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Lie properties of symmetric elements in group rings

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ABSTRACT

Let $*$ be an involution of a group G extended linearly to the group algebra KG. We prove that if G contains no 2-elements and K is a field of characteristic $p \neq 2$, then the $*$-symmetric elements of KG are Lie nilpotent (Lie n-Engel) if and only if KG is Lie nilpotent (Lie n-Engel).

1. Introduction

Let R be a ring with an involution $*$. Let $R^+ = \{r \in R \mid r^* = r\}$ be the set of symmetric elements of R under $*$ and $R^- = \{r \in R \mid r^* = -r\}$ the set of skew symmetrics. A general question of interest is which properties of R^+ or R^- can be lifted to R (see [10]). For example, a classical result of Amitsur [1] states that if R^+ (or R^-) satisfies a polynomial identity, then so does R.

Group rings are naturally endowed with an involution; the one obtained as a linear extension of the involution of G given by $g \mapsto g^{-1}$. We shall refer to this as the *classical involution*. For this particular involution, Giambruno and Sehgal [5] classified group algebras KG of groups with no 2-elements such that $(KG)^+$ is Lie nilpotent and G. Lee completed this work [12]. The implications of commutativity of $(KG)^+$ and $(KG)^-$ have also been investigated [2,3].

Recently, there has been a surge of activity in studying more general involutions of KG; namely, the maps obtained from arbitrary involutions of G, extended linearly to KG. Properties of $(KG)^+$ and

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$(KG)^-$ were considered in [4,11] and, recently, Gonçalves and Passman considered the existence of bicyclic units u in the integral group rings such that the group $\langle u, u^* \rangle$ is free [8]. Marciniak and Sehgal had proved that, with respect to the classical involution, $\langle u, u^* \rangle$ is always free if $u \neq 1$ [14]. Also, Gonçalves and Passman constructed free pairs of unitary units in group algebras [7]. Still another type of involution has been of interest, the *oriented involutions*, which are linear extensions of involutions of G, twisted by a homomorphism $G \to \{\pm 1\}$. The latter were introduced by Novikov [15], in the context of K-theory.

Throughout this paper * will denote an involution of KG obtained as a linear extension of an involution of G. We prove the following.

Theorem A. *Let G be a group with no 2-elements and K a field of characteristic $p \neq 2$. Then, $(KG)^+$ is Lie n-Engel if and only if KG is Lie m-Engel.*

Theorem B. *Let G be a group with no 2-elements and K a field of characteristic $p \neq 2$. Then, $(KG)^+$ is Lie nilpotent if and only if KG is Lie nilpotent.*

2. Some basic facts and notations

We collect important facts for use in later sections and set up some notation. For a given prime integer p, an element $x \in G$ will be called a p-element if its order is a power of p. We write

$$P = \{x \in G \mid x \text{ is a } p\text{-element}\},$$

$$Q = \left\{x \in G \mid x^q = 1, \text{ for some integer } q, \ (2p, q) = 1\right\},$$

$$G^+ = \left\{g \in G \mid g^* = g\right\}.$$

The following are some basic results. We recall that a group G is said to be p-abelian if G', the commutator group of G, is a finite p-group.

Theorem 2.1. *(See [19, Theorem V.6.1].) Let K be a field of characteristic $p > 0$. Then KG is Lie n-Engel if and only if G is nilpotent and contains a normal p-abelian subgroup A such that G/A is a finite p-group.*

Theorem 2.2. *(See Passi, Passman and Sehgal [16].) The group algebra KG is Lie nilpotent if and only if G is nilpotent and p-abelian.*

Theorem 2.3. *(See Giambruno and Sehgal [5].) Assume $\mathrm{char}(K) \neq 2$ and that G contains no 2-elements. If, with respect to the classical involution, $(KG)^+$ is Lie nilpotent then KG is Lie nilpotent.*

Lemma 2.4. *For any semiprime ring with involution R which is Lie n-Engel, with $2R = R$, we have $[R^+, R^+] = 0$ and R satisfies St_4, the standard identity in four variables.*

Proof. We first remark that R^+, being Lie n-Engel, satisfies a polynomial identity. Hence by a result of Amitsur [1], R is a PI-ring i.e., satisfies an ordinary polynomial identity.

Consider a prime ideal P of R. If $P^* \neq P$, then $S = (P + P^*)/P$ is a nonzero ideal of R/P and $a + a^* + P = a^* + P$, for any $a \in P$. Since R^+ is Lie n-Engel, for any $a, b \in P$, $[a^* + P, b^* + P, \ldots, b^* + P] = [a + a^*, b + b^*, \ldots, b + b^*] + P = P$. Hence the prime ring S is Lie n-Engel. Since S is also a prime PI-ring, by Posner's theorem [6, Theorem 1.11.13] its central localization $A = S \otimes_Z F$ is a finite dimensional simple algebra over F, where $Z \neq 0$ is the center of S and F is the field of fractions of Z. Moreover S and A satisfy the same polynomial identities, hence also A is Lie n-Engel. If we now apply Wedderburn theorem and then tensor with a splitting field F of A, we obtain $m \times m$ matrices over F being still Lie n-Engel. A direct inspection shows that $m = 1$ in this case. Hence A, and so S is commutative. Recalling that S is an ideal of the prime ring R/P we also get that R/P is commutative and, so, $[R, R] \subseteq P$.

In case $P^* = P$, the ring $\bar{R} = R/P$ is a prime PI-ring with induced involution. Moreover since $2R = R$, char $\bar{R} \neq 2$ and the symmetric elements of $\bar{R}$ are Lie n-Engel. By Posner's theorem the central localization A of $\bar{R}$ is a finite dimensional simple algebra with induced involution and A^+ is Lie n-Engel. After tensoring with the algebraic closure F of the center of A, we obtain $M_m(F)$, the algebra of $m \times m$ matrices over F with induced involution. Moreover $M_m(F)^+$ is Lie n-Engel. By [6, Theorem 3.6.8] we may assume that the involution on $M_m(F)$ is either the transpose or the symplectic involution and a direct inspection on $M_m(F)$ shows that the Lie n-Engel property forces $[M_m(F)^+, M_m(F)^+] = 0$ and $m \leqslant 2$. It follows that $\bar{R}$ satisfies St_4 and $[\bar{R}^+, \bar{R}^+] = 0$.

