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Codimensions of algebras and growth functions

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Abstract

Let A be an algebra over a field F of characteristic zero and let $c_n(A)$, $n = 1, 2, \dots$, be its sequence of codimensions. We prove that if $c_n(A)$ is exponentially bounded, its exponential growth can be any real number > 1. This is achieved by constructing, for any real number $\alpha > 1$, an F-algebra A_α such that $\lim_{n\to\infty} \sqrt[n]{c_n(A_\alpha)}$ exists and equals α. The methods are based on the representation theory of the symmetric group and on properties of infinite Sturmian and periodic words.

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1. Introduction

Let F be a field of characteristic zero. The theory of polynomial identities plays a significant role in the general theory of algebras over F. For instance, if F is an algebraically closed field, it turns out that any finite dimensional simple associative or Lie algebra is uniquely determined by its identities (see [17,20]). Also, in the associative case, the polynomial identities allow to establish a surprising link between finitely generated and finite dimensional algebras [11]. In

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general, the study of the polynomial identities and of the corresponding varieties is one of the most fruitful approaches to the investigation of some important classes of non-associative algebras [23].

On the other hand, the description of the identities of a given algebra is a very difficult problem in general. Even if $A = M_n(F)$ is the algebra of $n \times n$ matrices over F, the description of the identities is known only for $n = 2$. One of the effective ways of studying the identities of a given algebra is that of combining algebraic and analytical methods. The idea of applying numerical methods for investigating the identities was originally realized in the associative case (see for instance [1,18]) and the results obtained in recent years have given new impetus to the development of the theory (see [7]). The same analytical approach was also effectively applied in Lie theory (see [7,13]).

In general, given an algebra A over F, one can associate to A a numerical sequence $c_n(A)$, $n = 1, 2, \dots,$ called the sequence of codimensions of A (see next section for details). The sequence $c_n(A)$, $n = 1, 2, \dots,$ gives in some way a measure of the polynomial relations vanishing in the algebra A and in general has overexponential growth. For instance, if $F\{X\}$ is the free (non-associative) algebra on a set X, $|X| \geqslant 2$, $c_n(F\{X\}) = p_n n!$ where $p_n = \frac{1}{n}\binom{2n-2}{n-1}$ is the nth Catalan number. For the free associative algebra $F\langle X\rangle$ and the free Lie algebra $L\langle X\rangle$ we have $c_n(F\langle X\rangle) = n!$ and $c_n(L\langle X\rangle) = (n-1)!$, respectively.

A number of methods have been developed in the years in order to deal with codimension sequences without any further assumption (see for instance [16,18]). But the most significant results have been obtained in case $c_n(A)$ is exponentially bounded.

There is a wide class of algebras with exponentially bounded codimension growth. For instance, if $\dim A = d < \infty$, then $c_n(A) \leqslant d^n$ [3]. Also, any associative PI-algebra (algebra satisfying a non-trivial polynomial identity), any infinite dimensional simple Lie algebra of Cartan type [2] or any affine Kac–Moody algebra has exponentially bounded codimension growth [18,21].

In case the sequence of codimensions is exponentially bounded, say $c_n(A) \leqslant d^n$, one can construct the bounded sequence $\sqrt[n]{c_n(A)}$, $n = 1, 2, \dots,$ and it is an open problem if $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists.

In the 80s Amitsur conjectured that for any associative PI-algebra A, $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is a non-negative integer. This conjecture was recently confirmed in [5,6]. In [22] it was also shown that the same conclusion holds for any finite dimensional Lie algebra.

For associative algebras it was known long time before [10] that the sequence of codimensions cannot have intermediate growth, i.e., either $c_n(A) \geqslant 2^n$ asymptotically or $c_n(A)$ is polynomially bounded. A similar result for general Lie algebras was proved in [14]. Only recently it was shown that for any finite dimensional algebra A either $c_n(A)$ is polynomially bounded or $c_n(A) > \varphi(d)^n$ where $\dim A = d$ and φ is some explicit function with $\varphi(d) > 1$ [8].

There is only one known example of infinite dimensional Lie algebra L with $3.1 < \sqrt[n]{c_n(L)} < 3.9$ [15]. Nevertheless, even for this algebra it is unknown if $\lim_{n\to\infty} \sqrt[n]{c_n(L)}$ exists.

The first goal of this paper is to construct examples of (non-associative) algebras A such that $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is non-integer. Actually we prove that for any real number $t \geqslant 1$ there exists an algebra $A = A(t)$ such that $\lim_{n\to\infty} \sqrt[n]{c_n(A(t))} = t$ (Corollary 7.1). Then we discuss the asymptotics of the codimensions of finite dimensional algebras. Clearly, if F is countable, then $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ can take only a countable set of values. Hence not every real number greater than 1 can be realized as the exponential growth of the codimension sequence of a finite dimensional algebra. Nevertheless, the set

$$\left\{ \lim_{n\to\infty} \sqrt[n]{c_n(A)} \;\Big|\; \dim_F A < \infty \right\}$$

is a dense subset of $\mathbb{R}$ (Corollary 7.2).

The main tool in our study is the ordinary representation theory of the symmetric group. Also, by applying methods of algebraic combinatorics [12], to each algebra $A(t)$ we associate an infinite word w in the alphabet $\{0, 1\}$ and we relate the complexity of w to the sequence of codimensions of $A(t)$. It turns out that the basic properties of Sturmian and periodic words will allow us to prove our result.

The techniques developed in this paper allowed us to construct examples of infinite dimensional non-associative algebras whose growth of the codimensions is intermediate between polynomial and exponential [8]. As it was mentioned above, this phenomenon cannot occur in case of associative or Lie algebras.

2. Preliminaries and basic tools

Throughout F is a field of characteristic zero and $F\{X\} = F\{x_1, x_2, \ldots\}$ is the free non-associative algebra of countable rank over F. Given a polynomial $f = f(x_1, \ldots, x_n) \in F\{X\}$, we say that $f \equiv 0$ is a polynomial identity for an F-algebra A (or that A satisfies $f \equiv 0$) if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$. In case A satisfies a non-trivial polynomial identity, A is called a PI-algebra. Let

$$\mathrm{Id}(A) = \left\{ f \in F\{X\} \mid f \equiv 0 \text{ in } A \right\}$$

denote the subset of $F\{X\}$ of polynomial identities of A. It is clear that $\mathrm{Id}(A)$ is an ideal invariant under all endomorphisms (i.e., a T-ideal) of $F\{X\}$. We refer the reader to [4,7,20,23] for an account of the basic properties of PI-algebras.

For every $n \geqslant 1$, let P_n be the subspace of $F\{X\}$ of all multilinear polynomials in the variables $x_1, \ldots, x_n$. Notice that since the number of distinct arrangements of parentheses on a monomial of length n is the Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$, it readily follows that $\dim_F P_n = \binom{2n-2}{n-1}(n-1)!$.

Now let A be an arbitrary algebra and let $\mathrm{Id}(A)$ be its T-ideal of identities in the free algebra $F\{X\}$.

Definition 2.1. The non-negative integer

$$c_n(A) = \dim \frac{P_n}{P_n \cap \mathrm{Id}(A)}$$

is called the nth codimension of A.

As it was mentioned in the introduction, the sequence $c_n(A)$, $n = 1, 2, \ldots$, is exponentially bounded for a wide class of algebras. For such algebras one can define the following numerical invariant

Definition 2.2. Let A be an algebra over F and let the sequence $c_n(A)$, $n = 1, 2, \ldots$, be exponentially bounded. Then the PI-exponent of A is the real number

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)} \tag{1}$$

provided the limit on the right-hand side of (1) exists.

Since $\operatorname{char} F = 0$, by the well-known multilinearization process, every T-ideal is determined by its multilinear polynomials. Hence the T-ideal $\operatorname{Id}(A)$ is completely determined by the sequence of spaces $\{P_n \cap \operatorname{Id}(A)\}_{n\geqslant 1}$. These spaces are studied through the representation theory of the symmetric group.

For the convenience of the reader we recall the basic notions and constructions of this theory (see [9]).

Let S_n be the symmetric group on $\{1, \dots, n\}$. Since $\operatorname{char} F = 0$, by Maschke's theorem any finite dimensional S_n-representation is completely reducible. In other words, if M is a finite dimensional module for the group algebra FS_n, then M can be decomposed as

$$M = M_1 \oplus \cdots \oplus M_q \tag{2}$$

where $M_1, \dots, M_q$ are irreducible FS_n-modules. In the language of S_n-characters (2) can be rewritten in the form

$$\chi(M) = \chi(M_1) \oplus \cdots \oplus \chi(M_q), \tag{3}$$

where $\chi(M)$ is the character of M and $\chi(M_i)$ is the character of M_i, $i = 1, \dots, q$. The number q of irreducible summands in (2) and (3) is called the length $l(M)$ of M. Now, it is well known that there is a one-to-one correspondence between irreducible S_n-characters and partitions of n. Recall that a sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ of positive integers is called a partition of n, and we write $\lambda \vdash n$, if $\lambda_1 \geqslant \cdots \geqslant \lambda_r$ and $\lambda_1 + \cdots + \lambda_r = n$. To a partition $\lambda \vdash n$ one can associate a Young diagram D_λ which is an array of boxes such that the jth row of D_λ contains λ_j boxes. If we fill up the boxes of D_λ with the integers $1, \dots, n$, we obtain a Young tableau of shape λ.

