Mathematics of Control, Signals, and Systems manuscript No.
(will be inserted by the editor)

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Boolean-controlled systems via receding horizon and linear programming

Received: date / Revised: date

Abstract We consider dynamic systems controlled by boolean signals or decisions. We show that in a number of cases, the receding horizon formulation of the control problem can be solved via linear programming by relaxing the binary constraints on the control. The idea behind our approach is conceptually easy: a feasible control can be forced by imposing that the boolean signal is set to one at least one time over the horizon. We translate this idea into constraints on the controls and analyze the polyhedron of all feasible controls. We specialize the approach to the stabilizability of switched and impulsively-controlled systems.

Keywords Impulse Control · Inventory Control · Hybrid Systems

1 Introduction

Hybrid optimal control problems are, in general, difficult to solve (see, e.g., [10; 13; 29] and references therein). For this reason, a current research goal is to *isolate those problems that lead to tractable solutions* [10]. According to this aim, in this paper we identify among the larger set of hybrid optimal control problems dealt in [13], a special class of problems which are easy to solve. Easy to solve means that the solution algorithms are polynomial in time and therefore suitable to the on-line implementation in real-time

This work was supported by MURST-PRIN 2007ZMZK5T “Decisional model for the design and the management of logistics networks characterized by high interoperability and information integration”.

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problems. We do this by using a paradigm borrowed from the Operations Research field.

More precisely, this paper is one of the several recent attempts [1; 2; 4; 5; 13] to apply the tools of combinatorial optimization to hybrid optimal control problems. The general recipe is to take the standard continuous-time hybrid optimal control problem, discretize it, and reformulate and massage it until it is suitable for applying one of the several discrete optimization techniques [4; 5]. Integer programming, or one of its variations, seems to be specially favorite for this line of research.

In Section 2, we consider systems characterized by a continuous state, a binary state, and controlled through a binary control. The binary state describes the two operating modes of the system while the binary controls represent resets, impulses, or switches between modes. The problem consists in finding *feasible controls*, i.e., controls that satisfy certain stabilizability conditions.

Boolean control/decision spaces can be found in finite-alphabet control and in particular on-off control problems [16], impulsively-controlled systems (activate the impulse or not) [10; 20], or switching control (switches between active and passive modes) [21; 22; 26]. Applications include inventory with set up costs (reordering or not from a warehouse in order to meet a demand) [8], distributed computer systems (processing or not the assigned task) [15], air-conditioning systems control, economics and finance (see, e.g., [9] and references therein).

The main result of this work is stated in Section 3. There, we show that in many cases the receding horizon formulation of the problem can be solved via linear programming after relaxing the binary constraints on the control and exploiting the total unimodularity of the constraint matrix [11; 14; 24]. This is the case anytime a feasible solution derives from imposing that the control is set to one at least one time in the horizon window.

In Section 4, we specialize the approach to the (asymptotic) stabilizability of switched systems. Two cases are considered: time dependent slow switching controls and state dependent switching controls. As illustrative example, we simulate a switched oscillating system under different feasible solutions.

In Section 5, we extend the approach to the Input to State Stabilizability (ISS) of impulsively controlled systems, according to the definition provided in [20]. In particular, we focus on ISS systems with dwell time and reverse dwell time. As illustrative example, we simulate a first order system under different feasible solutions.

In Section 6, we extend the discussion to inventory applications and in Section 7, we draw some conclusions.

2 Problem formulation

2.1 Boolean-controlled systems

Consider the following systems characterized by a continuous state $x(t) \in \mathbb{R}^n$, a continuous disturbance $d(t) \in \mathbb{R}^m$, and a binary control $u(t) \in \{0, 1\}$.

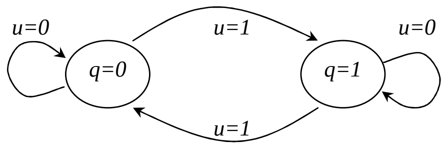

Fig. 1 Transitions between mode 1 and 2.

Equation (1) describes a switched system where $q \in \{0, 1\}$ is a binary state (the operating modes), function $f_q : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is the dynamics in mode q of $x(t)$, the binary control $u(t) \in \{0, 1\}$ for all $t \geq 0$ describes a switching control returning a switch whenever $u(t)$ is set to one. Transitions of the binary state from one mode to the other one are described by the automata displayed in Fig. 1:

$$\begin{aligned} \dot{x}(t) \quad &= f_q(x(t), d(t)), \\ q(t^+) &= \begin{cases} \quad q(t) & \text{if } u(t) = 0 \\ 1 - q(t) & \text{if } u(t) = 1. \end{cases} \end{aligned} \tag{1}$$

Equation (2) describes an impulsively-controlled system where function $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ is the dynamics of $x(t)$, $h(x(t), d(t))$ is the reset value, $u(t)$ is the impulse control law returning impulses whenever $u(t)$ is set to one:

$$\begin{array}{ll} \dot{x}(t) = f(x(t), d(t)) & \text{if } u(t) = 0 \\ x(t^+) = h(x(t), d(t)) & \text{if } u(t) = 1. \end{array} \tag{2}$$

Let an additional function $V : \mathbb{R}^n \mapsto \mathbb{R}$ be given. For instance, one may think $V(x(t))$ being a differentiable norm function. We wish to solve the following problem.

Problem 1 Find a control $u(t) \in \{0, 1\}$ for all $t \geq 0$ such that the following condition is satisfied

$$\psi(V(0), V(t), \dot{V}(t)) > 0, \tag{3}$$

where $\psi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is a generic function of $V(0)$,$V(t)$, and $\dot{V}(t)$.

We use $\dot{V}(t)$ to indicate the derivative of V at time t.

In Section 4, condition (3) reduces to $\dot{V}(t) < 0$ (negative derivative) as we focus on asymptotic stabilizability of a switched system (see also the *switching-based Lyapunov function* approach in [12]). Similarly in Section 5 condition (3) describes the Input to State Stabilizability (ISS) condition for an impulsively-controlled system (see the ISS conditions introduced in [20]).

2.2 Receding horizon

Let a finite set of times $\{r_0, \ldots, r_h\}$ be arbitrarily chosen and consider a receding horizon from time r_i to time r_{i+1}, with $i = 0, \ldots, h-1$ (control and prediction horizons coincide). Take a sample interval $\Delta t = \frac{r_i - r_{i-1}}{N}$ with the number of samples N chosen arbitrarily and extract the associated discrete times $r_i+k\Delta t$ with $k=0,\dots,N$. Let the discrete time continuous and binary states be $\xi(k)$, and $\zeta(k)$ respectively with the initial condition $\xi(0)=x(r_i)$ and $\zeta(0)=q(r_i)$. Also, let the discrete time control $\mu(k)$ and disturbance $\gamma(k)$ be obtained by sampling $u(t)$ and $d(t)$ at time $r_i+k\Delta t$, i.e., $\mu(k)=u(r_i+k\Delta t)$ and $\gamma(k)=d(r_i+k\Delta t)$.

Then, for $k=0,\dots,N-1$, the sampled counterpart of system (1) is

$$\begin{aligned} \xi(k+1) &= \xi(k) + w_{\zeta(k)}(\xi(k),d(t)), \quad && \xi(0)=x(r_i) \\ \zeta(k+1) &= \begin{cases} \zeta(k) & \text{if } \mu(k)=0 \\ 1-\zeta(k) & \text{if } \mu(k)=1 \end{cases} && \zeta(0)=q(r_i) \\ \mu(k) &\in \{0,1\}, \end{aligned} \tag{4}$$

where $w_{\zeta(k)}(\xi(k),d(t)) = \int_{r_i+k\Delta t}^{r_i+(k+1)\Delta t} f_{\zeta(k)}(x(t),d(t))dt$. Here we indicate $d(t)$ as explicit argument of $w_{\zeta(k)}(\cdot)$ to mean that $w_{\zeta(k)}(\cdot)$ depends on the whole function $d(t)$ over the interval from $r_i+k\Delta t$ to $r_i+(k+1)\Delta t$. Analogously to the continuous time case, the condition $\mu(k)=1$ means that a switch occurs at time $r_i+k\Delta t$ whereas $\mu(k)=0$ means that the binary state is unchanged.