The outcome of the above is that $St_4(r_1, \ldots, r_4) \in \bigcap P$, for all $r_1, \ldots, r_4$ in R, and $[R^+, R^+] \subset \bigcap P$ where the intersection runs over all prime ideals of R. Since R is semiprime, $\bigcap P = 0$ and the proof is complete. $\square$

We remind the reader that a group G is called LC if it is not commutative and for every pair of elements $g, h \in G$ we have that $gh = hg$ if and only if either g or h or gh is central in G. The LC groups with a unique nonidentity commutator are precisely those groups G with center $\mathcal{Z}(G)$ such that $G/\mathcal{Z}(G) \cong C_2 \times C_2$, a direct product of two cyclic groups of order 2 (see [9, Proposition III.3.6]).

We shall need the following.

Theorem 2.5. *(See Jespers and Ruiz [11].) Let K be a field of characteristic different from 2. Then $(KG)^+$ is commutative if and only if G is either abelian or an LC group with a unique nonidentity commutator, which is of order 2.*

The next lemma has also been proved by G. Lee [13] in a different manner.

Lemma 2.6. *Let K be a field of characteristic $p > 2$, G a finite group and J the Jacobson radical of KG. Suppose that KG/J is isomorphic to a direct sum of simple algebras of dimension at most four over their center. Then P is a subgroup.*

Proof. First we observe that if we extend the base field K to its algebraic closure $\bar{K}$, then $\bar{K}G/\bar{J}$ still satisfies the hypothesis. Actually $\bar{K}G/\bar{J}$ is isomorphic to a direct sum of copies of $\bar{K}$ and 2×2 matrices over $\bar{K}$. Hence we may assume that K is algebraically closed.

Next we claim that the hypothesis is inherited by subgroups and homomorphic images of G. In fact, by the Amitsur–Levitski theorem, KG/J satisfies St_4, the standard identity of degree four. Moreover, as KG is a finite dimensional algebra, J is nilpotent, say $J^k = 0$. But then St_4^k is a polynomial identity of KG. Let now H be a subgroup of G. As a subalgebra of KG, KH and also $KH/J(KH)$ still satisfies the polynomial identity. Now, $KH/J(KH)$, being semisimple decomposes as $KH/J(KH) = M_{n_1}(K) \oplus \cdots \oplus M_{n_t}(K)$ and each $M_{n_i}(K)$ satisfies St_4^k. Being a prime algebra, it follows that each $M_{n_i}(K)$ satisfies St_4. By the Amitsur–Levitski theorem this implies that $n_i \leqslant 2$. Hence $KH/J(KH)$ has the desired decomposition. A similar proof holds for homomorphic images of G.

Let $g, h \in G$ be p-elements and let H be the subgroup they generate. Our aim is to show that gh is a p-element. Since the hypothesis is inherited by subgroups, without loss of generality we may assume that $G = H$.

In case every irreducible representation of G is of degree one, i.e., KG/J is isomorphic to copies of K, then KG/J is commutative. Hence $\Delta(G, G') \subseteq J$ where $\Delta(G, G')$ is the kernel of the natural projection $KG \to K(G/G')$. Since J is nilpotent, $\Delta(G')$ is nilpotent and, so, G' is a p-group. This says that the p-elements of G form a subgroup and gh is a p-element, as desired.

Therefore we may assume that $G = \langle g, h \rangle$ has at least one irreducible representation of degree two. Under these hypotheses we shall reach a contradiction.

Let V be a 2-dimensional vector space over K on which G acts irreducibly. By taking the quotient with the kernel T of the corresponding representation, we may also assume that G acts faithfully on V (notice that G/T still satisfies the hypotheses of the lemma). Define the subspaces

$$V_g = \{v \in V \mid (g-1)v = 0\} \quad \text{and} \quad V_h = \{v \in V \mid (h-1)v = 0\}.$$

Thus $g-1 \in \mathrm{End}(V)$ is such that $\mathrm{Im}(g-1) = (g-1)V$ and $\ker(g-1) = V_g$. This says that $\dim V - \dim V_g = \dim(g-1)V$. Now, since $g^{p^n} = 1$, for some n, $(g-1)^{p^n} = 0$; since $\dim V = 2$, we get $(g-1)^2 = 0$ and, so, $(g-1)^2 V = 0$. It follows that $(g-1)V \subseteq V_g$ and by the above, $2 - \dim V_g \leqslant \dim V_g$. Thus $\dim V_g \geqslant 1$. Similarly $\dim V_h \geqslant 1$.

Suppose that $V_g \cap V_h \neq 0$. Since $(g-1)V_g = 0$, g acts trivially on V_g and, so on $V_g \cap V_h$. Similarly also h acts trivially on $V_g \cap V_h$. This implies that $G = \langle g, h \rangle$ acts trivially on $V_g \cap V_h$ and this contradicts the fact that G acts faithfully on V. Thus $V_g \cap V_h = 0$ and $V = V_g \oplus V_h$.

Notice that $g-1 : V_h \to V_g$ maps V_h into V_g. Also, if $v \in \ker(g-1)$, then $(h-1)v = 0$ and $(g-1)v = 0$ implies that $v \in V_g \cap V_h = 0$. Thus $g-1$ is an isomorphism. Similarly $h-1 : V_g \to V_h$ is an isomorphism.

Choose $v \in V_h$, $v \neq 0$. Then $(g-1)v \in V_g$ and is nonzero. This says that $\{(g-1)v, v\}$ is a basis of V. In this basis g and h have matrices $A_g = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $A_h = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$, respectively, where $\alpha \in K$ is nonzero.

