J. Math. Anal. Appl. 359 (2009) 780–785

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Three periodic solutions for perturbed second order Hamiltonian systems

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ARTICLE INFO

Article history:
Received 24 October 2008
Available online 21 June 2009
Submitted by R. Manásevich

Keywords:
Second order Hamiltonian systems
Periodic solutions
Critical point

ABSTRACT

In this paper we study the existence of three distinct solutions for the following problem

$$-\ddot{u}+A(t)u=\nabla F(t,u)+\lambda\nabla G(t,u) \quad \text{a.e. in } [0,T],$$
$$u(T)-u(0)=\dot{u}(T)-\dot{u}(0)=0,$$

where $\lambda\in\mathbb{R}$, T is a real positive number, $A:[0,T]\to\mathbb{R}^{N\times N}$ is a continuous map from the interval $[0,T]$ to the set of N-order symmetric matrices. We propose sufficient conditions only on the potential F. More precisely, we assume that G satisfies only a usual growth condition which allows us to use a variational approach.

1. Introduction

Consider the following problem

$$\begin{aligned}&-\ddot{u}+A(t)u=\nabla F(t,u)+\lambda\nabla G(t,u) \quad \text{a.e. in } [0,T],\\&u(T)-u(0)=\dot{u}(T)-\dot{u}(0)=0,\end{aligned} \tag{P_λ}$$

where $\lambda\in\mathbb{R}$, T is a real positive number, $A:[0,T]\to\mathbb{R}^{N\times N}$ is a continuous map from the interval $[0,T]$ to the set of N-order symmetric matrices, $F,G:[0,T]\times R^N\to\mathbb{R}$ are measurable with respect to t, for every $x\in\mathbb{R}^N$, continuously differentiable in x, for almost every $t\in[0,T]$ and satisfy the following standard summability condition:

$$\sup_{|x|\leqslant c}\left(\max\left\{\left|F(\cdot,x)\right|,\left|G(\cdot,x)\right|,\left|\nabla F(\cdot,x)\right|,\left|\nabla G(\cdot,x)\right|\right\}\right)\in L^1\left([0,T]\right) \tag{1.1}$$

for all $c>0$.

Note that the above condition is satisfied, for instance, simply assuming ∇F and ∇G continuous in $[0,T]\times\mathbb{R}^N$.

Moreover, without loss of generality, it is supposed that

$$F(t,0)=G(t,0)=0 \quad \left(\text{for a.e. } t\in[0,T]\right).$$

Then consider the space of functions

$$H^1_T=\left\{u:[0,T]\to\mathbb{R}^N \;\middle|\; u \text{ is absolutely continuous}, u(0)=u(T) \text{ and } \dot{u}\in L^2\left(0,T;\mathbb{R}^N\right)\right\}$$

endowed with the norm

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.06.049

$$\|u\| = \left(\int_0^T |u(t)|^2\,dt + \int_0^T |\dot{u}(t)|^2\,dt \right)^{\frac{1}{2}} = \left(\|u\|_2^2 + \|\dot{u}\|_2^2 \right)^{\frac{1}{2}},$$

where $\|\cdot\|_2 = (\int_0^T |\cdot|^2)^{\frac{1}{2}}$ is the canonical norm of $L^2(0,T;\mathbb{R}^N)$.

H_T^1 is a Hilbert space and, by embedding theorems, it is compactly embedded into $C^0(0,T;\mathbb{R}^N)$.

We recall that $u \in H_T^1$ is said to be a solution of (P_λ) if

$$\int_0^T (\dot{u}(t), \dot{v}(t))\,dt + \int_0^T \big(\big(A(t)u(t) - \nabla F(t,u(t)) - \lambda \nabla G(t,u(t)) \big), v(t) \big)\,dt = 0,$$

for all $v \in H_T^1$, where $(\cdot,\cdot)$ is the standard scalar product in $\mathbb{R}^N$.

