Generalized person-by-person optimization in team problems with binary decisions

Dario Bauso and Raffaele Pesenti

***Abstract*— In this paper, we extend the notion of person by person optimization to binary decision spaces. The novelty of our approach is the adaptation to a dynamic team context of notions borrowed from the pseudo-boolean optimization field as completely local-global or unimodal functions and sub-modularity. We also generalize the concept of pbp optimization to the case where the Decision Makers (DMs) make decisions sequentially in groups of m, we call it $m\mathbf{b}m$ *optimization*. The main contribution are certain sufficient conditions, verifiable in polynomial time, under which a pbp or an $m\mathbf{b}m$ optimization algorithm leads to the team-optimum. We also show that there exists a subclass of sub-modular team problems, recognizable in polynomial time, for which the convergence is guaranteed if the pbp algorithm is opportunely initialized.**

I. INTRODUCTION

Most fundamental results in team theory concern linear quadratic gaussian problems or, in general, problems with continuous decision spaces, where the cost is somehow convex in the strategies and the information structure is a "nice" one (see, e.g., partial nested structures) [1], [2], [13]. In such particular cases, it is well known that a simple solution idea consisting in a sequential optimization on the part of the Decision Makers (DMs), called *person by person optimization* (pbp), leads to the team-optimum [10], namely the argument minimizing the team objective function.

In this paper, on the same line of [7], we restrict our attention to boolean decision spaces. The novelty of our approach is the adaptation to a dynamic team context of notions borrowed from *pseudo-boolean optimization* [4], as Completely Local-Global (CLG) functions, Completely Unimodal (CU) functions (also known as acyclic unique sink orientations and abstract objective functions [12]) and sub-modular functions [5], [9].

Boolean decision spaces can be found in finite-alphabet control and in particular on-off control problems [8], impulsively-controlled systems (activate the impulse or not) [6], or switching control (switches between active and passive modes) [14]. Boolean decisions are encountered in many applications as inventory with set up costs (reordering or not from a warehouse in order to meet a demand) [3], distributed computer systems (processing or not the assigned task) [7], in air-conditioning systems control, in economics and finance (see, e.g., [4] and references therein).

This work was supported by MURST-PRIN "Analysis, optimization, and coordination of logistic and production systems"

D. Bauso is with Dipartimento di Ingegneria Informatica-DINFO, Università di Palermo, 90128 Palermo, Italy, dario.bauso@unipa.it

R. Pesenti is with Dipartimento di Matematica Applicata-DMA, Università di Venezia, 30123 Venezia, Italy, pesenti@unive.it

As first contribution, we generalize the concept of pbp optimization to the case where the Decision Makers (DMs) make decisions sequentially in groups of m, we call it mbm *optimization*.

The main contribution of this paper consists in providing certain sufficient conditions, verifiable in polynomial time, for the optimality of such pbp (respectively $m\mathrm{b}m$) optimization algorithms. Then we can frame our results in the literature on person by person algorithms in team theory, which has drawn the attention of the control audience since the '70s (see, e.g., [10]).

As a further contribution, we have paid special attention to problems with sub-modular team objective function (*sub-modular team problems*). Though sub-modularity alone does not guarantee the convergence of any pbp optimization algorithm, we show that there exists a special class of sub-modular team problems, recognizable in polynomial time, for which the convergence is guaranteed when the algorithm is opportunely initialized. This class is characterized by so-called *threshold strategies*.

This paper is organized as follows. In Section II, we introduce some notions from team theory [10] and pseudo-boolean optimization [4]. In Section III, we introduce the class of completely local-global functions and completely unimodal functions [5], and [9]. In Section IV, we address the $m\mathrm{b}m$ optimization. In Section V, we focus on sub-modular team problems. In Section VI we provide numerical examples. Finally, in Section VII, we discuss how to extend the obtained results.

II. DEFINITIONS AND PROBLEM STATEMENT

Consider a set N of n DMs making decisions x from a discrete hypercube $\mathbb{B}^n = \{0,1\}^n$. Decisions are made in order to optimize a common team objective function, $J(x) : \mathbb{B}^n \mapsto \mathbb{Z}$, where $\mathbb{Z}$ is the set of integer numbers.

Assumption 1: The team objective function $J(x)$ is injective and has the following *quadratic* form

$$J(x) = \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j. \tag{1}$$

with a_{ij} and b_i integer (this causes $J(x)$ assuming only integer values).

The following definitions are slightly modified from [7].

Definition 1: (Team-optimum) A point x^* is a *team-optimum* if

$$x^* = \arg\min_{x\in\mathbb{B}^n} J(x).$$

As the set $\mathbb{B}^n$ is finite, a team optimum x^* always exists. Furthermore, as $J(x)$ is injective, the team optimum is unique.

Definition 2: (pbp optimum) The point x^* is a *pbp optimum* if for any DM i the following condition holds

$$J(x_i^*, x_{-i}^*) < J(x_i, x_{-i}^*),\ \forall x_i \neq x_i^* \tag{2}$$

where $x_i \in \mathbb{B}$ is the decision of DM i and $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1} \dots, x_n)^T \in \mathbb{B}^{n-1}$ is a vector collecting decisions of all other DMs. From the above definitions we have that a team-optimum always implies pbp optimality but not vice versa.