Given a Young tableau T_λ let R_{T_λ} and C_{T_λ} be the subgroups of S_n stabilizing the rows and the columns of T_λ, respectively. Then write

$$\bar{R}_{T_\lambda} = \sum_{\sigma \in R_{T_\lambda}} \sigma, \qquad \bar{C}_{T_\lambda} = \sum_{\tau \in C_{T_\lambda}} (\operatorname{sgn} \tau)\tau$$

for the corresponding elements of the group algebra FS_n. Then the element $e_{T_\lambda} = \bar{R}_{T_\lambda}\bar{C}_{T_\lambda}$ is an essential idempotent of FS_n, i.e., $e_{T_\lambda}^2 = \alpha e_{T_\lambda}$, for some non-zero scalar α. It is well known that e_{T_λ} generates a minimal left ideal of FS_n, that is realizes an irreducible representation of S_n. Moreover for any two tableaux T_λ and T'_λ of the same shape λ the left modules $FS_n e_{T_\lambda}$ and $FS_n e_{T'_\lambda}$ are isomorphic. On the other hand, $FS_n e_{T_\lambda}$ and $FS_n e_{T_\mu}$ are not S_n-isomorphic as soon as $\lambda \neq \mu$. Recall also that $e_{T_\lambda} M \neq 0$ for any irreducible S_n-module M isomorphic to $FS_n e_{T_\lambda}$.

Since the characters of two equivalent representations coincide, by using standard notation, we write

$$\chi_\lambda = \chi(FS_n e_{T_\lambda})$$

the irreducible character of the module $FS_n e_{T_\lambda}$. Combining together similar summands, we can rewrite (3) in the form

$$\chi(M) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \tag{4}$$

where m_λ is a non-negative integer called the multiplicity of χ_λ in $\chi(M)$ ($m_\lambda = 0$ if the right-hand side of (2) does not contain summands isomorphic to $FS_n e_{T_\lambda}$). Recall that for any S_n-module Q, the degree of the character $\chi(Q)$ is $\deg \chi(Q) = \dim Q$. Then from (2), (3) and (4) it follows that

$$\dim M = \sum_{\lambda \vdash n} m_\lambda d_\lambda$$

where $d_\lambda = \deg \chi_\lambda$ is the degree of the character χ_λ and

$$l(M) = \sum_{\lambda \vdash n} m_\lambda.$$

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_t) \vdash n$, denote by $\lambda'_1, \ldots, \lambda'_r$ the heights of first, second, rth column of the Young diagram D_λ, respectively. Clearly $r = \lambda_1$, $\lambda'_1 + \cdots + \lambda'_r = n$ and $\lambda'_1 \geqslant \cdots \geqslant \lambda'_r$. Hence $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$ is a partition of n called the conjugate partition of λ.

In what follows we shall also use the hook formula for the dimension d_λ of the irreducible S_n-representations. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of n and let D_λ be the corresponding Young diagram. Denote by (i, j) the box of D_λ at the intersection of the ith row and the jth column and define the hook number

$$h_{ij} = \lambda_i - (j-1) + \lambda'_j - (i-1) - 1 = \lambda_1 + \lambda'_j - i - j + 1$$

where $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is the conjugate partition on λ. Then

$$d_\lambda = \frac{n!}{\prod_{(i,j) \in D_\lambda} h_{ij}}$$

is the hook formula for $d_\lambda = \deg \chi_\lambda$.

Now let the symmetric group S_n act on the left on P_n by requiring that for $\sigma \in S_n$ and $f(x_1, \ldots, x_n) \in P_n$,

$$\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Since for any PI-algebra A, the subspace $P_n \cap \mathrm{Id}(A)$ is S_n-invariant, this in turn induces a structure of S_n-module on the space $P_n(A) = \frac{P_n}{P_n \cap \mathrm{Id}(A)}$. The S_n-character of $P_n(A)$, denoted $\chi_n(A)$, is called the nth cocharacter of the algebra A and

$$c_n(A) = \deg \chi_n(A) = \dim_F P_n(A)$$

is the nth codimension of A. Then the nth cocharacter of the PI-algebra A decomposes as

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \tag{5}$$

where $m_\lambda \geqslant 0$ is the multiplicity of χ_λ in $\chi_n(A)$. In particular

$$c_n(A) = \sum_{\lambda \vdash n} m_\lambda d_\lambda. \tag{6}$$

The two sequences $\{\chi_n(A)\}_{n\geqslant 1}$ and $\{c_n(A)\}_{n\geqslant 1}$ will be the main object of our study.

Another important numerical sequence is the sequence of colengths. If the nth cocharacter of A has the decomposition given in (5), the nth colength of A is

$$l_n(A)=\sum_{\lambda\vdash n} m_\lambda. \tag{7}$$

The equalities (6) and (7) are very useful in the study of the asymptotic behavior of the sequence $c_n(A)$. For instance, if A is an associative PI-algebra, then all d_λ appearing in (6) with non-zero multiplicity m_λ, are exponentially bounded as functions of n, whereas $l_n(A)$ in (7) is polynomially bounded. In the following sections we shall construct a family of non-associative algebras sharing the same properties. This will allow us to reduce the study of the asymptotic behavior of $c_n(A)$ to the estimate of the asymptotic behavior of the degrees of the corresponding representations of the symmetric groups.

3. The algebra $A(K)$

Given a sequence of integers $K=\{k_i\}_{i\geqslant 1}$ such that $k_i\geqslant 2$ for all i, we define a (non-associative) algebra $A(K)$ that will be the main object of our investigation.

Definition 3.1. Let $K=\{k_i\}_{i\geqslant 1}$ be a sequence of positive integers $k_i\geqslant 2$. Then $A(K)$ is the algebra over F with basis

$$\{a,b\}\cup Z_1\cup Z_2\cup\cdots$$

where

$$Z_i=\left\{z_j^{(i)}\;\middle|\;1\leqslant j\leqslant k_i\right\},\quad i=1,2,\ldots,$$

and multiplication table given by

$$z_2^{(i)}a=z_3^{(i)},\ \ldots,\ z_{k_i-1}^{(i)}a=z_{k_i}^{(i)},\ z_{k_i}^{(i)}a=z_1^{(i)},\quad i=1,2,\ldots,$$
$$z_1^{(i)}b=z_2^{(i+1)},\quad i=1,2,\ldots,$$

and all the remaining products are zero.

Recall that, given elements $y_1,y_2,\ldots,y_n$ of a non-associative algebra, their left-normed product is defined inductively as $y_1\cdots y_n=(y_1\cdots y_{n-1})y_n$. From Definition 3.1 it easily follows that only left-normed products of basis elements of $A(K)$ may be non-zero. Moreover the only non-zero products are of the type $z_j^{(i)}f(a,b)$ for some left-normed monomial $f(a,b)$. Because of the multiplication table of $A(K)$, any other arrangement of the parentheses in $f(a,b)$ gives a zero value, hence there is no lost of generality if we view $f=f(a,b)$ as an associative monomial on a and b and we shall tacitly do this in what follows.

Let us denote by $\deg_a f$ and $\deg_b f$ the degree of f on a and b, respectively. Notice that all degrees are well defined only if we consider the elements $f(a,b)$ as words in the alphabet $\{a,b\}$

and do not compute their values in $A(K)$. It is clear that given any $z_j^{(i)}$ and $z_k^{(l)}$ such that $l > i$ or $l = i$ and $k > j$, there exists only one monomial $f(a,b)$ on a and b with

$$z_k^{(l)} = z_j^{(i)} f(a,b). \tag{8}$$

Some conclusions can be easily drawn about the cocharacter sequence of $A(K)$.

Recall that given a set $N \subseteq \{1,\dots,n\}$, a multilinear polynomial $g(x_1,\dots,x_n) \in P_n$ is alternating on the set of indeterminates $\{x_k \mid k \in N\}$, if

$$\sigma g(x_1,\dots,x_n) = (\operatorname{sgn}\sigma) g(x_1,\dots,x_n),$$

for all $\sigma \in S(N)$, where $S(N)$ is the symmetric group on the set N. The characteristic property of alternating polynomials is that if we evaluate the above g in an algebra A, i.e., if we substitute $x_i \to a_i \in A$, $i = 1,\dots,n$, then $g(a_1,\dots,a_n) = 0$ as soon as $a_i = a_j$ for some $i, j \in N$.

Now let $\lambda = (\lambda_1,\dots,\lambda_t) \vdash n$ be a partition of n and let $\lambda' = (\lambda'_1,\dots,\lambda'_r)$ denote the conjugate partition of λ. Denote by $h(\lambda)$ the height of D_λ, i.e., $h(\lambda) = \lambda'_1$.

Let T_λ be a λ-tableau and let e_{T_λ} be the corresponding essential idempotent of FS_n. Denote by N_j, $j = 1,\dots,r$, the integers contained in the jth column of T_λ. Then $\{1,\dots,n\} = N_1 \cup \dots \cup N_r$ is a disjoint union and

$$C_{T_\lambda} = S(N_1) \times \dots \times S(N_r). \tag{9}$$

Given a multilinear polynomial $f = f(x_1,\dots,x_n) \in P_n$, consider the polynomial $g = g(x_1,\dots,x_n) = e_{T_\lambda} f(x_1,\dots,x_n)$. From (9) and the definition of $\bar{C}_{T_\lambda}$ it follows that the polynomial $f' = \bar{C}_{T_\lambda} f$ is alternating on any set $\{x_k \mid k \in N_i\}$, $1 \leqslant i \leqslant r$.

Next we evaluate the polynomial f' in the algebra $A(K)$. We remark that since f' is a multilinear polynomial, it is enough to evaluate it into a linear basis of $A(K)$. Suppose first that $N_1 \geqslant 4$. Note that

$$I = \operatorname{span}\{z_j^{(i)} \mid 1 \leqslant j \leqslant k_i,\ i \geqslant 1\}$$

is a two-sided ideal of $A(K)$. Hence when evaluating f' in $A(K)$, we have to replace two variables x_i and x_j, $i, j \in N_1$ either with elements of I or both with a or both with b. In any case, since $I^2 = 0$ and f' is alternating on $\{x_k \mid k \in N_1\}$, it follows that f' is an identity of $A(K)$. Hence $g(x_1,\dots,x_n) = \bar{R}_{T_\lambda} f'(x_1,\dots,x_n)$ is also an identity of $A(K)$. We have proved that if $h(\lambda) = N_1 > 3$, then $e_{T_\lambda} f$ is an identity of $A(K)$. Similarly, if $N_1 = 3$ and $N_2 = 3$, i.e., $\lambda_3 > 1$ in $\lambda = (\lambda_1,\dots,\lambda_t)$, then also $\bar{C}_{T_\lambda} f \equiv 0$, and so $e_{T_\lambda} f \equiv 0$, in $A(K)$.