Then we consider the functionals $\Psi, \Phi : H_T^1 \to \mathbb{R}$ defined as follows

$$\Psi(u) = \frac{1}{2} \left(\int_0^T \big(|\dot{u}(t)|^2\,dt + \big(A(t)u(t), u(t)\big) \big)\,dt \right) - \int_0^T F(t,u(t))\,dt,$$

and

$$\Phi(u) = -\int_0^T G(t,u(t))\,dt.$$

Condition (1.1) implies that Ψ and Φ are well-defined, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous in H_T^1. By Corollary 1.1 in [7], the solutions to (P_λ) in H_T^1 are exactly the critical points of $\Psi + \lambda\Phi$.

By spectral theorem for compact self-adjoint operators on a Hilbert space (see [7, p. 89]), the differential operator $u \to -\ddot{u} + \mathbf{A}u$, with $(-\ddot{u} + \mathbf{A}u)(t) = -\ddot{u}(t) + A(t)u(t)$, has a sequence of eigenfunctions which is an orthogonal basis for H_T^1 and the following decomposition holds:

$$H_T^1 = H^+ \oplus H^- \oplus H^0,$$

where

$$\begin{aligned} H^+ &= \overline{\operatorname{span}}\big\{u \in H_T^1\colon -\ddot{u} + \mathbf{A}u = \lambda u \text{ with } \lambda > 0\big\}, \\ H^- &= \operatorname{span}\big\{u \in H_T^1\colon -\ddot{u} + \mathbf{A}u = \lambda u \text{ with } \lambda < 0\big\}, \\ H^0 &= \ker\{-\ddot{u} + \mathbf{A}u\}, \end{aligned}$$

and one has $\dim(H^-) < +\infty$ and $\dim(H^0) < +\infty$.

Denote by $\lambda_1(A)$ the lowest eigenvalue of $-\ddot{u} + \mathbf{A}u$, by Proposition VI.9 in [2], it can be characterized by

$$\lambda_1(A) = \inf_{u \in H_T^1,\ \|u\|=1} \left(\int_0^T \big(|\dot{u}(t)|^2 + \big(A(t)u(t), u(t)\big) \big)\,dt \right).$$

Under these settings, our main result, Theorem 3.1, assures that problem (P_λ) admits at least three distinct solutions, for λ in a suitable neighbourhood of zero, provided that the following further assumptions, only on the potential F, are satisfied:

(F_1) $\lim_{|x|\to+\infty} (\frac{1}{2}\lambda_1(A)|x|^2 - F(t,x)) = +\infty$, uniformly in $[0,T]$.

(F_2) There exists $\delta > 0$ such that $\frac{1}{2}\lambda_1(A)|x|^2 - F(t,x) > 0$, for all $x \in \mathbb{R}^N \setminus \{0\}$ with $|x| < \delta$ and a.e. $t \in [0,T]$.

(F_3) There exists $x_0 \in \mathbb{R}^N$ such that $\int_0^T (A(t)x_0, x_0)\,dt < \int_0^T F(t,x_0)\,dt$.

It is worth stressing out that such solutions belong to a ball of H_T^1 centered in the origin and with suitable radius which does not depend on λ.

When $\lambda_1(A)$ is positive, in order to verify (F_1) and (F_2) it is sufficient to suppose $F(t,\cdot)$ superquadratic at 0 and subquadratic at $+\infty$ uniformly with respect to $t \in [0,T]$.

Many authors dealt with multiplicity results for second order Hamiltonian systems. Focusing our attention on those ones which carried out their studies about the three periodic solutions, we cite Tang and Wu [9–11]. More recently other contributions to this topic has been given by Cordaro in [3] and Faraci in [5,6]. However, at our best knowledge, there are not many results of the type of Theorem 3.1 proposed here. All of the papers quoted above consider the problem without the perturbation term ∇G. We also note that, instead of [3,5], our multiplicity result is proved without assuming the positive definiteness of the matrix $A(\cdot)$ in $[0,T]$.

