Average flow constraints and stabilizability in

uncertain production-distribution systems

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Abstract

Consider multi–inventory systems with controlled flows and uncertain demands (disturbances) bounded within assigned compact sets. The system is modelled as a first order one integrating the discrepancy between controlled flows and demands at different sites/nodes. Thus, the buffer levels at the nodes represent the system state. Given a long–term average demand, we are interested in a control strategy that satisfies just one of the two requirements: i) meeting any possible demand at each time (worst case stability) or ii) achieving a pre-defined flow in the average (average flow constraints). Necessary and sufficient conditions

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for the achievement of both goals have been proposed by the authors. In this paper we face the case in which these conditions are not satisfied. We show that if we ignore the requirement on worst case stability, we can find a control strategy driving the expected value of the state to zero. On the contrary, if we ignore the average flow constraints, we can find a control strategy that satisfies worst case stability while optimizing any linear cost on the average control. In the latter case we provide a tight bound for the cost.

Keywords: Inventory control, Robust control, Stochastic stability.

1 Introduction

We consider a continuous time linear multi–inventory system with controlled flows and unknown demands (see, e.g., [1]). The controlled process matrix has more columns than rows and is full row rank. Flows and demands are bounded within assigned polytopic sets (see, e.g., [2, 3, 4]). The system is modelled as a first order one integrating the discrepancy between controlled flows and demands at different sites/nodes. When the discrepancy is null we say that the controlled flows "balance" the demands. Thus, the buffer levels at the nodes represent the system state while the controlled flows and the demands represent the vector of controls and disturbances respectively. It makes sense to choose controlled flows as function of the demand or of the buffer lengths. In this case we denote by control strategy such a

function.

Such systems arise in several applications, such as manufacturing [5, 6, 7, 8, 9], commu-
nications [10, 11], water distribution [12], logistics and traffic control [13].

With such systems, here and in several other works [2, 3, 4, 6, 14, 15, 16], a typical goal
is to find a control strategy that stabilizes in a robust sense (i.e., under any disturbance
realization) the state. Motivations derive from the benefits associated to keeping the state
and consequently also the inventory costs bounded.

Several authors use inventory costs to construct a performance index and then look for an
optimal [16, 17, 18] or near-optimal [19] control strategy. The idea of a performance index
and optimal control strategy is present also in this work.

Bounding polytopic sets for the controlled flows and demands have also appeared in a num-
ber of recent papers on robust network flows [20, 21, 22, 23, 24]. In particular, [21] is centered
around the idea of "adjusting" some of the variables to the outcome of the uncertainty. In
other words some variables are decided before the uncertainty realization while the rest are
decided after the uncertainty realization. Such a problem formulation is known under dif-
ferent names such as "Adjustable Robust Counterpart" (ARC) problem, "Two-stage Robust

optimization with recourse". In many cases, the adjustable variables are expressed affinely

on the uncertainty and the problem is renamed "Affinely Adjustable Robust Counterpart"

(AARC) problem.

There are interesting connections between this paper and the notions of "adjustable variables" in ARC, AARC. For instance, the controlled flows play the role of the adjustable

variables in the ARC set up and in most cases the strategy is affine in the uncertainty as in

AARC problems.

In particular, we focus on stabilizing control strategies meeting alternatively one of the

two following requirements:

AFC **Average Flow Constraints** — the control strategy must return an average control

that balances the average disturbance, the latter one being a-priori known. Henceforth,

we say *average control (disturbance)* to mean the control (disturbance) vector averaged

over time. Intuitively, this means that the discrepancy between the controlled flows and

the demands integrated on an infinite horizon must be finite. When this happens we

also say that the controlled flows satisfy/meet the demands.

WCS **Worst Case Stability** — the control strategy must drive the state to a hyperbox in

finite time and keep the state within it for the rest of the time under any disturbance realization. The hyperbox has pre-assigned dimension $\epsilon \geq 0$ and no assumption on the average disturbance is made.

Henceforth, the sentence "the control strategy satisfies AFC (WCS)" indicates that such a control strategy meets the first (second) requirement. Justifications for AFC derive from the observation that contracts, long-term agreements or even physical limitations of production machinery may fix the utilization level of the network on the long run. This immediately translates into constraints on the average flows.

In [25], we have found conditions for a control strategy to satisfy both AFC and WCS. This occurs only in some special cases as the obtained conditions seem to be strong. In this paper, we impose only one of the two basic requirements at the time. In this sense, the expression "relaxing WCS (AFC)" indicates instances where the requirement on WCS (AFC) is disregarded. Obviously, the resulting control strategy may not satisfy WCS (AFC). Only note that, when dealing with AFC, we will also require that the control strategy drives the state to zero in probability. This is the same of saying that the expected value of the state tends to zero for increasing times. When this occurs we say that the control strategy

achieves stochastic stability. We next summarize the two main results of this paper.