Let S any subset of N with m elements. We indicate this with $S \subseteq N$ with $|S| = m$, where $|V|$ means cardinality of V. Let $x_S \in \mathbb{B}^m$ be a vector collecting the decisions of all the DMs belonging to S, namely, $x_S = (x_i : i \in S)$. Analogously, let $x_{-S} \in \mathbb{B}^{n-m}$ be a vector collecting the decisions of all the other DMs, $x_{-S} = (x_i : i \in N \setminus S)$.

Definition 3: (mbm optimum) The point x^* is an *mbm optimum* if, for any subset $S \subseteq N$ with $|S| = m$, the following condition holds

$$J(x_S^*, x_{-S}^*) < J(x_S, x_{-S}^*),\ \forall x_S \neq x_S^*. \tag{3}$$

All the results stated in the following hold true for any value of the parameter m from 1 to n.

In agreement with [7] and [10], we define a pbp strategy as follows.

Definition 4: (strict pbp strategy) A strategy $\mu_i : \mathbb{B}^{n-1} \mapsto \mathbb{B}$ is *pbp strict* for DM i if, for any $x_{-i} \in \mathbb{B}^{n-1}$, we have

$$\mu_i(x_{-i}) = \arg \min_{\tilde{x}_i \in \{0,1\}} J(\tilde{x}_i, x_{-i}).$$

As $J(x)$ is injective, the above equation has a unique solution. Then, under a strict pbp strategy, a DM i changes decision from zero to one or vice versa only if such a change lets the team objective function decrease for fixed decisions of all other DMs $j \neq i$.

Definition 5: (Strict mbm strategy) A strategy $\mu_S : \mathbb{B}^{n-m} \mapsto \mathbb{B}^m$ is *mbm strict* for DMs in S where $S \subseteq N$ with cardinality $|S| = m$ if, for any $x_{-S} \in \mathbb{B}^{n-m}$, we have

$$\mu_S(x_{-S}) = \arg \min_{\tilde{x}_S \in \mathbb{B}^m} J(\tilde{x}_S, x_{-S}).$$

The above definition has the following geometric interpretation. For any $x \in \mathbb{B}^n$ and $S \subseteq N$, denote by $\Pi_S(x)$ as the the corresponding m-dimensional face $\{\tilde{x} = (\tilde{x}_S, x_{-S}) \in \mathbb{B}^n : x_{-S} \text{ fixed}\}$ of hypercube $\mathbb{B}^n$. Then, a strict mbm strategy means that either (x_S, x_{-S}) is the optimal vertex in $\Pi_S(x)$ or the DMs in S coordinate their decisions to find an optimal vertex in $\Pi_S(x)$.

With the above definitions in mind, we call *pbp optimization algorithm*, any algorithm that returns a sequence of decisions $x(0) \to x(1) \to \dots$ where, for each iteration t, we denote by $x(t) = \{x_1(t) \dots x_n(t)\}$ and $x_i(t)$ the vector of decisions and the decision of DM i respectively. We also require that each decision $x(t)$ is obtained from $x(t-1)$ by a unilateral improvement on the part of a single DM $i = \sigma(t)$, i.e., $x(t) = [\mu_i(x_{-i}(t-1)), x_{-i}(t-1)]$, where $\sigma : \mathbb{N} \mapsto N$, is a periodic surjective function, with period n, that returns a DM for each iteration t. For instance, $\sigma(1) = 2$, $\sigma(2) = 5 \dots$ means that at iteration 1, DM 2 plays the strict pbp strategy for fixed decisions of all other DMs, and similarly for DM 5 at iteration 2. We define an *mbm optimization algorithm* in a similar manner. Here, the function σ becomes $\sigma : \mathbb{N} \mapsto \mathcal{Q}$, with period $|\mathcal{Q}|$, where $\mathcal{Q}$ is the set of all subsets $S \subseteq N$ with $|S| = m$, and the vector of decisions at iteration t becomes $x(t) = [\mu_S(x_{-S}(t-1)), x_{-S}(t-1)]$.

We can now state the problem of interest.

Problem 1: Find conditions under which any pbp (respectively mbm) optimization algorithm converges to the team-optimum.

Throughout the paper, convergence means "from any generic $x(0)$", unless specified differently.

Remark 1: Any strict pbp (respectively mbm) optimization algorithm converges to a pbp (mbm) optimum in a finite number of iterations. Actually, the set $\mathbb{B}^n$ is finite and at each iteration t of the algorithm the value of objective function $J(x(t))$ decreases.

There is a vast literature on functions $f(x) : \mathbb{B}^n \mapsto \mathbb{Z}$ that map from a discrete hypercube $\mathbb{B}^n$ to the ordered field $\mathbb{Z}$ of integer numbers. They are usually referred to as *pseudo-boolean functions* [4].

In the following, we recall some notions and optimality conditions in the context of pseudo-boolean optimization that we use to prepare and motivate the results of the next sections.