In order to prove Theorem 3.1, we use a recent result, proved by Fan and Deng in [4], which shows a more convenient way to apply Theorem 1 of [8] in some concrete cases. It is also worth of stressing out that our proof relies on a general mountain pass lemma without (P.S.) condition, Theorem 2.8 of [12], which allows us to consider perturbations only satisfying the usual growth conditions.

2. Preliminary results

In this section we give some preliminary lemmas. The first concerns with the component H^0 of the space H_T^1. We omit the rather technical proof which can be found in [1] (see proof of Lemma 3.2).

Lemma 2.1. *For each $\epsilon > 0$ there exists a constant $M(\epsilon) > 0$ such that*

$$m\big(\big\{t \in [0,T]\colon \big|u^0(t)\big| < M(\epsilon)\big\|u^0\big\|\big\}\big) < \epsilon \quad \forall u^0 \in H^0,$$

where $m(\cdot)$ denotes the Lebesgue measure.

Lemma 2.2. *There exists $\tilde{\lambda} > 0$ such that*

$$\Psi_1(u) = \int_0^T \big|\dot{u}(t)\big|^2\,dt + \int_0^T \big(A(t)u(t), u(t)\big)\,dt \geqslant \tilde{\lambda}\|u\|^2$$

for every $u \in H^+$.

Proof. We argue by contradiction. Suppose that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq H^+$, with $\|u_n\| = 1$, such that $\lim_{n\to+\infty}\Psi_1(u_n) \leqslant 0$. Up to a subsequence which is denoted by $\{u_n\}$ again, there exists $u^* \in H^+$ such that $u_n \to u^*$ weakly as $n \to +\infty$. Exploiting the weakly sequentially lower semicontinuity of Ψ_1, one has

$$\Psi_1\big(u^*\big) \leqslant \liminf_{n\to+\infty}\Psi_1(u_n) \leqslant 0.$$

Consequently, being $\Psi(u^*) \geqslant 0$ since $u^* \in H^+$, it results that $\Psi(u^*) = 0$ hence $u^* = 0$. Now the compact embedding of H_T^1 into $C^0(0,T;\mathbb{R})$ assures us that $u_n \to 0$ strongly in $C^0(0,T;\mathbb{R})$, as $n \to +\infty$. Then

$$\lim_{n\to+\infty}\|\dot{u}_n\|_2^2 = \lim_{n\to+\infty}\left(\Psi_1(u_n) - \int_0^T \big(A(t)u_n(t), u_n(t)\big)\,dt\right) = 0.$$

So $\{u_n\}$ strongly converges to 0 in H_T^1 which is absurd since by hypothesis $\|u_n\| = 1$, for every $n \in \mathbb{N}$. □

Lemma 2.3. *If hypothesis (F_1) holds then Ψ is coercive.*

Proof. We first assume that $\lambda_1(A) = 0$.

So, let $\{u_n\}_{n\in\mathbb{N}} \subseteq H_T^1$ be a sequence such that $\lim_{n\to+\infty}\|u_n\| = +\infty$. We have $u_n = u_n^+ + u_n^0$, with $u_n^+ \in H^+$ and $u_n^0 \in H^0$, because, by definition, $\lambda_1(A) = 0$ implies that $\dim(H^-) = 0$. Hence, one has

$$\begin{aligned}
\Psi(u_n) &= \frac{1}{2}\left(\int_0^T \big(\big|\dot{u}_n(t)\big|^2\,dt + \big(A(t)u_n(t), u_n(t)\big)\big)\,dt\right) - \int_0^T F\big(t, u_n(t)\big)\,dt \\
&= \frac{1}{2}\left(\int_0^T \big(\big|\dot{u}_n^+(t)\big|^2\,dt + \big(A(t)u_n^+(t), u_n^+(t)\big)\big)\,dt\right) - \int_0^T F\big(t, u_n(t)\big)\,dt \\
&\geqslant \frac{\tilde{\lambda}}{2}\big\|u_n^+\big\|^2 - \int_0^T F\big(t, u_n(t)\big)\,dt, \qquad (2.2)
\end{aligned}$$

where the last inequality follows by Lemma 2.2.