First, we show that if we relax WCS we can always define a control strategy that satisfies AFC and achieves stochastic stability (i.e., it is asymptotically stable with probability one) provided that i) the demand and the buffer levels are not correlated, and that ii) there exists an average control internal to the bounding polytopic set that satisfy the average demand. Second, we prove that if we relax AFC, we can find a control strategy that satisfies WCS while optimizing a linear cost of the average control.

This paper is organized as follows. In Section 2, we describe the model. In Section 3 we recall some preliminary results. In Section 4, we study AFC while relaxing WCS. In Section 5, we study WCS while relaxing AFC. In Section 6, we provide a numerical example. Finally, in Section 7, we draw some conclusions.

2 Model formulation

We first introduce the multi–inventory model and then state formally the notion of WCS and AFC. In doing this, we also discuss the underlying assumptions on the topology of the system and on the disturbance realization. Many concepts introduced in this section can also

be found in [25].

Consider the continuous time system

$$\dot{x}(t) = Bu(t) - w(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the vector of the buffer levels, $u(t) \in \mathbb{R}^m$ are the controlled flows, $B \in \mathbb{R}^{n \times m}$ is the controlled process matrix and $w(t) \in \mathbb{R}^n$ is the vector of the demand. We also name $x(t)$, $u(t)$ and $w(t)$ simply "state", "control" and "disturbance" respectively. To model backlog $x(t)$ may be negative. Also note that a negative demand can be interpreted as a negative deviation from an expected value of w.

We assume that u and w are respectively subject to constraints

$$u(t) \in \mathcal{U} = \{u : u^- \leq u \leq u^+\}, \tag{2}$$

$$w(t) \in \mathcal{W}, \tag{3}$$

where u^- and u^+ are assigned vectors and $\mathcal{W}$ is a polytope. We also assume that matrix B is "fat", i.e., it has more columns than rows and is also full row rank.

Assumption 2.1 Matrix B is such that $m > n$ and also $rank(B) = n$.

Indeed, if B is not full row rank the system is unreachable and, as we will see momentarily, if B is square defining a strategy may become trivial.

Given a vector function of time $f : \mathbb{R}^+ \to \mathbb{R}^n$ we introduce the notation

$$Av[f] = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) \, dt \qquad (4)$$

to denote "the (deterministic) average" of f. Also we restrict our attention to disturbance realizations $w(t)$ having the property that the average $Av[w]$ always exists and satisfying the following assumption.

Assumption 2.2 All the possible realizations of the demands $w(t)$ have the same deterministic average $\bar{w} = Av[w]$ with $\bar{w}$ being in the relative interior of the bounding polytopic set $\mathcal{W}$. [1]

As an example, any ergodic process for $w(t)$ satisfies Assumption 2.2.

To formalize the notion of WCS we next define balancing and ϵ-stabilizing control strategies.

Definition 2.1 The function $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is a *static balancing strategy* if for $u(t) = \Phi(w(t))$,

$$Bu(t) = w(t),$$

and $u(t) \in \mathcal{U}$, for all $w(t) \in \mathcal{W}$, for all $t \geq 0$.

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[1] we mean that $\bar{w}$ is an interior point of $\mathcal{W}$ with respect to the smallest linear subspace including it, for instance given a vector $v \neq 0$, 0 is in the relative interior of a segment joining v and $-v$

The fact that, for $m = n$, the only static balancing strategy is trivially $u(t) = B^{-1}w(t)$ justifies Assumption 2.1.

When, a static balancing strategy is applied, we have $\dot{x}(t) = 0$ for all $t \geq 0$ and therefore we refer to this scenario as the static case/problem. Then, theoretically, the buffer level remains bounded since the controlled flows meet the demand at each time. However, this is not a feedback strategy and the resulting system is not stabilized as infinitesimal measurements errors on $w(t)$ may cause buffers overflow. Actually, our ultimate goal is solving the dynamic problem of steering the system buffer to the neighborhood of a prescribed level. To this end, we introduce the following definition.

Definition 2.2 Given $\epsilon > 0$ and a reference value $\bar{x}$, an *ϵ-stabilizing strategy* is a feedback control strategy $u(.)$ for which there exists a continuous positive function $\psi(t)$, monotonically decreasing and converging to 0 as $t \to \infty$ such that for all $w(t) \in \mathcal{W}$ and for all $x(0)$, the conditions $u(t) \in \mathcal{U}$ and

$$\|x(t) - \bar{x}\| \leq \max\{\|x(0) - \bar{x}\|\psi(t), \epsilon\}$$

hold true.

With in mind the definitions of balancing and ϵ-stabilizing control strategy, we can now

state formally the notion of WCS.