Analogously, the sampled counterpart of system (2) is

$$\begin{aligned} \xi(k+1) &= \xi(k) + w(\xi(k),d(t)) + \\ &\quad + (h(\xi(k),\gamma(k)) - \xi(k))\mu(k), \quad \xi(0)=x(r_i) \\ \mu(k) &\in \{0,1\}, \end{aligned} \tag{5}$$

where we denote by

$$w(\xi(k),d(t)) = \int_{r_i+k\Delta t}^{r_i+(k+1)\Delta t} f(x(t),d(t))dt. \tag{6}$$

Again, $d(t)$ as explicit argument of $w_{\zeta(k)}(\cdot)$ means that $w_{\zeta(k)}(\cdot)$ depends on the whole function $d(t)$ over the interval from $r_i+k\Delta t$ to $r_i+(k+1)\Delta t$. We can relax conditions (3) by considering the following discrete time counterpart, for $k=0,\dots,N-1$

$$\psi\left(V(\xi(0)),V(\xi(k+1)),V(\xi(k+1))-V(\xi(k))\right) > 0. \tag{7}$$

Feasible solutions for fixed horizon $[r_i,r_{i+1}]$, are $\mathbf{u}$, $\mathbf{d}$, $\mathbf{x}$, and $\mathbf{q}$ that satisfy (4) or (5), (6) and (7) where we define

$$\begin{aligned} \mathbf{u} = [\mu(0),\dots,\mu(N-1)] \ \mathbf{d} = [\gamma(0),\dots,\gamma(N-1)] \\ \mathbf{x} = [\xi(0),\dots,\xi(N)] \quad \mathbf{q} = [\zeta(0),\dots,\zeta(N)]. \end{aligned}$$

For a compact description, define the feasible solution set

$$\begin{aligned} \mathcal{F}(q(r_i),x(r_i)) = \{\mathbf{u},\mathbf{d},\mathbf{x},\mathbf{q} \in \{0,1\}^N \times \mathbb{R}^{N\times m} \times \\ \times \mathbb{R}^{(N+1)\times n} \times \{0,1\}^{N+1} : (4) \text{ or } (5), \text{ and } (7) \text{ satisfied}\}. \end{aligned}$$

Note that the feasible solution set depends on $q(r_i),x(r_i)$ because of the initial conditions on the discrete time state ξ and ζ. Also $q(r_i),x(r_i)$ are measured and full known at the beginning of the horizon and therefore they can be dealt with as known parameters.

Now, given the set $H = \{0, \dots, N\}$ of possible values of the index k spanning over the horizon window, consider a generic set of subsets $\{C_1, \dots, C_m\}$ such that $\bigcup_j C_j = H$ and each C_j is made by consecutive elements of H, i.e., given any pair $y, z \in C_j$ with $y < z$ this implies $v \in C_j$ for any $y < v < z$ and for all $j = 1, \dots, m$.

We claim that in a number of cases (some of these cases are discussed in Section 4 and 5) there exists a specific set of subsets $\{C_1, \dots, C_m\}$ with $m \leq N$ such that condition (7) is satisfied if at the initial time r_i of the horizon and under certain conditions on the initial states $q(r_i), x(r_i)$ of the horizon, we impose that the following constraints on the binary controls hold true

$$\sum_{k \in C_j} \mu(k) \geq l_j(q(r_i), x(r_i)), \quad \text{for all } j = 1, \dots, m \tag{8}$$

where function $l_j : \{0,1\} \times \mathbb{R}^n \to \{0,1\}$ models some logical conditions for $q(r_i)$ and $x(r_i)$. In all these situations we can get rid of $\mathbf{d}$, $\mathbf{x}$, and $\mathbf{q}$ and rewrite the feasible solution set in a simplified manner as shown below

$$\mathcal{F}(q(r_i), x(r_i)) = \{\mathbf{u} \in \{0,1\}^N : \text{(8) satisfied}\}.$$

Rewriting the solution set as above has the advantage of converting the original dynamic problem (because of the presence of the state variable) into a static one. This is possible as in a receding horizon setting, variables $q(r_i)$ and $x(r_i)$ once measured at time r_i enter as parameters in the right-hand side of (8).

To complete the formulation of the receding horizon problem, let the following vector of costs of the switching controls over the horizon be given

$$c = [c_0, \dots, c_{N-1}]^T.$$

The receding horizon problem is then

$$\min_{\mathbf{u} \in \mathcal{F}(q(r_i), x(r_i))} c^T \mathbf{u}. \tag{9}$$

Finally, once obtained the optimal sequence of discrete controls $\mu(0), \dots, \mu(N-1)$, we need to reconstruct the continuous time controls $u(t)$. We can do this through the following function $\theta : \{0,1\}^N \mapsto \{u(t), r_i \leq t < r_{i+1}\}$ returning, for each interval $[r_i + k\Delta t, r_i + (k+1)\Delta t)$, the control $u(r_i + k\Delta t) = \mu(k)$ and $u(t) = 0$ for all $t \in (r_i + k\Delta t, r_i + (k+1)\Delta t)$.

Figure 2 displays the closed-loop system with a switched static feedback. At time r_i the receding horizon controller selects and implements the controls $\mu(0), \dots, \mu(N-1)$ based on the current state $x(r_i)$, $q(r_i)$. The procedure is repeated at time r_{i+1} on the bases of the new state update $x(r_{i+1})$, $q(r_{i+1})$.

3 Main result

There is an important aspect that needs to be emphasized and represents the main result of this work (see also [1; 2]). The set of feasible solutions $\mathcal{F}(q(r_i), x(r_i))$ is a discrete set in the sense that it contains only integer

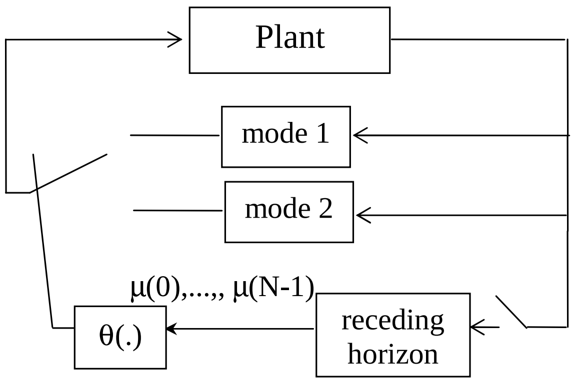

Fig. 2 Closed-loop system with switched static feedback: at time r_i the receding horizon controller selects and implements the controls $\mu(0), \ldots, \mu(N-1)$ based on the measured state $x(r_i)$, $q(r_i)$. The procedure is repeated at time r_{i+1} on the bases of the new state update $x(r_{i+1})$, $q(r_{i+1})$.

points. However we can replace the integrality constraints $\mathbf{u} \in \{0,1\}^N$ by the relaxed and more tractable constraints $0 \le \mathbf{u} \le 1$ and consider the resulting polytope

$$\mathcal{P}(q(r_i), x(r_i)) = \{\mathbf{u} \in \mathbb{R}^N : \\ \sum_{k \in C_j} \mu(k) \ge l_j(q(r_i), x(r_i)),\ \forall j = 1, \ldots, m,\ 0 \le \mathbf{u} \le 1\}\,.$$

We clarify this aspect more in details next. Let us rewrite the inequalities (8) in matrix form. We can do this by using a matrix $A \in \{0,1\}^{m \times N}$, with only entries 0 and 1, one row for each inequality of type (8), one column for each time k. Observe that the constraint matrix is an *interval matrix*, i.e., it has 0-1 entries and each row is of the form

$$(0, \ldots, 0\ \underbrace{1, \ldots\ldots\ldots, 1}_{\text{consecutive 1's}}\ 0, \ldots, 0).$$

It is well known from the literature [24] that each interval matrix is totally unimodular where we remind here that a matrix is *totally unimodular* if the determinant of any square sub-matrix is equal to -1, 0 or 1. We report next a simple proof.