Let us now associate to a binary vector $x \in \mathbb{B}^n$ its neighborhood $N_r(x)$ of *radius* r, defined as $N_r(x) = \{y : \rho_H(x, y) \leq r\}$, where $\rho_H(x, y)$ denotes the *Hamming distance* of the vectors x and y, defined as the number of components in which these two vectors differ. According to this definition, the neighborhood of radius n of each $x \in \mathbb{B}^n$ is equal to $\mathbb{B}^n$, that is $N_n(x) = \mathbb{B}^n$.

A vector x is a *local minimum* of a pseudo-boolean $f(.)$ if $f(y) \geq f(x)$ for all neighboring vectors $y \in N_1(x)$. It is a *global minimum* if $f(y) \geq f(x)$ for all vectors $y \in \mathbb{B}^n$.

Local minima can be determined by means of local search algorithms. In particular, [5] defines as a *single switch algorithm* any algorithm that at each iteration proceeds to a better neighbor of the current iterate, by changing one coordinate at a time, until a local optimum is found. Similarly, they define as a *multiple switch algorithm* of order m any algorithm that at each iteration proceeds to a next better iterate that differs from the current vertex in at most m coordinates.

Remark 2: The following correspondences hold:

i) The team objective function $J(x)$ is a pseudo-boolean function.
ii) Any pbp (respectively mbm) optimum is a local optimum in a neighborhood of radius one (respectively m).
iii) The team-optimum is a global optimum.
iv) Strict pbp (respectively mbm) strategies are single (respectively multiple) switch algorithms.

There is a large variety of techniques applied in the literature for solving problems that can be modelled by quadratic pseudo-Boolean functions optimization. As this last problem is NP-hard, many of the published algorithms are implicitly enumerative. However, specialized optimization algorithms have been developed for increasing or decreasing pseudo-Boolean functions.

We can associate to a pseudo-boolean function its first order ith *derivative*

$$\frac{\partial f}{\partial x_i}(x) = f(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n) + - f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n),$$

which will be used later on. If $f(.)$ is injective, $\frac{\partial f}{\partial x_i}(x) \neq 0$ for all $x \in \mathbb{B}^n$, for all $i \in N$. Let us finally introduce the following operation.

Definition 6: Given a function $f : \mathbb{B}^n \mapsto \mathbb{R}$, for any subset $S \subseteq N$, define *restriction* of f into S, $\mathcal{R}_S f(x) : \mathbb{B}^n \mapsto \mathbb{R}$ the function obtained from f by considering the only monomials and binomials including DMs in S and setting the values of the variables in S equal to 1

$$\mathcal{R}_S f(x) = \sum_{i\in S} b_i + \sum_{i,j\in S} a_{ij} + \sum_{k\notin S}\sum_{i\in S} a_{ik}x_k.$$

The above definition has the following geometric interpretation. Consider the face $\Pi_S(x) : \{x = (x_S, x_{-S}) \in \mathbb{B}^n : x_{-S}$ fixed$\}$ of $\mathbb{B}^n$ and extract two points $\overline{x} = (\mathbf{1}, x_{-S})$ and $\underline{x} = (\mathbf{0}, x_{-S})$ from it. Note that, for fixed x_{-S}, in $\overline{x}$ all DMs $i \in S$ set $x_i = 1$ while in $\underline{x}$ all DMs $i \in S$ set $x_i = 0$. Then, the restriction is the difference $J(\overline{x}) - J(\underline{x})$ of the team objective function computed on the two points. Also, note that for a singleton, $S = \{i\}$, then $\mathcal{R}_S f(x) = \frac{\partial f}{\partial x_i}(x)$.

III. Person by Person Optimization

In this section, we present sufficient conditions, verifiable in polynomial time, for the convergence of any pbp algorithm to the team-optimum.

Definition 7: (CLG-functions [9]) An injective function $f : \mathbb{B}^n \mapsto \mathbb{Z}$ is *Completely Local-Global* (CLG) if in $\mathbb{B}^n$ there is a unique local minimum.

Lemma 1: Any pbp optimization algorithm guarantees convergence to the team-optimum x^* if and only if $J(x)$ is a CLG-function.

Proof: (sufficiency) If $J(.)$ is a CLG-function then there is a unique pbp optimum which is also team-optimum. Any pbp optimization algorithm guarantees convergence to it.
(necessity) If $J(.)$ is not a CLG-function then there is a second pbp optimum $\bar{x}$ which is not team-optimum. Any pbp optimization algorithm starting at $\bar{x}$ cannot deviate from it and therefore does not reach the global optimum. ∎

The class of CLG-functions includes the class of completely unimodal functions.

Definition 8: (CU-functions) An injective function $f : \mathbb{B}^n \mapsto \mathbb{Z}$ is *Completely Unimodal* (CU) if f has a unique local minimum on every face of $\mathbb{B}^n$.

From the above lemma we can derive the following corollary.

Corollary 1: Any pbp optimization algorithm converges to the team-optimum x^* if $J(x)$ is a CU-function.

To the best of author's knowledge, recognizing CU-functions or CLG-functions is, in general, a difficult task. Actually, it involves an exponential number of conditions as shown next. Furthermore, even if f is a CLG or CU-function, strict pbp strategies may converge in exponential time.