Definition 2.3 We say that a strategy satisfies WCS if it is balancing in the static problem

or it is ϵ-stabilizing in the dynamic problem.

It is left to state formally the notion of AFC. On this purpose, let us consider the system

behaviour on the long run. If we assume that conditions (6) or (7) hold true, and if we apply

either a balancing or an ϵ-stabilizing strategy for any pre-assigned $\epsilon > 0$, then the state $x(t)$

remains constant or bounded and this implies, in turn, that (after integrating (1))

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T [Bu(t) - w(t)]\, dt = \lim_{T \to \infty} \frac{1}{T}\, [x(T) - x(0)] = 0.$$

The latter means that the average control balances the average demand, that is,

$$B\, Av[u] = Av[w]. \tag{5}$$

Now, because of the assumption on B fat, given an average demand $\bar{w}$, multiple controls $\bar{u}$

exist satisfying the property $B\bar{u} = \bar{w}$. Then we wish to find a control strategy $u(t)$ such that

$\bar{u} = Av[u(t)]$. Roughly speaking, AFC consist in selecting a-priori a desired value for $\bar{u}$ and

looking for a control strategy that returns an average control equal to such a value, whenever

$Av[w] = \bar{w} \in \mathcal{W}$. For sake of simplicity we can assume that the average demand is null:

Assumption 2.3 We assume that the average demand is $\bar{w} = 0$.

The above assumption is not restrictive as we can always translate the variable w. Then, we can formalize the notion of AFC as follows.

Definition 2.4 We say that a strategy satisfies AFC if it is such that whenever $Av[w] = 0$ then $Av[u] = 0$.

3 Preminary results

We report the necessary and sufficient conditions for the existence of balancing or ϵ-stabilizing strategies present in the literature [3].

Theorem 3.1 *Given system (1) with polytopic bounding sets (2) and (3),*

i) *there exists a static balancing strategy as in Definition 2.1 if and only if*

$$\mathcal{W} \subseteq B\mathcal{U}; \tag{6}$$

ii) *for any $\epsilon > 0$, there exists a feedback stabilizing strategy as in Definition 2.2 if and only if*

$$\mathcal{W} \subseteq int\{B\mathcal{U}\}. \tag{7}$$

The following result shows necessary and sufficient conditions for a control strategy to satisfy both WCS and AFC [25]. Let us denote by $w^{(r)}$ with $r \in Ext\{\mathcal{W}\}$ any vertex of the polytopic set $\mathcal{W}$. To be more precise, $Ext\{\mathcal{W}\}$ is the set of indices of all vertices of $\mathcal{W}$.

Theorem 3.2 *[25] There exists a strategy which satisfies both WCS and AFC if and only if there exists a matrix $D \in \mathbb{R}^{m \times n}$ such that*

$$BD = I \tag{8}$$

$$u^- \leq Dw^{(r)} \leq u^+, \quad r \in Ext\{\mathcal{W}\}. \tag{9}$$

In the static case, the theorem also shows that the balancing control strategy is linear

$$u(t) = Dw(t). \tag{10}$$

In the dynamic case, the ϵ-stabilizing control strategy is obtained after some mathematical manipulations which we report in the remaining part of this section. Complete matrices B and D with matrices C and F such that

$$\begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} D & F \end{bmatrix} = I. \tag{11}$$

Consider the augmented system

$$\dot{x}(t) = Bu(t) - w(t)$$

$$\dot{y}(t) = Cu(t).$$

(12)

Consider the new variable $z(t)$ defined as

$$z(t) = \begin{bmatrix} D & F \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} B \\ C \end{bmatrix} z(t).$$

The augmented system becomes

$$\dot{z}(t) = u(t) - Dw(t).$$

(13)

It is decoupled in its state variable, then componentwise we have

$$\dot{z}_i(t) = u_i(t) - D_i w(t),$$

(14)

where D_i the ith row of D and $u_i^- \leq u_i \leq u_i^+$. On the basis of such a decomposition in [25]

we have proved that a possible ϵ-stabilizing strategy is

$$u_i(t) = sat_{[u_i^-, u_i^+]}[-\kappa z_i]$$

where $\kappa > 0$ and

$$sat_{[u_i^-,u_i^+]}[\xi] \doteq \begin{cases} u_i^- & \text{if } \xi > u_i^+ \\ u_i^+ & \text{if } \xi < u_i^- \\ \xi & \text{if } u_i^- \leq \xi \leq u_i^- \end{cases}.$$

Finally, we derive a (discontinuous) switching ϵ-stabilizing strategy y letting $\kappa \to \infty$.

$$u_i = \begin{cases} u_i^- & \text{if } z_i > 0 \\ u_i^+ & \text{if } z_i < 0 \\ 0 & \text{if } z_i = 0 \end{cases}. \tag{15}$$

4   Average flow constraints and stochastic stability

In this section, we show that if we relax WCS we can always find a strategy that satisfies AFC and achieves stochastic stability. We recall that the system is almost surely asymptotically stable if $x(t) \to 0$ with probability one [26].