Recall that if M is an S_n-module,

$$M = M_1 \oplus \dots \oplus M_q,$$

with $M_1,\dots,M_q$ irreducible S_n-modules and, say, $\chi(M_1) = \chi_\lambda$, for some $\lambda \vdash n$, then $e_{T_\lambda} M_1 \neq 0$, for any λ-tableau T_λ. Hence, by the complete reducibility of S_n-representations, the S_n-module P_n can be decomposed into the sum of two S_n submodules

$$P_n = M \oplus \bigl(P_n \cap \operatorname{Id}(A(K))\bigr).$$

As it was shown above, $e_{T_\lambda} f$ is an identity of $A(K)$ as soon as $h(\lambda) > 3$ or $\lambda_3 > 1$. Hence M does not contain irreducible S_n-submodules with character χ_λ where $h(\lambda) > 3$ or $\lambda_3 > 1$. Since

$$P_n\big(A(K)\big) = \frac{P_n}{P_n \cap \mathrm{Id}(A(K))} \cong M,$$

we immediately obtain the following

Lemma 3.1.

$$\chi_n\big(A(K)\big) = m_{(n)}\chi_{(n)} + \sum_{\lambda=(\lambda_1,\lambda_2)\vdash n} m_\lambda \chi_\lambda + \sum_{\lambda=(\lambda_1,\lambda_2,1)\vdash n} m_\lambda \chi_\lambda.$$

In order to investigate the asymptotics of the degrees of the irreducible S_n-characters we introduce the real valued function

$$\Phi(x) = \frac{1}{x^x(1-x)^{1-x}}$$

defined on the interval $(0, \frac{1}{2}]$.

The proof of the next lemma is based on standard arguments and is left to the reader.

Lemma 3.2. *The function $\Phi(x)$ is continuous in the interval $(0, \frac{1}{2}]$, and $\Phi(a) < \Phi(b)$ whenever $a < b$. Moreover $\lim_{x\to 0^+} \Phi(x) = 1$ and $\Phi(\frac{1}{2}) = 2$.*

Next we need to estimate the degrees of the irreducible S_n-characters χ_λ for the three types of partitions: $\lambda = (n)$ or (λ_1, λ_2) or $(\lambda_1, \lambda_2, 1)$. If $\lambda = (n)$ then $\deg \chi_\lambda = 1$. In case $\lambda = (n-k, k)$, by the hook formula (see Section 2), we obtain

$$\deg \chi_\lambda = \frac{n!}{k!(n-k)!} \cdot \frac{n-2k+1}{n-k+1}$$

and, so,

$$\frac{1}{n}\binom{n}{k} \leqslant \deg \chi_\lambda \leqslant \binom{n}{k}.$$

Now, by Stirling's formula [19], for some $0 < \theta_n < 1$ we have

$$n! = \sqrt{2\pi n}\,\frac{n^n}{e^n}\,e^{\frac{\theta_n}{12n}}.$$

Since $k \leqslant n-k$, write $k = \alpha n$ where $0 < \alpha = \frac{k}{n} \leqslant \frac{1}{2}$. Then we have

$$\binom{n}{k} = \sqrt{\frac{2\pi n}{2\pi k \cdot 2\pi(n-k)}}\,\frac{n^n \gamma_{n,k}}{k^k(n-k)^{n-k}} = \frac{\gamma_{n,k}}{\sqrt{2\pi\alpha(1-\alpha)n}}\Phi(\alpha)^n$$

where

$$\gamma_{n,k} = \frac{e^{\frac{\theta_n}{12n}}}{e^{\frac{\theta_k}{12k}} \cdot e^{\frac{\theta_{n-k}}{12(n-k)}}}.$$

Notice that

$$\frac{\gamma_{n,k}}{\sqrt{2\pi\alpha(1-\alpha)n}} = \frac{\gamma_{n,k}}{\sqrt{2\pi k(1-\alpha)}} \leqslant \frac{e^{\frac{1}{12}}}{\sqrt{\pi}} < \sqrt{\frac{e}{\pi}} < 1$$

and

$$\frac{\gamma_{n,k}}{\sqrt{2\pi\alpha(1-\alpha)n}} > \frac{1}{e^{\frac{1}{12}}} \cdot \frac{1}{e^{\frac{1}{12}}} \cdot \frac{1}{\sqrt{2\pi\frac{n}{4}}} > \frac{1}{\sqrt{\pi n}}.$$

Hence we have proved the following

Lemma 3.3. *Let $\lambda = (\lambda_1, \lambda_2)$ be a partition of n. Then*

$$\frac{1}{\sqrt{\pi n^3}} \Phi(\alpha)^n < \deg \chi_\lambda < \Phi(\alpha)^n,$$

where $\alpha = \frac{\lambda_2}{n}$ and $\Phi(\alpha) = \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}}$.

A similar result can be proved about partitions of the type $(\lambda_1, \lambda_2, 1)$, but we shall reduce all calculations for such partitions to the case of partitions with only two parts.

4. Complexity of infinite words and colength sequence

In this section we shall bound from above the colength sequence of the algebra $A(K)$ for some special types of sequences K associated to infinite words in the alphabet $\{0, 1\}$.

We start with a non-difficult but important remark on the bound of the multiplicities in the cocharacter of any PI-algebra. For every $d \leqslant n$, let $W_n^{(d)}$ be the subspace of the free algebra $F\{X\}$ of homogeneous polynomials in $x_1, \dots, x_d$ of degree n. Given any PI-algebra A, define

$$W_n^{(d)}(A) = \frac{W_n^{(d)}}{W_n^{(d)} \cap \mathrm{Id}(A)}.$$

Lemma 4.1. *Let A be a PI-algebra with nth cocharacter $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$. Then, for every $\lambda \vdash n$ with $h(\lambda) \leqslant d$, we have that $m_\lambda \leqslant \dim W_n^{(d)}(A)$.*

Proof. Let $\lambda \vdash n$ with $h(\lambda) \leqslant d$ and for short, write $\lambda = (\lambda_1, \dots, \lambda_d)$ even if $h(\lambda) < d$. Recall that $\chi_n(A)$ is the S_n-character of the module $P_n(A) = \frac{P_n}{P_n \cap \mathrm{Id}(A)}$. Hence, since χ_λ participates in $\chi_n(A)$ with multiplicity m_λ, $P_n(A)$ contains a submodule

$$M = M_1 \oplus \cdots \oplus M_q$$

with $q = m_\lambda$, where for $i = 1, \dots, q$, each M_i has character $\chi(M_i) = \chi_\lambda$.

For $i \geqslant 1$, write $\bar{x}_i = x_i + \mathrm{Id}(A) \in \frac{F\{X\}}{\mathrm{Id}(A)}$. Now, let T_λ be the Young tableau of shape λ obtained from the diagram D_λ by filling the boxes of the first row from left to right with the integers

$1, \dots, \lambda_1$, the second row with $\lambda_1 + 1, \dots, \lambda_1 + \lambda_2$, and so on. It is well known (see Section 2) that $e_{T_\lambda} M_i \neq 0$, for all $i = 1, \dots, q$. Given $1 \leqslant i \leqslant n$, let $g_i \in M_i$ be a multilinear polynomial such that $\tilde{f_i} = \tilde{f_i}(\bar{x}_1, \dots, \bar{x}_n) = e_{T_\lambda} g_i \neq 0$. By the structure of the essential idempotent e_{T_λ}, it follows that $\tilde{f_i}$ is symmetric on each of the sets $\{\bar{x}_1, \dots, \bar{x}_{\lambda_1}\}$, $\{\bar{x}_{\lambda_1+1}, \dots, \bar{x}_{\lambda_1+\lambda_2}\}$, etc.

Let $F\{x_1, \dots, x_d\}$ denote the free algebra on the set $\{x_1, \dots, x_d\}$ and consider the homomorphism

$$\varphi : \frac{F\{X\}}{\mathrm{Id}(A)} \to \frac{F\{x_1, \dots, x_d\}}{F\{x_1, \dots, x_d\} \cap \mathrm{Id}(A)}$$

such that

$$\begin{gathered}
\varphi(\bar{x}_1) = \cdots = \varphi(\bar{x}_{\lambda_1}) = y_1, \\
\varphi(\bar{x}_{\lambda_1+1}) = \cdots = \varphi(\bar{x}_{\lambda_1+\lambda_2}) = y_2, \\
\cdots \\
\varphi(\bar{x}_{\lambda_1+\cdots+\lambda_{d-1}+1}) = \cdots = \varphi(\bar{x}_n) = y_d
\end{gathered}$$

where for $1 \leqslant j \leqslant d$, $y_j = x_j + F\{x_1, \dots, x_d\} \cap \mathrm{Id}(A)$.

Clearly $\varphi(M) \subseteq W_n^{(d)}(A)$. Denote $f_1 = \varphi(\tilde{f_1}), \dots, f_q = \varphi(\tilde{f_q})$. It is well known that $\tilde{f_1}, \dots, \tilde{f_q}$ are, up to non-zero scalars, the complete linearizations of $f_1, \dots, f_q$, respectively. Since $\tilde{f_1}, \dots, \tilde{f_q} \notin \mathrm{Id}(A)$, also $f_1, \dots, f_q \notin \mathrm{Id}(A)$.

Suppose that the elements $f_1, \dots, f_q$ are linearly dependent over F. Then the elements $\tilde{f_1}, \dots, \tilde{f_q}$ obtained by complete linearization of $f_1, \dots, f_q$ respectively, are still linearly dependent over F. But $\tilde{f_1} \in M_1, \dots, \tilde{f_q} \in M_q$, and this is a contradiction. Thus

$$m_\lambda = q \leqslant \dim \varphi(M_1 \oplus \cdots \oplus M_q) \leqslant \dim W_n^{(d)}(A). \qquad \square$$

At this stage, in order to bound the colength sequence of $A(K)$, we need to specialize the sequence K.