Lemma 1 *Any interval matrix A is totally unimodular.*

Proof We need to show that any generic square sub-matrix $R \in \{0,1\}^{p \times p}$ of A is such that $det(R) \in \{0, \pm 1\}$. Take the incidence matrix of a directed chain graph with p nodes (its determinant is equal to one)

$$Z := \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

and compute the matrix $L := ZR^T$. One can see that L is still an incidence matrix of a directed graph as each column of L has either only one non null element (1 or -1) or just a 1 and -1 and the rest are zero elements. To see this, take the jth column of L,

$$L_{\bullet j} = (Z_{1\bullet} R^T_{\bullet j}, \ldots, Z_{p\bullet} R^T_{\bullet j})^T$$

and observe that the jth column of R^T has the following structure

$$R^T_{\bullet j} = (0, \ldots, 0, \underbrace{1}_{\bar{\imath}\text{th element}}, \ldots, \underbrace{1}_{\bar{\jmath}\text{th element}}, 0, \ldots, 0)^T.$$

Then, we derive that $L_{(\bar{\imath}-1)j} = Z_{(\bar{\imath}-1)\bullet} R^T_{\bullet j} = -1$, $L_{\bar{\jmath}j} = Z_{\bar{\jmath}\bullet} R^T_{\bullet j} = 1$ and $L_{ij} = Z_{i\bullet} R^T_{\bullet j} = 0$ for all $i \neq \bar{\imath} - 1, \bar{\jmath}$. With a little abuse of notation the same argument can be used to prove that the column $L_{1\bullet}$ has only one non null element equal to -1 (here note that $\bar{\imath} = 1$). This proves that L is still an incidence matrix of a directed graph also that $det(L) \in \{0, \pm 1\}$ or which is the same that L is totally unimodular. Then we can conclude that $det(R) = det(Z)det(R^T) = det(L) \in \{0, \pm 1\}$. $\square$

This means that the polytope $\mathcal{P}$ obtained from $\mathcal{F}$ by replacing the integrality constraints $\mu(k) \in \{0, 1\}$ with the linear constraint $0 \leq \mu(k) \leq 1$ is an integral polyhedron. As a consequence we have that the linear relaxation of the receding horizon problem (9) has an integral optimal solution as established in the next theorem. Let the vector of logical conditions be defined as $l = [l_1(q(r_i), x(r_i)), \ldots, l_m(q(r_i), x(r_i))]^T$.

Theorem 1 *Solving the receding horizon problem (9) is equivalent to solving the linear programming problem*

$$\min_{\mathbf{u}} \quad \mathbf{c}^T \mathbf{u} \tag{10}$$

$$s.t. \quad A\mathbf{u} \geq l \tag{11}$$

$$0 \leq \mathbf{u} \leq 1. \tag{12}$$

Proof Apply a standard technique in linear programming to turn the constraints (11) into equalities of type

$$[A \quad I] \begin{bmatrix} \mathbf{u} \\ \mathbf{s} \end{bmatrix} = l \tag{13}$$

where $\mathbf{s} \in \mathbb{R}^m$ is the surplus vector and $I \in \mathbb{R}^{m \times m}$ is the identity matrix. From the properties of total unimodular matrices one knows that if A is totally unimodular then also $[A \quad I]$ is totally unimodular. Then, take a generic square sub-matrix $R \in \mathbb{R}^{m \times m}$ and observe that $det(R) \in \{0, \pm 1\}$. Any admissible base solution of (13) is of the form $\bar{v} = R^{-1} l = \frac{adj(R)}{det(R)} l$ where $adj(R)$ is the adjoint matrix of R. Hence, because of the integrality of l and $det(R)$ we have that $\bar{v}$ is integer. This means that constraints (11)-(12) define an integral polyhedron, and that the optimal solution of the linear programming problem (10)-(12) is also integer. $\square$

4 Switched systems

In this section, we show that the paradigmatic problem (10)-(12) suits to the (asymptotic) stabilizability of switched systems. To do this, we can take for $V(x(t))$ any differentiable norm function and simplify equation (3) as $\dot{V} < 0$. Two cases are considered next: time dependent slow switching controls and state dependent switching controls. For the switched system (1) we wish to solve the following problem.

Problem 2 Find a switching control $u(t) \in \{0, 1\}$ for all $t \geq 0$ such that the origin is asymptotically stable.

4.1 Time dependent slow switching controls

Let us focus on systems that are stabilizable through a time dependent slow switching control. In particular, assume that there exists a (minimum) *dwell time* T. Minimum dwell time means that for guaranteeing $\dot{V} < 0$ it suffices that the time interval between two successive switches is at least T. Details on how to compute T for specific classes of problems can be found in [21; 22].

Consider the sampled counterpart (4) starting at time r_i, take for simplicity $\Delta t = 1$, and assume that $r_i - \hat{k}$ is the time of the last switch. The following linear programming problem of type (10)-(12) returns a switching control satisfying the above dwell time condition:

$$\begin{aligned} &\min_{\mathbf{u}} c^T \mathbf{u}, \quad \text{s.t.} \quad 0 \leq \mathbf{u} \leq 1, \\ &\underbrace{\begin{bmatrix} \overbrace{1 \ldots\ldots 1}^{b} \, 0 \ldots 0 \\ 0 \ldots\ldots 0 \, 1 \ldots 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \mu(0) \\ \vdots \\ \mu(N-1) \end{bmatrix}}_{\mathbf{u}} \geq \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{l}, \end{aligned} \tag{14}$$

where $b = T - \hat{k}$. The above problem derives from taking $C_1 = \{1, \ldots, T - \hat{k}\}$, and $C_2 = \{(T - \hat{k}) + 1, \ldots, N\}$. Note that the above constraint matrix A does not exclude multiple switchings between $(T - \hat{k}) + 1$ and N which possibly violate the dwell time condition. However such solutions though admissible, are not optimal for problem (10)-(12) as multiple switchings increase the cost.

The receding horizon process repeats at time $r_{i+1} = r_i + N\Delta t$ (regular starting times) or at time $r_{i+1} = r_i + (\tilde{k} + 1)\Delta t$ where $\tilde{k}$ is the last switching time returned by the problem solved at time r_i (time-varying starting times).

4.2 State dependent switchings

We now move to addressing systems that are stabilizable through (hysteresis-based) state dependent switchings. In other words, we can guarantee the condition $\dot{V} < 0$, if switches occur anytime the state is in a specific region.

In particular, let us consider two open conic regions Π_1 and Π_2 which overlap and such that $\Pi_1 \bigcup \Pi_2 = \mathbb{R}^n \setminus \{0\}$.

Assume that there exists a *hysteresis-based* stabilizing switching of the type described in [22] and recalled next. For each $t > 0$, if $q(t^-) = 0$ and $x(t) \in \Pi_1$, keep $q(t) = 0$. Differently, if $q(t^-) = 0$ but $x(t) \notin \Pi_1$, then assign $q(t) = 1$. Analogously, if $q(t^-) = 1$ and $x(t) \in \Pi_2$, keep $q(t) = 1$. Differently, if $q(t^-) = 1$ but $x(t) \notin \Pi_2$, then assign $q(t) = 0$.

Now, we can approximate the above hysteresis-based switching control by using the linear programming problem (14). To see this, consider the sampled counterpart (4) starting at time r_i, and let $r_i + b\Delta t$ as the expected time to cross a pre-defined surface $\mathcal{S} \in \Pi_1 \bigcap \Pi_2$.

The optimal solution of (14) returns no switches between 0 and b, and only one switch between $b+1$ and $N-1$, i.e.,

$$\mu(0) = \ldots = \mu(b) = 0, \quad \mu(b+1) + \ldots + \mu(N-1) = 1. \tag{15}$$

The length of the interval $[b+1, N]$ describes how long it takes for a switch to occur once the state has crossed the surface $\mathcal{S}$. In the special case where $N = b+1$, the surface $\mathcal{S}$ turns out to be a switching surface as a switch occurs any time the surface $\mathcal{S}$ is crossed.

4.2.1 Oscillating systems

Consider the second order systems

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\kappa(q) & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $q \in \{0,1\}$ is the mode and with spring coefficient $\kappa(0) = 1$ (passive control) and $\kappa(1) = 2$ (aggressive control). The binary state transition function is as in (1).

It is well known that switching law of type (16) asymptotically stabilizes the system at the origin [22]:

$$q(t) = \begin{cases} 0 \text{ if } x_1(t)x_2(t) \leq 0 \\ 1 \text{ if } x_1(t)x_2(t) > 0 \end{cases}. \tag{16}$$

This corresponds to having two switching surfaces $\mathcal{S}_1 = \{x \in \mathbb{R}^2 : x_1 = 0\}$ and $\mathcal{S}_2 = \{x \in \mathbb{R}^2 : x_2 = 0\}$ (see, e.g., Fig. 3).