To see why completely unimodality involves an exponential number of conditions consider that for existing two local minima on a 2-face containing x_i and x_j, it must hold

$$\left.\frac{\partial f(x)}{\partial x_i}\right|_{x_j=0} \cdot \left.\frac{\partial f(x)}{\partial x_i}\right|_{x_j=1} < 0 \quad (4)$$

$$\left.\frac{\partial f(x)}{\partial x_j}\right|_{x_i=0} \cdot \left.\frac{\partial f(x)}{\partial x_j}\right|_{x_i=1} < 0. \quad (5)$$

Then for f to be CU it is necessary that, on each 2-face and for all x, the above conditions are not satisfied, which implies an exponential number of verifications.

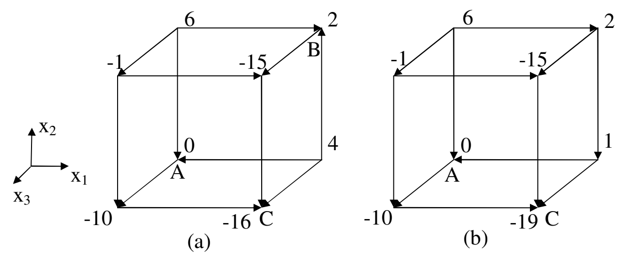

Fig. 1. Unit 3-dimensional cubes: oriented arcs indicate decreasing directions for $J(x)$ when (a) $J(x)$ is CLG-function or (b) $J(x)$ is CU-function. Solutions $x = (0,0,0)$ and $x = (1,1,0)$ (point A and B in (a)) are two local minima for the 2-face x_1-x_2 with $x_3 = 0$. In both cases, the global minimum is $x = (1,0,1)$ (point C).

Example 1: Consider the set $\mathbb{B}^3 = \{0,1\}^3$ and the team objective function $J(x) : \mathbb{B}^3 \mapsto \mathbb{Z}$, taking on the values displayed in Fig. 1.a. The explicit expression of the function J according to the formula (1) is

$$J(x) = \underbrace{4x_1^2 + 4x_2^2 - 8x_1x_2 + 2x_2}_{\mathcal{J}(x_1,x_2)} - 10x_3 - 10x_1x_3 + 3x_2x_3,$$

where we denote by $\mathcal{J}(x_1, x_2)$ the function obtained considering the only terms in x_1 and x_2. In Fig. 1.a, the oriented arcs indicate the decreasing directions for the team objective function $J(x)$. Function $J(x)$ is a CLG-function as it has a unique local (global) minimum in $\mathbb{B}^3$ which is $x = (1,0,1)$ (point C in the figure). However note that $\mathcal{J}(x_1,x_2)$ is not a CLG-function as it has two local minima in $\mathbb{B}^2$. For instance, see the 2-face x_1-x_2 with $x_3 = 0$ which has two local minima in $x = (0,0,0)$ and $x = (1,1,0)$ (point A and B). We complete the example by considering a different function $\hat{J}(x) : \mathbb{B}^3 \mapsto \mathbb{Z}$, taking on the values displayed in Fig. 1.b. The explicit expression is

$$\hat{J}(x) = \underbrace{x_1^2 + 4x_2^2 - 5x_1x_2 + 2x_2}_{\hat{\mathcal{J}}(x_1,x_2)} - 10x_3 - 10x_1x_3 + 3x_2x_3,$$

where again $\hat{\mathcal{J}}(x_1,x_2)$ is obtained considering the only terms in x_1 and x_2. In Fig. 1.b, the unique global minimum is again $x = (1,0,1)$ (point C in the figure) but differently from before function $J(x)$ is a CU-function in $\mathbb{B}^3$ as it has a unique local minimum on each 2-face. In correspondence to such a situation we also have that $\hat{\mathcal{J}}(x_1, x_2)$ is a CLG-function on $\mathbb{B}^2$ as it has a unique local minimum in $\mathbb{B}^2$ (see the 2-face x_1-x_2 with $x_3 = 0$ which has a local minimum in $x = (0, 0, 0)$ (point A)).

A special case of completely unimodality is when $f(.)$ is monotonic along any single direction, which corresponds to being both left hand side of (4) and (5) positive. Now, $f(.)$ is monotonic along any single direction, when for all $i = 1\ldots, n$, one of the following mutually exclusive conditions holds true

$$\max_{x\in\mathbb{B}^n} \frac{\partial J(x)}{\partial x_i} < 0 \tag{6}$$

$$\min_{x\in\mathbb{B}^n} \frac{\partial J(x)}{\partial x_i} > 0. \tag{7}$$

We can specialize Corollary 1 to such a particular case.

Lemma 2: (Sufficient conditions) If the team objective function $J(x)$ is such that, for all $i \in N$, one between (6) or (7) hold, then

1) the team optimum is

$$x_i^* = \begin{cases} 1 & \text{if } \max_{x\in\mathbb{B}^n} \frac{\partial J(x)}{\partial x_i} < 0 \\ 0 & \text{if } \min_{x\in\mathbb{B}^n} \frac{\partial J(x)}{\partial x_i} > 0 \end{cases}$$

2) the team optimum x^* is also the unique pbp optimum,
3) any pbp optimization algorithm converges to the team optimum x^* in at most n iterations.