Fix $L > 0$ arbitrarily, by (F_1), there exists $K = K(L) > 0$ such that

$$F(t,x) \leqslant -L, \quad \text{for a.e. } t \in [0,T] \text{ and all } |x| > K. \tag{2.3}$$

Denote by $\{|u| \leqslant K\}$ the set $\{t \in [0,T]\colon\ |u(t)| \leqslant K\}$ and by $\{|u| > K\}$ its complement in $[0,T]$. Moreover, put $b_K(t) = \sup_{|x|\leqslant K} |F(t,x)|$ for all $t \in [0,T]$. By assumption (1.1) we know that $b_K \in L^1([0,T])$. One has

$$\begin{aligned}
\Psi(u_n) &\geqslant \frac{\tilde{\lambda}}{2}\|u_n^+\|^2 - \int\limits_{\{|u_n|\leqslant K\}} F\big(t,u_n(t)\big)\,dt - \int\limits_{\{|u_n|>K\}} F\big(t,u_n(t)\big)\,dt \quad \text{by (1.1)}\\
&\geqslant \frac{\tilde{\lambda}}{2}\|u_n^+\|^2 - \int_0^T b_K(t)\,dt - \int\limits_{\{|u_n|>K\}} F\big(t,u_n(t)\big)\,dt \quad \text{by (2.3)}\\
&\geqslant \frac{\tilde{\lambda}}{2}\|u_n^+\|^2 - \int_0^T b_K(t)\,dt,
\end{aligned} \tag{2.4}$$

for every $n \in \mathbb{N}$.

So, if $\lim_{n\to+\infty}\|u_n^+\| = +\infty$, from (2.4), it follows that

$$\lim_{n\to+\infty} \Psi_1(u_n) \geqslant \lim_{n\to+\infty}\left(\frac{\tilde{\lambda}}{2}\|u_n^+\|^2 - \int_0^T b_K(t)\,dt\right) = +\infty,$$

and the thesis is proved.

At this point, it remains to consider the possibility that $\{u_n^+\}$ is bounded. In this case, since $\|u_n\| \to +\infty$, we must have $\|u_n^0\| \to +\infty$.

Let $0 < \epsilon < \frac{T}{4}$ be chosen small enough such that,

$$\int_A b_K(t)\,dt \leqslant \frac{LT}{2}, \tag{2.5}$$

for every measurable $A \subseteq [0,T]$, with $m(A) < \epsilon$.

By Lemma 2.1, there exists $M(\epsilon) > 0$ such that, if we set

$$A_n = \big\{t \in [0,T]\colon\ \big|u_n^0(t)\big| \geqslant M(\epsilon)\big\|u_n^0\big\|\big\}, \quad \text{for every } n \in \mathbb{N},$$

it results that $m([0,T]\setminus A_n) < \epsilon$. Then, we also have

$$\big|u_n(t)\big| \geqslant \big|u_n^0(t)\big| - \big|u_n^+(t)\big| \geqslant M(\epsilon)\big\|u_n^0\big\| - c, \quad \text{for a.e. } t \in A_n,$$

where $c > 0$ is a constant such that, $\max_{t\in[0,T]}|u_n^+(t)| \leqslant c$, which exists due to the boundedness of $\{u_n^+\}$ in H_T^1. Hence, there exists $\nu \in \mathbb{N}$ such that

$$A_n \subseteq \big\{|u_n| > K\big\}, \quad \text{for every } n \geqslant \nu. \tag{2.6}$$

Then, for $n > \nu$, one has

$$\begin{aligned}
\Psi(u_n) &\geqslant \frac{\tilde{\lambda}}{2}\|u_n^+\|^2 - \int\limits_{\{|u_n|\leqslant K\}} F\big(t,u_n(t)\big)\,dt - \int\limits_{\{|u_n|>K\}} F\big(t,u_n(t)\big)\,dt\\
&\geqslant -\int\limits_{\{|u_n|\leqslant K\}} b_K(t)\,dt + Lm\big(\big\{|u_n| > K\big\}\big) \quad \text{by (2.5) and (2.6)}\\
&\geqslant -\frac{LT}{2} + L(T-\epsilon) = \frac{LT}{4}.
\end{aligned} \tag{2.7}$$

Owing to the arbitrariness of $L > 0$, by (2.7), the thesis follows.