Let $w = w_1 w_2 \dots$ be an infinite (associative) word in the alphabet $\{0, 1\}$. Given an integer $m \geqslant 2$, let $K_{m,w} = \{k_i\}_{i \geqslant 1}$ be the sequence defined by

$$k_i = \begin{cases} m, & \text{if } w_i = 0, \\ m+1, & \text{if } w_i = 1 \end{cases}$$

and write $A(m, w) = A(K_{m,w})$.

Recall that, given an infinite word w in a finite alphabet, the complexity Comp_w of w is the function $\mathrm{Comp}_w : \mathbb{N} \to \mathbb{N}$, where $\mathrm{Comp}_w(n)$ is the number of distinct subwords of w of length n (see [12, Chapter 1]).

Lemma 4.2. *For any $m \geqslant 2$ and for any word w, the algebra $A = A(m, w)$ has nth colength satisfying*

$$l_n(A) \leqslant 3(m+1)n^3 \,\mathrm{Comp}_w(n).$$

Proof. Consider the quotient algebra $R = \frac{F\{x_1,x_2,x_3\}}{\mathrm{Id}(A)}$ and denote by $y_i = x_i + \mathrm{Id}(A)$, $i = 1, 2, 3$, the canonical generators of R. Recall that $R = F(y_1, y_2, y_3)$ is the relatively free algebra of the variety generated by the algebra A.

Now, if $\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda$ is the nth cocharacter of A, by Lemma 3.1, all partitions $\lambda \vdash n$ with $m_\lambda \neq 0$ are of the type $\lambda = (n)$, $\lambda = (\lambda_1, \lambda_2)$ or $\lambda = (\lambda_1, \lambda_2, 1)$. In particular $h(\lambda) \leqslant 3$ and, by Lemma 4.1, it follows that $m_\lambda \leqslant \dim W_n^{(3)}(A)$, where in our notation, $W_n^{(3)}(A)$ is the subspace of R of homogeneous polynomials of degree n in y_1, y_2, y_3.

Since $l_n(A) = \sum_{\lambda\vdash n} m_\lambda$ and the number of partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3) \vdash n$ with $\lambda_3 \leqslant 1$ does not exceed n^2, we obtain that $l_n(A) \leqslant n^2(\dim W_n^{(3)}(A))$. Therefore, in order to prove the lemma, it is enough to show that

$$\dim W_n^{(3)}(A) \leqslant 3(m+1)n \operatorname{Comp}_w(n). \tag{10}$$

Fix n and make the following auxiliary construction. Let $F\langle a, b\rangle$ be the free associative algebra in the elements a and b. Let M be the free right $F\langle a, b\rangle$-module on the set $X = \{x_1, x_2, x_3\}$. We make M into a $F\langle a, b\rangle$-bimodule by requiring that $F\langle a, b\rangle M = 0$. Hence any element of M can be written as a linear combination of elements $x_i f(a, b)$ where $f(a, b)$ is a monomial with coefficient 1, i.e., a word in the alphabet $\{a, b\}$.

Recall that a monomial of the type $(((x_{i_1}x_{i_2})x_{i_3})\cdots x_{i_n}) = x_{i_1}\cdots x_{i_n}$ is called left normed. Now, since $(x_1x_2)(x_3x_4) \equiv 0$ is an identity of A, one can choose a basis of R consisting of left normed monomials in y_1, y_2, y_3. Hence

$$\{y_{i_1}\cdots y_{i_k} \mid k \geqslant 1,\ i_1, \ldots, i_k \in \{1, 2, 3\}\}$$

generate R as a vector space.

Let now $\sigma : R \to A$ be the evaluation map such that

$$\sigma(y_i) = \sum_{s,j} \lambda_{ijs} z_j^{(s)} + \alpha_i a + \beta_i b.$$

Then, by the multiplication table of A we obtain

$$\sigma(y_{i_1}\cdots y_{i_k}) = \left(\sum_{s,j} \lambda_{i_1 js} z_j^{(s)}\right)(\alpha_{i_2}a + \beta_{i_2}b)\cdots(\alpha_{i_k}a + \beta_{i_k}b).$$

It follows that σ can be considered as the composition of two linear maps

$$R \xrightarrow{\psi} M \xrightarrow{\varphi} A$$

where ψ and φ are defined on generators by the rule

$$\psi(y_{i_1}\cdots y_{i_k}) = x_{i_1}(\alpha_{i_2}a + \beta_{i_2}b)\cdots(\alpha_{i_k}a + \beta_{i_k}b)$$

and

$$\varphi\big(x_i f(a, b)\big) = \left(\sum_{s,j} \lambda_{ijs} z_j^{(s)}\right) f(a, b).$$

Actually φ is a homomorphism of $F\langle a,b\rangle$-modules.

Denote by I the intersection of the kernels of all such maps $\varphi : M \to A$. Let also $M^{(n)}$ denote the space generated by all elements of the form $x_i f(a,b)$, $i=1,2,3$, where f is a monomial of degree $n-1$. Clearly,

$$\dim W_n^{(3)}(A) \leqslant \dim \frac{M^{(n)}}{I \cap M^{(n)}}. \tag{11}$$

Hence in order to complete the proof, we only need to show that the codimension of $I \cap M^{(n)}$ in $M^{(n)}$ does not exceed $3(m+1)n \operatorname{Comp}_w(n)$.

For $k=1,2,3$, let M_k be the $F\langle a,b\rangle$-submodule of M generated by x_k and let $I_k = I \cap M^{(n)} \cap M_k$.

For $k=1$, if $\varphi_{ij} : M \to A$ denotes the linear map such that $\varphi_{ij}(x_1) = z_j^{(i)}$, then

$$I_1 = \bigcap_{i,j} \operatorname{Ker} \varphi_{ij}.$$

We next bound the codimension of the kernel of any fixed map φ_{ij}. Later we shall compare the kernels of different maps.

Let $x_1 f(a,b) \notin \operatorname{Ker} \varphi_{ij}$ for some monomial $f = f(a,b)$. Then from the multiplication table of A it follows that

$$f = a^{i_0} b a^{i_1} b \dots b a^{i_{r+1}}$$

with $0 \leqslant i_0, i_{r+1} \leqslant m$ and $i_1, \dots, i_r \in \{m-1, m\}$. More precisely, the structure of f is closely related to the subword $w(i+1, r+1) = w_{i+1} w_{i+2} \dots w_{i+r+1}$ of $w = w_1 w_2 \dots$ in the following way. Since we must have

$$z_j^{(i)} a^{i_0} = z_1^{(i)},\ z_1^{(i)} b = z_2^{(i+1)},\ z_2^{(i+1)} a^{i_1} = z_1^{(i+1)},\ \dots,$$

define the word $\tilde{f}$ on $\{0,1\}$ by the rule $\tilde{f} = \tilde{f}_1 \tilde{f}_2 \dots \tilde{f}_r$ where

$$\tilde{f}_s = \begin{cases} 0, & \text{if } i_s = m-1, \\ 1, & \text{if } i_s = m. \end{cases}$$

Then $\varphi_{ij}(x_1 f(a,b)) \neq 0$ if and only if

(1) $\tilde{f} = w(i+1, r)$,
(2) $i_0 = 0$ in case $j=1$ and $j + i_0 = m + w_i$ otherwise,
(3) $i_{r+1} \leqslant m - 1 + w_{i+r+1}$.

In this case it can be checked that $\varphi_{ij}(x_1 f(a,b)) = z_{2+i_{r+1}}^{i+r+1}$.

Notice that distinct monomials of the type $x_1 f(a,b)$ with $\varphi_{ij}(x_1 f(a,b)) \neq 0$ are automatically linearly independent modulo I_1. Moreover $\operatorname{Ker} \varphi_{ij} = \operatorname{Ker} \varphi_{lj}$ as soon as $w(i, r+1) = w(l, r+1)$. Recalling that the number of distinct subwords of w of length r is $\operatorname{Comp}_w(r)$, we obtain that the number of subspaces $\operatorname{Ker}\varphi_{ij}$, for fixed r, is at most $(m+1)\operatorname{Comp}_w(r+2)$. Since $1 \leqslant r+2 \leqslant n$ and Comp is a monotone increasing function, this implies that

$$\dim \frac{M_1 \cap M^{(n)}}{I_1} \leqslant n(m+1)\operatorname{Comp}_w(n).$$

Similarly, the codimension of I_k in $M_k \cap M^{(n)}$ $(k = 2, 3)$, is also bounded by $n(m+1) \times \operatorname{Comp}_w(n)$. Thus, since $M = M_1 \oplus M_2 \oplus M_3$,

$$\dim W_n^{(3)}(A) \leqslant \dim \frac{M^{(n)}}{I \cap M^{(n)}} \leqslant 3(m+1)n\operatorname{Comp}_w(n)$$

and the proof of the lemma is complete. $\square$

5. Sturmian or periodic words and real exponential growth < 2

In this section we shall further specialize the algebra $A(m, w)$ by choosing the word w in a suitable way.

Recall that an infinite word $w = w_1 w_2 \cdots$ in the alphabet $\{0, 1\}$ is periodic with period T if $w_i = w_{i+T}$ for $i = 1, 2, \ldots$. It is easy to see that for any such word $\operatorname{Comp}_w(n) \leqslant T$. Moreover, it is known that $\operatorname{Comp}_w(n) \geqslant n + 1$ for any aperiodic word and an infinite word w is called a *Sturmian* word if $\operatorname{Comp}_w(n) = n + 1$ for all $n \geqslant 1$ (see [12]).

For a finite word x, the height $h(x)$ of x is the number of letters 1 appearing in x. Also, if $|x|$ denotes the length of the word x, the *slope* of x is defined as $\pi(x) = \frac{h(x)}{|x|}$. In some cases this definition can be extended to infinite words in the following way. Let w be some infinite word and let $w(1, n)$ denote its prefix subword of length n. If the limit

$$\pi(w) = \lim_{n\to\infty} \frac{h(w(1, n))}{n}$$

exists then $\pi(w)$ is called the slope of w. It is easy to give examples of infinite words for which the slope is not defined. Nevertheless for periodic words and Sturmian words the slope is well defined. In the next proposition we give the basic properties of these words that we shall need in the sequel.