We can approximate a switching law of type (16) as shown next. First consider the sampled counterpart of the above system. To do this denote by $\omega(q) = \sqrt{\kappa(q)}$. Sampling is possible after defining the following two components vector,

$$w_{\zeta(k)}(\xi(k)) = \begin{bmatrix} \xi_1(k)\cos(\omega(\zeta(k))\Delta t) + \\ +(\frac{\xi_2(k)}{\omega(\zeta(k))})\sin(\omega(\zeta(k))\Delta t) - \xi_1(k) \\ \\ -\xi_1(k)\omega\sin(\omega(\zeta(k))\Delta t) + (\frac{\xi_2(k)}{\omega(\zeta(k))})\cdot \\ \cdot\omega(\zeta(k))\cos(\omega(\zeta(k))\Delta t) - \xi_2(k) \end{bmatrix},$$

with initial condition $\xi_i(0) = x_i(r_i)$, $i = 1, 2$, and where the two components predict the evolution of position and velocity respectively.

Now, let $r_i + b\Delta t$ be the expected time to cross the surface $\mathcal{S}_1$ if $q(r_i) = 0$. Similarly, let $r_i + b\Delta t$ be the expected time to cross the surface $\mathcal{S}_2$ if $q(r_i) = 1$. To approximate the switching law of type (16) it suffices to take $N = b + 1$ where b has the meaning discussed above. Actually, the linear programming problem (14) returns a switch any time one of the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ is crossed.

A slight modification in the choice of N may return the asymptotically stabilizing hysteresis-based switching control discussed next (see, Fig. 3). Take for a small enough $\alpha > 0$

$$\begin{array}{l} \Pi_1 = \{x \in \mathbb{R}^2 : x_1 x_2 < 0\} \bigcup \{x \in \mathbb{R}^2 : x_2 > \alpha x_1\}, \\ \Pi_2 = \{x \in \mathbb{R}^2 : x_1 x_2 > 0\}. \end{array} \tag{17}$$

The overlapping region is $\Pi_1 \bigcap \Pi_2 = \{x \in \mathbb{R}^2 : x_1 x_2 > 0, x_2 > \alpha x_1\}$ (grey region), the surface $\mathcal{S}_3 = \{x_2 = \alpha x_1\}$ describes the boundary of Π_1 while the surface $\mathcal{S}_2$ describes the boundary of Π_2. Now take r_i such that $x(r_i) \in \Pi_1$ and $q(r_i) = 0$. Compute $r_i + b\Delta t$ as the expected time to cross the surface $\mathcal{S}_1$ or take $b = 0$ if the state has already crossed $\mathcal{S}_1$ (that is, $x(r_i)$ is already in $\Pi_1 \bigcap \Pi_2$). Also take N as the expected time to cross the surface $\mathcal{S}_3$. Then the linear programming problem (14) returns only one switch when the state is in the overlapping region $\Pi_1 \bigcap \Pi_2$. Let the time of the switch be $r_i + e\Delta t$. Note that the resulting switching control is again of type (15). Fig. 3 top, displays the predicted state trajectory (dashed line) from point $\xi(0) = x(r_i)$ on the x_1-axes (also surface $\mathcal{S}_2$) to point $\xi(N)$ on surface $\mathcal{S}_3$. In evidence point $\xi(b) = x(r_i + b\Delta t)$ on the x_2-axes (also surface $\mathcal{S}_1$), and point $\xi(e) = x(r_i + e\Delta t)$ when the switch occurs. At the next iteration of the receding horizon procedure, we take $r_{i+1} := r_i + (e+1)\Delta t$. Now, we have $x(r_{i+1}) \in \Pi_2$ and $q(r_{i+1}) = 1$. Compute $r_{i+1} + b\Delta t$ as the expected time to cross the surface $\mathcal{S}_2$ and take $N = b + 1$. Then, the linear programming problem (14) returns only one switch immediately after the state has crossed $\mathcal{S}_2$. Fig. 3 bottom, displays the current state trajectory from $x(r_i)$ to $x(r_{i+1})$ (solid line) and predicted state trajectory (dashed line) from point $\xi(0) = x(r_{i+1})$ to point $\xi(N)$. In evidence point $\xi(b) = x(r_i + b\Delta t)$ on the x_1-axes (also surface $\mathcal{S}_2$) when the switch occurs. We can repeat the procedure when the state is in the second and third quadrant. The resulting hysteresis-based switching control is such that there exists a periodically decreasing quadratic function $V(x)$ (if we keep the passive control mode, i.e., $q(t) = 0$ for all t, the trajectory describes an ellipsoid with main axes parallel to the x_1x_2-axes). Actually, from Fig. 3 it is evident that $V(x(r_{i+2})) - V(x(r_i)) < -\beta x(r_i)$ for a small enough $\beta > 0$. The above condition is a sufficient condition for the asymptotical stability of the system at the origin.

In Fig. 4 we simulate six different switching times to the “active control” when the state is in the first and third quadrant. Reading the figure from top-left to bottom-right the switching times increase which means that the system remains in the passive control for a longer time in the first and third quadrant and the converging time increases as well. In the simulation at the bottom-right, the system switches to the “active control” just before leaving

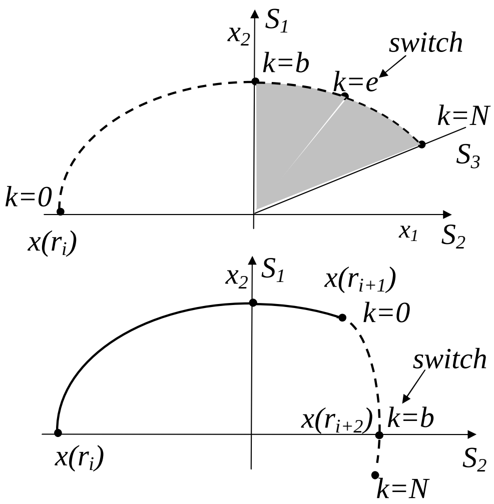

Fig. 3 Two successive iterations of the receding horizon procedure: (top) predicted trajectory (dashed line) at the ith iteration; (bottom) current trajectory (solid line) and predicted trajectory (dashed line) at the $i+1$st iteration. Function $V(x(t))$ is periodically decreasing.

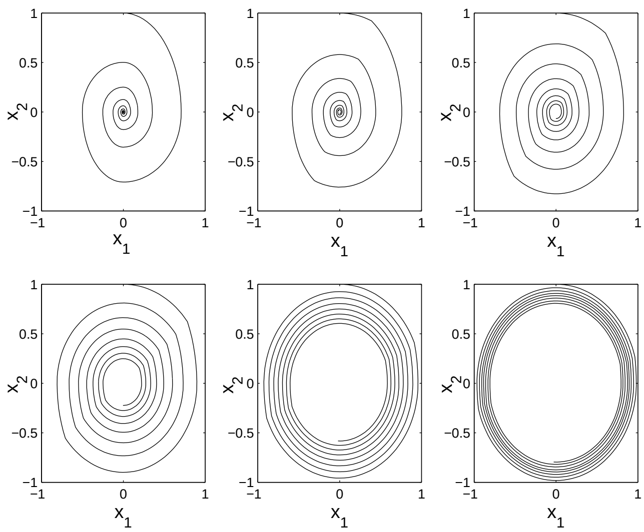

Fig. 4 Trajectories in the x_1x_2-plane under different feasible solutions.

the first and third quadrant. The six feasible controls are obtained by simply changing the costs of the controls.

5 Impulsively-controlled systems

In this section, we show that the paradigmatic problem (10)-(12) suits to the Input to State Stabilizability (ISS) of impulsively controlled systems, according to the definition provided in [20].

To do this, we can take for $V(x(t))$ any differentiable norm function and consider equation (3) as a reformulation of condition (2) in [20].

In particular, we focus next on ISS systems with dwell time and reverse dwell time. For the impulsively-controlled system (2) we wish to solve the following problem.

Problem 3 Find an impulse control law $u(t)$ (i.e., a sequence of impulse times $t_1, t_2 \dots, t_k, \dots$) such that system (2) is input to state stable (ISS) according to the definition of [20].

5.1 (Reverse) dwell time

In [20] it has been shown that for a number of systems Problem 3 can be solved by any impulse control law $u(t)$ satisfying some so-called (reverse) dwell time conditions. A typical dwell time condition requires that intervals between consecutive impulses must be no shorter than T time units. All these cases can be dealt with exactly as shown in Section 4.1. On the contrary, a typical reverse dwell time condition requires that intervals between consecutive impulses must be no longer than T time units. We can generalize the approach by considering m different dwell times T_1, $T_2, \dots, T_m$ over the horizon as shown next.