Then, let us relax the conditions of Theorem 3.1 or 3.2 by simply assuming as follows.

Assumption 4.1 Assume that $0 \in int\{B\mathcal{U}\}$.

Note that the validity of just one from conditions (6) or (7) or (9) implies $0 \in int\{B\mathcal{U}\}$ but not viceversa.

Only for this section, we need to make the following additional assumption.

Assumption 4.2 The demand $w(t)$ is a mean ergodic stochastic process and it is not correlated with the buffer levels.

The above assumption implies that, at each time, the expected value of w coincides with the long term average, i.e., $E[w] = Av[w] = \bar{w}$, and that $E[x^T w] = 0$.

Note that in principle this is a restriction since the demand might be actually affected by the buffer levels (for instance a customer can choose a supply node or another based on their congestion state). However, in many situations the assumption is quite reasonable. This assumption is reasonable whenever the customers have no information on the state of suppliers. This situation occurs as an example at the retailer level, where the final customer typically has no idea about the retailer' inventory position.

The next result concerns the stochastic stability of the system.

Theorem 4.1 *Under Assumptions 2.1-4.2, there exists a control strategy such that the system is asymptotically stable with probability one. Furthermore, such a strategy satisfies AFC. A possible control is*

$$u = \arg\min_{u \in \mathcal{U}} x^T B u. \tag{16}$$

Proof. The condition $0 \in int\{B\mathcal{U}\}$, implies that for $x \neq 0$

$$\min_{u \in \mathcal{U}} \frac{x^T}{\|x\|} B u \leq -\beta$$

for some $\beta > 0$. Consider the Lyapunov function $V(x) = x^T x / 2$. The expected value of the derivative is

$$E[\dot{V}] = E[x^T B u] + E[x^T w] = E[x^T B u] \leq -\beta \|x\|,$$

which implies stability with probability one (see for instance [26]). The proposed strategy does imply stochastic stability but it does not satisfy necessarily AFC. To enforce AFC we can use the decomposition (14) of (13)

$$\dot{z}_i(t) = u_i(t) - \delta_i(t),$$

where $\delta_i(t) = D_i w(t)$, where now D_i is the ith row of any matrix D satisfying (8) but not necessarily (9). Note that $E[\delta_i] = D_i E[w(t)] = 0$. If we consider the Lyapunov function $z_i^2 / 2$,

we see that the control $u(t)$ obtained from (16) is equivalent to the control (15). This means that the z_i subsystem is stable with probability one. Then,

$$\frac{1}{T}\int_0^T [u_i(t) - \delta_i(t)]dt \to 0 \Rightarrow \frac{1}{T}[x(T) - x(0)] \to 0,$$

with probability one.

$\square$

Control (16) generalizes the results proposed in [25]. There, we define a linear strategy only for the situations in which a matrix D exists that satisfies all the constraints (9). Here, we drop this restrictive requirement on D and, indeed, we obtain a control (16) that in general is not linear but can be applied to a greater class of systems. As a side effect, we also get a complexity reduction in the computation of D.

It is interesting to note that, as long as we are able to characterize the statistics of the input δ and to characterize the variance of the variable z, we can estimate the variance of variable x as follows

$$E[\|x\|^2] = E[\|Bz\|^2] \leq \|B\|^2 E[\|z\|^2] = \|B\|^2 \sum_i E[\|z_i\|^2].$$

The values $E[\|z_i\|^2]$ can be computed easily from the analysis of the single dimensional process $z_i(t)$ under quite general assumptions on the type of signals δ. Note also that $E[\|z\|^2]$ is

affected by $E[||\delta||^2]$, and then it makes sense to optimize the choice of the augmenting matrix

D in order to minimize the variance of $\delta = Dw$ if we assume that the covariance matrix

$E[ww^T] = W$ is known. Indeed we have

$$tr E[\delta\delta^T] = tr[Dww^T D^T] = tr[DWD^T].$$

Then we can choose D by solving the linear quadratic problem

$$\min tr[DWD^T], \quad s.t. \quad BD = I.$$

5   Worst case stability and optimal average control

In this section we show that if we relax AFC, then we can find a control strategy satisfying

WCS while optimizing a linear cost of the average control. Differently from the previous

section, we now need condition (6) or (7) to be valid whereas we can disregard Assumption

4.2. Also, we can dispense with mentioning explicitly Assumption 4.1, as the latter is trivially

implied by condition (6) or (7) and Assumptions 2.2-2.3.