Proof: Item 3 is straightforward from item 2. To prove item 1 and 2 consider that if $\max \frac{\partial J(x)}{\partial x_i} < 0$, then $\frac{\partial J(x)}{\partial x_i} < 0$ for all x. Analogously, if $\min \frac{\partial J(x)}{\partial x_i} > 0$ then $\frac{\partial J(x)}{\partial x_i} > 0$ for all x. ∎

Let us finally observe that verifying whether (6) or (7) holds is easy (polynomial in n), as we just have to find the maxima, respectively the minima, of the n functions $\frac{\partial J(x)}{\partial x_i}$ linear in $x \in \mathbb{B}^n$.

IV. Generalization to mBm Optimization

Let us now generalize the results established in the preceding section to the case where DMs make decisions sequentially in groups of m.

Theorem 1: (Sufficient conditions) Let $x^* = \mathbf{1}$ be an $(m-1)$b$(m-1)$ optimum, if the team objective function $J(.)$ is such that for all $S \subseteq N$ with $|S| = m$ it holds

$$\max_{x\in\mathbb{B}^n} \mathcal{R}_S J(x) < 0 \tag{8}$$

then

1) x^* is the team-optimum
2) x^* is also the unique mbm optimum,
3) any mbm optimization algorithm converges to the team-optimum x^*.

Proof: Item 3 is straightforward from item 2. To prove item 1 and 2, let us assume by contradiction that there exists a team optimum value $x^* \neq \mathbf{1}$. Let $V = \{i : x_i^* = 0\}$. The cardinality of V cannot be greater than or equal to m. Indeed consider $S \subseteq V$ with $|S| = m$, since $\mathcal{R}_S J(x^*) < 0$ implies $J(x^\circ) < J(x^*)$, where $x^\circ \in \mathbb{B}^n$ differs from x^* only for the components in S, i.e., $x_i^\circ = 0$ if $i \in V \setminus S$, $x_i^\circ = 1$ otherwise. Then x^* should be within an Hamming distance strictly less than m from $\mathbf{1}$, but this situation cannot occur since $\mathbf{1}$ by definition is optimum within its neighborhood of radius $m - 1$. ∎

Example 2: Consider the team objective function $J(x) = x_1 + x_2 - 3x_3 - 5x_1x_2 + x_1x_3 + x_2x_3$. The solution $x^* = \mathbf{1}$ is a pbp optimum as, for all i, $b_i + \sum_{k\neq i} a_{ik} < 0$. Since for all S, with $|S| = 2$ condition (8) holds (for $i = 1$ and $j = 2$, we have $b_1 + b_2 + a_{12} + \max_{x\in\mathbb{B}^n}(a_{13} + a_{23})x_3 = -1$), then $x^* = \mathbf{1}$ is also team-optimum.

Remark 3: In the above lemma, the assumption $x^* = \mathbf{1}$ is without loss of generality. Actually, if the team problem has a unique team optimum $x^* \neq \mathbf{1}$ then the following transformation can be applied to the decision space such that the new team optimum is $\hat{x}^* = 1$:

$$\hat{x}_i = \begin{cases} x_i & \text{if } x_i^* = 1 \\ 1 - x_i & \text{it } x_i^* = 0. \end{cases} \tag{9}$$

Let us finally observe that verifying whether (8) holds is, for fixed m, polynomial in n although exponential in m, as we just have to find the maxima of the $\binom{n}{m}$ functions $\mathcal{R}_S J(x)$ linear in $x \in \mathbb{B}^n$.

V. Sub-modular Team Problems

In the past sections we have provided conditions for the convergence from any initial state $x(0)$. Now, we show that we can recognize in polynomial time a special class of sub-modular team problems, which do not meet the aforementioned conditions and for which the convergence is guaranteed at least when the pbp algorithm is opportunely initialized. This class is characterized by so-called *threshold strategies*.

Let us call *sub-modular team problems*, all team problems with sub-modular team objective function. From [4], we remind from that i) a pseudo-Boolean function $f(.)$ is *sub-modular* if $f(v) + f(w) \leq f(vw) + f(v \vee w)$ ii) a quadratic pseudo-Boolean function $f(.)$ is submodular iff its quadratic terms are nonpositive. However, from the following example, it is apparent that sub-modularity alone does not guarantee the convergence of any pbp optimization algorithm.

Example 3: Consider the sub-modular team objective function $J(x) = x_1 + x_2 - 3x_1x_2$ and take $x(0) = (0, 0)$. The team optimum is $(1, 1)$ but observe that at iteration 1, no DM alone benefits from changing its decision from 0 to 1. Hence the pbp optimization algorithm starts and terminates in $(0, 0)$.

