Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Nonlinear Analysis 68 (2008) 3879–3883
Nonlinear Analysis
www.elsevier.com/locate/na

Three solutions for a perturbed Dirichlet problem

Giuseppe Cordaro[a,*], Giuseppe Rao[b]

[a] Department of Mathematics, University of Messina, 98166 Sant'Agata-Messina, Italy
[b] Department of Mathematics, University of Palermo, via Archirafi, 34 - 90123 Palermo, Italy

Received 2 April 2007; accepted 18 April 2007

Abstract

In this paper we prove the existence of at least three distinct solutions to the following perturbed Dirichlet problem:

$$\begin{cases} -\Delta u = f(x,u) + \lambda g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set with smooth boundary $\partial\Omega$ and $\lambda \in \mathbb{R}$. Under very mild conditions on g and some assumptions on the behaviour of the potential of f at 0 and $+\infty$, our result assures the existence of at least three distinct solutions to the above problem for λ small enough. Moreover such solutions belong to a ball of the space $W_0^{1,2}(\Omega)$ centered in the origin and with radius not dependent on λ.

MSC: 35J20

Keywords: Weak solutions; Critical points; Weakly sequentially lower semicontinuity

1. Introduction and statement of the result

In this paper we present a multiplicity result for the following perturbed problem:

$$\begin{cases} -\Delta u = f(x,u) + \lambda g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, with boundary $\partial\Omega$ smooth enough, λ is a real number, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying the following growth conditions:

(f) There exist $q > \frac{N}{2}$, $a_1 \in L^q(\Omega)$, $a_2 > 0$ and $s > 1$, with $s < \frac{N+2}{N-2}$ if $N > 2$, such that:
When $N \geq 2$:

$$|f(x,t)| \leq a_1(x) + a_2|t|^s, \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

* Corresponding author.
E-mail addresses: cordaro@dipmat.unime.it (G. Cordaro), rao@math.unipa.it (G. Rao).

0362-546X/$ - see front matter
doi:10.1016/j.na.2007.04.027

When $N = 1$:

$$\sup_{|t|\le M} |f(\cdot, t)| \in L^1(\Omega), \quad \text{for every } M > 0 \text{ and } t \in \mathbb{R}.$$

(g) There exist $a_3 \in L^{\frac{2N}{N+2}}(\Omega)$, $a_4 > 0$ and $p > 1$, with $p < \frac{N+2}{N-2}$ if $N > 2$, such that:
When $N \ge 2$:

$$|g(x, t)| \le a_3(x) + a_4|t|^p, \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

When $N = 1$:

$$\sup_{|t|\le M} |g(\cdot, t)| \in L^1(\Omega), \quad \text{for every } M > 0 \text{ and } t \in \mathbb{R}.$$

The above growth conditions allow us to introduce the following functionals:

$$\Psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega \left(\int_0^{u(x)} f(x, t)\mathrm{d}t\right)\mathrm{d}x$$

and

$$\Phi(u) = -\int_\Omega \left(\int_0^{u(x)} g(x, t)\mathrm{d}t\right)\mathrm{d}x$$

defined on the Sobolev space $W_0^{1,2}(\Omega)$, endowed with the norm of gradient $\|\cdot\| = \int_\Omega |\nabla(\cdot)|^2$. By standard results, it is well known that such functionals are well defined, continuously differentiable and weakly sequentially lower semicontinuous on $W_0^{1,2}(\Omega)$. The critical points of $\Psi + \lambda\Phi$ are the weak solutions of problem (P_λ). We recall that a weak solution of (P_λ) in $W_0^{1,2}(\Omega)$ is any $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u(x)\nabla v(x)\mathrm{d}x - \int_\Omega (f(x, u(x)) + \lambda g(x, u(x)))v(x)\mathrm{d}x = 0,$$

for every $v \in W_0^{1,2}(\Omega)$.

Let us denote by λ_1 the first eigenvalue of the Dirichlet problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We recall the variational characterization of λ_1:

$$\lambda_1 = \inf_{u\in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}.$$

Our result is the following theorem:

Theorem 1.1. *Assume that, besides* (f) *and* (g)*, the following conditions are satisfied:*

(i) $\limsup_{|t|\to+\infty} \frac{\int_0^t f(x,s)\mathrm{d}s}{|t|^2} < \frac{1}{2}\lambda_1$ *uniformly in* $x \in \Omega$.

(ii) $\limsup_{|t|\to 0} \frac{\int_0^t f(x,s)\mathrm{d}s}{|t|^2} < \frac{1}{2}\lambda_1$, *uniformly in* $x \in \Omega$.

(iii) *There exists* $u_1 \in W_0^{1,2}(\Omega)$ *such that* $\Psi(u_1) < 0$.

Then, there exist $\lambda^* > 0$ *and* $r > 0$ *such that, for each* $\lambda \in]-\lambda^*, \lambda^*[$, *problem* (P_λ) *has at least three distinct weak solutions whose norms are less than* r.

