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EXTENSIONS OF POSITIVE LINEAR FUNCTIONALS ON A TOPOLOGICAL *-ALGEBRA

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ABSTRACT. The family of all extensions of a nonclosable hermitian positive linear functional defined on a dense *-subalgebra $\mathfrak{A}_0$ of a topological *-algebra $\mathfrak{A}[\tau]$ is studied with the aim of finding extensions that behave regularly. Under suitable assumptions, special classes of extensions (positive, positively regular, absolutely convergent) are constructed. The obtained results are applied to the commutative integration theory to recover from the abstract setup the well-known extensions of Lebesgue integral and, in noncommutative integration theory, for introducing a generalized non absolutely convergent integral of operators measurable with respect to a given trace σ.

1. Introduction. In many fields of pure and applied mathematics there are several problems that can be abstractly formulated in the following way.

Let $\mathfrak{A}$ be a topological *-algebra, with topology τ and continuous involution *, and let $\mathfrak{A}_0$ be a dense *-subalgebra of $\mathfrak{A}$. Given a positive linear functional ω on $\mathfrak{A}_0$ (i.e., $\omega(a^*a) \geq 0$, for every $a \in \mathfrak{A}_0$) is it possible to extend ω to some elements of $\mathfrak{A}$?

For instance, if we take as $\mathfrak{A}$ the *-algebra of Lebesgue measurable functions on $X = [0, 1]$ with the topology of convergence in measure, as $\mathfrak{A}_0$ the *-algebra of continuous functions on X, and as ω the Riemann integral, then the Lebesgue integral provides an extension of ω. Further extensions of ω were found by Denjoy, Perron, Khintchine, Henstock, Kurzweil, Foran and many others (see, e.g., [**3, 4, 5, 6, 9, 13, 16**]). These extensions of the Lebesgue integral do not share with it a relevant property: a measurable function f can be integrable without $|f|$ being integrable. For this reason, they are usually called *non absolutely convergent* extensions.

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Another rather familiar situation where the problem of extending positive linear functionals arises is that of the so-called *noncommutative integration*. In this case the starting point is a von Neumann algebra $\mathfrak{M}$ which admits a normal semifinite faithful trace σ. In the 1950's Segal [**14**] first formulated the notion of a *measurable* operator and defined the space $L^1(\sigma)$. Some years later, Nelson proposed an alternative (but fully equivalent) approach by defining $L^p(\sigma)$ as the completion of the ideal

$$\mathcal{J}_p = \{A \in \mathfrak{M}; \sigma(|A|^p) < \infty\},$$

with respect to the norm

$$\|A\|_p = (\sigma(|A|^p))^{1/p}.$$

This more abstract procedure leads (up to an identification) to Banach spaces of operators affiliated with $\mathfrak{M}$ which are measurable in Segal's sense. The problem of further extensions on the noncommutative integral (which is, of course, nothing but the problem of extending the trace σ to larger families of measurable operators) has not been investigated, as far as we know.

From the point of view of applications, the problem of extending positive linear functionals also arises, in a natural way, in quantum physics. In the Haag-Kastler formulation of quantum statistical mechanics, one defines the so called **-algebra of local observables* $\mathfrak{A}_0$, which is, roughly speaking, the union of the net of the *-algebras of observables in regions of finite measure of the configuration space. In the original Haag-Kastler approach all these algebras were supposed to be C*-algebras so that the *-algebra of observables of the system, i.e., the uniform completion $\mathfrak{A}$ of $\mathfrak{A}_0$ is a C*-algebra too. A *state* of the system is defined as a normalized positive linear functional on $\mathfrak{A}_0$ and then extended to $\mathfrak{A}$. But in this case, the extension is immediately obtained since every positive linear functional is automatically continuous! However, the assumption that all the algebras under consideration were C*-algebras was too strong and revealed not to be satisfied by many concrete physical models. For this reason, in the 1970's it was proposed (first by Lassner, [**10, 11**]) of completing $\mathfrak{A}_0$ not with respect to the uniform topology but with respect to a locally convex topology, called *physical*, obtaining in this way a new structure called locally convex quasi *-algebra, which is endowed only of a partial multiplication (see, also, [**1**, Chapter 10, 11]). At this point the problem of extending states of the system has not necessarily a trivial solution, since they can be discontinuous or even worse (nonclosable).