Proposition 5.1. *(See [12, Section 2.2].) Let w be a Sturmian or periodic word. Then there exists a constant C such that*

(1) $|h(x) - h(y)| \leqslant C$, *for any finite subwords x, y of w with $|x| = |y|$*;
(2) *the slope $\pi(w)$ of w exists*;
(3) *for any non-empty subword u of w,*

$$\left|\pi(u) - \pi(w)\right| \leqslant \frac{C}{|u|};$$

(4) *for any real number $\alpha \in (0, 1)$ there exists a word w with $\pi(w) = \alpha$ and w is Sturmian or periodic according as α is irrational or rational, respectively.*

In case w is Sturmian we can take $C = 1$, and if w is periodic of period T, then $\pi(w) = \frac{h(w(1,T))}{T}$.

Our aim is to prove that in case w is a periodic or a Sturmian word, then for the algebra $A = A(m, w)$, $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and any real number in the interval $(1, 2)$ can be realized in this way. We start by bounding from below the nth codimensions of such algebra.

Lemma 5.1. *Let w be a Sturmian or periodic word with slope $\pi(w) = \alpha$, let $A = A(m, w)$ and let $\beta = \frac{1}{m+\alpha}$. Then, given any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for all $n \geqslant N$, the $(n+1)$th codimension of A satisfies*

$$c_{n+1}(A) \geqslant \frac{1}{2^{m+1}\sqrt{\pi n^3}} \Phi(\beta - \varepsilon)^n.$$

Proof. Let $w(1, j) = w_1 \cdots w_j$ be the prefix subword of w of length j. Suppose first that there exists r such that $n = mr + \pi(w_1 \cdots w_r)r$ where $\pi(w_1 \cdots w_r)$ is the slope of the word $w_1 \cdots w_r$. Hence $\pi(w_1 \cdots w_r)r = w_1 + \cdots + w_r$. If we set $i_1 = m - 1 + w_2, \ldots, i_r = m - 1 + w_{r+1}$, then clearly

$$z_1^{(1)} b a^{i_1} b a^{i_2} \cdots b a^{i_r} = z_1^{(r+1)} \neq 0. \tag{12}$$

Consider the partition $\lambda = (\lambda_1, \lambda_2, 1) \vdash n + 1$ such that $\lambda_1 = i_1 + \cdots + i_r$, $\lambda_2 = r$ and let

$$T_\lambda = \begin{array}{|c|c|c|c|c|c|c|} \hline j_1+1 & j_2+1 & \cdots & j_r+1 & * & \cdots & * \\ \hline j_1 & j_2 & \cdots & j_r & \multicolumn{3}{c}{} \\ \cline{1-4} 1 & \multicolumn{6}{c}{} \\ \cline{1-1} \end{array}$$

be a Young tableau such that $j_1, \ldots, j_r$ are the positions of b in the monomial on the left-hand side of (12), i.e., $j_1 = 2$, $j_2 = i_1 + 3$, $j_3 = i_1 + i_2 + 4$, and the remaining boxes on the first row of T_λ are filled up with the remaining integers in $\{1, \ldots, n+1\}$. Let e_{T_λ} be the essential idempotent corresponding to the tableau T_λ. Then the evaluation

$$\begin{aligned} &\varphi(x_1) = z_1^{(1)}, \quad \varphi(x_{j_1}) = \cdots = \varphi(x_{j_r}) = b, \quad \text{and} \\ &\varphi(x_j) = a \quad \text{if } j \notin \{j_1, \ldots, j_r, 1\}, \end{aligned}$$

maps $e_{T_\lambda}(x_1 \cdots x_{n+1})$, where $x_1 \cdots x_{n+1}$ is a left-normed monomial, to $r!(n-r)!z_1^{(r+1)}$. Hence $e_{T_\lambda}(x_1 \cdots x_{n+1})$ is not an identity of A and this says that χ_λ participates with multiplicity $m_\lambda \neq 0$ in the decomposition of $\chi_{n+1}(A)$. Therefore $c_{n+1}(A) \geqslant \deg \chi_\lambda$.

Let $\mu = (\lambda_1, \lambda_2) \vdash n$. Since $\deg \chi_\lambda \geqslant \deg \chi_\mu$, by Lemma 3.3 we obtain

$$c_{n+1}(A) \geqslant \deg \chi_\mu > \frac{1}{\sqrt{\pi n^3}} \Phi(\gamma)^n$$

where

$$\gamma = \frac{\lambda_2}{n} = \frac{r}{n} = \frac{r}{mr + \pi(w_1 \cdots w_r)r} = \frac{1}{m + \pi(w_1 \cdots w_r)}.$$

Clearly if w is a periodic word, $\lim_{r\to\infty}\pi(w_1\cdots w_r)=\alpha$. Moreover the same conclusion holds for Sturmian words by Proposition 5.1. Hence, given $\varepsilon>0$, there exists $N=N(\varepsilon)$ such that

$$\gamma=\frac{1}{m+\pi(w_1\cdots w_r)}\geqslant\frac{1}{m+\alpha}-\varepsilon=\beta-\varepsilon$$

as soon as $n=mr+\pi(w_1\cdots w_r)r\geqslant N$. Hence

$$c_{n+1}(A)>\frac{1}{\sqrt{\pi n^3}}\Phi(\beta-\varepsilon)^n \tag{13}$$

and we are done in this case.

If n cannot be written as $mr+\pi(w_1\cdots w_r)r$ then one can find r such that

$$n_0=mr+\pi(w_1\cdots w_r)r<n<m(r+1)+\pi(w_1\cdots w_{r+1})(r+1).$$

In this case, since $n_0<n$ and $\pi(w_1\cdots w_{r+1})(r+1)-\pi(w_1\cdots w_r)r\leqslant 1$ we obtain $n-n_0<m+1$. By the first part of the proof, the codimension $c_{n_0+1}(A)$ satisfies (13). Moreover, by the structure of the algebra A, if a multilinear polynomial $p(x_1,\dots,x_i)$ of degree i is not an identity of A then $p(x_1,\dots,x_i)x_{i+1}\not\equiv 0$ on A. This implies that $c_{i+1}(A)\geqslant c_i(A)$ for all i. Hence by (13) we have

$$c_{n+1}(A)\geqslant c_{n_0+1}(A)>\frac{1}{\sqrt{\pi n_0^3}}\Phi(\beta-\varepsilon)^{n_0}.$$

Now, $n_0<n$ implies $\frac{1}{\sqrt{\pi n_0^3}}>\frac{1}{\sqrt{\pi n^3}}$ and since $\Phi(\beta-\varepsilon)<2$,

$$\Phi(\beta-\varepsilon)^{n_0}>\Phi(\beta-\varepsilon)^{n-m-1}>2^{-m-1}\Phi(\beta-\varepsilon)^n.$$

Thus $c_{n+1}(A)\geqslant\frac{1}{2^{m+1}\sqrt{\pi n^3}}\Phi(\beta-\varepsilon)^n$ as wished. $\square$

In the next lemma we get some information on the nth cocharacter of $A(m,w)$. Roughly speaking we shall prove that all characters χ_λ whose diagram λ has long second row, do not participate in $\chi_n(A(m,w))$. This fact will be exploited in the next lemma in order to get an upper bound for $c_n(A(m,w))$.

Lemma 5.2. *Let w be a Sturmian or periodic word with slope α, let $A=A(m,w)$ and let $\beta=\frac{1}{m+\alpha}$. Given any $\varepsilon>0$ there exists $N=N(\varepsilon)$ such that for all $n\geqslant N$ and for all $\lambda\vdash(n+1)$, the character χ_λ appears with zero multiplicity in $\chi_{n+1}(A)$ whenever $\frac{\lambda_2}{n}>\beta+\varepsilon$.*

Proof. Let $g=g(a,b)$ be an associative monomial on a and b, and suppose that $z_j^{(i)}g(a,b)\neq 0$ in A for some i,j. Then, as in Lemma 4.2,

$$g=a^{i_0}ba^{i_1}b\cdots ba^{i_{r+1}}$$

with $0 \leqslant i_0, i_{r+1} \leqslant m$ and $i_1, \dots, i_r \in \{m-1, m\}$. Moreover, $i_1 + \dots + i_r = (m-1)r + r\pi(w_{i+1} \cdots w_{i+r})$ where $\pi(w_{i+1} \cdots w_{i+r})$ is the slope of the subword $w_{i+1} \cdots w_{i+r}$ of w. By Proposition 5.1,

$$\left|\pi(w_1 \cdots w_r) - \pi(w_{i+1} \dots w_{i+r})\right| \leqslant \frac{C}{r}, \quad \alpha - \pi(w_1 \cdots w_r) \leqslant \frac{C}{r}$$

and, so, $r\pi(w_{i+1} \cdots w_{i+r}) \geqslant r\pi(w_1 \cdots w_r) - C \geqslant \alpha r - 2C$. It follows that

$$\begin{aligned} \deg g = n &= i_0 + i_{r+1} + i_1 + \dots + i_r + r + 1 \\ &\geqslant mr + r\pi(w_{i+1} \cdots w_{i+r}) + 1 \geqslant (m+\alpha)r - (2C-1). \end{aligned}$$

Hence we obtain

$$\begin{aligned} \frac{\deg_b g}{\deg g} &= \frac{r+1}{n} \leqslant \frac{r+1}{(m+\alpha)r - (2C-1)} \\ &= \frac{1}{m+\alpha - \frac{m+\alpha+2C-1}{r+1}} < \frac{1}{m+\alpha - \frac{C_1}{n}}, \end{aligned} \tag{14}$$

where $C_1 = 6m(m+\alpha+2C-1)$ and the last inequality in (14) follows from the relations

$$i_1 + \dots + i_r \leqslant (m-1)r + r = mr$$

and

$$n = i_0 + i_{r+1} + i_1 + \dots + i_r + r + 1 \leqslant i_0 + i_{r+1} + (m+1)r + 1 \leqslant 3m + 2mr \leqslant 6m(r+1).$$