When $\lambda_1(A) \neq 0$, we can argue as above by replacing the matrix A with $\hat{A}(t) = A(t) - \lambda_1(A)I$ and the potential F with $\hat{F}(t,x) = F(t,x) - \frac{\lambda_1(A)}{2}|x|^2$. In fact, we note that $\lambda_1(\hat{A}) = 0$. $\square$

3. Main result

Now we can state and prove our main result:

Theorem 3.1. *Assume that the potential F satisfies (F_1), (F_2) and (F_3). Then there exist $\lambda^* > 0$ and $r > 0$ such that, for every $\lambda \in]-\lambda^*, \lambda^*[$, problem (P_λ) admits at least three distinct solutions which belong to $B(0,r) \subseteq H^1_T$.*

Proof. By Lemma 2.3, condition (F_1) implies that the functional Ψ is coercive.

Now we prove that Ψ has a strict local minimum at 0.

By the compact embedding of H^1_T into $C^0(0,T;\mathbb{R}^N)$, there exists a constant $c_1 > 0$ such that

$$\max_{t\in[0,T]} \left|u(t)\right| \leqslant c_1 \|u\|, \quad \text{for all } u \in H^1_T.$$

So, chosen $r_\delta < \frac{\delta}{c_1}$, it results that

$$\overline{B(0,r_\delta)} = \left\{u \in H^1_T\colon \|u\| \leqslant r_\delta\right\} \subseteq \left\{u \in H^1_T\colon \max_{t\in[0,T]} \left|u(t)\right| < \delta\right\}.$$

Hence, for every $u \in B(0,r_\delta) \setminus \{0\}$, from (F_2) it follows that

$$\begin{aligned}
\Psi(u) &= \frac{1}{2}\left(\int_0^T \left(\left|\dot u(t)\right|^2 dt + \left(A(t)u(t), u(t)\right)\right) dt\right) - \int_0^T F\left(t, u(t)\right) dt \\
&= \frac{1}{2}\left(\int_0^T \left(\left|\dot u(t)\right|^2 + \left(\left(A(t) - \lambda_1(A)I\right)u(t), u(t)\right)\right) dt\right) \\
&\quad + \int_0^T \left(\frac{1}{2}\lambda_1(A)\left|u(t)\right|^2 - F\left(t,u(t)\right)\right) dt \quad \text{since } \lambda_1\left(A - \lambda_1(A)I\right) = 0 \\
&\geqslant \int_0^T \left(\frac{1}{2}\lambda_1(A)\left|u(t)\right|^2 - F\left(t,u(t)\right)\right) dt \\
&> \Psi(0) = 0,
\end{aligned} \tag{3.8}$$

that is the function $v = 0$ is a strict local minimum of Ψ in H^1_T.

Condition (F_3) assures that 0 is not a global minimum.

At this point, we can apply Theorem 3.8 of [4] taking Φ and $-\Phi$ as perturbing terms. Then, for every $\rho_1, \rho_2, \varepsilon \in \mathbb{R}$, with $\inf_{H^1_T} \Psi < \rho_1 < 0$, $\rho_2 > 0$ and $0 < \varepsilon \leqslant r_\delta$, there exists $\tilde\lambda > 0$ such that, for each $\lambda \in]-\tilde\lambda, \tilde\lambda[$, $\Psi + \lambda\Phi$ has two distinct local minima $u_1^{(\lambda)} \in \Psi^{-1}(]-\infty, \rho_1[)$ and $u_2^{(\lambda)} \in \Psi^{-1}(]-\infty, \rho_2[) \cap B(0,\varepsilon)$.