In this paper we will discuss first from an abstract point of view the general problem of extending a given hermitian positive linear functional ω defined on a dense subalgebra $\mathfrak{A}_0$ of a topological *-algebra $\mathfrak{A}$ to some subspaces of $\mathfrak{A}$ (of course, if $\mathfrak{A}_0$ has a unit the assumption of hermiticity of ω may be omitted). In particular, in Section 2, we will focus our attention to the case were ω is *nonclosable*, since in the opposite case the problem has a unique (and well-known) solution. The starting point of our discussion will be the notion of *slight* extension, which is treated for general linear maps in Köthe's book [**8**]. Our first result proves that there exists infinitely many positive extensions of ω whose graph is contained in the closure of the graph of ω. We also characterize the maximal ones. Then we consider a special kind of extensions which we call *positively regular* whose definition closely reminds that of the Lebesgue integral for measurable functions and proceed, under suitable assumptions on $\mathfrak{A}$ and ω, to the construction of *non absolutely convergent* extensions.

In Section 3, we will prove that the Henstock-Kurzweil integral is just one of the extensions defined in an abstract way in Section 2 of the Riemann integral. Among all possible extensions of the integral, we show that the Lebesgue one is the maximal absolutely convergent.

Section 4 is devoted to noncommutative integration. We prove that extensions of the trace σ beyond the space $L^1(\sigma)$ do really exist and we explicitly construct one of these extensions that behaves in a similar fashion to the Henstock-Kurzweil integral in the commutative case.

2. Slight extensions of positive linear functionals.

2.1. The simplest cases. As mentioned in the Introduction the problem of extending a given hermitian positive linear functional ω defined on a dense *-subalgebra $\mathfrak{A}_0$ of a topological *-algebra $\mathfrak{A}[\tau]$, with continuous involution *, may have, in some situations, easy solutions, namely when ω is τ-continuous or closable. To begin with, we shortly discuss these two cases.

Case 1. ω is τ-continuous. This is a trivial case, since ω can be extended to $\mathfrak{A}$ by continuity.

Case 2. ω is closable.

This means that one of the two equivalent statements which follow is satisfied. Define

$$G_\omega = \{(a, \omega(a)) \in \mathfrak{A}_0 \times \mathbf{C};\ a \in \mathfrak{A}_0\}.$$

- If $a_\alpha \to 0$ with respect to τ and $\omega(a_\alpha) \to \ell$, then $\ell = 0$.
- $\overline{G_\omega}$, the closure of G_ω, does not contain couples $(0, \ell)$ with $\ell \neq 0$.

In this case, we define

$$D(\overline{\omega}) = \{a \in \mathfrak{A} : \exists \{a_\alpha\} \subset \mathfrak{A}_0,\ a_\alpha \to a \text{ and } \omega(a_\alpha) \text{ is convergent}\},$$

and

$$\overline{\omega}(a) = \lim_\alpha \omega(a_\alpha), \quad a \in D(\overline{\omega}).$$

The closability of ω implies that $\overline{\omega}$ is well defined. The functional $\overline{\omega}$ is linear and is the minimal closed extension of ω (i.e., $G_{\overline{\omega}}$ is closed).

Remark 2.1. If ω is closable, the continuity of the involution implies that $\overline{\omega}$ is hermitian, i.e.,

$$\overline{\omega}(a^*) = \overline{\overline{\omega}(a)}, \quad \text{for all } a \in D(\overline{\omega}).$$

Hence, $\overline{\omega}(a)$ is real for every $a \in D(\overline{\omega})$ with $a = a^*$.

Example 2.2. Let $X = [0, 1]$, $\mathfrak{A}$ be the *-algebra of Lebesgue measurable functions on X, τ the topology of convergence in measure, $\mathfrak{A}_0 = C(X)$ the *-algebra of all continuous functions on X and

$$\omega(f) = \int_0^1 f(x)\, dx.$$

Then ω is not closable. Indeed, the sequence $f_n(x) = nx(1 - x^2)^n \to 0$ pointwise and then in measure. But

$$\int_0^1 f_n(x)\, dx = \frac{n}{2(n+1)} \longrightarrow \frac{1}{2}.$$

This example shows that it really makes sense to consider the problem of extending ω, even if ω is not closable. This will be done in the subsection 2.2.