The above theorem belongs to the class of multiplicity results for perturbed problems with minimal assumptions on the perturbation term g. To the best of our knowledge the first paper where the authors proposed a result of this type is [6]. In that paper, Li and Liu obtained the existence of multiple solutions for problem (P_λ) where g is supposed to be only continuous on $\overline{\Omega} \times \mathbb{R}$ and f is required to be odd in the second variable t uniformly in x. The possibility of considering functions f with no symmetric properties has been already widely investigated; see for instance [1–5,8]. Theorem 1.1 gives a contribution in this direction. We propose some assumptions on the non-perturbed term f of the nonlinearity in order to obtain the existence of at least three distinct solutions to (P_λ), for λ small enough. It is worth noticing that such solutions satisfy a stability property because they belong to a fixed ball centered at the origin when the parameter λ varies in a suitable interval.

2. Proof Theorem 1.1

The first step of the proof is to apply Theorem 3.8 of [4] which is a consequence of the more general results established in [7].

From (i), choosing $\gamma \in \mathbb{R}$ with

$$\limsup_{|t|\to+\infty} \frac{\int_0^t f(x,s)\mathrm{d}s}{|t|^2} < \gamma < \frac{1}{2}\lambda_1, \tag{2.1}$$

there exists $M > 0$ such that, for all $|t| > M$ and a.e. $x \in \Omega$, one has

$$\int_0^t f(x,s)\mathrm{d}s < \gamma t^2.$$

Define $\{|u| \le M\} = \{x \in \Omega : |u(x)| \le M\}$ and denote by $\{|u| > M\}$ its complement in Ω.
Hence it results that

$$\begin{aligned}\Psi(u) &= \frac{1}{2}\|u\|^2 - \int_{\{|u|\le M\}} \left(\int_0^{u(x)} f(x,s)\mathrm{d}s\right)\mathrm{d}x + - \int_{\{|u|>M\}} \left(\int_0^{u(x)} f(x,s)\mathrm{d}s\right)\mathrm{d}x \\ &\ge \frac{1}{2}\|u\|^2 - \gamma \int_\Omega |u(x)|^2\mathrm{d}x - c \\ &\ge \left(\frac{1}{2} - \frac{\gamma}{\lambda_1}\right)\|u\|^2 - c,\end{aligned} \tag{2.2}$$

for all $u \in W_0^{1,2}(\Omega)$. The existence of a constant $c > 0$ such that

$$\left|\int_{\{|u|\le M\}} \left(\int_0^{u(x)} f(x,s)\mathrm{d}s\right)\mathrm{d}x\right| \le c,$$

follows from growth condition (f).
From (2.1) and (2.2) it follows that $\lim_{\|u\|\to+\infty} \Psi(u) = +\infty$.

Now we prove that $u_0 \equiv 0$ is a strict local minimum of Ψ.
By (ii), we can choose $\beta \in \mathbb{R}$ and $\delta > 0$ such that

$$\frac{\int_0^t f(x,s)\mathrm{d}s}{|t|^2} < \beta < \frac{1}{2}\lambda_1, \tag{2.3}$$

for all $0 < |t| < \delta$ and a.e. $x \in \Omega$. So, for each $u \in W_0^{1,2}(\Omega)$, one has

$$\begin{aligned}\Psi(u) &= \frac{1}{2}\|u\|^2 - \int_{\{|u|<\delta\}} \left(\int_0^{u(x)} f(x,s)\mathrm{d}s\right)\mathrm{d}x - \int_{\{|u|\ge\delta\}} \left(\int_0^{u(x)} f(x,s)\mathrm{d}s\right)\mathrm{d}x \\ &\ge \left(\frac{1}{2} - \frac{\beta}{\lambda_1}\right)\|u\|^2 - \int_{\{|u|\ge\delta\}} \left(\int_0^{u(x)} f(x,s)\mathrm{d}s\right)\mathrm{d}x.\end{aligned} \tag{2.4}$$

In order to estimate the last term in a suitable neighbourhood of zero in $W_0^{1,2}(\Omega)$, we distinguish the cases $N = 1$ and $N > 1$.

In the first case, exploiting the compact embedding of $C(\overline{\Omega})$ into $W_0^{1,2}(\Omega)$, one can find $r_\delta > 0$ such that

$$\max_{x\in\overline{\Omega}} |u(x)| < \delta,$$

for all $u \in W_0^{1,2}(\Omega)$ with $\|u\| < r_\delta$. So, if $\|u\| < r_\delta$ it follows that

$$\int_{\{|u|\geq\delta\}} \left(\int_0^{u(x)} f(x,s)\mathrm{d}s\right)\mathrm{d}x = 0. \tag{2.5}$$

In the case $N > 1$, from (f), we have

$$\left|\int_{\{|u|\geq\delta\}} \left(\int_0^{u(x)} f(x,s)\mathrm{d}s\right)\mathrm{d}x\right| \leq \int_{\{|u|\geq\delta\}} a_1(x)|u(x)| + \int_{\{|u|\geq\delta\}} \frac{a_2}{s+1}|u(x)|^{s+1}\mathrm{d}x. \tag{2.6}$$

