$$

and the proof of the lemma in complete in case $B_m \neq F \oplus cF$.

Now let $B_m = F \oplus cF$, i.e. $G(B_m) \simeq G = G^{(0)} \oplus G^{(1)}$. It is well known that the polynomial $[x_1, x_2, x_3]$ generates the T-ideal $\mathrm{Id}(G)$ (see for instance [12, Theorem 4.1.8]). It follows that any multilinear proper polynomial of odd degree lies in $\mathrm{Id}(G)$ and, so, $c^p_{2n+1}(G) = 0$, for $n \geqslant 1$. Also, since $[x, y_1][z, y_2] \equiv -[x, y_2][z, y_1] \pmod{\mathrm{Id}(G)}$, it is easily checked that the polynomial $[x_1, x_2] \cdots [x_{2n-1}, x_{2n}]$ spans $\Gamma_{2n} \bmod \Gamma_{2n} \cap \mathrm{Id}(G)$. In conclusion we have

$$c_n^p(G) = \dim \Gamma_n(G) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

The final part of the proof is similar to the previous one. Given positive integers N and i, we denote $\lambda_m = (1, \dots, 1) \vdash 2iN$ and consider the standard polynomial St_{2iN} which is a proper polynomial not vanishing on $G(B_m) \simeq G$. Then, as above, we take for $B_1, \dots, B_{m-1}$ partitions $\lambda_1 \vdash n_1, \dots, \lambda_{m-1} \vdash n_{m-1}$ such that

$$h(k_j, l_j, 2t_j - k_j - l_j) \leqslant \lambda_j \leqslant h(k_j, l_j, 2t_j), \quad j = 1, \dots, m-1,$$

$t_{m-1} = 2iN$ and $t_1, \dots, t_{m-1}$ satisfy (21). Then we glue the tableaux $T_{\lambda_1}, \dots, T_{\lambda_m}$ as above. Clearly the corresponding partition $\rho = \rho(i) \vdash N_i = n_1 + \dots + n_{m-1} + 2iN$ satisfies

$$h(k, l, 2t - q) \leqslant \rho \leqslant h(k, l, 2t) \tag{29}$$

where $t = t_1$, q is a constant not depending on i and $N_{i+1} - N_i$ is bounded by some constant also not depending on i. Recall that

$$d_{h(k,l,t)} \underset{n \to \infty}{\simeq} a n^b (k + l)^n$$

for some constants a, b (see, for instance, [12, Lemma 6.2.5]) where $n = (k + l)t + kl$. From (29) it follows that

$$N_i - (2t - q)(k + l) - kl \leqslant (k + l)q.$$

Applying Lemma 4 to $d_{h(k,l,2t-p)}$ we conclude that the inequality

$$d_{h(k,l,2t-q)} \geqslant \frac{1}{(k + l)^{kl}} \frac{1}{N_i^{(k+l)^3}} \left(k + l - \frac{1}{2}\varepsilon\right)^{N_i - (k+l)q}$$

holds for all N_i large enough, for any fixed $\varepsilon > 0$. Hence if we take our initial N sufficiently large, the inequality

$$c^p_{N_i + 3m - 3}(A) \geqslant \frac{1}{(k + l)^{kl}} \frac{1}{N_i^{(k+l)^3}} (k + l - \varepsilon)^{N_i}$$

holds and the proof of the lemma is complete. □

Note that Lemma 9 holds also for $B = F \oplus cF$ but in the proof presented here, we have used the function $\Phi(\lambda)$ which is defined only in case $k + l \geqslant 2$.

Theorem 2. *Let $B = B^{(0)} \oplus B^{(1)}$ be a minimal superalgebra, $B \neq F \oplus cF$, with $\dim B_{ss}^{(0)} = k + 1$, $\dim B_{ss}^{(1)} = l$ and let $A = G(B)$. Then $\exp^p(A)$ exists and equals $k + l$.*

Proof. By Lemma 9, for any fixed $\varepsilon > 0$ there exist $N_1 < N_2 < \cdots$ such that $N_{i+1} - N_i < b$ and

$$c^p_{N_i+3m-3}(A) > \frac{1}{CN_i^a}(k+l-\varepsilon)^{N_i} \tag{30}$$

for some constants a, b, C not depending on ε. Here m is the number of simple components of the semisimple superalgebra B_{ss}.