2.2. Nonclosable functionals.

2.2.1. Slight extensions: preliminaries. We specialize to the case of hermitian positive linear functionals the notion of *slight* extension given in [**8**, Chapter 7, Section 36.7] for arbitrary linear maps and we give without proving them the basic properties.

If ω is not closable, $\overline{G_\omega}$ contains couples $(0, \ell)$ with $\ell \neq 0$.

Let $\mathcal{S}_\omega$ denote the collection of all subspaces H of $\mathfrak{A} \oplus \mathbf{C}$ such that

(g1) $G_\omega \subseteq H \subset \overline{G_\omega}$;

(g2) $(0, \ell) \in H$ if, and only if, $\ell = 0$.

Then, to every $H \in \mathcal{S}_\omega$, an extension corresponds to ω_H, to be called a *slight* extension of ω, defined on

$$D(\omega_H) = \{a \in \mathfrak{A} : (a, \ell) \in H\}$$

by

$$\omega_H(a) = \ell,$$

where ℓ is the unique complex number such that $(a, \ell) \in H$.

Moreover, by applying Zorn's lemma to the family $\mathcal{S}_\omega$, one has

Proposition 2.3. *ω admits a maximal slight extension.*

Put

$$\mathcal{K}_\omega = \{a \in \mathfrak{A} : (a, \ell) \in \overline{G_\omega}, \text{ for some } \ell \in \mathbf{C}\}.$$

$\mathcal{K}_\omega$ is a subspace of $\mathfrak{A}$ with the property that $a \in \mathcal{K}_\omega$ implies $a^* \in \mathcal{K}_\omega$.

Proposition 2.4. *For every maximal slight extension $\breve{\omega}$ of ω, $D(\breve{\omega}) = \mathcal{K}_\omega$.*

Remark 2.5. The previous proposition says, in other words, that all maximal slight extensions have the same domain $\mathcal{K}_\omega$; thus, they only differ for their values on elements which do not belong to $\mathfrak{A}_0$. On the other hand, it is clear that if a slight extension has $\mathcal{K}_\omega$ as domain then it is maximal.

Also the following proposition is a simple adaptation of the one given in [**8**].

Proposition 2.6. *If ω is not closable and $\mathfrak{A}_0$ is a proper subspace of $\mathcal{K}_\omega$, then ω admits infinitely many maximal slight extensions.*

2.2.2. Hermitian and positive extensions. The slight extensions defined above are neither hermitian nor positive, in general. As we are looking for positive extensions, it is natural to begin with considering the problem of the existence of hermitian slight extensions.

If $\mathfrak{A}$ is an arbitrary *-algebra, we put

$$\mathcal{P}(\mathfrak{A}) = \left\{ \sum_{i=1}^{n} a_i^* a_i ;\ a_i \in \mathfrak{A} \right\}.$$

Elements of $\mathcal{P}(\mathfrak{A})$ are called *positive.*

Definition 2.7. Let $\widehat{\omega}$ be a slight extension of ω, and let $D(\widehat{\omega})$ be its domain. We say that $\widehat{\omega}$ is *positive* if $\widehat{\omega}(x) \geq 0$, for every $x \in D(\widehat{\omega}) \cap \mathcal{P}(\mathfrak{A})$.

From Proposition 2.4, it follows, in particular, that a maximal slight extension $\widehat{\omega}$ is positive, if $\widehat{\omega}(x) \geq 0$, for every $x \in \mathcal{K}_\omega \cap \mathcal{P}(\mathfrak{A})$.

2.2.3. Hermiticity. Let ω be a nonclosable hermitian positive linear functional and let $(a, \ell) \in \overline{G_\omega}$. Then there exists a net $\{a_\alpha\}$ such that $a_\alpha \xrightarrow{\tau} a$ and $\omega(a_\alpha) \to \ell$. The τ-continuity of the involution and the hermiticity of ω on $\mathfrak{A}_0$ imply that $a_\alpha^* \xrightarrow{\tau} a^*$ and $\omega(a_\alpha) = \overline{\omega(a_\alpha)} \to \overline{\ell}$. Hence, if $(a, \ell) \in \overline{G_\omega}$, then $(a^*, \overline{\ell}) \in \overline{G_\omega}$. In particular, if $(a, \ell) \in \overline{G_\omega}$ and $a = a^*$, then there exists $\ell' \in \mathbf{R}$ such that $(a, \ell') \in \overline{G_\omega}$. Indeed, $(a, \ell) \in \overline{G_\omega}$ implies that $(a, \overline{\ell}) \in \overline{G_\omega}$; hence, taking into account that $\overline{G_\omega}$ is a vector space, $(a, \Re\ell) \in \overline{G_\omega}$, where $\Re\ell$ denotes the real part of ℓ.