Given a vector $c^T = [c_1 \ldots, c_m]$, we wish to minimize the linear cost

$$J = c^T Av[u].$$

Let us start by considering the static case. It is reasonable that to minimize the cost we have

to choose $u(t)$ as function of $w(t)$ according to the following optimal criterion

$$u = \arg\min_{u \in \mathcal{U}} c^T u \quad s.t. \quad Bu - w = 0.$$

Trivially, this is the optimal balancing strategy.

In the following, we show how to estimate the worst case cost. For each vertex $w^{(r)}$ with

$r \in Ext\{\mathcal{W}\}$, consider the minimum balancing flow

$$u^{(r)} = \arg\min_{u \in \mathcal{U}} c^T u \quad s.t. \quad Bu - w^{(r)} = 0. \tag{17}$$

Define

$$J^* = \max_{\sum_{r \in Ext\{\mathcal{W}\}} \alpha_r = 1, \ \alpha_r \geq 0} \sum_{r \in Ext\{\mathcal{W}\}} \alpha_r (c^T u^{(r)}) \quad s.t. \quad \sum_{r \in Ext\{\mathcal{W}\}} \alpha_r w^{(r)} = 0. \tag{18}$$

Then we have the following result.

Theorem 5.1 *Under Assumptions 2.1-2.3 and condition (6), we have that the optimal bal-*

ancing strategy is such that

$$J = c^T Av[u] \leq J^*$$

Moreover the bound is tight, namely for any $\epsilon > 0$ there exists $w(t)$ such that $J > J^* - \epsilon$.

Proof. For all $w(t) \in \mathcal{W}$ we write

$$w(t) = \sum_{r \in Ext\{\mathcal{W}\}} \alpha_r(t) w^{(r)}, \quad \alpha_r \geq 0, \quad \sum_{r \in Ext\{\mathcal{W}\}} \alpha_r = 1.$$

Here, with a little abuse of notation, the coefficients $\alpha_r(.)$ with $r \in Ext\{\mathcal{W}\}$ depend on time t. Owing to condition (6) an optimal balancing strategy $\Phi(w)$ exists (remind $\Phi(w)$ balancing means that $B\Phi(w) - w = 0$ for all $t \geq 0$) and at a generic time t can be chosen according to

$$J_1 = \min_{u \in \mathcal{U}} c^T u, \quad s.t. \ Bu = \sum_{r \in Ext\{\mathcal{W}\}} \alpha_r w^{(r)}.$$

The value $u = \sum_{r \in Ext\{\mathcal{W}\}} \alpha_r u^{(r)}$ is a feasible solution of this problem. Since J^* is achieved by maximizing over the α_r the instantaneous cost is $J_1 \leq J^*$ and therefore the average $J = c^T Av[u] \leq J^*$.

To prove that the bound is tight, consider the values α_r^* which solves the maximization problem (18) and the following T–periodic demand piecewise constant in intervals of length $T_i = \alpha_r^* T$

$$w(t) = w^{(r)}, \quad \sum_{i=0}^{r-1} T_i \leq t \leq \sum_{i=0}^{r} T_i, \quad r \in Ext\{\mathcal{W}\}.$$

Note that $Av[w] = 0$. The optimal cost is necessarily achieved by applying $u^{(r)}$ in any interval, but this turns out to be J^*.

$\square$

It is worth pointing out (for reasons that will be clear later) what follows.

Remark 5.1 The long term cost remains unchanged if we replace the constraints $Bu(t) - w(t) = 0$ by the constraint $Bu(t) - w(t - \tau) = 0$. for any fixed $\tau > 0$.

To handle the dynamic problem we consider a sampled–data strategy in which $u(t) = u(k)$, $t \in [k\tau, (k+1)\tau)$ [2]. Denote by $x(k) = x(k\tau)$, $w(k) = \int_{k\tau}^{(k+1)\tau} w(\sigma) \, d\sigma$, and obtain

$$
\begin{aligned}
x(k+1) &= x(k) + \tau Bu(k) - \int_{k\tau}^{(k+1)\tau} w(\sigma) \, d\sigma \\
&= x(k) + \tau Bu(k) - w(k)
\end{aligned}
$$

and note that $w(k) \in \mathcal{W}$. Then introduce the new variable

$$
\tilde{x}(k+1) = x(k) + \tau Bu(k)
$$

and again note that $x(k) - \tilde{x}(k) = -w(k-1)$. Let ρ be such that $\mathcal{W}$ is inside the ρ–ball, i.e., $\|w\| \le \rho$ for all $w \in \mathcal{W}$. Then, depending on whether $\|x(k)\| > \tau\rho$ or $\|x(k)\| \le \tau\rho$ choose $u(k)$, respectively, as

$$
u(k) = \arg\min_{u \in \mathcal{U}} \ \|x(k) + \tau Bu\| \tag{19}
$$

or as

$$
u(k) = \arg\min_{u \in \mathcal{U}} \ c^T u \quad s.t. \quad \tau Bu + x(k) - \tilde{x}(k) = \tau Bu - w(k-1) = 0. \tag{20}
$$

We next prove that this strategy is i) robust stabilizing as it drives the state to the ball of radius $\tau\rho$ and, once the state is in the ball ii) is optimal as it compensates at each interval the demand of the previous interval with the minimum balancing flow. The challenging and original part of the proof is the second one, as to prove the first result we can simply refer to [2].

