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HYBRID EQUILIBRIUM ELEMENTS FOR ACCURATE STRESS ANALYSIS

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Abstract. *In the present paper hybrid stress elements are proposed as alternative to standard finite elements for linear and non linear analysis. Hybrid stress formulation is developed in a rigorous mathematical setting. The proposed approach is qualitatively compared to displacement and mixed methods by linear elastic analysis of some structural examples, well known in literature.*

1 INTRODUCTION

Finite element (FE) formulation based on stress fields satisfying local equilibrium condition are known since 1964 by the pioneering work of de Vebeuke [1-2]. Equilibrated FEs were initially proposed [3-4] as numerical tool for error estimation of classical displacement based FE analyses; as in fact stress and displacement formulations produce respectively upper and lower bound, with respect to the exact solution, in terms of elastic strain energy.

Equilibrium elements can be developed in a hybrid framework with independent stress fields in each element [5-6], giving a solution satisfying strong equilibrium condition on the domain with codiffusive traction balance at each element edge. In hybrid formulation, traction equilibrium condition is enforced by the use of independent displacement at each edge, as Lagrangian parameters, for which compatibility condition is not imposed *a priori*. Strong traction equilibrium condition requires the same interpolation order for stress and displacement fields, but, as a consequence, it can produce spurious kinematic modes (SKM), which are displacement modes with null traction at edges. SKMs can heavily corrupt the elastic solution either in terms of displacement, or of stress, and they need to be restrained.

In the present paper an hybrid equilibrium formulation is proposed as valid alternative to the classical displacement based approach. A superior stress accuracy is then achieved

2 MODIFIED MINIMUM COMPEMENTARY ENERGY PRICIPLE

Let us consider an elastic domain Ω with boundary $\Gamma = \Gamma_u \cup \Gamma_t$, where Γ_u and Γ_t are the constrained and free boundary, respectively. The body is subjected to volume forces $\mathbf{b}$ in Ω, to tractions $\mathbf{t}$, on Γ_t, and to imposed displacements $\mathbf{u}$, on Γ_u.. The solution of elastostatic problem can be approached in a weak form by minimization of the following complementary energy functional

$$\Pi_c = \frac{1}{2}\int_{\Omega}\sigma_{ij}D_{ijhk}\sigma_{hk}d\Omega - \int_{\Gamma_u}\sigma_{ij}n_j\bar{u}_i d\Gamma \tag{1}$$

where the stress field σ_{ij} is assumed to implicitly satisfy strong equilibrium equations

$$\sigma_{ij,j} + b_j = 0 \quad \text{in } \Omega\,; \qquad \sigma_{ij}n_j = t_i \quad \text{on } \Gamma_t \tag{2,3}$$

It is well known that the stress field, which makes stationary the functional Π_c is the solution of the elastostatic problem.

3 HYBRID EQUILIBRIUM ELEMENT

In the FE context, application of minimum complementary energy approach requires the formulation of stress fields in equilibrium with body forces, in each element domain Ω_e. Moreover, inter-element equilibrium condition must be enforced at boundaries between neighbouring elements and boundary condition (3) must be enforced at element edges of free surface. Inter-element and boundary equilibrium conditions can be imposed by Lagrangian method, by considering the following modified complementary energy functional

$$\overline{\Pi}_c = \sum_{e=1}^{elm}\left[\frac{1}{2}\int_{\Omega_e}\sigma_{ij}^e D_{ijhk}\sigma_{hk}^e d\Omega - \int_{\Gamma_t^{e,s}} t_i v_i^s d\Gamma - \sum_{s=1}^{sides}\int_{\Gamma^e}\sigma_{ij}^e n_j^s v_i^s d\Gamma\right] \tag{4}$$

where v_i^s is the Lagrangian vector function, which consists on an independent displacement function on each element edge Γ_s^e and which is assigned to $v_i^s = \bar{u}_i$ on the constrained boundary. Hybrid Stress based finite elements are defined by:

- Stress field satisfying strong equilibrium condition (2);
- Independent displacement field at each element edge (face in 3-D elements).