Suppose first that B is a simple superalgebra, i.e., $m = 1$. We will show that $c^p_{n+j}(A) \geqslant c^p_n(A)$ for any $j \geqslant 2$. Indeed, let $c^p_n(A) = k$ and let $f_1, \ldots, f_t$ be multilinear proper polynomials on $x_1, \ldots, x_n$ linearly independent modulo $\Gamma_n \cap \mathrm{Id}(A)$. Since $B \neq F \oplus cF$, $A = G(B)$ is not Lie nilpotent, that is $g = [y_1, \ldots, y_j]$ is not an identity of A. Hence, since A generates a prime variety [14], for any set of scalars $\alpha_1, \ldots, \alpha_t$, not all zero, the polynomial

$$(\alpha_1 f_1 + \cdots + \alpha_t f_t)g$$

is not an identity of A. In particular, $c^p_{n+j}(A) \geqslant c^p_n(A)$. Hence by (30), for any $N_i \leqslant n \leqslant N_{i+1}$, we have

$$\begin{aligned} c^p_n(A) &\geqslant c^p_{N_{i-1}}(A) > \frac{1}{CN_{i-1}^a}(k+l-\varepsilon)^{N_{i-1}} \\ &\geqslant \frac{1}{Cn^a}(k+l-\varepsilon)^n \frac{1}{(k+l-\varepsilon)^{2b}} \geqslant \frac{1}{Cn^a}\frac{1}{(k+l)^{2b}}(k+l-\varepsilon)^n, \end{aligned}$$

since $n - N_{i-1} < 2b$. In particular,

$$\liminf_{n\to\infty} \sqrt[n]{c^p_n(A)} = k + l. \tag{31}$$

Since by [1]

$$\limsup_{n\to\infty} \sqrt[n]{c^p_n(A)} = \exp(A) - 1 = k + l,$$

the proof is complete in case $m = 1$.

Let now $m \geqslant 2$. We shall first prove that $c^p_{n+1}(A) \geqslant c^p_n(A)$. Again, let $c^p_n(A) = t$ and let $f_1, \ldots, f_t$ be multilinear proper polymomials on $x_1, \ldots, x_n$ linearly independent mod $\Gamma_n \cap \mathrm{Id}(A)$. Given a set of not all zero scalars $\alpha_1, \ldots, \alpha_t$, the polynomial

$$f = \alpha_1 f_1 + \cdots + \alpha_t f_t$$

is not an identity of A. Then by [12, Lemma 8.3.1], for some $p \in G$, the element $w_{1m} \otimes p$ is a linear combination of values of f in A. But since

$$[w_{1m} \otimes p, e_m \otimes q] = w_{1m}e_m \otimes pq = w_{1m} \otimes pq \neq 0,$$

for some $q \in G$, the proper polynomial $[f, y]$ is not an identity of A and therefore $c^p_{n+1}(A) \geqslant c^p_n(A)$.

From the latter inequality we get that for any $N_i + 3m - 3 \leqslant n \leqslant N_{i+1} + 3m - 3$,

$$c_n^p(A) \geqslant \frac{1}{CN_i^a}(k+l-\varepsilon)^{N_i} \geqslant \frac{1}{C'n^a}(k+l-\varepsilon)^n,$$

for some constant C' not depending on i. In particular,

$$\liminf_{n\to\infty} \sqrt[n]{c_n^p(A)} \geqslant k+l-\varepsilon$$

for any $\varepsilon > 0$. As before we conclude that

$$\liminf_{n\to\infty} \sqrt[n]{c_n^p(A)} = \limsup_{n\to\infty} \sqrt[n]{c_n^p(A)} = k+l. \qquad \square$$

6. Computing the proper exponent

In this section we shall prove the existence of the proper exponent for all PI-algebras with the exception of some algebras strictly related to the Grassmann algebra and whose proper exponent does not exist.