Now, the matrices A_g and A_h generate the group $SL(2,q)$, for some q a power of p. It follows that the hypotheses of the lemma hold for $SL(2,q)$ and also for its subgroup $SL(2,p)$. But it is known that $SL(2,p)$ has irreducible representations of degree $1, 2, \ldots, p$ and this is a contradiction since $\mathrm{char}\, K > 2$. □

Lemma 2.7. *Let G be a finite group, K a field of characteristic $p > 2$ and * an involution on G. If $(KG)^+$ is Lie n-Engel then P is a subgroup.*

Proof. Let J be the Jacobson radical of KG. Then $R = KG/J$ is a semisimple algebra with induced involution and R^+ is Lie n-Engel. By Lemma 2.4 R satisfies St_4. Hence if we write KG/J as a sum of simple algebras A_i, each A_i satisfies St_4. Since any simple algebra of dimension m^2 over its center does not satisfy any identity of degree less than $2m$, we deduce that R is isomorphic to a direct sum of simple algebras of dimension at most four over their center. Therefore the group G satisfies the hypotheses of the previous lemma and P is a subgroup. □

Lemma 2.8. *Let K be a field of characteristic $p > 0$. If $(KG)^+$ is Lie n-Engel, then for every symmetric element g of G, g^{p^n} is central.*

Proof. Let $g \in G^+$. If $x \in G^+$ then $[x, g, \ldots, g] = 0$ implies $xg^{p^n} = g^{p^n}x$. So assume that $x \neq x^*$. Then $[x + x^*, g^{p^n}] = 0$ and

$$(x + x^*)g^{p^n} = g^{p^n}(x + x^*).$$

Hence, either $xg^{p^n} = g^{p^n}x$ or $xg^{p^n} = g^{p^n}x^*$.

Suppose $xg^{p^n} = g^{p^n}x^*$. Then $xg^{p^n} \in G^+$ and by the first part $(xg^{p^n})g^{p^n} = g^{p^n}(xg^{p^n})$. Then, we can cancel g^{p^n} on the right and obtain again that $xg^{p^n} = g^{p^n}x$, as desired. □

Lemma 2.9. *Assume A is an abelian group with no 2-elements and let $^* : A \to A$ be an automorphism of order 2. Then*

$$A^2 \subset A_1 \times A_2,$$

where $A_1 = \{a \in A \mid a^* = a\}$ and $A_2 = \{a \in A \mid a^* = a^{-1}\}$.

Proof. Given $b = a^2 \in A^2$, write

$$b = (aa^*)\big(a(a^*)^{-1}\big).$$

This gives the required decomposition. □

Corollary 2.10. *If A is an abelian torsion group with no 2-elements and $* : A \to A$ is an automorphism of order 2 then*

$$A = A_1 \times A_2,$$

where $A_1 = \{a \in A \mid a^* = a\}$ and $A_2 = \{a \in A \mid a^* = a^{-1}\}$.

Lemma 2.11. *Let G be any group, K a field of characteristic $p > 2$ and $*$ an involution on G. Let A be a torsion abelian normal subgroup of G, without elements of order 2, and let $x \in G \setminus A$ be an element such that $x^* = x^{-1}c$ with $c \in A$. Then, there exists a symmetric element $b \in A$ such that $(xb)^* = (xb)^{-1}$.*

Proof. Write $A = A_1 \times A_2$, with $A_1 = \{a \in A \mid a^* = a\}$ and $A_2 = \{a \in A \mid a^* = a^{-1}\}$ as in Corollary 2.10. Notice that $xx^* = c$ is in A and is symmetric, so actually $c \in A_1$. Also $x^{-1}cx \in A_1$.

As A_1 has no elements of order 2, we can find $b \in A_1$ such that $b^2 = x^{-1}c^{-1}x$. This means that $b^{-1}x^{-1} = bx^{-1}c$ and thus $(xb)^{-1} = bx^{-1}c = (xb)^*$, as desired. □

Theorem 2.12. *Let G be a finite group of odd order, K a field of characteristic $p > 2$ and $*$ an involution on G. If $(KG)^+$ is Lie n-Engel, then KG is Lie nilpotent.*

Proof. By Lemma 2.7, P is a subgroup of G. Since $(|G/P|, |P|) = 1$ by the theorem of Schur–Zassenhaus we can write $G = P \rtimes X$ with X a p'-group. Since $(KG)^+$ is Lie n-Engel, by Lemma 2.4 and Theorem 2.5 X is abelian. It follows that G is a p-abelian group and by Theorem 2.2, in order to complete the proof it is enough to show that G is nilpotent. Now, since P is nilpotent, it is actually enough to prove that G/P' is nilpotent.

If $P' \neq 1$ we are done, by induction. Hence, we may assume that $P' = 1$ and, thus, that P is abelian. Suppose that there exists an invariant subgroup $H = H^* \neq 1$ contained in ζ, the center of G. Since $(K(G/H))^+$ and $(KH)^+$ are both n-Engel, by induction we get that G/H and H are nilpotent. Hence, G is nilpotent and we are done. Therefore, without loss of generality, we may assume that G contains no central element $z \neq 1$, as $\langle z, z^*\rangle$ would then give a central subgroup invariant under $*$.

Since $|X^*| = |X|$, it follows that X^* is another complement to P so, by Schur–Zassenhaus, X^* is conjugate to X; i.e. $X^* = X^y$ for some $y \in P$. For an element $x \in X$, let $x_1 \in X$ be such that

$$x^* = y^{-1}x_1y.$$

Then $xx^* = xy^{-1}x_1y = xx_1(x_1, y)$ and $(x_1, y) \in P$. It follows that $(xx^*)^{p^n} = (xx_1)^{p^n}d$ for some element $d \in P$. Thus, by Lemma 2.8, $[(xx_1)^{p^n}, P] = 0$. Hence $xx_1 \in \zeta$. This implies $xx_1 = 1$; i.e. $x_1 = x^{-1}$. Thus, $x^* = y^{-1}x^{-1}y$, for all $x \in X$. So, we can write $x^* = x^{-1}xy^{-1}x^{-1}y = x^{-1}c$ with $c \in P$.

We shall prove that, for any fixed element $x \in X$, we have that $(x, P) = 1$. As P is of odd order, by Lemma 2.9, we have $P = A_1 \times A_2$ where $A_1 = \{a \in P \mid a^* = a\}$ and $A_2 = \{a \in P \mid a^* = a^{-1}\}$.

By Lemma 2.11 there exists an element $b \in A_1$ such that $(xb)^* = (xb)^{-1}$.

Since $(xb, P) = 1$ implies $(x, P) = 1$, we may assume hereafter, that $x^* = x^{-1}$. Since $(KG)^+$ is Lie n-Engel, $[a + a^*, (x + x^{-1})^{p^n}] = 0$ for all $a \in P$. Hence $[a + a^*, x^{p^n} + (x^{-1})^{p^n}] = 0$. Notice that $x^{p^n} = (x^{-1})^{p^n} \pmod P$ implies $x^{2p^n} = 1 \pmod P$ and thus also $x = 1 \pmod P$. So, x and x^{-1} belong to different cosets of P in G, and we get that $[a + a^*, x^{p^n}] = 0$. Since x is a p'-element, we get $[a + a^*, x] = 0$. If $xa \neq ax$ we must have $a^*x = xa$ and $ax = xa^*$. From this we have $xax^{-1} = a^*$ and $xa^*x^{-1} = a$. Combining, we conclude that x^2 commutes with a. Thus $xa = ax$ for all $a \in P$.