In such a hybrid element, displacement degrees of freedom are not related to nodes, but they belongs to the edges, each of which connects two adjacent elements. In the present paper, hybrid equilibrium element formulation is limited to bi-dimensional membrane elastostatic problems. Implementation and numerical results are proposed only for triangular element with quadratic stress field, which is hereafter developed.

Let a triangular finite element of domain Ω_e be considered with straight edges and referred to a local Cartesian reference (*x, y*) parallel to the global one (X, Y), but with origin at vertex 1, as shown in fig. 1.

Planar stress field are defined by the following polynomial functions

$$\begin{aligned}
\sigma_x &= a_1 + a_2 y + a_3 y^2 - a_9 x - b_x x - a_{10} x^2/2 - 2a_{12}xy \\
\sigma_y &= a_4 + a_5 x + a_6 x^2 - a_8 y - b_y y - a_{10} y^2/2 - 2a_{11}xy \\
\tau_{xy} &= a_7 + a_8 x + b_y x + a_9 y + b_x y + a_{10}xy + a_{11}x^2 + a_{12}y^2
\end{aligned} \tag{5 a, c}$$

and in compact notation $\boldsymbol{\sigma} = \mathbf{S}\cdot\mathbf{a}$ where $\mathbf{a}$ collects generalized stress parameters $a_1, a_2, \ldots, a_{12}$. Stresses in eqs. (5) implicitly verify strong equilibrium condition in eq. (2).

Displacement fields at element edges are reported in the local natural reference, with tangential and normal axes related to the oriented edge, (see fig. 1) and, in matrix notation are

$$\mathbf{v}(\xi) = \mathbf{N}(\xi)\mathbf{u} \tag{6}$$

where $\mathbf{u}$ collects tangential and normal displacement at ends and at the midside points of each edge, $\mathbf{N}(\xi)$ is the matrix of classical quadratic shape functions

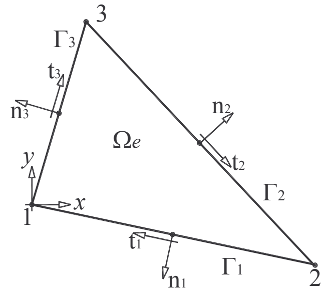

Figure 1: local references for element stress fields and for edge displacement laws.

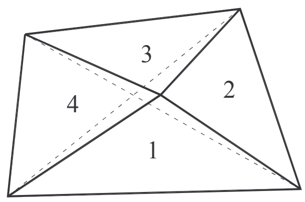

Figure 2: quadrilateral patch mesh of four triangular element with internal vertex not lying at diagonal intersection.

Finite element formulation of eq. (4) gives

$$\overline{\Pi}_c = \sum_{e=1}^{elm}\left[\frac{1}{2}\mathbf{a}_e^T\mathbf{K}_{aa}^e\mathbf{a}_e - \sum_{s=1}^{3}\mathbf{a}_e^T\mathbf{K}_{av}^{e,s}\mathbf{u}^s\right] - \sum_{\Gamma_s\in\Gamma_t}\mathbf{t}_s^T\mathbf{u}^s \tag{7}$$

where $\mathbf{K}_{aa}^e = \int_{\Omega_e}\mathbf{S}^T\mathbf{D}\mathbf{S}d\Omega$, $\mathbf{K}_{av}^{e,s} = \int_{\Gamma_s}\mathbf{S}_e^T\mathbf{g}_s^T\mathbf{N}_s d\Gamma$, $\mathbf{g}_s$ is the edge rotation matrix and $\mathbf{t}_s$ collects the traction components at interpolation points of the free boundary edges.