We start with the following technical lemmas. Recall from Section 2 that $B^\sharp$ denotes the algebra obtained from an algebra B by adjoining a unit element.

Lemma 10. *For any superalgebra B,* $\mathrm{Id}(G(B^\sharp)) = \mathrm{Id}(G(B)^\sharp)$.

Proof. Clearly $G(B)^\sharp = G(B) + F$ and $G(B^\sharp) = G(B) + G^{(0)} \supseteq G(B)^\sharp$. Hence $G(B)^\sharp$ satisfies all the identities of $G(B^\sharp)$. On the other hand, $G^{(0)}$ lies in the center of $G(B^\sharp)$ and therefore $\mathrm{Id}(G(B^\sharp)) = \mathrm{Id}(G(B) + F)$. $\square$

Lemma 11. *For any finite dimensional superalgebra B we have* $\exp(G(B^\sharp)) = \exp(G(B))$ *or* $\exp(G(B)) + 1$.

Proof. As we remarked before, we may assume that F is algebraically closed. Let $B = C_1 \oplus \cdots \oplus C_m + J$ be the Wedderburn–Malcev decomposition of B where $C_1, \dots, C_m$ are simple superalgebras and J is the Jacobson radical of B. By [8] (see also [12]),

$$\exp\bigl(G(B)\bigr) = \max \dim(C_{i_1} \oplus \cdots \oplus C_{i_k}),$$

where $C_{i_1}, \dots, C_{i_k} \in \{C_1, \dots, C_m\}$ are distinct and satisfy $C_{i_1} J C_{i_2} J \cdots J C_{i_k} \neq 0$.

Since $C_1 \oplus \cdots \oplus C_m + F$ is a maximal semisimple subalgebra of $B^\sharp$, we obtain that $\exp(G(B^\sharp))$ is equal to $\exp(G(B))$ or to $\exp(G(B)) + 1$. $\square$

Given an algebra A let $\mathrm{var}(A)$ be the variety of algebras generated by A. Suppose that the field F is algebraically closed. Recall that a finite dimensional superalgebra A is called reduced if A has a Wedderburn–Malcev decomposition $A = A_1 \oplus \cdots \oplus A_r + J$ where $A_1, \dots, A_r$ are simple superalgebras, J is the Jacobson radical and $A_1 J A_2 J \cdots J A_r \neq 0$ (see [12, Definition 9.4.2]).

We say that two algebras A and B are PI-equivalent and we write $A \sim_{\mathrm{PI}} B$ if $\mathrm{Id}(A) = \mathrm{Id}(B)$. We recall the following result that we shall use in the next theorem.

Theorem 3. *(See [12, Theorem 9.4.3].) If A is a PI-algebra, there exist a finite number of reduced superalgebras $B_1, \dots, B_t$ and a finite dimensional superalgebra D such that*

$$A \sim_{\mathrm{PI}} G(B_1) \oplus \cdots \oplus G(B_t) \oplus G(D),$$

where $\exp(A) = \exp(G(B_1)) = \cdots = \exp(G(B_t))$ *and* $\exp(G(D)) < \exp(A)$.

It is known that for any given integer $d \geqslant 2$ there are only finitely many so-called minimal varieties of exponent d. Recall that a variety $\mathcal{V}$ with $\exp(\mathcal{V}) = d \geqslant 2$ is called minimal if any proper subvariety of $\mathcal{V}$ has exponent strictly less than d, i.e., at most $d-1$ (see [5,6,10]). For $d=2$ the minimal varieties are $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$ where UT_2 is the algebra of 2×2 upper triangular matrices. In particular, if $\exp(\mathcal{V}) = 2$ then $\mathcal{V}$ contains either UT_2 or G or both of them (see for instance [12, Lemma 8.1.5]). We shall make use of this result below.