This says that X is a normal subgroup of G, so $G = P \times X$ and G is nilpotent. □

The following is an easy but useful observation.

Lemma 2.13. *Let K be any field of characteristic different from 2 and G any group. For any element $c \in G$ not of even order, the element $1 + c \in KG$ is not a zero divisor.*

Proof. We first remark that if $1 + c$ is a zero divisor in KG, then it is also a zero divisor in $K\langle c\rangle$. If $o(c) = \infty$ then $K\langle c\rangle$ contains no zero divisors. So, assume that $o(c) = p^k q$ with $(p, q) = 1$, where $p = \operatorname{char}(K) > 2$. Then c^{p^k} has order q, $(q, 2) = 1$ so, in each component of the Wedderburn decomposition of $K\langle c\rangle$ it maps to a root of unity different from -1. Consequently, $1 + c^{p^k}$ is a unit in $K\langle c\rangle$. Thus $(1 + c)^{p^k} = 1 + c^{p^k}$ is a unit and so $1 + c$ is also a unit. A similar argument holds if $\operatorname{char}(K) = 0$. □

Remark 2.14. Let G be a finite group and K a field of characteristic $p > 2$ such that $(KG)^+$ is Lie n-Engel. Then it follows from Lemma 2.7 that P is a subgroup, so Lemma 2.4 gives that $(KG/P)^+$ is commutative and then Theorem 2.5 shows that G/P is either abelian or an LC group with a unique nonidentity commutator, which is an element of order 2.

We shall need the following generalization of Theorem 2.12 where we do not require that $|G|$ is odd.

Lemma 2.15. *Let G be a finite group and K of characteristic $p > 2$ such that $(KG)^+$ is Lie n-Engel. If G/P is abelian, then G is nilpotent.*

Proof. Notice that, if $|G|$ is odd, we already know, from Theorems 2.12 and 2.2 that G is nilpotent. We shall give a proof for arbitrary finite groups G, by induction on $|G|$.

If G has a central element $z \neq 1$, as before we have that $\langle z, z^*\rangle$ is a central subgroup invariant under * so, by induction, $G/\langle z, z^*\rangle$ is nilpotent and we are done. We show that this is always the case by proving that, if $\zeta = 1$ then $G = 1$.

Since G/P is abelian, it can be written in the form $G/P = (M/P) \times (N/P)$ where M/P is of odd order and N/P is a 2-group. Clearly, M is invariant under *, so $(KM)^+$ is Lie n-Engel and M is nilpotent. Thus, we can write $M = P \times Q$, with $(|Q|, p) = 1$.

We claim that the elements of Q also commute with 2-elements. In fact, take $q \in Q$ and let t be a 2-element. Then $(q, t) \in Q$ and, since G/P is commutative, $G' \subset P$ so actually $(q, t) \in Q \cap P = 1$, as desired. This shows that Q is central, so $Q = 1$. Consequently, we can write $G = P \rtimes T$, where T is a 2-group.

Pick $x \in T$ such that $x^2 = 1$ (and thus, also $(x^*)^2 = 1$). Then $(xx^*)^{p^n} \in \zeta$ so $(xx^*)^{p^n} = 1$ and $xx^* \in P$. Consequently, $x^* = xc$ for some element $c \in P$ and, $x = x^{**} = c^*xc$, $c^* = xc^{-1}x^{-1}$. As $c = xx^*$ is symmetric, we have $cx = xc^{-1}$. We compute

$$0 = \left[c, \left(x + x^*\right)^{p^n}\right] = \left[c, \left(x(1 + c)\right)^{p^n}\right],$$

thus $c(x(1 + c))^{p^n} = (x(1 + c))^{p^n} c$. As

$$\left(x(1 + c)\right)^{p^n} = x^{p^n}(1 + c)\left(1 + c^{-1}\right)(1 + c)\cdots\left(1 + c^{-1}\right)(1 + c),$$

we get

$$\left(cx^{p^n} - x^{p^n}c\right)(1 + c)\left(1 + c^{-1}\right)(1 + c)\cdots\left(1 + c^{-1}\right)(1 + c) = 0$$

and Lemma 2.13 shows that $x^{p^n}c = cx^{p^n}$, so also $xc = cx$. Hence $c^2 = 1$ so $c = 1$ and $x^* = x$ is symmetric. By Lemma 2.8, $x^{p^n} \in \zeta$ so $x^{p^n} = 1$ and, as $x^2 = 1$ we get $x = 1$. This shows that $G = P$ is abelian and $G = 1$ □

3. $(KG)^+$ Lie n-Engel

In this section, we shall assume throughout that $\mathrm{char}(K) = p > 2$ and that G has no 2-elements. We shall also assume that $(KG)^+$ is Lie n-Engel. This implies that KG satisfies a $*$-polynomial identity, so it also satisfies a polynomial identity by a theorem of Amitsur [1]. It then follows from a theorem of Passman [17, p. 197] that G has a normal p-abelian subgroup A of finite index. We can assume A is $*$ invariant by replacing it by $A \cap A^*$; i.e. we have the following.

Remark 3.1. If KG^+ is Lie n-Engel, then there exists a normal subgroup A of G, which is $*$-invariant and such that G/A is finite and A' is a finite p-group.

Proposition 3.2. *P is a subgroup and G/P is abelian.*

Proof. We wish to prove that if $x, y \in P$ then $xy \in P$. We can assume, without loss of generality, that $G = \langle x, y, x^*, y^* \rangle$. Since G is finitely generated and A is of finite index in G, we have that A is also finitely generated. Since A' is invariant under $*$, we may factor by A' and assume that A is abelian. Then, we have $A = F \times T$, where F is finitely generated free abelian and T is finite. We write $|T| = p^m s$, with $(p, s) = 1$.

Set $A_1 = A^{p^m s} = F^{p^m s}$ and consider G/A_1. Then G/A_1 is of finite order, say $p^\ell t$ with $(p, t) = 1$. Now $(xy)^{p^\ell}$ is an element of order dividing $t \pmod{A_1}$ and by Lemma 2.7 it is a p-element, so $(xy)^{p^\ell} = 1 \pmod{A_1}$.