Let now λ be a partition of $n+1$ such that $\frac{\lambda_2}{n} > \beta + \varepsilon$. Given a multilinear polynomial $f = f(x_1, \dots, x_{n+1})$ and a Young tableau T_λ, consider the value of $f' = e_{T_\lambda} f$ under any evaluation $\varphi : \{x_1, \dots, x_{n+1}\} \to \{a, b, z_j^{(i)}\}$. Recall that $e_{T_\lambda} = \bar{R}_{T_\lambda} \bar{C}_{T_\lambda}$ and the polynomial $\bar{C}_{T_\lambda} f$ is alternating on λ_2 disjoint subsets of variables of order $\lambda'_1, \dots, \lambda'_t \geqslant 2$, respectively. Therefore the same property holds for the polynomial $\sigma \bar{C}_{T_\lambda} f$, for any $\sigma \in S_n$. Hence f' is a linear combination of polynomials of the type $f''(x_1, \dots, x_{n+1})$ where f'' is alternating on λ_2 disjoint subsets of variables each of order at least two (see Section 3). It follows that, in order to get a non-zero value of f', we need to replace one of the x_k's with $z_j^{(i)}$ and at least $\lambda_2 - 1$ of the x_k's with b. In this case $\varphi(f')$ will be a sum of monomials of type $z_j^{(i)} g(a, b)$ with $\deg g = n$ and $\deg_b g \geqslant \lambda_2 - 1$. But then

$$\frac{\deg_b g}{\deg g} = \frac{\deg_b g}{n} \geqslant \frac{\lambda_2 - 1}{n} \geqslant \beta + \varepsilon - \frac{1}{n} = \frac{1}{m+\alpha} + \varepsilon - \frac{1}{n}. \tag{15}$$

Clearly, the inequality (15) contradicts (14), provided n is large enough. Thus $e_{T_\lambda} f \equiv 0$ is an identity of A for any multilinear polynomial f and this says that $m_\lambda = 0$ in $\chi_{n+1}(A)$. □

Lemma 5.3. *Let w be a Sturmian or periodic word with slope α and let $A = A(m, w)$. If $\beta = \frac{1}{m+\alpha}$, then asymptotically*

$$c_{n+1}(A) \leqslant 3(m+1)(n+2)^5 \big(\Phi(\beta) + \nu\big)^n,$$

for any $\nu > 0$.

Proof. Let $\nu > 0$ be an arbitrary real number. Since $\Phi(x)$ is a continuous function (see Lemma 3.2) there exists $\varepsilon > 0$ such that $|\Phi(x) - \Phi(\beta)| < \nu$ as soon as $|x - \beta| < \varepsilon$. By Lemma 5.2 there exists N such that χ_λ has zero multiplicity in $\chi_{n+1}(A)$ for all $\lambda \vdash (n+1)$, as soon as $n \geqslant N$ and $\frac{\lambda_2}{n} > \beta + \varepsilon$.

Consider $\lambda \vdash (n+1)$ with $m_\lambda \neq 0$. Then $\frac{\lambda_2}{n} = \alpha \leqslant \beta + \varepsilon$. If $\lambda_3 = 0$ then by Lemma 3.3,

$$\deg \chi_\lambda \leqslant \Phi(\alpha)^{n+1} \leqslant \big(\Phi(\beta) + \nu\big)^{n+1}.$$

If $\lambda_3 \neq 0$ then $\lambda_3 = 1$, and by the hook formula (see Section 2), we obtain

$$\deg \chi_\lambda = (\deg \chi_\mu) \frac{\lambda_2(\lambda_1 + 1)(n+1)}{(\lambda_2 + 1)(\lambda_1 + 2)} < (n+1) \deg \chi_\mu,$$

where $\mu = (\lambda_1, \lambda_2) \vdash n$. Hence by Lemma 3.3,

$$\deg \chi_\lambda < (n+1)\Phi(\alpha)^n$$

where $\alpha = \frac{\lambda_2}{n}$. It follows that in any case

$$\deg \chi_\lambda < (n+1)\big(\Phi(\beta) + \nu\big)^n, \tag{16}$$

for all $\lambda \vdash (n+1)$ such that $m_\lambda \neq 0$. Now, from (16) and Lemma 4.2, we obtain

$$\begin{aligned} c_{n+1}(A) &= \sum_{\lambda \vdash n} m_\lambda \deg \chi_\lambda < l_{n+1}(A)(n+1)\big(\Phi(\beta) + \nu\big)^n \\ &\leqslant 3(m+1)(n+1)^4(n+2)\big(\Phi(\beta) + \nu\big)^n \\ &\leqslant 3(m+1)(n+2)^5\big(\Phi(\beta) + \nu\big)^n. \qquad \square \end{aligned}$$

Recall that by Definition 2.2, the PI-exponent of an algebra A is $\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ in case such limit exists.

Putting together Lemmas 5.1 and 5.3 it is clear that for the algebras $A(m, w)$ the PI-exponent exists and equals $\Phi(\beta)$. We record this in the following.

Theorem 5.1. *Let w be an infinite Sturmian or periodic word with slope α, $0 < \alpha < 1$. If $m \geqslant 2$ then for the algebra $A = A(m, w)$ the PI-exponent exists and* $\exp(A) = \Phi(\beta)$ *where $\beta = \frac{1}{m+\alpha}$.*

Recalling that the function $\Phi : \mathbb{R} \to \mathbb{R}$ defined by $\Phi(x) = \frac{1}{x^x(1-x)^{1-x}}$ is continuous and $\Phi((0, \frac{1}{2})) = (1, 2)$, we immediately obtain.

Corollary 5.1. *For any real number d, $1 < d < 2$, there exists an algebra A such that* $\exp(A) = d$.

All algebras $A(m, w)$ constructed above are infinite dimensional. In case the word w is periodic we can actually construct a finite dimensional algebra B such that $\mathrm{Id}(B) = \mathrm{Id}(A(m, w))$ (and $\exp(B) = \exp(A(m, w))$). The construction is the following. Recall that given an infinite word on $\{0, 1\}$ and $m \geqslant 2$, the sequence $K_{m,w} = \{k_i\}_{i\geqslant 1}$ is defined by

$$k_i = \begin{cases} m, & \text{if } w_i = 0, \\ m+1, & \text{if } w_i = 1. \end{cases}$$

Definition 5.1. Let $K = K_{m,w} = \{k_i\}_{i\geqslant 1}$ be a sequence such that $m \geqslant 2$ and w is an infinite periodic word of period T. Then $B(K)$ is the algebra over F with basis

$$\{a, b\} \cup Z_1 \cup Z_2 \cup \cdots \cup Z_T$$

where

$$Z_i = \{z_j^{(i)} \mid 1 \leqslant j \leqslant k_i\}, \quad i = 1, 2, \ldots, T,$$

and multiplication table given by

$$z_2^{(i)} a = z_3^{(i)}, \ \ldots, \ z_{k_i-1}^{(i)} a = z_{k_i}^{(i)}, \ z_{k_i}^{(i)} a = z_1^{(i)}, \quad i = 1, 2, \ldots,$$
$$z_1^{(i)} b = z_2^{(i+1)}, \quad i = 1, 2, \ldots, (T-1)$$

and

$$z_1^{(T)} b = z_2^{(1)}.$$

All the remaining products are zero.

Proposition 5.2. *The algebras $A(K)$ and $B(K)$ satisfy the same identities.*

Proof. In order to distinguish between the elements of the basis of $A(K)$ and those of the basis of $B(K)$, we rename the elements $a, b, z_j^{(i)}$ of $B(K)$ as $\bar{a}, \bar{b}, \bar{z}_j^{(i)}$, respectively.

Let T be the period of w. It is easy to see that the linear map $\varphi : A(K) \to B(K)$ defined by

$$\varphi(a) = \bar{a}, \qquad \varphi(b) = \bar{b}, \qquad \varphi(z_j^{(i)}) = \bar{z}_j^{(i')}$$

where $1 \leqslant i' \leqslant T$ and $i \equiv i' \pmod T$, is an epimorphism of algebras. Hence $B(K)$ satisfies all the identities of $A(K)$.

Conversely, let $f = f(x_1, \ldots, x_n) \equiv 0$ be an identity of $B(K)$. Since $\operatorname{char} F = 0$, it is enough to prove that f is an identity of $A(K)$ in case f is multilinear. Consider the algebra $\bar{B} = B(K) \otimes_F F[t]$, where $F[t]$ is the polynomial ring in the indeterminate t. Then clearly $\bar{B}$ still satisfies $f \equiv 0$. If we let

$$\bar{A} = \operatorname{span}\{\bar{a} \otimes 1, \bar{b} \otimes t, \bar{z}_j^{(i)} \otimes t^{i-1+lT} \mid l \geqslant 0, \ 1 \leqslant i \leqslant T, \ 1 \leqslant j \leqslant k_i\},$$

then it is readily seen that $\bar{A}$ is actually a subalgebra of $\bar{B}$. Moreover the map $\varphi : \bar{A} \to A(K)$ such that

$$\varphi(\bar{a} \otimes 1) = a, \qquad \varphi(\bar{b} \otimes t) = b, \qquad \varphi\big(\bar{z}_j^{(i)} \otimes t^{i-1+lT}\big) = z_j^{i+lT},$$

extends to an isomorphism of algebras. Since $A(K)$ is isomorphic to a subalgebra of $\bar{B}$, it must satisfy the identity $f \equiv 0$, and the proof is complete. $\square$

Note that if w is an infinite periodic word, then its slope α is a rational number. Conversely, any positive rational number can be realized as the slope of an infinite periodic word.

The following result is an obvious consequence of Proposition 5.2 and Theorem 5.1.

Corollary 5.2. *For any rational number β, $0 < \beta \leqslant \frac{1}{2}$, there exists a finite dimensional algebra B such that* $\exp(B) = \Phi(\beta)$.