Theorem 5.2 *Under Assumptions 2.1-2.3 and condition (6), strategy (19,20) is ϵ-stabilizing with $\epsilon = \tau\rho$ and guarantees an average cost $J \leq J^*$.*

Proof. The fact that the strategy drives $x(k)$ to the ball of radius $\epsilon = \tau\rho$ has been proven in [2].

Let us consider the following delayed cost

$$
\begin{aligned}
J_D &= \min_{u(t)\in\mathcal{U}} \quad c^T A v[u] \\
\\
&\text{s.t.} \quad Bu(t) - w(t - \tau) = 0
\end{aligned}
\tag{21}
$$

and according to Remark 5.1 we have $J_D \leq J^*$. Consider the next problem

$$
\begin{aligned}
J_R &= \min_{u(t)\in\mathcal{U}} \quad c^T A v[u] \\
\\
&\text{s.t.} \quad \int_{k\tau}^{(k+1)\tau}[Bu(t) - w(t - \tau)]dt = 0, \\
\\
&\quad k = 0, 1, \ldots
\end{aligned}
\tag{22}
$$

and note that since the constraint of this new optimization problem is a relaxed version of

(21) we have $J_R \leq J_D$. On the other hand, the integral constraint of (22) is equivalent to

$$\int_{k\tau}^{(k+1)\tau} Bu(t)dt - w(k-1) = 0.$$

Thus the optimal solution of problem (22) is constant in each interval $[k\tau, (k+1)\tau]$ and turns

out to be (20) which therefore provides a cost $J = J_R \leq J_D \leq J^*$.

$\square$

Remark 5.2 The bound J^* is tight for the optimal strategy (19,20). To prove this fact,

consider a long discrete–time interval partitioned in sub-intervals, that are proportional to

the α_r^*. On each sub-interval define a constant demand equal to a vertex of $\mathcal{W}$ and apply

the result of the above theorem. It is trivial to see that, the longest the discrete interval the

closer J_R and J_D come to J^*.

6 Example

The proposed investigation method can be applied to high dimensional systems without particular problems since it is based on a standard and efficient algorithm. For the sake of comprehension, we consider the very simple example proposed in [25]

$$\dot{x}(t) = v_1(t) + v_2(t) - d(t)$$

with

$$0 \le v_1 \le 5, \quad 0 \le v_2 \le 3, \quad 1 \le d \le 7$$

with $Av[d] = 4$. Consider the nominal values

$$\bar{v}_1 = 3.5, \quad \text{and} \quad \bar{v}_2 = 0.5$$

as assigned desired average values for the two inputs. Define $u_1 = v_1 - \bar{v}_1$, $u_2 = v_2 - \bar{v}_2$ and $w = d - \bar{d}$. Then the feasible flows and demand are

$$-3.5 \le u_1 \le 1.5, \quad -0.5 \le u_2 \le 2.5, \quad -3 \le w \le 3$$

and $Av[w] = 0$. The set of feasible flows is the rectangle (solid line) A-D-F-L displayed in Fig. 1. Segments NB and GE include all feasible flows that balance the minimum demand $w = -3$ and the maximum demand $w = 3$ respectively, i.e., $u_1 + u_2 = \pm 3$. Also, the segment

HC is the set of feasible flows that balance a demand $w = 0$. It is easy to see that there no exist any matrix D that satisfies the conditions of Theorem 3.2 and therefore, according to the theory developed in [25], this average flow $\bar{u} = 0$ is not achievable. In other words, there are realizations of w with $Av[w] = 0$ that cannot be compensated by flows with 0-average. Indeed all the achievable averages are in the dotted polygon B-E-G-N in Fig. 1. Given the fact that $\bar{u} = 0$ is not achievable, we can either achieve stochastic stability by guaranteeing the 0–average or optimize an average linear flow cost.