A similar argument shows that, for any positive integer r such that $(r, p) = 1$, replacing A_1 by A_1^r we get that $(xy)^{p^\ell} = 1 \pmod{A_1^r}$. Hence $(xy)^{p^\ell} \in \bigcap_r A_1^r = 1$. Consequently, we have $(xy)^{p^\ell} = 1$, as claimed.

Since $(KG/P)^+$ is Lie n-Engel, it follows from Lemma 2.4 that $(KG/P)^+$ is commutative. By Theorem 2.5 then G/P is either abelian or an LC group with a unique nonidentity commutator, which is an element of order 2. Since G/P has no 2 elements, it follows that it is abelian. $\square$

Corollary 3.3. *$T = \{x \in G \mid o(x)$ is finite$\}$ is a subgroup and $T = P \times Q$ where Q is central in G.*

Proof. By Proposition 3.2, G/P is abelian and, thus, T is a subgroup of G. By using Remark 3.1 we see that T is locally finite.

Hence, by the finite case, it follows that $T = P \times Q$. Further, for $x \in G$ we have that $(x, Q) \subset P \cap Q$ as $Q \lhd G$ and G/P is abelian. Hence $(x, Q) = 1$, as claimed. $\square$

Proposition 3.4. *Suppose that there exists a normal subgroup A of G, invariant under $*$, such that A is abelian and G/A is of odd order. Let $x \in G$ be an element whose order, modulo A is $q \neq 2$, relatively prime to p. Then $(x, A) = 1$.*

Proof. We know, from Proposition 3.2 that G/P is abelian, so $G' \subset P$. Take $a \in A$; we want to show that $(x, a) = 1$. Thus, we can assume that $G = \langle x, a, x^*, a^* \rangle$ is finitely generated. In this case, A is a normal subgroup of finite index in a finitely generated group, so it is finitely generated and we can write

$$A = F \times T, \quad \text{where } F \text{ is torsion free and } T \text{ is finite.}$$

Suppose $|T| = p^m \ell$, with $(2p, \ell) = 1$ and set $A_1 = A^{p^m \ell} = F^{p^m \ell}$. Since $x^q \in A$ we see that $x^{qp^m \ell} \in A_1$ so the order of x^{p^m}, modulo A_1, divides $q\ell$. As G/A_1 has no 2-elements, Corollary 3.3 shows that $(x^{p^m}, a) = 1 \pmod{A_1}$. Also, as $x^q \in A$, we have $(x^{q\ell}, a) = 1$.

Thus, $(x, a) = 1 \pmod{A_1}$ and, since $G' \subseteq P$, $(x, a) = 1 \pmod{P \cap A_1}$. Since $(P \cap A_1) = (P \cap F^{p^m \ell}) = 1$, the proposition is proved. $\square$

Proposition 3.5. *Assume that there exists a normal subgroup A of G, invariant under *, such that G/A and A' are both finite p-groups and A' is abelian. Then G acts as a finite p-group of automorphisms on A'.*

Proof. As A' is finite, normal, G does act as a finite group of automorphisms, by conjugation. We need to show that every element $x \in G$ acts as a p-element. Since G/A is a finite p-group, we may assume that $x \in A$. It will suffice to show that

$$\left(x^{p'}, A'\right) = 1 \quad \text{implies} \quad (x, A') = 1 \quad \text{for any prime } p \neq p'. \tag{1}$$

It should be mentioned that it follows from Theorems 2.12 and 2.2 that if G is finite then G is nilpotent and thus the proposition is true in this case.

If $(x^{p'}, A') = 1$ then $((x^*)^{p'}, A') = 1$ and thus also $((xx^*)^{p'}, A') = 1$. We know, by Lemma 2.8 that $(xx^*)^{p^n}$ is central. This implies that $(xx^*, A') = 1$.

We shall handle separately the cases when $p' \neq 2$ and when $p' = 2$.

(a) Let $p' = q \neq 2$.

Define $B = \{y \in A \mid (y^q, A') = 1\}$.

It is easy to see that B is a *-invariant normal subgroup containing x and x^* and clearly B/A' is abelian. By Lemma 2.9, for any $x \in B$ we can write $x^2 = x_1x_2$ where $x_1^* = x_1 \pmod{A'}$ and $x_2^* = x_2^{-1} \pmod{A'}$. We shall prove separately that $(x_1, A') = 1$ and $(x_2, A') = 1$.

Since, $x_1^* = x_1c$, with $c \in A'$, the group $H = \langle x_1, A'\rangle$ is *-invariant. Set $\zeta = \zeta(H)$. The group H/ζ is finite as $x_1^q \in \zeta$. Thus, in this factor group we get $(\overline{x_1}, \overline{A'}) = 1$ and therefore, for any $a \in A'$ we have $x_1^{-1}ax_1 = az$, with $z \in \zeta$. Consequently, $a = x_1^{-q}ax_1^q = az^q$, so $z^q = 1$ and thus also $z = 1$. It follows that $(x_1, A') = 1$.

Now, consider $x_2^* = x_2^{-1}c$, with $c \in A'$. Then, by Lemma 2.11, we can modify x_2 by an element of A' and assume that $x_2^* = x_2^{-1}$. From the hypothesis, for any $a \in A'$ we have

$$0 = \left[\left(x_2 + x_2^{-1}\right)^{p^n}, a + a^*\right] = \left[x_2^{p^n} + x_2^{-p^n}, a + a^*\right].$$

If $x_2^{p^n}A' = x_2^{-p^n}A'$ then $\langle x_2, A'\rangle$ is a finite, *-invariant group, and we are done. So, let $x_2^{p^n}A' \neq x_2^{-p^n}A'$. Consequently

$$\left[x_2^{p^n}, a + a^*\right] = 0.$$

Hence, either $x_2^{p^n}a = ax_2^{p^n}$ and, thus $x_2a = ax_2$ (as $x_2^qa = ax_2^q$), or $x_2^{p^n}a = a^*x_2^{p^n}$ and $x_2^{p^n}a^* = ax_2^{p^n}$ which implies $(x_2^{2p^n}, a) = 1$. Since also $(x_2^q, A') = 1$ we conclude that $(x_2, A') = 1$.