Now let $\mathcal{H}_\omega$ denote the collection of all subspaces $H \in \mathcal{S}_\omega$ for which the following additional condition holds

(h3) $(a, \ell) \in H$ implies $(a^*, \overline{\ell}) \in H$.

From (g2) and (h3) it follows

(h4) $(a, \ell) \in H$ and $a = a^*$, implies ℓ is real.

Every $H \in \mathcal{H}_\omega$ defines a slight extension ω_H of ω on the domain

$$D(\omega_H) = \{a \in \mathfrak{A} : (a, \ell) \in H\}.$$

The functional ω_H is, first, defined on an element $a = a^* \in D(\omega_H)$ by $\omega_H(a) = \ell$, where ℓ is the unique real number such that $(a, \ell) \in H$. An arbitrary element $a \in D(\omega_H)$ can be written as $a = b + ic$, with $b = (a + a^*)/2$ and $c = (a - a^*)/2i$. Since H is a vector space, (h3) implies that $b, c \in D(\omega_H)$. Whence,

$$\omega_H(a^*) = \omega_H(b^* - ic^*) = \omega_H(b) - i\omega_H(c) = \overline{\omega_H(a)}.$$

Therefore ω_H is hermitian. Moreover,

Proposition 2.8. *The following statements hold.*

(i) *ω admits a maximal hermitian slight extension.*

(ii) *Let $\breve{\omega}$ be a maximal hermitian slight extension of ω. Then $D(\breve{\omega}) = \mathcal{K}_\omega$.*

(iii) *If ω is not closable and $\mathfrak{A}_0$ is a proper subspace of $\mathcal{K}_\omega$, then ω admits infinitely many maximal hermitian slight extensions.*

Proof. (i) $\mathcal{H}_\omega$ satisfies the assumptions of Zorn's lemma. Then it has a maximal element $\breve{H}$.

(ii) As it is clear, for every hermitian slight extension $\breve{\omega}$, one has $D(\breve{\omega}) \subseteq \mathcal{K}_\omega$. Let $a \in \mathcal{K}_\omega \setminus D(\breve{\omega})$. Then also $a^* \in \mathcal{K}_\omega \setminus D(\breve{\omega})$, since if $a^* \in D(\breve{\omega})$, then $a \in D(\breve{\omega})$ and this contradicts the assumption. Now put $b = (a + a^*)/2$ and let $\ell \in \mathbf{R}$ be such that $(b, \ell) \in \overline{G_\omega}$. Consider $G_{\breve{\omega}} \oplus \langle (b, \ell) \rangle$, where $\langle (b, \ell) \rangle$ denotes the subspace generated by (b, ℓ). Clearly, $G_{\breve{\omega}} \oplus \langle (b, \ell) \rangle \subseteq \mathcal{H}_\omega$ and this contradicts the maximality of $\breve{\omega}$.

(iii) Since ω is not closable, $\overline{G_\omega}$ contains couples $(0, m)$ with $m \neq 0$. We may assume that $m \in \mathbf{R}$. Let $b \in \mathcal{K}_\omega \setminus \mathfrak{A}_0$ with $b^* = b$, and let $\ell \in \mathbf{R}$

be such that $(b, \ell) \in \overline{G_\omega}$. Then there exists a maximal hermitian slight extension $\breve{\omega}$ such that $G_{\breve{\omega}} \supset G_\omega \oplus \langle (b, \ell) \rangle$. Clearly $(b, \ell) + \lambda(0, m) \in \overline{G_\omega}$, for every $\lambda \in \mathbf{R} \setminus \{0\}$. For each λ, there exists a maximal hermitian slight extension whose graph contains $(b, \ell) + \lambda(0, m)$. It is clear that, for different values of λ, the corresponding maximal hermitian extensions are different. □

Remark 2.9. From (ii) of the previous proposition it follows that every maximal hermitian extension is maximal.