We can generalize the above reasoning to show that sub-modularity alone does not guarantee the convergence of any mbm optimization algorithm. On this purpose, note that if the team objective function is sub-modular, then condition (8) reduces to

$$\sum_{i\in S} b_i + \sum_{i,j\in S} a_{ij} < 0, \quad \text{for all } S, \text{ with } |S| = m. \tag{10}$$

We derive the above result by reminding that all quadratic terms are nonpositive and therefore $\max_x \sum_{k\neq i,j}(a_{ik} + a_{jk})x_k \leq 0$ with the equality verified in $x = 0$.

Example 4: Consider the sub-modular team objective function $J(x) = 2x_1 + 2x_2 + 2x_3 - 3x_1x_2 - 3x_1x_3 - 3x_2x_3$ and take $x(0) = (0, 0)$. The team optimum is again $(1, 1)$ but observe that at iteration 1, no pairs i and j of DMs alone benefits from changing their decisions from 0 to 1. Note that condition (10) for $m = 2$ becomes $b_i + b_j + a_{ij} < 0$ and there is no pair i and j that satisfies such a condition. Hence the mbm optimization algorithm starts and terminates in $(0, 0)$.

A. A Special Class with Threshold Strategies

Threshold strategy means that a DM i chooses $x_i = 1$ if and only if at least other l_i DMs do the same. The following simple example shows that when players (DMs) have threshold strategies the team objective function is sub-modular. The team objective function is as in (1). We say that player i has a threshold strategy with threshold $l_i = k$, if its strict pbp strategy is

$$\mu_i(x_{-i}) = \begin{cases} 1 & \text{if } \|x_{-i}\|_1 \geq k \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

Lemma 3: If all players have threshold strategies then the team objective function $J(x)$ must be sub-modular.

Proof: Observe that player i has a threshold strategy with $l_i = k$. Denote by $\mathcal{S}(k)$ the set of all subsets of N, which do not contain DM i and have cardinality less than k. Now, for a generic subset $S \in \mathcal{S}(k)$, take x_{-i} such that $x_j = 1$ for all $j \in S$ and $x_j = 0$ for all $j \in N \setminus (S \bigcup \{i\})$ and observe that from (11) it must hold that $\mu_i(x_{-i}) = 0$. But this means that the following condition holds true

$$b_i + \sum_{j \in S} a_{ij} \geq 0 \quad \text{for all } S \in \mathcal{S}(k). \quad (12)$$

Repeat the same reasoning considering a generic subset $S \subseteq N \setminus \mathcal{S}(k)$, and take x_{-i} such that $x_j = 1$ for all $j \in S$ with $j \neq i$ and $x_j = 0$ for all $j \in N \setminus S$. Observe that from (11) it must hold that $\mu_i(x_{-i}) = 1$ which implies that the following condition hold true

$$b_i + \sum_{j \in S} a_{ij} < 0 \quad \text{for all } S \subseteq N \setminus \mathcal{S}(k). \quad (13)$$

Now, consider two sets $S^1 \in \mathcal{S}(k)$ with $|S^1| = k - 1$ and $S^2 = S^1 \cup \{j\} \in N \setminus \mathcal{S}(k)$. Observe that S^2 has cardinality $|S^2| = k$ as it is obtained from S^1 by adding a single DM j. We complete the proof by observing that for (12) and (13) to be valid it must be $a_{ij} < 0$ for all i and j. Then $J(.)$ has all quadratic terms negative which proves that $J(.)$ is sub-modular. ■

This special class of sub-modular team problems is interesting as i) threshold structures can be recognized in polynomial time and ii) any pbp optimization algorithm initialized at $x(0) = \mathbf{1}$ converges to the team-optimum x^*, in general different from $\mathbf{1}$, as established in the following theorem.

Theorem 2: There exists a polynomial algorithm that verifies conditions (12) and (13) in $O(n^2 \log n)$. In case of positive answer, any pbp optimization algorithm initialized at $x(0) = \mathbf{1}$ converges to the team-optimum.

Proof: (Complexity) Given a DM i, consider all DMs except i in the order $\sigma(1), \ldots, \sigma(n)$ with $a_{i\sigma(1)} \leq \ldots \leq a_{i\sigma(n)}$. We remind here that the ordering process has a complexity $O(n \log n)$. Now, conditions (12) and (13) are verified if and only if $b_i + a_{i\sigma(1)} + \ldots + a_{i\sigma(k-1)} \geq 0$ and $b_i + a_{i\sigma(n-k)} + \ldots + a_{i\sigma(n)} < 0$. We can limit ourselves to verify the latter two conditions for any possible value of the threshold l_i from 1 to n. Such a procedure is carried out via a dicotomic search and has a complexity of $O(\log n)$. Then, for fixed i the total complexity is $O(n \log n) + O(\log n)$, and as $O(n \log n)$ dominates (is always greater than) $O(\log n)$ the total complexity simply reduces to the cost of the ordering process $O(n \log n)$. We conclude our proof by noticing that the ordering process must be repeated n times (one for all DM i) and therefore the resulting complexity is $O(n^2 \log n)$.