For any $x \in B$, we have shown that $(x^2, A') = 1$. Since also $(x^q, A') = 1$, we can conclude again that $(x, A') = 1$, proving (a).

Now, let us prove case (b) when $p' = 2$. Using Corollary 2.10, we can write $A' = B_1 \times B_2$ where $b_1^x = b_1$ and $b_2^x = b_2^{-1}$ for all $b_1 \in B_1$, $b_2 \in B_2$. We claim that $B_2 = 1$. In fact, take $b \in B_2$. We have

$$xx^*b = bxx^* = xb^{-1}x^*.$$

Thus

$$x^*b\left(x^*\right)^{-1} = b^{-1}, \qquad x^{-1}b^*x = \left(b^*\right)^{-1}.$$

This implies that $b^* \in B_2$ and $x^*b^*(x^*)^{-1} = (b^*)^{-1}$. As $(KG)^+$ is Lie n-Engel, we have

$$\left[bb^*, \left(x + x^*\right)^{p^n}\right] = 0 \quad \text{and so} \quad bb^*\left(x + x^*\right)^{p^n} = \left(x + x^*\right)^{p^n}bb^*.$$

Since x and x^* both invert b and b^* and because p is odd, we get

$$\left(x+x^*\right)^{p^n}\left(bb^*\right)^{-1} = \left(x+x^*\right)^{p^n} bb^*.$$

Since $x+x^*$ is not a zero divisor by Lemma 2.13, we have $(bb^*)^{-1} = bb^*$ and, as G contains no 2-elements, we conclude that $bb^* = 1$, thus $b^* = b^{-1}$.

We need still another calculation:

$$xb + (xb)^* = xb + b^*x^* = xb + b^{-1}x^* = xb + x^*b = \left(x+x^*\right)b.$$

Thus $[(x+x^*)b, (x+x^*)^{p^n}] = 0$. Consequently

$$\left(x+x^*\right)b\left(x+x^*\right)^{p^n} = \left(x+x^*\right)^{p^{n+1}} b$$

and

$$\left(x+x^*\right)^{p^{n+1}} b^{-1} = \left(x+x^*\right)^{p^{n+1}} b.$$

As above, $b^2 = 1$ and also $b = 1$. This proves (1) and the proposition. □

Lemma 3.6. *Assume that there exists a normal subgroup A of G, invariant under $*$, such that A is abelian and G/A is a finite p-group. Then, A^{p^n} is central.*

Proof. Take $x \in G$, $a \in A$. Then

$$0 = \left[x+x^*, \left(a+a^*\right)^{p^n}\right] = \left[x+x^*, b+b^*\right],$$

where $b = a^{p^n}$. If $xA \neq x^*A$, then $[x, b+b^*] = 0$. Otherwise, $x^* = xc$, with $c \in A$ so $x + x^* = x(1+c)$ and $[x(1+c), b+b^*] = 0$. Since $1+c$ is not a zero divisor, we obtain again that $[x, b+b^*] = 0$.

We claim that $xb = bx$. If not, $xb = b^*x$ and $xb^* = bx$. Thus $xbx^{-1} = b^*$, $xb^*x^{-1} = b$ and $x^2bx^{-2} = b$. We have that $(x^2, b) = 1$. Also $x^{p^k} \in A$, for some integer k; therefore $(x, b) = 1$. This proves our statement. □

Lemma 3.7. *Assume that there exists a normal subgroup A of G, invariant under $*$, such that A is abelian and G/A is finite. If there exists an element $x \in G$ such that $x^2 \in A$ and $xA = x^*A$, then $(x, A) = 1$.*

Proof. Consider $A_1 = \{a \in A \mid a^x = a\}$ and $A_2 = \{a \in A \mid a^x = a^{-1}\}$. From the hypothesis, we have that $x^* = xc$, with $c \in A$. For an element $a \in A_1$ we compute $xa^*x^{-1} = x^*a^*(x^{-1})^* = (a^x)^* = a^*$, which shows that A_1 is $*$-invariant. We set $\bar{G} = G/A_1$.

Since Lemma 2.9 shows that $A^2 \subset A_1 \times A_2$ we have $\bar{A}^2 \subset \overline{A_2}$ and thus $\bar{a}^{\bar{x}} = \bar{a}^{-1}$, for all $a \in A^2$.

We claim that $\bar{A}^2 = 1$. Take $a \in A^2$. As Lemma 2.9 also applies to $*$, we shall consider separately the cases when $a^* = a$ and $a^* = a^{-1}$.

If $a^* = a$ we have that $[a, (x+x^*)^{p^n}] = 0$ and so $a(x+x^*)^{p^n} = (x+x^*)^{p^n}a$. This implies $(x+x^*)^{p^n}a^{-1} = (x+x^*)^{p^n}a$ and, as Lemma 2.13 shows that $(x+x^*)$ is not a zero divisor, we get $a^2 = 1$, and thus also $a = 1$, as desired.

If $a^* = a^{-1}$, then $xa + (xa)^* = xa + a^*x^* = xa + x^*(a^*)^{-1} = xa + x^*a = (x+x^*)a$. Then, by hypothesis, $[(x+x^*)a, (x+x^*)^{p^n}] = 0$. Therefore,

$$\left(x+x^*\right)^{p^{n+1}} a = \left(x+x^*\right)a\left(x+x^*\right)^{p^n}.$$

Since $a(x+x^*) = (x+x^*)a^{-1}$ we get $(x+x^*)^{p^{n+1}}a = (x+x^*)^{p^{n+1}}a^{-1}$.

Again, $(x + x^*)$ is not a zero divisor, by Lemma 2.13, so $a = a^{-1}$ which implies $a = 1$.

This proves our claim. Hence, we see that $A^2 \subset A_1$ so $(x, a^2) = 1$ for all $a \in A$; since A is abelian, this implies that $(x, a)^2 = 1$ and since G contains no 2-elements, we get $(x, a) = 1$ for all $a \in A$. □

Proof of Theorem A. By Remark 3.1, if $\mathrm{char}(K) = p > 2$, G has no 2-elements and $(KG)^+$ is Lie n-Engel, then G contains a normal p-abelian subgroup A of finite index which is *-invariant. Choose A maximal among the groups with these properties. We shall show that G/A is a finite p-group.