Consider the sampled counterpart (5) starting at time r_i, take for simplicity $\Delta t = 1$, and assume that $r_i - 1$ is the time of the last impulse. The following linear programming problem of type (10)-(12) returns a switching control satisfying the above reverse dwell time condition:

$$\min_{\mathbf{u}} c^T \mathbf{u}, \quad \text{s.t.} \quad 0 \leq \mathbf{u} \leq 1,$$
$$\underbrace{\begin{bmatrix} \overbrace{1 \dots 1}^{T_1} & 0 \dots 0 \dots \overbrace{0 \dots 0}^{T_m} \\ 0\,1 \dots\dots\dots 1 & \dots\ 0 \dots 0 \\ \vdots & \ddots \quad \vdots \\ 0 \dots 0 & 0 \dots 0 \dots 1 \dots 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \mu(0) \\ \vdots \\ \mu(N-1) \end{bmatrix}}_{\mathbf{u}} \geq \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{l}. \tag{18}$$

The above problem derives from taking $C_1 = \{0, \dots, T_1\}$, $C_2 = \{1, \dots, 2 + T_2\}$, $\dots$, $C_i = \{i-1, \dots, i+T_i\}$, $\dots$, $C_m = \{m-1, \dots, N\}$. In the next section, we apply the approach to a first-order system.

5.1.1 First order system

Consider system

$$\dot{x} = \underbrace{a(t)x(t) + rand(-1,1)}_{f(x(t),d(t))} + sat(-x^-(t))u(t)$$

with rate $a(t) > 0$ and where $rand(-1, 1)$ is a random disturbance uniformly distributed in the interval between -1 and 1, and $sat(.)$ is the typical linear saturated (at -1 or 1) function. The $sat(.)$ function derives from taking

$$h(.,.) = \begin{cases} x(t) - sign(x(t)) & \text{if } |x(t)| > 1 \\ 0 & \text{if } |x(t)| \leq 1 \end{cases}.$$

We simulate a one step (from $r_i = 0$ to r_{i+1}) receding horizon procedure. The number of sets C_i is $m = 27$.

We take as initial state $x(r_i) = x(0) = 9$. Let us take as time unit the value $-log((\epsilon - 1)/\epsilon) = 0.0953$ where $\epsilon = 11$ is an upper bound of $|x(t)|$. This value derives from the fact that between two consecutive impulses we can guarantee the condition $|x(r_{i+1})| \leq \frac{\epsilon-1}{\epsilon}|x(r_i)|$ at least on the average because of the random disturbance $rand(-1, 1)$ (the same condition is always guaranteed in absence of a random disturbance). The sample interval is $\frac{1}{10}$ of the time unit $-log((\epsilon - 1)/\epsilon)$, i.e., $\Delta t = -log((\epsilon - 1)/\epsilon)0.1 = 0.00953$.

Now, the rate is $a(t) = 0.1$ for $0 \leq t < 14$ (during intervals C_i, with $i = 1, \dots, 15$) and $a(t) = 0.2$ for $14 \leq t < 20$ (during intervals C_i, with $i = 16, \dots, 27$). The reverse dwell time is computed following the procedure in [20] as $T_j = \frac{\text{\# of steps for time unit}}{\text{rate in } C_j}$ and the result is $T_j = \frac{10}{0.1} = 100$, for $j = 1, \dots, 15$ and $T_j = \frac{10}{0.2} = 50$ for $j = 16, \dots, 27$. It follows that the horizon length is $N = 14 \cdot 100 + 12 \cdot 50 = 2100$.

Figure 5 shows the time plot of $x(t)$ when impulses occur at the beginning of each interval (solid line). This happens when costs are increasing, that is, $c_1 < c_2 \leq \ldots \leq c_{N-1}$, or also when costs are increasing over each interval $C_i = \{\sum_{j=1}^{i-1} T_j + 1, \dots, \sum_{j=1}^{i-1} T_j + T_i\}$, i.e., $c_r < \ldots < c_s$ with $r = \sum_{j=1}^{i-1} T_j + 1$ and $s = \sum_{j=1}^{i-1} T_j + T_i$. The figure also shows the time plot of $x(t)$ when impulses occur in the middle of each interval (dotted line). This happens when on each interval $C_i = \{\sum_{j=1}^{i-1} T_j + 1, \dots, \sum_{j=1}^{i-1} T_j + T_i\}$ we have $c_r < c_s$ for all $r \neq s$ with $r = \sum_{j=1}^{i-1} T_j + \frac{T_i}{2}$. Finally, the figure shows the time plot of $x(t)$ when impulses occur at the end of each interval (dash-dot line). This happens, for instance, when costs are decreasing, that is, $c_1 \geq c_2 \geq \ldots > c_{N-1}$ or also when costs are decreasing over each interval $C_i = \{\sum_{j=1}^{i-1} T_j + 1, \dots, \sum_{j=1}^{i-1} T_j + T_i\}$, i.e., $c_r \geq \ldots > c_s$ with $r = \sum_{j=1}^{i-1} T_j + 1$ and $s = \sum_{j=1}^{i-1} T_j + T_i$. In all of the three cases the system is ISS stable as the state $x(t)$ is driven in a neighborhood of the origin whose sizes depend on the maximal amplitude of the random disturbance.

6 Inventory examples

The following examples are borrowed from the inventory theory. However they can be generalized to any storage system, such as bank account [19], water tank. Examples from the economic and financial world are investments research projects in the natural resource industry [23], target zone models

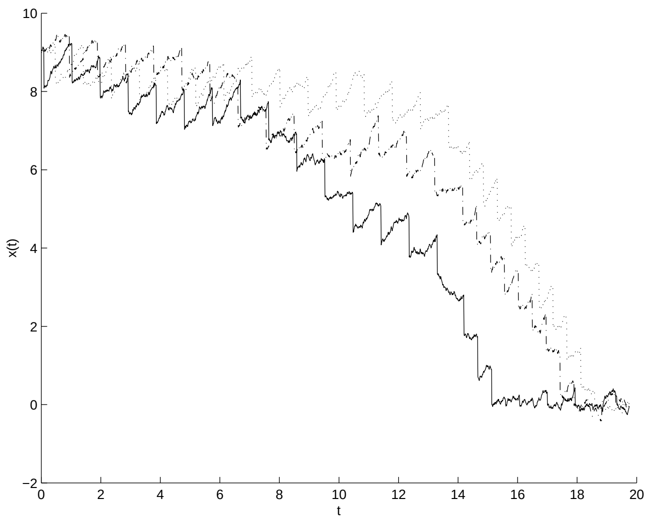

Fig. 5 Time plot of $x(t)$ when impulses occur at the beginning of each interval C_i, $i = 1, \ldots, 27$ (solid line), in the middle of each interval (dotted line), and at the end of each interval (dash-dot line).

for the exchange rate [17; 18] (see, e.g., [6] for an exhaustive list of applications). The first example can be found also in [6] and [10], Example 3.5, and in [2]. The second example provides a more generic interpretation of the impulsively-controlled system (2), which is now used to describe a controlled switched multi–inventory system (see, e.g., [1]).

6.1 Inventory ([6] and [10], Example 3.5)

With in mind the impulsively-controlled system (2), let the state $x(t) \in \mathbb{R}$ describe the inventory level, let function $V(t) = x(t)$ and use (3) to impose the condition $x(t) \geq s$, where s is the lower inventory threshold. Resets $u(t)$ describe the choice of the retailer of reordering, in which case the inventory is restored at level $S = h(x(t), d(t))$, or not reordering. Costs of resets c are transportation costs. Let $f(x(t), d(t)) = -d(t)$ where $d(t) \in \mathcal{D} \subseteq \mathbb{R}$ is the (nonnegative) demand faced by the retailer. Then, equation (9) is the minimization of transportation costs, equation (2) is the evolution of the inventory and equation (3) prevents the inventory to be lower than threshold s. The sample interval is the minimum time occurring between two consecutive reorders.

This inventory problem is closely related to some well-studied dynamic lot-sizing problems with the only difference that demand is now unknown but bounded [7]. As a consequence of this, the well known (low-order) polynomial time algorithms for lot-sizing problems (see e.g., [25; 27; 28]) do not apply straightforwardly to our case. New optimization methods have been recently proposed which cope with bounded uncertainties [8].