(Convergence of pbp) Assume DMs ordered by increasing thresholds, i.e., $l_1 \leq \ldots \leq l_n$. Starting at $x(0) = \mathbf{1}$ any pbp optimization algorithm converges to the pbp optimum nearest to $\mathbf{1}$ (in terms of Hamming distance), call it $\hat{x}$. In other words $\hat{x} = \arg\min\{\|x - \mathbf{1}\| : x \text{ is pbp-opt.}\}$. We must show that $\hat{x}$ is also the team-optimum. To prove this fact corresponds to proving that, if there exists a second pbp optimum, call it $\tilde{x}$, it must hold

$$\begin{aligned} J(\hat{x}) - J(\tilde{x}) &= \mathcal{R}_S J(\tilde{x}) = \\ &= \sum_{i \in S} b_i + \sum_{i,j \in S} a_{ij} + \sum_{r \notin S} \sum_{i \in S} a_{ir} \tilde{x}_r \leq 0, \end{aligned}$$

where S is the set of components which are zero in $\tilde{x}$ and one in $\hat{x}$. Now note that $\sum_{i,j \in S} a_{ij} + \sum_{r \notin S} \sum_{i \in S} a_{ir} \tilde{x}_r = \sum_{i \in S} \sum_{r=1}^{n} a_{ir} \hat{x}_r$ and therefore we can rewrite the above inequality as

$$J(\hat{x}) - J(\tilde{x}) = \sum_{i \in S} (b_i + \sum_{r=1}^{n} a_{ir} \hat{x}_r) = \sum_{i \in S} (b_i + \sum_{r \in \bar{S}} a_{ir}) \leq 0, \quad (14)$$

where we denote by $\bar{S}$ the set of components which are one in $\hat{x}$. Then we need to prove the validity of (14). Now, note that if DMs are ordered by increasing thresholds, it must hold $\tilde{x} \leq \hat{x}$ component-wise. Hence, as $\hat{x}$ is a pbp optimum then each $i \in S$ has threshold $l_i < \|\hat{x} - \mathbf{0}\| = \|\hat{x}\|$ which in turns implies that $\sum_{i \in S} (b_i + \sum_{r \in \bar{S}} a_{ir}) \leq 0$ and therefore (14) hold true. ■

VI. Numerical Example

In this first example we simulate a pbp optimization and show that the algorithm converges to the team optimum. Consider the following team objective function

$$\begin{aligned} J(x) &= -x_1 + x_2 + x_3 + x_4 + 5x_5 - 2x_1x_2 + 4x_1x_3 + \\ &+ 2x_1x_4 - 4x_1x_5 - 6x_2x_3 - 2x_2x_4 - 7x_4x_5 \end{aligned}$$

By direct verification, it can be proved that the above function is a CLG-function as it has a unique local minimum in $(1, 1, 1, 1, 1)$. Similarly, we can see that it is not a CU-function as, for instance, on the 2-face x_1-x_3 with $x_2 = x_4 = x_5 = 0$, conditions (4)-(5) are both verified. The function is not submodular because of the presence of positive quadratic terms.

TABLE I

SEQUENCE OF DMS' DECISIONS: BLOCKS ON THE LEFT, MIDDLE AND RIGHT DESCRIBE THE FIRST, SECOND AND THIRD ROUND OF OPTIMIZATION.

DM	x_i	x	$J(x)$		DM	x_i	x	$J(x)$		DM	x_i	x	$J(x)$
1	1*	(1,0,0,0,0)	-1		1	0*	(0,1,1,0,0)	-4		1	1*	(1,1,1,1,1)	-7
2	1*	(1,1,0,0,0)	-2		2	1	(0,1,1,0,0)	-4		2	1	(1,1,1,1,1)	-7
3	1*	(1,1,1,0,0)	-3		3	1	(0,1,1,0,0)	-4		3	1	(1,1,1,1,1)	-7
4	0	(1,1,1,0,0)	-3		4	1*	(0,1,1,1,0)	-5		4	1	(1,1,1,1,1)	-7
5	0	(1,1,1,0,0)	-3		5	1*	(0,1,1,1,1)	-6		5	1	(1,1,1,1,1)	-7

TABLE II

SEQUENCE OF DECISIONS: FIRST AND SECOND ROUND OF PBP OPTIMIZATION (LEFT AND MIDDLE BLOCKS), 2B2 OPTIMIZATION (RIGHT BLOCK).

DM	x_i	x	$J(x)$		DM	x_i	x	$J(x)$		DM	x_i	x	$J(x)$
1	0	(0,0,0,0,0)	0		1	0	(0,0,1,0,0)	-3		1-2	$1^* - 1^*$	(1,1,0,0,0)	-3
2	0	(0,0,0,0,0)	0		2	0	(0,0,1,0,0)	-3		3-4	$1^* - 1^*$	(1,1,1,1,0)	-11
3	1*	(0,0,1,0,0)	-3		3	1	(0,0,1,0,0)	-3		5-1	$1^* - 1$	(1,1,1,1,1)	-23
4	0	(0,0,1,0,0)	-3		4	0	(0,0,1,0,0)	-3		2-3	1-1	(1,1,1,1,1)	-23
5	0	(0,0,1,0,0)	-3		5	0	(0,0,1,0,0)	-3		4-5	1-1	(1,1,1,1,1)	-23

Start from the decision vector $x = 0$ and assume that the DMs make their decision in the following order: DM 1, DM 2, . . ., DM 5. Table I reports the sequence of DMs' decisions (decisions are starred when they change with respect to the previous round). Blocks on the left describe the first and second round of optimization while block on the right describes the third round of optimization.