First we note that, as A' is a finite p-group, we may factor by A' and assume that A is abelian. Set $H = G/A$ and let P denote the p-Sylow subgroup of H. By Remark 2.14, H/P is either abelian or an LC group with a unique nonidentity commutator of order 2.

We shall first handle the case when H/P is abelian. Then, by Corollary 3.3, $H = P \times Q \times T$ where T is a 2-group, $|Q| = q$ is odd and relatively prime to p and both Q and T are abelian. Thus, there exists a subgroup L such that $A \lhd L \lhd G$, $L/A \cong Q$, $G/L \cong P \times T$ and L is *-invariant.

For any $x \in L$ we have that $(x^q, A) = 1$ and, by Proposition 3.4, also $(x, A) = 1$. Thus, A is central in L and the index of L over its center is a divisor of q. It follows, by Schur's theorem [19, Theorem I.4.2] that $|L'|$ is a divisor of q. Also $L' \subset G'$, so L' is a p-group; thus $L' = 1$ and L is abelian, which implies $L = A$; hence $Q = 1$.

Now we have $H = P \times T$ so we can find a subgroup N such that $A \lhd N \lhd G$, $G/N \cong P$ and $N/A \cong T$.

Let $N_1 = \{b \in N \mid b^2 \in A\}$. Take $x \in N_1$, $x \neq 1$. Then $x^2 = 1 \pmod{A}$, $(x^*)^2 = 1 \pmod{A}$ and, as N/A is abelian, also $(xx^*)^2 = 1 \pmod{A}$. Moreover, by Lemma 2.8, $(xx^*)^{p^n}$ is central. It follows that $(xx^*, A) = 1$. Then $\langle xx^*, A\rangle$ is abelian and, as $A \subset \langle xx^*, A\rangle$, from the maximality of A we get $xx^* \in A$.

Thus $x^*A = x^{-1}A = xA$. Then, by Lemma 3.7, $(x, A) = 1$ and again, from the maximality of A, it follows that $x \in A$. We conclude that $N_1 = 1$, so also $N = 1$ and $H = P$, as desired.

Now we consider the remaining case, namely when H/P is LC with a unique nonidentity commutator, which is of order 2. Let M be the subgroup of G containing A such that $M/A = P$, the p-Sylow subgroup of $G/A = H$. Let $z \in G$ be an element such that zM is the unique commutator of order 2 in G/M. Then, $z^* = z \pmod{M}$. Consider $L = \langle M, z\rangle$. By the abelian case $z \in A$. Hence, we have $A \lhd M \lhd G$ with G/M abelian. Again from the previous case, it follows that G/A is a p-group.

To complete the proof of necessity, after Theorem 2.1, we need to prove that G is nilpotent. Since A' is a finite p-group it is nilpotent, so it will suffice to prove that G/A'' is nilpotent. Hence, we may assume that A' is an abelian finite p-group.

It follows from Proposition 3.5 that G acts as a finite p-groups of automorphisms on A'. Hence, by [19, Lemma V.4.1], we have that $A' \subset \zeta_r(G)$, for some positive integer r. Thus, we can assume that $A' = 1$ and that A is abelian.

By Lemma 3.6, A^{p^n} is central, so we may factor by A^{p^n} and assume that A is of bounded p-power exponent. As G/A is a finite p-group, G acts as a finite p-group of automorphisms on A, so we can use again [19, Lemma V.4.1] to obtain that $A \subset \zeta_s(G)$ for some positive integer s. Since G/A is a finite p-group, it is nilpotent, and hence G is nilpotent, as desired.

The case when $\mathrm{char}(K) = 0$ is easy to see. The hypothesis implies that $(\mathbb{Z}/p\mathbb{Z})G$ is Lie n-Engel, for any prime integer p, so G' is a p-group. Since p is arbitrary, it follows that $G' = 1$.

The converse is trivial. □

4. Lie nilpotency

In this section, we shall prove our second main result, namely that if G is a group with no 2-elements, K is a field of characteristic $p \neq 2$, $(KG)^+$ is Lie nilpotent, then KG is Lie nilpotent.

Since Lie nilpotency implies n-Engel, for some positive integer n, we can apply Theorem A. Thus, we can assume that $p > 2$ and we know that G is nilpotent and that it contains a p-abelian, normal, subgroup A, which is invariant under *, such that G/A is a finite p-group. Hence, according to Theorem 2.2, we are left only to prove that G' is a finite p-group and we already know, from Proposition 3.2 that G' is a p-group.

We shall now prove a crucial special case of Theorem B.

Theorem 4.1. *Let G be a group with no 2-elements and K a field of characteristic $p > 2$, such that $(KG)^+$ is Lie nilpotent. Suppose that G contains a normal subgroup A, invariant under $*$, such that G/A is a cyclic p-group. Then G' is a finite p-group.*

Proof. Since the theorem is clearly true for finite groups, we shall assume that A is infinite.

Let $x \in G$ be such that $G/A = \langle xA \rangle$. It suffices to prove that the group $(x, A) \subset A$ is finite. Since G contains no 2-elements, it will be enough to prove that (x, A^2) is finite. We recall that, by Lemma 2.9, we have $A^2 \subset A_1 \times A_2$, where $A_1 = \{a \in A \mid a^* = a\}$ and $A_2 = \{a \in A \mid a^* = a^{-1}\}$.

Our first aim is to reduce to the case when $A_2 = 1$.

So, let $A_2 \neq 1$. If $A_2 \cap \zeta$ is infinite then, by the arguments in [5, p. 4257], KG is Lie nilpotent and G' is finite. Thus, we may assume that $A_2 \cap \zeta$ is finite. Since A is bounded modulo the center by Lemma 3.6, we conclude that A_2 is bounded. Write

$$B = (x, A_2) = \{(x, a_2) \mid a_2 \in A_2\}.$$

As $1 = (x, a_2^{p^m}) = (x, a_2)^{p^m}$, we see that B is a group of bounded p-exponent. Suppose B is infinite. Then, by [18, Theorem 4.3.5], $B = \prod_i B_i$, an infinite direct product of cyclic groups. For an arbitrary positive integer s, we are going to produce elements $a_i \in A_2$ such that, after a possible renumbering of the indices, $(x, a_i) \in B_i$, $1 \leqslant i \leqslant s$, so that

$$e = [x + x^*, a_1 + a_1^*, a_2 + a_2^*, \ldots, a_s + a_s^*] \neq 0.$$

This will be a contradiction proving B is finite. Notice that

$$e = [x, a_1 + a_1^*, a_2 + a_2^*, \ldots, a_s + a_s^*] + [x^*, a_1 + a_1^*, a_2 + a_2^*, \ldots, a_s + a_s^*]$$

vanishes if and only if each of the two summands vanishes as can be seen by considering the two cases $xA = x^*A$ and $xA \neq x^*A$. It will therefore suffice to find $a_1, a_2, \ldots, a_s$ so that

$$[x, a_1 + a_1^*, a_2 + a_2^*, \ldots, a_s + a_s^*] \neq 0.$$

For $s = 1$ we pick $a_1 \in A_2$ such that $1 \neq (x, a_1) \in B_1$. Then, it can be checked directly that $[x, a_1 + a_1^*] \neq 0$.