Since $v = 0$ is a strict local minimum of Ψ, by Theorem 3.6 of [4], the above ε can be chosen such that $\gamma = \inf_{\|u\|=\varepsilon} \Psi(u) > 0$.

Now let $r_1 > 0$ be such that

$$B(0,r_1) \supseteq \Psi^{-1}\left(]-\infty, \rho_1[\right) \cup B(0,\varepsilon),$$

and put $b = \sup_{\|u\|\leqslant r_1} |\Psi(u)|$. Owing to the coerciveness of Ψ, there exists $r_2 > r_1$ such that $\inf_{\|u\|=r_2} \Psi(u) = d > b$.

Hence, for every $u \in H^1_T$ with $\|u\| = r_2$, one has

$$\begin{aligned}
\Psi(u) + \lambda\Phi(u) &\geqslant d - |\lambda| \sup_{\|u\|\leqslant r_2} \left|\Phi(u)\right| \\
&> \frac{d+b}{2},
\end{aligned} \tag{3.9}$$

and when $\|u\| \leqslant r_1$

$$\Psi(u) + \lambda\Phi(u) \leqslant b + |\lambda| \sup_{\|u\|\leqslant r_2} \left|\Phi(u)\right| < b + \frac{d-b}{2} = \frac{d+b}{2}, \tag{3.10}$$

because $\lambda \in \mathbb{R}$ can be chosen with

$$|\lambda| < \frac{d-b}{2\sup_{\|u\|\leqslant r_2}|\Phi(u)|},$$

due to the sequentially weakly continuity of Φ which implies that

$$\sup_{\|u\|\leqslant r_2}\left|\Phi(u)\right| < +\infty.$$

Hence, it is easily seen that, $\tilde{\lambda}$ can be chosen small enough that the following conditions

$$\Psi(u_1)+\lambda\Phi(u_1) < 0,$$
$$\Psi(u_2)+\lambda\Phi(u_2) < \frac{\gamma}{2},$$
$$\inf_{\|u\|=\varepsilon}\big(\Psi(u)+\lambda\Phi(u)\big) \geqslant \frac{\gamma}{2},$$

and (3.9), (3.10) hold, for every $\lambda \in]-\tilde{\lambda},\tilde{\lambda}[$.

For a given λ in the interval above, define the set of paths going from u_1 to u_2

$$\mathcal{A} = \big\{\alpha \in C\big([0,1],E\big)\colon\ \alpha(0)=u_1,\ \alpha(1)=u_2\big\},$$

and consider the real number $c = \inf_{\alpha\in\mathcal{A}}\sup_{t\in[0,1]}(\Psi(\alpha(t))+\lambda\Phi(\alpha(t)))$. Since $u_1 \neq B(0,\varepsilon)$ and each path α goes through $\partial B(0,\varepsilon)$, one has $c \geqslant \frac{\gamma}{2}$.

So, taking into account (3.9) and (3.10), there exists a sequence $\{\alpha_n\} \subset \mathcal{A}$, with $\alpha_n([0,1]) \subset B(0,r_2)$ for every $n\in\mathbb{N}$, such that

$$\lim_{n\to\infty}\sup_{t\in[0,1]}\big(\Psi\big(\alpha_n(t)\big)+\lambda\Phi\big(\alpha_n(t)\big)\big) = c.$$

Applying Theorem 2.8 of [12], there exists a sequence $\{u_n\} \subset B(0,r_2)$ which satisfies $\Psi(u_n)+\lambda\Phi(u_n) \to c$ and $\Psi'(u_n)+\lambda\Phi'(u_n) \to 0$ as $n\to\infty$. Hence $\{u_n\}$ is a bounded $(PS)_c$ sequence and, taking into account the fact that $\Psi'+\lambda\Phi'$ is an (S_+) type mapping, admits a convergent subsequence to some u_3. So, such u_3 turns to be a critical point of $\Psi+\lambda\Phi$, with $\Psi(u_3)+\lambda\Phi(u_3)=c$, hence different from u_1 and u_2 and $u_3 \neq 0$. □

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