6.2 Switched multi–inventory systems

Consider the family of continuous time linear multi–inventory systems

$$\dot{x}(t) = B_i u_c(t) - d(t), \quad i \in \{1, \dots, \mathcal{Q}\} \tag{19}$$

where $x(t) \in \mathbb{R}^n$ is a vector whose components are the buffer levels, $u_c(t) \in \mathbb{R}^m$ is the controlled flow vector, $B_i \in \mathbb{Q}^{n \times m}$ is the controlled process matrix and $d(t) \in \mathbb{R}^n$ is the unknown demand. To model backlog $x(t)$ may be less than zero. Controls and demands are bounded within polytopes according to

$$u_c(t) \in \mathcal{U}_c = \{u_c \in \mathbb{R}^m : u_c^- \leq u_c \leq u_c^+\} \tag{20}$$

$$d(t) \in \mathcal{D} = \{d \in \mathbb{R}^n : d^- \leq d \leq d^+\}, \tag{21}$$

where u_c^-, u_c^+, d^-, and d^+ are assigned vectors. We also assume that matrix B_i is a "fat matrix" and has full row rank.

For the above switched multi–inventory system, the notion of unstable mode can be reviewed as follows. We say that each system $\dot{x}(t) = B_i u_c(t) - d(t)$ (henceforth simply system B_i), is an unstable mode if there no exists *feedback stabilizing strategies* [3], that is, strategies able to drive the state within a neighborhood of a reference value x_{ref} of radius ϵ in finite time. This is true if the polytope of demand is not contained in the image of the polytope of controls via B_i, i.e., $\mathcal{D} \not\subseteq int\{B_i \mathcal{U}\}$. Henceforth, assume that only $B_{\mathcal{Q}}$ satisfies the above condition, namely, $\mathcal{D} \subseteq int\{B_{\mathcal{Q}} \mathcal{U}\}$ and that $\mathcal{D} \not\subseteq int\{B_i \mathcal{U}\}$, $i = 1, \dots, \mathcal{Q} - 1$. In the following, we refer to systems $B_1, \dots, B_{\mathcal{Q}-1}$ as the unstable modes whereas we refer to system $B_{\mathcal{Q}}$ as the stable mode.

After introducing the stable and unstable modes, the switched multi–inventory system is alternatively in one of the $\mathcal{Q}$ modes as described by the following dynamics

$$\begin{array}{c} \dot{x} = \sum_{i=1}^{\mathcal{Q}} \alpha_i(t)(B_i u_c(t) - d(t)) \\ \sum_{i=1}^{\mathcal{Q}} \alpha_i(t) = 1, \quad \text{binary.} \end{array} \tag{22}$$

Transitions between successive unstable modes are autonomous and in accordance to a given sequence (only unstable modes are in the sequence) whereas transitions from an unstable mode to the stable mode are controlled by the switching signal $\mu(k) \in \{0, 1\}$ for $k = 0, \dots, N-1$. More precisely, let us call kth *epoch* the time interval between $t(k)$ and $t(k+1)$ where

$$t(k+1) = t(k) + \mathcal{T}\mu(k) + \Delta t, \tag{23}$$

the latter meaning that the epoch has a constant size Δt if no switch has occurred, $\mu(k) = 0$, and size $\Delta t + \mathcal{T}$ if a switch has occurred, $\mu(k) = 1$. Then, define function $\sigma : \{0, \dots, N-1\} \to \{1, \dots, \mathcal{Q} - 1\}$ which associates unstable modes to epochs, i.e., $\sigma(k)$ returns the mode at the generic kth epoch. For instance, given the mode sequence (assume $\mathcal{Q} > 3$) B_3 - B_1 - B_2 - B_2 - ... then $\sigma(0) = 3$, $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 2$ and so on. Then, if we introduce logical conditions in square brackets and denote by $\wedge$ and $\sim$

the two logical operators "and" and "not", transitions are governed by the following expressions. For all $k = 0, \ldots, N-1$

$$[t = t(k)] \wedge [\mu(k) = 1] \wedge [\alpha_{\sigma(k-1)}(t) = 1] \rightarrow [\alpha_{\mathcal{Q}}(t^+) = 1] \tag{24}$$

$$[t = t(k)] \wedge [\mu(k) = 0] \wedge [\alpha_{\sigma(k-1)}(t) = 1] \rightarrow [\alpha_{\sigma(k)}(t^+) = 1] \tag{25}$$

$$[t = t(k) + \mathcal{T}] \wedge [\alpha_{\mathcal{Q}}(t) = 1] \rightarrow [\alpha_{\sigma(k)}(t^+) = 1]. \tag{26}$$

Conditions (24) and (25) describe transitions at the beginning of the kth epoch from mode $B_{\sigma(k-1)}$ to $B_{\mathcal{Q}}$ if $\mu(k) = 1$ and to $B_{\sigma(k)}$ (successive mode in the sequence) if $\mu(k) = 0$, respectively. Condition (26) describes the transition from mode $B_{\mathcal{Q}}$ to $B_{\sigma(k)}$ after a time interval of $\mathcal{T}$ (throughout this paper we always assume that, for the stable mode, a time interval of length $\mathcal{T}$ is large enough to drive the state x within a neighborhood of zero). Finally, the following condition says that in all the other circumstances no transitions occur:

$$\sim \big([t = t(k)] \wedge [\alpha_{\sigma(k)}(t) = 1]\big) \wedge \sim \big([t = t(k) + \mathcal{T}] \wedge [\alpha_{\mathcal{Q}}(t) = 1]\big) \rightarrow [\alpha_i(t^+) = \alpha_i(t)]. \tag{27}$$

Figure 6 displays controlled and uncontrolled transitions (arcs) among modes (nodes) for $\mathcal{Q} = 3$. Figure 7 displays the time plot for a one dimensional state. In evidence the epoch (sampling interval) between $t(3)$ and $t(4)$ which has a size of $\Delta t + \mathcal{T}$.

To put the above model in the form (5), we need to slightly modify the definition of the state dependent disturbance in (6) and the sampling rule introduced in Section 2.2. Actually, now, the disturbance is no longer state dependent, and sampling occurs at the beginning of each epoch, where the epochs are defined as in (23). Given this, we only need to change the notation $w(\xi(k), d(t))$ into $w(k, u_c(t), d(t))$ and also to specify that if at time $t(k)$ a switch occurs $\mu(k) = 1$ then one extreme of the integral (6) is shifted forward of $\mathcal{T}$, i.e.,

$$w(k, u_c(t), d(t)) = \int_{t(k)+\mathcal{T}\mu(k)}^{t(k+1)} (B_{\sigma(k)} u_c(t) - d(t))dt. \tag{28}$$

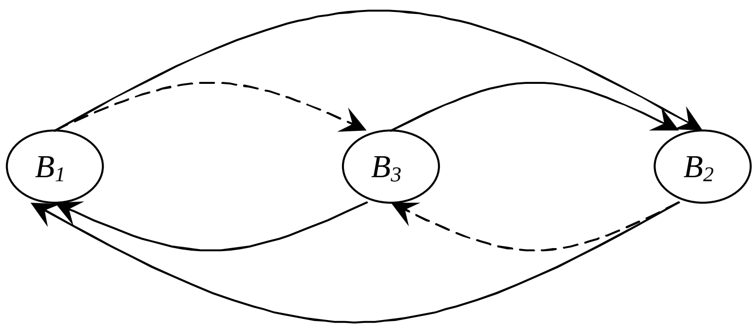

Fig. 6 Transitions among different modes: nodes represent unstable modes B_1 and B_2 and stable mode B_3. Arcs describe controlled transitions (dashed) to the stable mode B_3 and uncontrolled transitions (solid) to the unstable modes B_1 and B_2.

Henceforth, we simply write $w(k, u_c, d)$ instead of $w(k, u_c(t), d(t))$.

Then, the problem of interest consists in finding the optimal schedule of the switches among unstable and stable modes thus to maintain the system in a safe operating region, while minimizing a function related to the cost of the switches. In this case, the cost of a switch represents the cost of driving the state to the origin, once a transition to the stable mode has occurred (in the assumption that such a cost is independent of the value of the state at the time of the transition). The decision variables are thus binary (whether to switch to the stable mode at a given time instant or not).

With in mind the sampling rule (23) and the new definition of $w(k, u_c, d)$ as in (28) the sampled multi–inventory model reduces to

$$\xi(k+1) = \xi(k) + w(k, u_c, d) + (x_{\text{ref}} - \xi(k))\mu(k) \quad \mu(0) = 1, \qquad (29)$$

the above equation being a specialization of (5).