Let us suppose that we already have $a_1, \ldots, a_{s-1}$ as stated. Let N be the normal closure of $\langle a_1, \ldots, a_{s-1} \rangle$. Then N is a finite abelian group as A_2 is of bounded exponent and every element has a finite number of conjugates. Remember that, as $(x, ab) = (x, a)(x, b)$, every element of B is a commutator. Thus there exists an index s so that $B_s \cap N = 1$. Choose $a_s \in A_2$ so that $1 \neq (x, a_s) \in B_s$. Then also $(x, a_s^2) = (x, a_s)^2 \neq 1$ and so $a_s^2 \notin N$.

We know already, by induction that

$$[x, a_1 + a_1^*, a_2 + a_2^*, \ldots, a_{s-1} + a_{s-1}^*] = x\alpha \neq 0, \quad \alpha \in KN.$$

Therefore

$$\begin{aligned} e &= [x, a_1 + a_1^*, a_2 + a_2^*, \ldots, a_s + a_s^*] \\ &= [x, a_s + a_s^{-1}]\alpha = x(a_s + a_s^{-1} - a_s^x - a_s^{-x})\alpha. \end{aligned}$$

We claim that a_sN is not equal to $a_s^{-1}N$, a_s^xN or $a_s^{-x}N$. In the first case $a_s^2 \in N$ and in the second $(x, a_s) \in N$, both contradictions. If $a_sN = a_s^{-x}N$, then $x^{-1}a_s^{-1}x = a_s \pmod{N}$, so $x^{-2}a_sx^2 = a_s \pmod{N}$. Also a p-power of x commutes with a_s, consequently $(x, a_s) \in N$, again a contradiction. Thus $xa_s\alpha \neq 0$ as $\alpha \neq 0$, and $e \neq 0$ as desired.

We have proved that $B = (x, A_2)$ is finite. Let $\bar{B}$ be the finite group which is the normal and * closure of B. Then $(KG/\bar{B})^+$ is Lie nilpotent. Also, the image of A_2 in $G/\bar{B}$ is central. Thus, A_2 can be assumed to be finite modulo $\bar{B}$ (otherwise $(G/\bar{B})'$ is finite and so is G'). Consequently, A_2 is finite. Factoring by the normal and * closure of A_2 we can assume that A is infinite and that $A = A_1$; i.e. $a^* = a$ for all $a \in A$ as we wanted.

Since G is nilpotent, by induction on its class of nilpotency, we can assume that $(G/\zeta)'$ is finite. Thus $G'\zeta/\zeta \cong G'/\zeta \cap G'$ is finite. If $\zeta \cap G'$ is finite, we are done. Suppose $B = (A, x) \cap \zeta$ is infinite. We shall show that this leads to a contradiction, completing the proof.

Again B is of bounded exponent, as $1 = (a^{p^m}, x) = (a, x)^{p^m}$ and we can write $B = \prod_i B_i$, an infinite direct product of cyclic groups.

We observe

$$\bigl[x + x^*, a\bigr] = [x, a] + \bigl[x^*, a\bigr] = xa\bigl(1 - (a, x)\bigr) + x^*a\bigl(1 - \bigl(a, x^*\bigr)\bigr).$$

Let us choose $a_i \in A$ such that $1 \neq (a_i, x) \in B_i$. Then

$$\bigl[x + x^*, a_1, \ldots, a_s\bigr] = xa_1 \cdots a_s\bigl(1 - (a_1, x)\bigr) \cdots \bigl(1 - (a_s, x)\bigr) + x^*a_1 \cdots a_s\bigl(1 - \bigl(a_1, x^*\bigr)\bigr) \cdots \bigl(1 - \bigl(a_s, x^*\bigr)\bigr).$$

This expression is not zero for any s, as can be seen by considering the two cases $xA = x^*A$ or $xA \neq x^*A$. This contradiction proves the theorem. □

Proof of Theorem B. Since $(KG)^+$ is Lie nilpotent, it is also Lie n-Engel, for some positive integer n. Thus, by Theorem A, also KG is Lie n-Engel and, by Theorem 2.1, G is nilpotent and there exists a normal subgroup A of G such that both G/A and A' are finite p-groups. By replacing A by $A \cap A^*$, we can assume that A is normal and *-invariant.

We wish to prove that G' is finite. It is a p-group, as P is a subgroup and G/P is abelian by Proposition 3.2.

By induction on the class of nilpotency of G we may assume that G is metabelian. Therefore, G/A is metabelian.

If G/A is cyclic, we are done by Theorem 4.1. Let us first assume that G/A is abelian. Then $G/A = G_1/A \times G_2/A$ where $g_1^* = g_1 \pmod{A}$ and $g_2^* = g_2^{-1} \pmod{A}$, for all $g_1 \in G_1$, $g_2 \in G_2$. Then (g_1, A) is finite and so (G_1, A) is finite. Similarly, (G_2, A) is finite. Thus (G, A) is finite. Factoring by (G, A) we conclude that A is central and G, being central by finite, has a finite derived group.

Let us now consider the general case, namely, assume there exists a normal subgroup L of G containing A such that both G/L and L/A are abelian and L can be assumed *-invariant. Then, by the abelian case, L' is finite. Factoring by L', we may assume that L is abelian, and we are done by the special case. Again, the converse is trivial. □

5. Note added in proof

The classification in Theorems A and B has now been completed for all groups by G. Lee, E. Spinelli and S.K. Sehgal in "Lie properties of symmetric elements in group rings II" which is to appear in the Journal of Pure and Applied Algebra.

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