2.2.4. Positivity. We begin with defining

$$P_\omega = \{(a, \omega(a)) : a \in \mathcal{P}(\mathfrak{A}_0)\}.$$

Clearly, $P_\omega \subset G_\omega$ and therefore $\overline{P_\omega} \subseteq \overline{G_\omega}$. If $(a, \ell) \in \overline{P_\omega}$, then there exists a net $\{(a_\alpha, \omega(a_\alpha))\} \subset P_\omega$ such that

$$a_\alpha \xrightarrow{\tau} a \text{ and } \omega(a_\alpha) \longrightarrow \ell.$$

Hence, $a \in \overline{\mathcal{P}(\mathfrak{A}_0)}$ and $\ell \geq 0$. Therefore

$$\overline{P_\omega} \subseteq \{(a, \ell) \in \overline{G_\omega} : a \in \overline{\mathcal{P}(\mathfrak{A}_0)},\ \ell \geq 0\}.$$

The converse inclusion does not hold in general. Clearly,

$$\{a \in \mathfrak{A} : (a, \ell) \in \overline{P_\omega}\} \subseteq \mathcal{K}_\omega \cap \overline{\mathcal{P}(\mathfrak{A}_0)} \subseteq \mathcal{K}_\omega \cap \mathcal{P}(\mathfrak{A}).$$

Remark 2.10. If $\mathcal{K}_\omega \cap \mathcal{P}(\mathfrak{A}) = \{a \in \mathfrak{A} : (a, \ell) \in \overline{P_\omega}\}$, then every maximal slight extension is positive (and, therefore, hermitian). This seems, however, to be a very strong condition.

Let $\mathcal{P}_\omega$ denote the collection of all subspaces $K \in \mathcal{H}_\omega$ satisfying the following additional condition

(p5) $(a, \ell) \in K$ and $a \in \mathcal{P}(\mathfrak{A})$, implies $\ell \geq 0$.

Then, to every $K \in \mathcal{P}_\omega$, it corresponds a positive slight extension ω_K of ω, defined on

$$D(\omega_K) = \{a \in \mathfrak{A} : (a, \ell) \in K\}$$

by

$$\omega_K(a) = \ell, \quad a \in D(\omega_K).$$

Then, by (p5) ω_K is a positive slight extension of ω.

Proposition 2.11. *The following statements hold.*

(i) *ω admits a maximal positive slight extension.*

(ii) *Let $\breve{\omega}$ be a maximal positive slight extension of ω. Then*

$$D(\breve{\omega}) \cap \mathcal{P}(\mathfrak{A}) = \mathcal{K}_\omega^\ddagger := \{a \in \mathcal{K}_\omega \cap \mathcal{P}(\mathfrak{A}) : (a, \ell) \in \overline{G_\omega},\ \textit{for some}\ \ell \geq 0\}.$$

(iii) *Let ω be nonclosable, and let $\mathcal{P}(\mathfrak{A}_0) \neq \mathcal{K}_\omega^\ddagger$. Then ω admits infinitely many positive slight extensions.*

Proof. (i) $\mathcal{P}_\omega$ satisfies the assumptions of Zorn's lemma. Then it has a maximal element $\breve{H}$.

(ii) Let $a \in \mathcal{K}_\omega^\ddagger$; then $(a, \ell) \in \overline{G_\omega}$, for some $\ell \geq 0$. Assume that $a \notin D(\breve{\omega})$. Then $G_{\breve{\omega}} \subset G_{\breve{\omega}} \oplus \langle (a, \ell) \rangle \in \mathcal{P}_\omega$. Hence, $\breve{\omega}$ is not maximal positive. This shows that $\mathcal{K}_\omega^\ddagger \subseteq D(\breve{\omega}) \cap \mathcal{P}(\mathfrak{A})$. Vice versa, if $a \in D(\breve{\omega})$ with $a \in \mathcal{P}(\mathfrak{A})$, then $\breve{\omega}(a) \geq 0$, by the positivity of $\breve{\omega}$. Since $(a, \breve{\omega}(a)) \in \overline{G_\omega}$, then $a \in \mathcal{K}_\omega^\ddagger$.