Now, assume that we wish to keep the state within a neighborhood of a reference value x_{ref} of size ϵ. Assume $x(0)$ is already in the neighborhood and take $x_{\text{ref}} = 0$ without loss of generality. We can formalize the above concept by considering, for instance, $V(x) = \|x\|_\infty$. Let $\mathcal{X} = \{x \in \mathbb{R}^n : V(x) \leq \gamma\}$ for a given threshold $\gamma \in \mathbb{R}_+$ with $\mathbb{R}^n_+$ the positive orthant. Condition (7) is then simply

$$V(\xi(k+1)) \leq \gamma, \quad k = 0, \ldots, N-1. \qquad (30)$$

With this in mind, the generic set C_j used in (8) can be defined as follows.

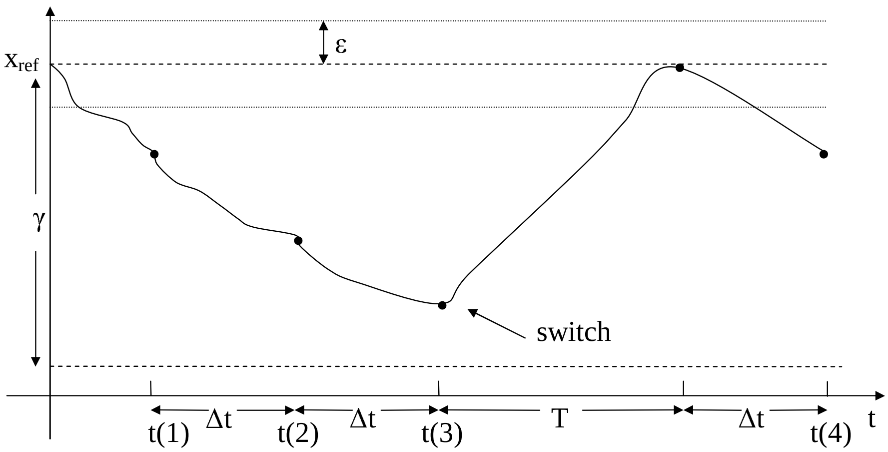

Fig. 7 Time plot for a one dimensional state. In evidence the sampling interval between $t(3)$ and $t(4)$ of different size $\Delta t + \mathcal{T}$.

Definition 1 Set $C_j = \{\bar{k}, \bar{k}+1, \ldots, \tilde{k}-1, \tilde{k}\}$ is made up by consecutive time instants such that, for $x(\bar{k}) = x_{\text{ref}}$, it holds

$$\min_{u_c(.)\in\mathcal{U}_c} \max_{d(.)\in\mathcal{D}} V\left(\sum_{k=\bar{k}}^{\tilde{k}} w(k, u_c, d)\right) > \gamma \tag{31}$$

$$\min_{u_c(.)\in\mathcal{U}_c} \max_{d(.)\in\mathcal{D}} V\left(\sum_{k=\bar{k}}^{\tilde{k}-1} w(k, u_c, d)\right) \leq \gamma. \tag{32}$$

Sets C_j's define the minimal time intervals which must be "covered" by at least one reset, for a solution $\mathbf{u}$ to be feasible under the worst disturbance. Difficulties in computing the sets C_j's are discussed in the next section.

6.2.1 Computation of sets C_j's

With in mind the fact that the disturbance is, for this system, state independent (see, e.g., (28)) and in the assumption that, for the computation of sets C_j's, no switch occurs between times $\bar{k}$ and $\tilde{k}$, we must solve a min-max optimization problem of type

$$z^* = \min_{u_c(.)\in\mathcal{U}_c} \max_{d(.)\in\mathcal{D}} V\left(\sum_{k=\bar{k}}^{\tilde{k}} w(k, u_c, d)\right). \tag{33}$$

Note that (33) is the same as the first term of (31). Also, we can immediately observe that the cost $V\left(\sum_{k=\bar{k}}^{\tilde{k}} w(k, u_c, d)\right)$ is convex. Now, in state of solving (33) consider the following greedy min-max problem (the index g reminds "greedy"). Find the solution of

$$z^g = \sum_{k=\bar{k}}^{\tilde{k}} \left(\min_{u_c(.)\in\mathcal{U}_c} \max_{d(.)\in\mathcal{D}} V\left(w(k, u_c, d)\right)\right). \tag{34}$$

Note that the greedy problem (34) is obtained from (33) by simply inverting the "min max" with the "sum". Denote by $(u_c^*(t), d^*(t))$, the solution of (33), i.e.,

$$u_c^*(t) = \arg \min_{u_c(.)\in\mathcal{U}_c} V\left(\sum_{k=\bar{k}}^{\tilde{k}} w(k, u_c, \phi(u_c))\right) \tag{35}$$

$$d^*(t) = \phi(u_c^*(t)), \tag{36}$$

where the function $\phi(u_c) = \arg\max_{d(.)\in\mathcal{D}} V\left(\sum_{k=\bar{k}}^{\tilde{k}} w(k, u_c, d)\right)$. Analogously, denote by $(u_c^g(t), d^g(t))$, the solution of each term of the sum in (34), that is

$$u_c^g(t) = \arg \min_{u_c(.)\in\mathcal{U}_c} V\left(w(k, u_c, \phi(u_c))\right) \tag{37}$$

$$d^g(t) = \phi(u_c^g(t)), \tag{38}$$

where for each interval $t(k) \leq t \leq t(k+1)$, the function

$$\phi^g(u_c) = \arg \max_{d(.) \in \mathcal{D}} V\left(w(k, u_c, d)\right). \tag{39}$$

Associate to $(u_c^*(t), d^*(t))$ and $(u_c^g(t), d^g(t))$ the corresponding disturbance and state

$$w^*(k) = \int_{t(k)}^{t(k+1)} (B_{\sigma(k)} u_c^* - d^*) dt, \; \xi^*(k) = \textstyle\sum_{r=\bar{k}}^{k-1} w_j^*(r), \tag{40}$$

$$w^g(\xi(k)) = \int_{t(k)}^{t(k+1)} (B_{\sigma(k)} u_c^g - d^g) dt, \; \xi^g(k) = \textstyle\sum_{r=\bar{k}}^{k-1} w^g(r). \tag{41}$$

In other words, (40)-(41) are the disturbances and states for the two problems (33) and (34) respectively.

Note that, we have denoted the first extreme in the two above integrals simply by $t(k)$ rather than $t(k) + \mathcal{T}u(k)$ as for the computation of sets C_j's, we always assume that no switch occurs between times $\bar{k}$ and $\tilde{k}$. Finally, let $t(\hat{k}) = \arg\max_{t(r), r=1,\ldots,N} t(k) \leq t$. Then, the continuous time state can be obtained as

$$\xi^*(t) = \xi^*(\hat{k}) + \int_{t(k)}^{t} (B_{\sigma(k)} u_c^* - d^*) d\tau \tag{42}$$

$$\xi^g(t) = \xi^g(\hat{k}) + \int_{t(k)}^{t} (B_{\sigma(k)} u_c^g - d^g) d\tau. \tag{43}$$

We are now ready to establish the exactness of the greedy computation.

Theorem 2 *For the multi–inventory system of Section 6.2 it holds* $z^g = z^*$.