To achieve stochastic stability consider the following augmentation

$$
\begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \mu & \nu \end{bmatrix}
$$

whose inverse is

$$
\begin{bmatrix} B \\ C \end{bmatrix}^{-1} = \frac{1}{\mu - \nu} \begin{bmatrix} -\nu & 1 \\ \mu & -1 \end{bmatrix}
$$

The system is equivalent to

$$
\begin{aligned}
\dot{z}_1(t) &= u_1(t) + \tfrac{-\nu}{\mu - \nu} w(t) \\
& \\
\dot{z}_2(t) &= u_2(t) + \tfrac{\mu}{\mu - \nu} w(t).
\end{aligned}
\tag{23}
$$

It is easy to see that the set of all vectors of the form

$$
\begin{bmatrix} \tfrac{-\nu}{\mu - \nu} \\[2mm] \tfrac{\mu}{\mu - \nu} \end{bmatrix} w = \begin{bmatrix} \tfrac{-\nu/\mu}{1 - \nu/\mu} \\[2mm] \tfrac{1}{1 - \nu/\mu} \end{bmatrix} w, \quad w \in [-3, 3]
$$

is a 0–symmetric segment (see the dash-dot segments in Fig. 1). No matter how μ and ν are

taken, such a segment is never included in the rectangle defining the constraints for u_1 and

u_2. This last consideration is in agreement with the fact that the augmented system cannot

be stabilized in the worst case. Indeed it can be shown that the extrema of this interval

are on the lines including segments NB and GE and that if one of the extrema is inside

the rectangle of feasible flows A-D-F-L the other one is outside. However both equations

in (23) can be written in the form $\dot{z}_i = u_i + \delta_i$. Since $E[w] = 0$, we have $E[\delta_i] = 0$, and so

the system can be stabilized in the stochastic sense with the provided control. Note that if

$z_1(t), z_2(t) \to 0$ (or they are bounded), then $y(t) \to 0$ and $x(t) \to 0$ (or they are bounded),

so we have

$$\frac{1}{T} \int_0^T (\mu u_1 + \nu u_2)dt = \frac{1}{T}[y(T) - y(0)]dt \to 0$$

namely

$$\mu Av[u_1] = -\nu Av[u_2]$$

Repeating the same argument for the variable x we have

$$Av[u_1] + Av[u_2] = Av[w]$$

so that for $Av[w] = 0$ we are actually achieving the desired average values $(0,0)$ for u. This condition is stochastically assured. Note that if, as a special case, we choose $\mu = 1$ and $\nu = -7$ (or any proportional value), then the possible average values of v_1 and v_2 turn out to be proportional to the desired values 3.5 and 0.5 (even when $Av[w] \neq 0$). This fact is interesting if the values 3.5 and 0.5 come out from a "desired workload partition" between the arcs 1 and 2 (in the sense that arc 1 should be loaded 7 times as much as arc 2).

We simulated the transient assuming $\mu = 1$ and $\nu = -7$ and the buffer in an initial condition of backlog $x(0) = -3$. In Fig. 2 we show the transient of variables x and y. Note the initial undershoot of variable y caused by a "mismatch" from the desired workload partition between u_1 and u_2. Indeed, both controls initially saturate to fix the backlog. Fig.

3 reports the finite–time average values $Avt[u_i] := (\int_0^t u_i(t)d\sigma)/t$. There, observe that the

aforementioned mismatch is eventually recovered, y bounded, while the averages values of u_1

and u_2 converge to zero.

Figures 2 and 3 have a different time scale since the transient of the average of u_1 and

u_2 is slower than the transient of x and y. The final ripples are a well known effect of the

discretization.

Let us now, retain worst case stability and optimize the average. On this purpose, first,

consider the case where the state is within the ball of radius $\tau\rho$, $\|x(k)\| \leq \tau\rho$, and the

control compensates at each interval the demand of the previous interval according to (20).

Distinguish between the two opposite cases where u_1 has a (unitary) cost lower/greater than

u_2. Denote by $c^T = [c_1 \, c_2]$ and refer to Fig. 1. If $c_1 < c_2$ then the level surfaces for the linear

cost $c^T u = K$, where K is a parameter, are parallel to line a. If $w(k-1) = 0$, then line

HC is the set of feasible flows for (20), namely satisfying $\tau Bu - w(k-1) = 0$. In this case,

minimizing according to (20) corresponds to translating the surface level until we intersect

the extreme point C of HC which represents the minimum. If we repeat the same procedure

when demand is minimum, $w(k-1) = -3$ or maximum, $w(k-1) = 3$, the set of feasible

flows is described by line NB and GE and the corresponding minima are points B and E respectively.

It is easy to see that for any $w(k-1) \in \mathcal{W}$ minimization (20) returns all points of line BD, if $1 \leq w(k-1) \leq 5$ and DE if $5 \leq w(k-1) \leq 7$.

Consider the opposite case where $c_1 > c_2$. Then the level surfaces are parallel to line b and minimization (20) returns all points of line NL, if $1 \leq w(k-1) \leq 3$ and LG if $3 \leq w(k-1) \leq 7$.