If $N > 2$, since $q > \frac{N}{2}$, there exists $m \in \mathbb{R}$ with $\frac{2q}{q-1} < m < \frac{2N}{N-2}$. When $N = 2$, it is enough to choose $m > \frac{2q}{q-1}$. Setting $l = \frac{m(q-1)}{q}$, one has

$$\begin{aligned}\int_{\{|u|\geq\delta\}} a_1(x)|u(x)| &\leq \int_{\{|u|\geq\delta\}} \frac{a_1(x)}{\delta^{l-1}}|u(x)|^l\mathrm{d}x \\ &\leq \frac{1}{\delta^{l-1}}\left(\int_\Omega |a_1(x)|^q\right)^{\frac{1}{q}}\left(\int_\Omega |u(x)|^m\mathrm{d}x\right)^{\frac{q-1}{q}} \\ &\leq C_1\|u\|^l. \end{aligned}\tag{2.7}$$

Here $C_1 > 0$, constant with respect to u, exists because of the embedding theorems. Analogously there exists $C_2 > 0$, not dependent on u, such that

$$\int_{\{|u|\geq\delta\}} \frac{a_2}{s+1}|u(x)|^{s+1}\mathrm{d}x \leq C_2\|u\|^{s+1}. \tag{2.8}$$

From (2.4)–(2.7) and (2.8) it follows that

$$\Psi(u) \geq \left(\frac{1}{2} - \frac{\beta}{\lambda_1}\right)\|u\|^2 - C_1\|u\|^l - C_2\|u\|^{s+1},$$

for all $u \in W_0^{1,2}(\Omega)$, with $\|u\| < r_\delta$. Since $l > 2$, $s + 1 > 2$ and $\frac{\beta}{\lambda_1} < \frac{1}{2}$, Ψ has a strict local minimum at $u_0 \equiv 0$.

By condition (iii), u_0 is not a point of global minimum for Ψ.

At this point we apply Theorem 3.8 of [4] twice, taking as the perturbing term Φ and $-\Phi$. So, choose $r_1 > 0$ such that $u_0 \equiv 0$ is a strict global minimum of Ψ in $\overline{B(0,r_1)}$, where $B(0,r_1)$ is the open ball in $W_0^{1,2}(\Omega)$ centered at the origin and with radius r_1. For any $\rho_1, \rho_2 \in \mathbb{R}$ with $\inf_{W_0^{1,2}(\Omega)} \Psi < \rho_1 < 0$ and $\rho_2 > 0$, there exists $\tilde{\lambda} > 0$ such that $\Psi + \lambda\Phi$ has two distinct local minima $u_1^{(\lambda)} \in \Psi^{-1}(]-\infty,\rho_1[)$ and $u_2^{(\lambda)} \in \Psi^{-1}(]-\infty,\rho_2[) \cap B(0,r_1)$, for all $\lambda \in]-\tilde{\lambda},\tilde{\lambda}[$.

Hence, arguing as in the proof of Theorem 4.2 in [4] and applying a mountain pass lemma without a (P.S.) condition, Theorem 2.8 of [9], there exist $r > 0$ and $\lambda^* \in \mathbb{R}$ with $0 < \lambda^* < \tilde{\lambda}$, such that $\Psi + \lambda\Phi$ has a third critical point $u_3^{(\lambda)}$ distinct from $u_1^{(\lambda)}$ and $u_2^{(\lambda)}$, and $u_1^{(\lambda)}, u_2^{(\lambda)}, u_3^{(\lambda)} \in B(0,r)$, for all $\lambda \in]-\lambda^*,\lambda^*[$. So the theorem is proved.

References

[1] G. Anello, Existence and multiplicity of solutions to a perturbed Neumann problem, Math. Nachr. (in press).

[2] G. Anello, G. Cordaro, Perturbation from Dirichlet problem involving oscillating nonlinearities, J. Differential Equations 234 (2007) 80–90.

[3] G. Cordaro, Multiple solutions to a perturbed Neumann problem, Studia Math. 178 (2) (2007) 167–175.

[4] X. Fan, S.-G. Deng, Remarks on Ricceri's variational principle and applications to $p(x)$-Laplacian equations, Nonlinear Anal. 67 (11) (2007) 3064–3075.

[5] F. Faraci, Multiple solutions for two nonlinear problems involving the p-Laplacian, Nonlinear Anal. 63 (2005) e1017–e1029.
[6] S.J. Li, Z.L. Liu, Perturbations from symmetric elliptic boundary value problems, J. Differential Equations 185 (2002) 271–280.
[7] B. Ricceri, Sublevel sets and global minima of coercive functional and local minima of their perturbations, J. Nonlinear Convex Anal. 52 (2004) 157–168.
[8] B. Ricceri, A multiplicity theorem for the Neumann problem, Proc. Amer. Math. Soc. 134 (2006) 1117–1124.
[9] M. Willem, Minimax Theorems, Birkhauser, Boston, 1996.