6. Gluing PI-algebras

We next wish to extend Theorem 5.1 and Corollary 5.1 to all real numbers > 1, i.e., we want to construct, for any real number $\alpha > 1$ an algebra A such that $\exp(A) = \alpha$. We shall accomplish this by constructing an appropriate algebra B and then by gluing, in an appropriate way, B to one of the algebras $A(m, w)$ constructed in the previous section.

Given any positive integer d we define a non-associative algebra $B = B(d)$ as follows: B has basis $\{u_1, \dots, u_d,\ s_1, \dots, s_d\}$ with multiplication table given by

$$s_1u_1 = u_2,\ \dots,\ s_{d-1}u_{d-1} = u_d,\ s_du_d = u_1,$$

and all other products are zero.

Now given a sequence of integers $K = \{k_i\}_{i\geqslant 1}$ let $A(K)$ be the algebra given in Definition 1. Starting with $A(K)$ and B, we next define an algebra $A(K, d)$ which will contain both $A(K)$ and B as subalgebras.

Definition 6.1. Let W be the vector space spanned by the set $\{w_{ij}^{(t)} \mid 1 \leqslant i \leqslant d,\ j \geqslant 1,\ t \geqslant 1\}$ and let $A(K, d)$ be the algebra which is the vector space direct sum of $A(K)$, B and W,

$$A(K, d) = A(K) \oplus B \oplus W.$$

The multiplication in $A(K, d)$ is induced by the multiplication in $A(K)$, B and $u_s z_j^i = w_{sj}^{(i)}$, $1 \leqslant s \leqslant d$, $1 \leqslant j \leqslant k_i$, $i \geqslant 1$, and all other products are zero.

We start by studying the identities of $B = B(d)$.

Lemma 6.1. *The algebra B satisfies the right-normed identity*

$$y_1\big(x_1 \cdots \big(x_{d-1}(y_2x_d)\big) \dots\big) \equiv y_2\big(x_1 \cdots \big(x_{d-1}(y_1x_d)\big) \dots\big) \tag{17}$$

and the left-normed identity $x_1x_2x_3 \equiv 0$.

Proof. The second statement is obvious. Now take an evaluation $x_i \mapsto \bar{x}_i \in B$, $i = 1, \dots, d$, $y_j \mapsto \bar{y}_j \in B$, $j = 1, 2$, and let $v_1 = \bar{y}_1(\bar{x}_1 \cdots (\bar{x}_{d-1}(\bar{y}_2\bar{x}_d))\dots)$ and $v_2 = \bar{y}_2(\bar{x}_1 \cdots (\bar{x}_{d-1}(\bar{y}_1\bar{x}_d))\dots)$. We will show that $v_1 = v_2$ and, since we are dealing with multilinear polynomials, we may restrict ourselves to substitutions into elements of a basis. If $v_1 = 0$ and $v_2 = 0$ then $v_1 = v_2$. Suppose one of the monomials, say v_1 is non-zero. Then $\bar{x}_d = u_i$ for some $1 \leqslant i \leqslant d$ and $\bar{y}_2 = s_i$, $\bar{x}_{d-1} = s_{i+1}, \bar{x}_{d-2} = s_{i+2}, \dots, \bar{x}_1 = s_{i-1}$ where all the indices of the s_j's are reduced modulo d. Also $\bar{y}_1 = s_i$ and $v_1 = u_{i+1}$. But then $\bar{y}_1 = \bar{y}_2$ and $v_2 = u_{i+1}$ follows. Therefore we are done. $\square$

It is clear that modulo the identity (17) any right-normed monomial of degree $n+1$ can be written in the following form:

$$x_{i_n}\big(x_{i_{n-1}}\big(\cdots(x_{i_1}x_j)\big)\big) \tag{18}$$

where

$$\begin{gathered} i_1 \leqslant i_{d+1} \leqslant i_{2d+1} \leqslant \cdots, \\ i_2 \leqslant i_{d+2} \leqslant i_{2d+2} \leqslant \cdots, \\ \vdots \end{gathered}$$

Write $n = qd + r = (q+1)r + q(d-r)$, with $0 \leqslant r < d$. Consider the following decomposition of $\{1, 2, \dots, n\}$:

$$\{1, 2, \dots, n\} = I_1 \cup \cdots \cup I_d$$

where

$$\begin{gathered} I_1 = \{i_1, i_{d+1}, i_{2d+1}, \dots, i_{qd+1}\}, \\ I_2 = \{i_2, i_{d+2}, i_{2d+2}, \dots, i_{qd+2}\}, \\ \vdots \\ I_d = \{i_d, i_{2d}, \dots, i_{qd}\}. \end{gathered}$$

and $|I_1| = \cdots = |I_r| = q+1$, $|I_{r+1}| = \cdots = |I_d| = q$. Denote by $m_j(I_1, \dots, I_d)$ the monomial (18).

The interesting property of the monomials $m_j(I_1, \dots, I_d)$ is given in the next lemma.

Lemma 6.2. *The monomials $m_j(I_1, \dots, I_d)$ are linearly independent modulo* $\mathrm{Id}(B)$*, the T-ideal of identities of B.*

Proof. Clearly any evaluation $\phi : X \to B$, such that $\phi(x_j) = u_1$, $\phi(x_i) = s_k$ maps $m_j(I_1, \dots, I_d)$ to a non-zero value and all other monomials $m_{j'}(I'_1, \dots, I'_d)$ with $\{j', I'_1, \dots, I'_d\} \neq \{j, I_1, \dots, I_d\}$ to zero, as soon as $i \in I_k$. In fact in this case $\phi(m_j(I_1, \dots, I_d)) = \cdots (s_1(s_d \cdots (s_1u_1))) \neq 0$. $\square$

We next compute the nth codimension of B.

Lemma 6.3. *Let $n = qd + r$, $0 \leqslant r < d$. Then*

(1) $c_{n+1}(B) = (n+1)\binom{n}{k_1,\dots,k_d}$ *where* $\binom{n}{k_1,\dots,k_d} = \frac{n!}{k_1!\cdots k_d!}$ *is the generalized binomial coefficient and* $k_1 = \cdots = k_r = q+1$, $k_{r+1} = \cdots = k_d = q$;

(2) *for any multilinear polynomial* $f(x_1,\dots,x_{n+1})$ *not vanishing on* B *there exists an evaluation* $\phi_f : X \to B$ *such that* $\phi_f(f) = u_k$, *for some* $1 \leqslant k \leqslant d$. *Moreover for any multilinear polynomial* f' *in* $x_1,\dots,x_{n+1}$, $\phi_f(f') = \lambda(f')u_k$, *for some* $\lambda(f') \in F$.

Proof. By Lemma 6.1 any multilinear polynomial is a linear combination of right-normed monomials of type $m_j(I_1,\dots,I_d)$ and, by Lemma 6.2 these monomials are linearly independent modulo $\mathrm{Id}(B)$. Since the number of such monomials is $(n+1)\binom{n}{k_1,\dots,k_d}$, the first part of the lemma is proved.

In order to prove (2), as in the proof of Lemma 6.2, notice that any evaluation $\phi(x_j) = u_1$, $\phi(x_i) = s_k$ for all $i \in I_k$, $k = 1,\dots,d$, maps any monomial of the given type, except $m_j(I_1,\dots,I_d)$, to zero. The latter monomial is mapped to some u_k, $1 \leqslant k \leqslant d$. Clearly, any linear combination of basic monomials evaluates to $\lambda \cdot u_k$, for some scalar λ. $\square$

Next we want to estimate the codimensions of the algebra $A(K,d)$. Fix n and consider the space P_n of multilinear polynomials in $x_1,\dots,x_n$. Denote for short any right-normed product $y_1(\cdots(y_{m-1}y_m)\cdots)$ by $[y_1\cdots y_{m-1}y_m]$.

In the next lemma we find a set of generators of $P_n(A(K,d))$.

Lemma 6.4. *Let $A(K,d)$ be the algebra defined above. Then*

$$P_n = \bigoplus_{I\subseteq\{1,\dots,n\}} V_I \quad \bigl(\mathrm{mod}\ \mathrm{Id}\bigl(A(K,d)\bigr)\bigr)$$

where V_I is the subspace spanned by all monomials of the type

$$[x_{i_1}\cdots x_{i_k}](x_{j_1}\cdots x_{j_{n-k}}) \tag{19}$$

with $\{i_1,\dots,i_k\} = I$ *and* $\{j_1,\dots,j_{n-k}\} = \{1,\dots,n\}\setminus I$.

Proof. It is readily checked that the algebra $A(K)$ satisfies the identity $[x_1x_2x_3] \equiv 0$ and the algebra $B + W$ satisfies the identity $x_1x_2x_3 \equiv 0$. It follows that all monomials except the ones in (19) are identities of $A(K,d)$. Hence

$$P_n = \sum_{I\subseteq\{1,\dots,n\}} V_I \quad \bigl(\mathrm{mod}\ \mathrm{Id}\bigl(A(K,d)\bigr)\bigr).$$

Suppose that

$$f = \sum_{I\subseteq\{i_1,\dots,i_k\}} f_I \in \mathrm{Id}\bigl(A(K,d)\bigr),$$

where $f_I \in V_I$. Fix a subset I and show that f_I is also an identity of $A(K,d)$. Let $\phi : X \to A(K,d)$ be any evaluation such that $\phi(x_i)$ is some element in a fixed basis of $A(K,d)$, for all

$i=1,\dots,n$. If $\phi(x_j)\notin B+W$ for at least one $j\in I$, then $\phi(V_I)=0$ and $\phi(f_I)=0$. On the other hand, if $\phi(x_j)\in B+W$ for all $j\in I$ then, by the above, $\phi(V_{I'})=0$ for all $I'\neq I$. Hence also in this case $\phi(f_I)=\phi(f)=0$.