Proof First, observe that for the multi–inventory system of Subsection 6.2, $V(.)$ is convex and $w(k, u_c, d)$ is linear on $d(t)$. As a consequence of this, the worst demand d^* and the greedy demand d^g are on a vertex of $\mathcal{D}$. We specialize the proof to the case where the state trajectory is in the negative orthant. We wish to show that $\xi^g(k) = \xi^*(k)$ for all $k = \bar{k}, \ldots, \tilde{k}$. First note that proving $\xi^g(k) \geq \xi^*(k)$ for all $k = \bar{k}, \ldots, \tilde{k}$ derives straightforwardly by the definition of d^* and d^g. Now, we prove by induction that $\xi^g(k) \leq \xi^*(k)$ for all $k = \bar{k}, \ldots, \tilde{k}$. Actually, for $k = \bar{k}$, we have $\xi(\bar{k}) = 0$ and $w^g(\bar{k}) \leq w^*(\bar{k})$. The two latter conditions imply $\xi^g(\bar{k}+1) \leq \xi^*(\bar{k}+1)$. Now, for any $k < \tilde{k}$, assume $\xi^g(k) \leq \xi^*(k)$ and prove $\xi^g(k+1) \leq \xi^*(k+1)$. To do this, observe that, if $\xi^g(k+1) > \xi^*(k+1)$ then there must exist a time t with $t(k) < t \leq t(k+1)$ such that $\xi^g(t) = \xi^*(t)$ and $\dot{\xi}^g(t) > \dot{\xi}^*(t)$. But this is possible only if $d^* > d^g = d^+$ which contradicts the assumption $d^- \leq d \leq d^+$. We can conclude that $\xi^g(k) = \xi^*(k)$ for all $k = \bar{k}, \ldots, \tilde{k}$ and also $z^g = z^*$. □

A last comment concerns the complexity of computing d^g. It is worth to be noted that the maximization over each component d_i is independent of the other components and therefore to find d^g, it suffices to make a number of n comparisons of type $\xi_i(t(k)) + d_i^+$ and $\xi_i(t(k)) + d_i^-$, one for each component. Such a property derives from the special structure of the multi–inventory system.

6.2.2 Numerical Example

See the switched multi–inventory system in Fig. 8. Brackets $[0, 8]$ indicate the minimum and maximum demand at the nodes. Maximum capacity for each arc is specified below the name, i.e., "$a, 5$" means that arc a has maximum capacity equal to 5. Minimum capacity is 0 for all arcs. Topology 1 has only arc $a, b, \ldots, l$, topology 2 has the additional arc m and topology 3 has additional arcs m, n and p. The corresponding incidence matrices are as follows (B_2 is obtained from B_1 adding the only column of arc m)

$$B_3 = \begin{array}{c@{}c} \overbrace{\hphantom{1 \; -1 \; 0 \; -1 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0}}^{B_1} & \overbrace{\hphantom{0 \; 0 \; 0 \; 0}}^{m,n,p} \\ \left[\begin{array}{ccccccccc} 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right. & \left.\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \end{array}$$

Let us also assume that the total demand (summed over all 6 nodes) is at most equal to 8, that is, $\sum_{i=1,\ldots,6} d_i \leq 8$. Then, from (38) the greedy demand is $d^g = [0, 0, 0, 0, 8, 0]^T$ for topology 1 and $d^g = [0, 0, 0, 0, 0, 8]^T$ for topology 2. Also, from (37) the greedy flows are $u_c^g = [3, 0, 0, 3, 0, 0, 3, 0, 0]^T$ and $u_c^g = [1, 0, 3, 1, 0, 3, 1, 1, 3]^T$ for topology 1 and 2 respectively. Assume $\Delta t = 1$, from (41), we obtain $w^g(k) = [0, 0, 0, 0, 5, 0]^T$ for topology 1 (namely, for k such that $\sigma(k) = 1$), and $w^g(k) = [0, 0, 0, 0, 0, 4]^T$ for topology 2 (namely, for k such that $\sigma(k) = 2$).

Observe that only system B_3 is ϵ-stabilizable (it satisfies $\mathcal{D} \subseteq int\{B_3\mathcal{U}\}$), whereas systems B_1 and B_2 are not.

Assume that the system switches autonomously between topology 1 and 2 according to a uniformly distributed random sequence.

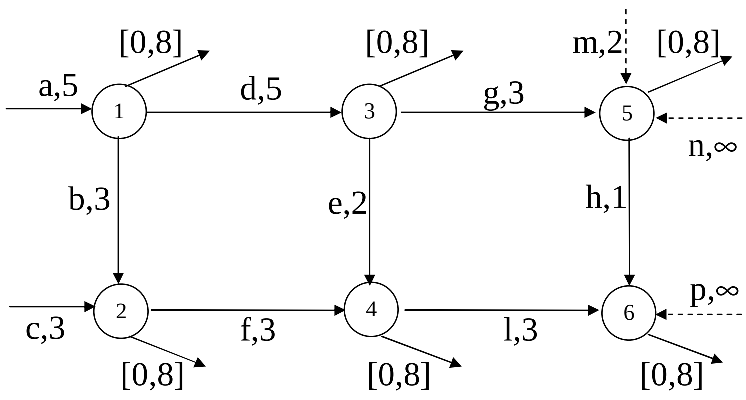

Fig. 8 Switched multi–inventory system: topology 1 has only arcs $a, b, \ldots, l$; topology 2 has additional arc m; topology 3 has additional arcs m, n, p.

Also, assume that at any reset the system switches on B_3 so that the buffer length can be driven within a neighborhood of 50. We also force the buffer to be non negative (no backlog). Then set $x_{\text{ref}} = 50$ and $\gamma = 50$. Over the horizon of length $N = 100$, the costs of the resets are increasing (controlled switches to B_3 occur at the end of each interval) in the first case and decreasing in the second case (controlled switches to B_3 occur at the beginning of each interval).

Once the system has switched to B_3, the controlled flow in the additional arcs n and p is of type $u_{c,11}(t) = x_{\text{ref}} - x_5(t)$ and $u_{c,12}(t) = 0.2(x_{\text{ref}} - x_6(t))$ respectively. For sake of simplicity we assume that when mode B_3 is active, the demand is of type $d(t) = [0, 0, 0, 0, 8, 0]$. With the above choice of $u_{c,11}(t)$ and $d(t)$, the system is stabilizable within a neighborhood of size 8 (equal to the maximal demand at node 5). The initial state is $x_5(t) = x_6(t) = x_{\text{ref}} = 50$.

If $V(.)$ is the ∞-norm, we can compute $\|w^g(k)\|_\infty$ for each k. From (34) we can compute the sets C_j's, and derive a set of 100 inequalities of type (8) that enter as constraints in the linear programming problem (10)-(12) returning the optimal control sequence (controlled switches). Such a sequence is used to simulate the state evolution of Fig. 9 in the two cases of decreasing (solid line) and increasing (dashed line) costs.

7 Conclusions

Hybrid optimal control problems are, in general, difficult to solve. A current research goal is to isolate those problems that lead to tractable solutions. In this paper, we have identified among the larger set of hybrid control problems a special class of optimal control problems which are easy to solve. Easy to solve means that solution algorithms are polynomial in time and therefore suitable to the on-line implementation in real-time problems. We have done this by using a paradigm borrowed from the Operations Research field.

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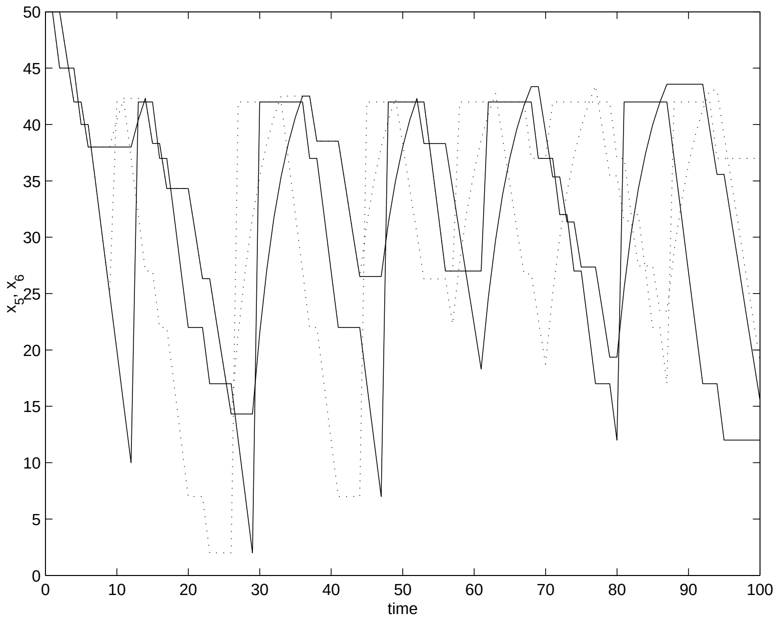

Fig. 9 Time plot of $x_5(t)$ and $x_6(t)$ in the cases of increasing (solid line) and decreasing (dotted line) costs. Any time a controlled switch to mode B_3 occurs, the state is driven to about 42, which is nothing but the boundary of a neighborhood of size 8 centered in 50. The size of 8 derives from considering a maximum demand at node 5 equal to 8 and a feedback controlled flow at arc n of type $u_{c,11}(t) = x_{\mathrm{ref}} - x_5(t)$

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