It is left to consider the case where $\|x(k)\| > \tau\rho$, and the state is driven into the ball of radius $\tau\rho$ in finite time under control (19). It is easy to see that depending on whether $x(k) < 0$ or $x(k) > 0$, the optimal control (19) spans the triangle A-B-N and E-F-G respectively. Finally, observe that the cost for driving the state into the ball in finite time does not affect the long-term average cost.

7 Conclusions

In a recent paper [25] we have provided necessary and sufficient conditions for achieving simultaneously worst–case stability and average flow constraints. This work studies the case

where the aforementioned conditions are not valid. We answer to the question whether at least one between worst–case stability or average flow constraints can be still guaranteed. For this reason this work represents a continuation of [25]. Along this line of research, at least two other issues draw our attention for the near future: i) the specialization of LMI techniques to multi-inventory applications (see, e.g., the recent work [27]), and ii) the adaptation of randomized algorithms [28] to the robust optimization of inventory control strategies.

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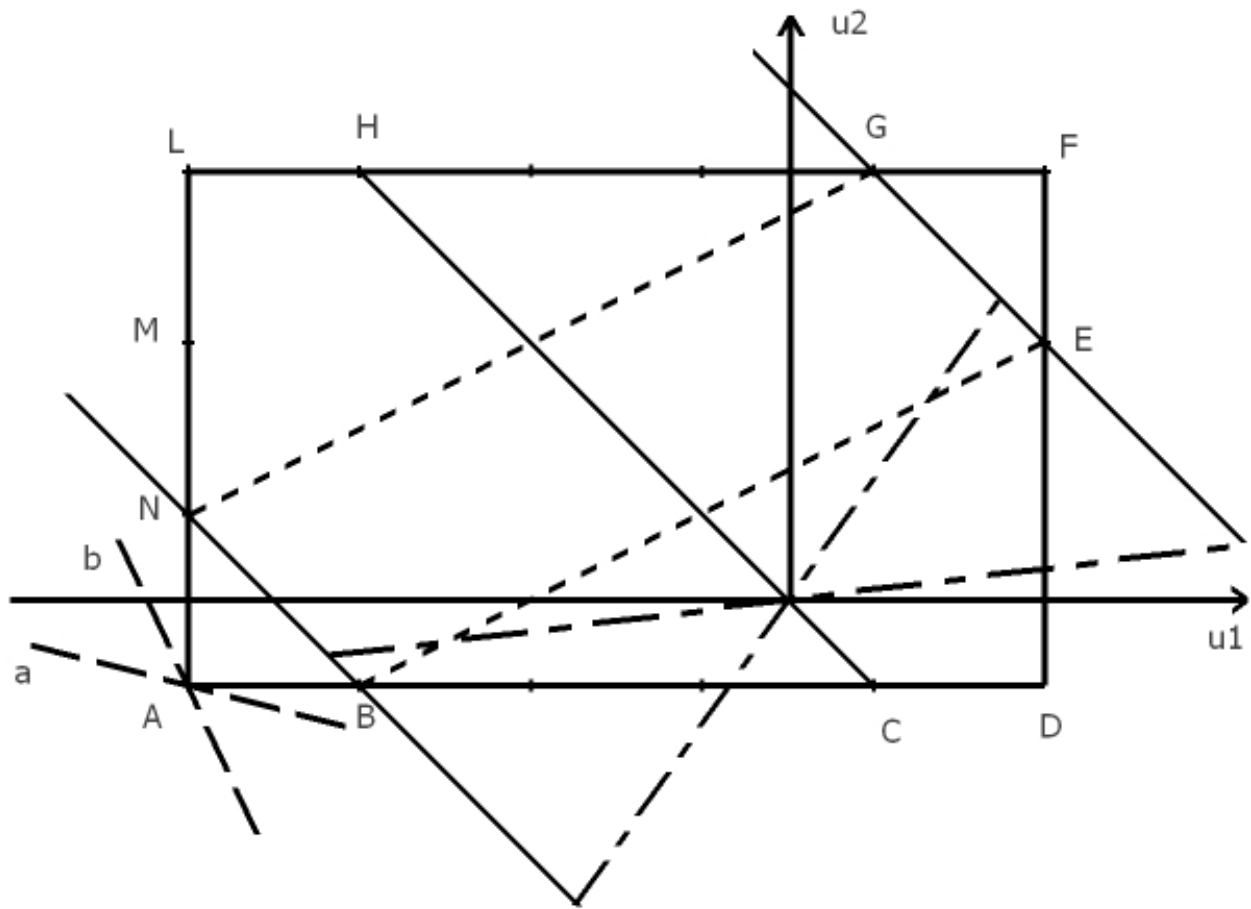
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