It follows that modulo $\mathrm{Id}(A(K,d))$ the sum $\sum_I V_I$ is direct and the proof is complete. □

Lemma 6.5. *Let $I\subseteq\{1,\dots,n\}$, $|I|=k$, and let $c_k(B)$ and $c_{n-k}(A(K))$ be the codimensions of B and $A(K)$, respectively. Then*

$$\dim V_I=c_k(B)\cdot c_{n-k}\big(A(K)\big).$$

Proof. Write $p=c_k(B)$, $q=c_{n-k}(A(K))$ and suppose for short that $I=\{1,\dots,k\}$. By Lemma 6.2 all monomials $m_j(J_1,\dots,J_d)$, $J_1\cup\dots\cup J_d=\{1,\dots,k\}$, $1\leqslant j\leqslant k$, form a basis of P_k modulo $\mathrm{Id}(B)$. Rename these monomials as $m'_1,\dots,m'_p$ and fix some multilinear polynomials $m''_1,\dots,m''_q$ in $x_{k+1},\dots,x_n$ linearly independent modulo $\mathrm{Id}(A(K))$. Then V_I is spanned modulo $\mathrm{Id}(A(K,d))$, by all products $m'_i\cdot m''_j$, $1\leqslant i\leqslant p$, $1\leqslant j\leqslant q$. Let us check that all these products are linearly independent modulo $\mathrm{Id}(A(K,d))$.

Suppose

$$\sum_{i,j}\lambda_{ij}m'_im''_j=h(x_1,\dots,x_n)=h$$

is an identity of $A(K,d)$ and let one of the λ_{ij}'s, say λ_{11} be non-zero. We can write h as

$$h=f_1m''_1+\dots+f_qm''_q$$

where

$$f_j=f_j(x_1,\dots,x_k)=\sum_i\lambda_{ij}m'_i$$

and f_1 is not an identity of B. By Lemma 6.3 there exists an evaluation $\phi:\{x_1,\dots,x_k\}\to B$ such that $\phi(f_1)=u_i$ and $\phi(f_j)=\gamma_ju_i$ for all $j=2,\dots,n$. Since h is an identity of $A(K,d)$, then

$$u_i\psi\big(m''_1\big)+\gamma_2u_i\psi\big(m''_2\big)+\dots+\gamma_qu_i\psi\big(m''_q\big)=0$$

for any evaluation $\psi:\{x_{k+1},\dots,x_n\}\to A(K)$. Thus

$$u_i\psi\big(m''_1+\gamma_2m''_2+\dots+\gamma_qm''_q\big)=0. \tag{20}$$

Since the polynomials $m''_1,\dots,m''_q$ are linearly independent modulo $\mathrm{Id}(A(K))$, the polynomial $f'(x_{k+1},\dots,x_n)=m''_1+\gamma_2m''_2+\dots+\gamma_qm''_q$ is not an identity of $A(K)$. Hence there exists a non-zero evaluation of f' in $A(K)$. If $n-k\geqslant 2$ then any non-zero value of f' is of the form

$$\sum_{r,j}\alpha_{rj}z_j^{(r)}.$$

If $n-k=1$ then $f'(x_n)=\alpha x_n$, for some $\alpha\in F$, and clearly $z_1^{(1)}$ is one of the values of f'. In all cases (20) takes the form

$$u_i\left(\sum_{r,j}\alpha_{rj}z_j^{(r)}\right)=\sum_{r,j}\alpha_{rj}w_{ij}^{(r)}=0,$$

a contradiction, since one of the α_{rj}'s is non-zero.

We have proved that the elements $m_i'm_j''$ are linearly independent and span V_I modulo $\mathrm{Id}(A(K,d))$. Hence $\dim V_I = pq = c_k(B)c_{n-k}(A(K))$ and the proof of the lemma is complete. □

7. Algebras with real exponential growth > 1

We start with an easy technical lemma.

Lemma 7.1. *Let $\alpha_n, \beta_n, \gamma_n$, $n = 1, 2, \ldots$, be three sequences of real numbers such that*

(1) *there exist constants $C_1, \ldots, C_4, d_1, \ldots, d_4, q_2, q_4 > 0$ and $q_1, q_3 < 0$, such that*

$$C_1 n^{q_1} d_1^n \leqslant \alpha_n \leqslant C_2 n^{q_2} d_2^n,$$
$$C_3 n^{q_3} d_3^n \leqslant \beta_n \leqslant C_4 n^{q_4} d_4^n,$$

for all $n \geqslant 1$;

(2) $\gamma_n = \sum_{k=0}^{n}\binom{n}{k}\alpha_k\beta_{n-k}$.

Then

$$C_1C_3 n^{q_1+q_3}(d_1+d_3)^n \leqslant \gamma_n \leqslant C_2C_4 n^{q_2+q_4}(d_2+d_4)^n.$$

Proof. Clearly,

$$\begin{aligned}\gamma_n &\leqslant \sum_{k=0}^{n}\binom{n}{k}C_2C_4k^{q_2}(n-k)^{q_4}d_2^k d_4^{n-k}\\ &\leqslant C_2C_4n^{q_2+q_4}\sum_{k=0}^{n}\binom{n}{k}d_2^k d_4^{n-k} = C_2C_4n^{q_2+q_4}(d_2+d_4)^n.\end{aligned}$$

The lower bound is computed similarly. □

In order to apply Lemma 7.1 we need to bound the codimensions of B.

Lemma 7.2. *For any $\varepsilon > 0$ there exists N such that for all $n > N$, the nth codimension of $B = B(d)$ satisfies the inequalities*

$$\left(\frac{1}{6e}\right)^d \frac{1}{n^d} d^n < c_n(B) < 6n^d(d+\varepsilon)^n.$$

Proof. Write $n = qd + r$, with $0 \leqslant r < d$. By Lemma 6.3, since $k_1 = \cdots = k_r = q + 1$ and $k_{r+1} = \cdots = k_d = q$, we obtain

$$c_n(B) = n\binom{n-1}{k_1, \ldots, k_d} = n\frac{(n-1)!}{((q+1)!)^r (q!)^{d-r}} = \frac{n!}{((q+1)!)^r (q!)^{d-r}}.$$

Using Stirling's formula we obtain

$$\frac{n^n}{e^n} < n! < 6n\frac{n^n}{e^n},$$
$$\frac{q^{qd}}{e^{qd}} < (q!)^d < (6q)^d\frac{q^{qd}}{e^{qd}}.$$

Hence, since $qd = n - r$,

$$c_n(B) < 6n\left(\frac{n}{q}\right)^n \cdot \frac{q^r}{e^r} < 6n^{r+1}\left(d + \frac{rd}{n-r}\right)^n.$$

For n large enough we get $\frac{rd}{n-k} < \varepsilon$, and $c_n(B) < 6n^d(d+\varepsilon)^n$ as required. Similarly,

$$c_n(B) > \left(\frac{1}{6q}\right)^d \left(\frac{n}{q}\right)^n \frac{q^r}{e^r} > \left(\frac{1}{6ne}\right)^d \left(\frac{n}{q}\right)^n \geqslant \left(\frac{1}{6e}\right)^d \frac{1}{n^d} d^n$$

and the proof of the lemma is complete. □

Combining all previous results we can now prove the main theorem of this section. Recall that if $K_{m,w}$ is the sequence defined by the integer $m \geqslant 2$ and by the periodic or Sturmian word w, then the algebra $A(K_{m,w}) = A(m, w)$ satisfies the conclusion of Corollary 5.1.

Theorem 7.1. *Let $m \geqslant 2$ and let w be a periodic or Sturmian word. Then the PI-exponent of the algebra $A(K_{m,w}, d)$ exists and $\exp(A(K_{m,w}, d)) = d + \delta$ where $\delta = \exp(A(K_{m,w}))$.*

Proof. Let α be the slope of w, $0 < \alpha < 1$ and let $\beta = \frac{1}{m+\alpha}$. If $\delta = \Phi(\beta)$ then by Lemmas 5.1 and 5.3 we have that asymptotically

$$C_1 n^{q_1}(\delta - \varepsilon)^n \leqslant c_n\big(A(m, w)\big) \leqslant C_2 n^{q_2}(\delta + \varepsilon)^n$$

for any $\varepsilon > 0$. Also by Lemma 7.2, $C_3 n^{q_3} d^n \leqslant c_n(B) \leqslant C_4 n^{q_4}(d+\varepsilon)^n$ for some constants C_i, q_i, $i = 1, \ldots, 4$.

Since by Lemmas 6.4 and 6.5,

$$c_n\big(A(K_{m,w}), d\big) = \sum_{k=0}^{n} \binom{n}{k} c_k(B) c_{n-k}\big(A(K_{m,w})\big),$$

we can apply Lemma 7.1 and obtain that

$$C n^q (d + \delta - \varepsilon)^n \leqslant c_n\big(A(K_{m,w}), d\big) \leqslant C' n^{q'}(d + \delta + 2\varepsilon)^n,$$

for some constants C, C', q, q' and for any $\varepsilon > 0$.

This readily implies that $\exp(A(K_{m,w})) = \lim_{n\to\infty} \sqrt[n]{c_n(A(K_{m,w}), d)} = d + \delta$ and the proof is complete. □

As an immediate consequence of Theorems 5.1 and 7.1 we obtain.

Corollary 7.1. *For any real number $t \geqslant 1$ there exists an algebra R such that* $\exp(R) = t$.

Another consequence of the previous theorem together with Corollary 5.2 is the following.

Corollary 7.2. *For any $1 \leqslant \alpha < \beta$ there exists a finite dimensional algebra R such that $\alpha < \exp(B) < \beta$.*

Recall that the PI-exponent of any finite dimensional associative or Lie algebra always exists and is an integer [5,22]. In [8] we showed that for general non-associative finite dimensional algebra A either $c_n(A)$ is polynomially bounded or asymptotically $c_n(A) \geqslant \delta^n$ where δ is an explicit function of $\dim A$. At the light of Corollary 7.2 and recalling the results about associative and Lie algebras it is worth asking if the PI-exponent exists for any finite dimensional algebra. Also, is the set of all possible values of $\exp(A)$, $\dim A < \infty$, countable?

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