Next theorem is the main result of this paper about the existence of the proper exponent. We remark that if $A = G \oplus B \oplus N$ where B is a finite dimensional algebra with Jacobson radical of codimension 1 and N is a nilpotent algebra, then, for n large enough $c_n^p(A) = c_n^p(G)$. Hence $\exp^p(A)$ does not exist in this case. Next we shall prove that actually this is the only exception.

Theorem 4. *Let A be a PI-algebra with* $\exp(A) = d \geqslant 1$. *Then* $\exp^p(A)$ *exists, is an integer and* $\exp^p(A) = d$ *or* $d-1$, *unless* $\exp(A) = 2$ *and A is PI-equivalent to an algebra of the type* $G \oplus B \oplus N$ *where B is a finite dimensional algebra whose Jacobson radical is of codimension* 1 *and N is a nilpotent algebra.*

Proof. At the light of Proposition 1, we may assume that $\exp(A) \geqslant 2$.

Suppose first that either $\exp(A) \geqslant 3$ or $\exp(A) = 2$ and $UT_2 \in \mathrm{var}(A)$. As we mentioned at the beginning of Section 2, we may assume that the base field F is algebraically closed. Then, by [12, Lemma 8.1.5], there exists a minimal superalgebra B such that $G(B) \in \mathrm{var}(A)$ and $d = \exp(A) = \dim B_{ss} = \exp(G(B))$. If $\exp(A) \geqslant 3$, then $B \neq F \oplus cF$. On the other hand if $\exp(A) = 2$, by hypothesis $UT_2 \in \mathrm{var}(A)$ and we may take $B = UT_2$ with trivial grading. Hence also in this case $B \neq F \oplus cF$, and by Theorem 2 we conclude that $\exp^p(G(B))$ exists and equals $d-1$.

Now, if A is a unitary algebra, then by [1], $\overline{\exp}^p(A) = \exp(A) - 1 = d - 1$. On the other hand, $\underline{\exp}^p(A) \geqslant \exp^p(G(B)) = d-1$. Hence $\exp^p(A)$ exists and equals $d-1$.

If A is not a unitary algebra, by Lemmas 10 and 11 for the algebra $A^\sharp$ we have $\exp(A^\sharp) = d$ or $d+1$. If $\exp(A^\sharp) = d$ then by the first part of the proof, $\exp^p(A^\sharp) = d-1$. Since A and $A^\sharp$ have the same proper identities, we get $\exp^p(A) = d-1$ and we are done. If $\exp(A^\sharp) = d+1$, then $\exp^p(A^\sharp) = d$ since $A^\sharp$ is a unitary algebra. Since $\Gamma_n(A) = \Gamma_n(A^\sharp)$ for all n, $\exp^p(A)$ exists and equals d.

Therefore in order to finish the proof we may assume that $\exp(A) = 2$ and $UT_2 \notin \mathrm{var}(A)$. We shall prove that in this case $\exp^p(A)$ exists provided A is not PI-equivalent to an algebra of the type stated in the theorem.

By Theorem 3, $A \sim_{\mathrm{PI}} G(B_1) \oplus \cdots \oplus G(B_t) \oplus G(D)$, for some reduced superalgebras $B_1, \dots, B_t$ and a finite dimensional superalgebra D, where $\exp(G(B_1)) = \cdots = \exp(G(B_t)) = 2$ and $\exp(G(D)) \leqslant 1$. Now, by the defining property of the exponent, it follows that for $1 \leqslant i \leqslant t$, either $B_i = F_1 \oplus F_2 + J$ where $F_1 \cong F_2 \cong F$ and $F_1 \oplus F_2 \subseteq B_i^{(0)}$, or $B_i = F \oplus cF$, $c^2 = 1$. In the first case, $F_1 J F_2 \neq 0$ implies that $UT_2 \in \mathrm{var}(G(B_i)) \subseteq \mathrm{var}(A)$, a contradiction. Thus $B_i = F \oplus cF$, and $G(B_i) = G$. In conclusion $A \sim_{\mathrm{PI}} G \oplus G(D)$ and $\exp(G(D)) \leqslant 1$.