Stationary conditions of functional $\overline{\Pi}_c$ give the following equations

$$\begin{aligned}
&\frac{\partial\overline{\Pi}_c}{\partial\mathbf{a}_e} = \mathbf{K}_{aa}^e\mathbf{a}_e - \mathbf{K}_{av}^{e,s1}\mathbf{u}^{e,s1} - \mathbf{K}_{av}^{e,s2}\mathbf{u}^{e,s2} - \mathbf{K}_{av}^{e,s3}\mathbf{u}^{e,s3} = 0\\
&\frac{\partial\overline{\Pi}_c}{\partial\mathbf{v}_s} = -\mathbf{a}_{e1}^T\mathbf{K}_{av}^{e1,s} - \mathbf{a}_{e2}^T\mathbf{K}_{av}^{e2,s} = 0 \quad \Gamma_s\in\Gamma_{\text{int}}\\
&\frac{\partial\overline{\Pi}_c}{\partial\mathbf{v}_s} = -\mathbf{a}_e^T\mathbf{K}_{av}^{e,s} - \mathbf{t}_s = 0 \quad \Gamma_s\in\Gamma_t
\end{aligned} \tag{8 a, c}$$

where $s1$, $s2$, $s3$ are the edges of element e in eq. (8a), $e1$ and $e2$ are the two elements adjacent to edge s in eq. (8b); moreover, Γ_{int} collects internal edges and Γ_t collects free boundary edges. Equations (8 b,c) strongly enforce respectively inter-element and boundary equilibrium conditions. After assembling operation, hybrid equilibrium approach provides the following equation system

$$\begin{bmatrix}\mathbf{K}_{aa} & \mathbf{K}_{av}\\ \mathbf{K}_{va} & \mathbf{K}_{vv}\end{bmatrix}\begin{bmatrix}\mathbf{A}\\ \mathbf{U}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{T}\end{bmatrix} \tag{10}$$

where generalized stress knowns can be condensed out at element level.

4 NUMERICAL TESTS

The present formulation has been implemented in a FE code and two tests have been performed: the first one is the so called Cook's membrane (fig. 3) and second one is a short cantilever (fig. 4); both of them are subjected to vertical load at the right end; details of the tests are in [7]. In order to avoid presence of SKMs, as proposed in [5-6], the meshes are composed of quadrilateral patches of four triangular elements with internal vertex not lying at diagonals intersection (see fig. 2). The results are compared in literature in terms of vertical displacement of the loaded ends and are reported in Table 1, respectively for: the proposed approach (EQ), four nodes displacement element (Q4), nine node displacement element (Q9), improved four node Hellinger–Reissner element (HR) and reference solution.

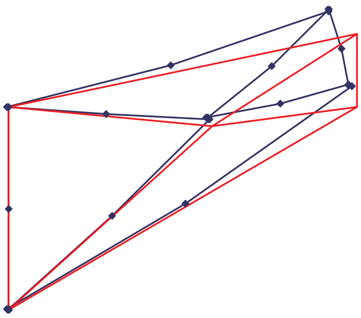

Figure 3: Cook’s membrane subjected to vertical load at right end.

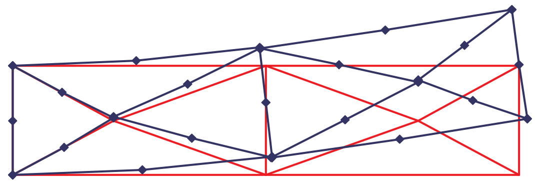

Figure 4: short cantilever subjected to vertical load at right end.

	EQ	Q4	Q9	HR	Ref
Cook's membrane	24.04	18.23	23.29	21.353	23.81
Short cantilever	102.78	87.01	101.71	98.261	102.6

Table 1: vertical displacement of loaded ends for the two performed tests; results are compared with displacement based approach and mixed approach.

5 CONCLUSIONS

The hybrid equilibrium element carried out the best solution in both tests in terms of displacement. This kind of elements can be extremely useful for nonlinear analyses were accurate stress solution are required. In facts, in case of strain localization problems or cohesive fracture mechanics problems, an accurate stress solution inside the element is a paramount requisite for accurate, and sometime meaningful numerical solution.

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