If we consider only the vectors x that change from a decision to another one we obtain the sequence

$$\begin{aligned}\sigma \;=\;& (1,0,0,0,0),(1,1,0,0,0),(1,1,1,0,0),(0,1,1,0,0),\\ & (0,1,1,1,0),(0,1,1,1,1),(1,1,1,1,1).\end{aligned}$$

In this second example we simulate the pbp and the 2b2 optimization for the following team objective function and show that only in the second case we converge to the team optimum:

$$\begin{aligned}J(x) = x_1 + x_2 - 3x_3 + x_4 + x_5 - 5x_1x_2 + x_1x_3 + x_2x_3 +\\ -4x_1x_4 - 4x_1x_5 - 4x_2x_4 - 4x_2x_5 - 5x_4x_5.\end{aligned}$$

First observe that the solution $x^* = \mathbf{1}$ is a pbp optimum as, for all i, $b_i + \sum_{k\neq i} a_{ik} < 0$. Furthermore, since for all S, with $|S| = 2$ condition (8) holds, then $x^* = \mathbf{1}$ is also team-optimum. The pbp optimization is carried out as in the previous example and decisions are reported in Table II (left blocks describe the first and second round). Convergence is on $x = (0,0,1,0,0) \neq x^*$. Differently, the 2b2 optimization converges to x^* as evident from the sequence of decisions listed in the right block.

VII. CONCLUDING REMARKS

In future works, we wish to extend the obtained results to consensus problems. Actually, consensus problems have been recently reinterpreted as special potential games [11]. For these games there exist algorithms, very similar in spirit to pbp algorithms and called *best response path algorithm*, that guarantee the distributed convergence to Nash equilibria.

A second line of research aims at providing a parallel between mbm and self organizing/Kohonen maps, since both are optimization methods that can be applied to boolean spaces with decreasing goal functions that in each iteration modify a subset of decision variables.

REFERENCES

[1] B. Bamieh, F. Paganini, and M. A. Dahleh, "Distributed control of spatially invariant systems", *IEEE Transactions on Automatic Control*, vol. 47, no. 7, pp. 1091-1107, July 2002.

[2] B. Bamieh and P. G. Voulgaris. "A convex characterization of distributed control problems in spatially invariant systems with communication constraints", *Systems and Control Letters*, vol. 54, no. 6, pp. 575-583, June 2005.

[3] D. Bertsimas, A. Thiele, "A Robust Optimization Approach to Inventory Theory", *Operations Research*, vol. 54, no. 1, January-February 2006, pp. 150-168.

[4] E. Boros and P. L. Hammer, "Pseudo-Boolean optimization", *Discrete Applied Mathematics*, vol. 123, pp. 155-225, 2002.

[5] H. Björklund and S. Sandberg and S. Vorobyov, "Optimization on completely unimodal hypercubes", *Technical Report 2002-018*, Department of Information Technology, Uppsala University, May 2002.

[6] M. S. Branicky, V. S. Borkar and S. K. Mitter, "A Unified Framework for Hybrid Control: Model and Optimal Control Theory", *IEEE Trans. on Automatic Control*, vol. 43, no. 1, pp. 3145, 1998.

[7] P. R. De Waal and J. H. Van Schuppen, "A class of team problems with discrete action spaces: optimality conditions based on multimodularity", *SIAM Journal on Control and Optimization*, vol. 38, pp. 875–892, 2000.

[8] G.C. Goodwin and D.E. Quevedo, "Finite alphabet control and estimation", *International Journal of Control, Automation, and Systems*, vol.1, no.4, pp. 412–430, 2003.

[9] P. L. Hammer and B. Simeone and T. Liebling and D. de Werra, "From linear separability to unimodality: a hierarchy of pseudo-boolean functions, *SIAM Journal of Discrete Mathematics*, vol. 1, pp. 174–184, 1988.

[10] Y.-C. Ho, "Team decision theory and information structures", *Proceedings IEEE*, vol. 68, pp. 644–654, 1980.

[11] J.R. Marden, G. Arslan, and J.S. Shamma, "Connections between cooperative control and potential games illustrated on the consensus problem", *European Control Conference*, Kos, Greece, July 2-5, 2007, pp. 4604-4611.

[12] J. Matousek, "The number of unique-sink orientations of hypercube", *Combinatorica*, vol. 26, no. 1, pp. 91–99, Feb. 2006.

[13] A. Rantzer, "Linear quadratic team theory revisited", *Proceedings of the 2006 American Control Conference* Minneapolis, Minnesota, USA, June 14-16, pp. 1637–1641, 2006.

[14] D. C. Tarraf, A. Megretski and M. A. Dahleh, "A Framework for Robust Stability of Systems Over Finite Alphabets", *IEEE Transactions on Automatic Control*, To appear. Also available as Technical Report LIDS-P-2680, February 2006.