By [12, Theorem 7.2.12], $G(D) \sim_{\mathrm{PI}} B_1 \oplus \cdots \oplus B_m$ where for $1 \leqslant i \leqslant m$, B_i is a finite dimensional algebra and $\dim B_i/J(B_i) \leqslant 1$.

First suppose that $B_1, \ldots, B_m$ are nilpotent or unitary algebras. Let, for example, $B_1, \ldots, B_k$ be unitary and $B_{k+1}, \ldots, B_m$ nilpotent. Then $N = B_{k+1} \oplus \cdots \oplus B_m$ is a nilpotent algebra.

Let $e_1, \ldots, e_k$ be the unit elements of $B_1, \ldots, B_k$ and $J_1, \ldots, J_k$ the Jacobson radicals of $B_1, \ldots, B_k$, respectively. Denote by L the one-dimensional subspace spanned by $e = e_1 + \cdots + e_k$. Then $L \simeq F$. We will show that the algebra

$$B = L + J_1 \oplus \cdots \oplus J_k$$

is PI-equivalent to $B_1 \oplus \cdots \oplus B_k$. If $f = f(x_1, \ldots, x_n)$ is a multilinear polynomial and, say, $a_1, \ldots, a_j \in J_t$ for some $1 \leqslant t \leqslant k$ then

$$f(a_1, \ldots, a_j, e_t, \ldots, e_t) = f(a_1, \ldots, a_j, e, \ldots, e).$$

That is if f is not an identity of B_t then f is not an identity of B. Since $B \in \mathrm{var}(B_1 \oplus \cdots \oplus B_k)$ we get: $B \sim_{\mathrm{PI}} B_1 \oplus \cdots \oplus B_k$. Hence $A \sim_{\mathrm{PI}} G \oplus B \oplus N$, where N is a nilpotent algebra and $B = F + J$ is an algebra with 1, the desired conclusion.

Hence we may assume that some B_i is a non-unitary algebra which is not nilpotent. Write $B_i = F + J$ and decompose $J = J_{11} \oplus J_{10} \oplus J_{01} \oplus J_{00}$ as in Proposition 1. Then, since B_i is non-unitary, $J_{10} \oplus J_{01} \neq 0$. As in Proposition 1, it follows that $[x_1, \ldots, x_n]$ is not an identity of B_i, for all $n \geqslant 2$. Hence $c_n^p(B_1 \oplus \cdots \oplus B_m) \geqslant c_n^p(B_i) \geqslant 1$ for all $n \geqslant 2$. In conclusion, $\exp^p(A) = \exp^p(G \oplus B_1 \oplus \cdots \oplus B_m) = 1$ and the proof is complete. □

We conclude the paper by remarking that one cannot expect to extend the equality $\exp^p(A) = \exp^L(A)$ from finitely generated PI-algebras to arbitrary algebras. In fact, the following is an example of a (unitary) PI-algebra A such that $\exp^L(A) < \exp^p(A) = \exp(A) - 1$.

Let

$$A = M_{1,1}(G) = G\big(M_{1,1}(F)\big) = \begin{pmatrix} G^{(0)} & G^{(1)} \\ G^{(1)} & G^{(0)} \end{pmatrix}.$$

It is easily checked that A satisfies the Lie identity $[[x_1, x_2], [x_3, x_2], x_5] \equiv 0$, i.e., A is a central metabelian Lie algebra. Also $c_n^L(A) \geqslant 1$ since A is not Lie nilpotent. It can be shown (see [15] or [12]) that $c_n^L(A)$ is polynomially bounded. Actually it is not difficult to see that asymptotically $c_n^L(A) \leqslant n^3$. Hence $\exp^L(A) = 1$. On the other hand, it is well known that $\exp(A) = 4$ (see for instance [12]); hence by Theorem 4, $\exp^p(A) = 3$.

We finally remark that the question whether the Lie exponent of an associative PI-algebra exists and is an integer is still open in general. In case of Lie algebras of so-called non-associative type, there is an example of a Lie algebra L such that $3.1 < \underline{\exp}(L) \leqslant \overline{\exp}(L) < 3.9$ [16].

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