(iii) Since ω is not closable, $\overline{G_\omega}$ contains couples $(0, m)$ with $m \neq 0$. We may assume that $m > 0$, since $\{(0, k) : (0, k) \in \overline{G_\omega}\}$ is a subspace of $\overline{G_\omega}$ stable under involution. Let $a_0 \in \mathcal{K}_\omega^\ddagger \setminus \mathcal{P}(\mathfrak{A}_0)$, and let $\ell \in \mathbf{R}^+$ be such that $(a_0, \ell) \in \overline{G_\omega}$. Then there exists a maximal positive slight extension $\breve{\omega}$ such that $G_{\breve{\omega}} \supset G_\omega \oplus \langle (a_0, \ell) \rangle$. Clearly $(a_0, \ell) + \lambda(0, m) \in \overline{G_\omega}$, for every $\lambda \in \mathbf{R}^+$. Therefore, for every λ, there exists a maximal positive slight extension whose graph contains $(a_0, \ell) + \lambda(0, m)$. Also in this case, the extensions corresponding to different values of λ are different. □

Remark 2.12. All maximal positive slight extensions are defined on the same set of positive elements. However, if $\breve{\omega}$ satisfies $D(\breve{\omega}) \cap \mathcal{P}(\mathfrak{A}) = \mathcal{K}_\omega^\ddagger$, $\breve{\omega}$ need not be maximal positive. Finally notice that a maximal positive extension needs not be maximal.

Definition 2.13. Let $\mathfrak{A}$ be a *-algebra and $\mathfrak{A}_h = \{b \in \mathfrak{A} : b = b^*\}$. We say that $\mathfrak{A}$ has the property (D) if, for every $a \in \mathfrak{A}_h$, there exists a unique pair (a_+, a_-) of elements of $\mathfrak{A}$, with $a_+, a_- \in \mathcal{P}(\mathfrak{A})$ and $a_+a_- = a_-a_+ = 0$, such that

(D1) $a = a_+ - a_-$;

(D2) $(a+b)_+ \leq a_+ + b_+$, for all $a, b \in \mathfrak{A}_h$ with $ab = ba$;

(D3) $(\lambda a)_+ = \lambda a_+$, for all $a \in \mathfrak{A}_h$, $\lambda \in \mathbf{R}^+$;

(D4) if $a \in \mathfrak{A}_h \cap \mathfrak{A}_0$, then $a_+ \in \mathcal{P}(\mathfrak{A}_0)$.

Remark 2.14. From (D2) it follows that $(a+b)_- \leq a_- + b_-$, for all $a, b \in \mathfrak{A}_h$ with $ab = ba$. Indeed,

$$(a+b)_- = (a+b)_+ - (a+b) \leq a_+ + b_+ - a - b = a_- + b_-.$$

If (D) holds, and $a = a^*$, we put $|a| := a_+ + a_-$. For a generic element $a \in \mathfrak{A}$, $|a|$ is, in general, not defined. From (D2), if $ab = ba$, we get

$$|a+b| \leq |a| + |b|, \text{ for all } a, b \in \mathfrak{A}_h.$$

Definition 2.15. We say that a positive slight extension $\widetilde{\omega}$ has property (I) if

$$a \in D(\widetilde{\omega}) \Longrightarrow |a| \in D(\widetilde{\omega}) \text{ and } |\widetilde{\omega}(a)| \leq \widetilde{\omega}(|a|),$$
$$\text{for all } a \in D(\widetilde{\omega}), \text{ with } a = a^*.$$

In what follows, a special role will be played by the condition

$$a \in D(\widetilde{\omega}) \Longleftrightarrow a_+, a_- \in D(\widetilde{\omega}). \tag{1}$$

For this reason we give the following

Definition 2.16. Let $\mathfrak{A}$ satisfy condition (D). A slight positive extension $\widetilde{\omega}$ of ω is called *absolutely convergent* if (1) holds.

Proposition 2.17. *Assume that $\widetilde{\omega}$ is an absolutely convergent slight extension of ω. Then $a \in D(\widetilde{\omega})$ if, and only if, $|a| \in D(\widetilde{\omega})$. Moreover, $\widetilde{\omega}$ has the property* (I).

Proof. The first statement is clear. Moreover, let a be an hermitian element of $D(\widetilde{\omega})$, then it easily follows that

$$|\widetilde{\omega}(a)| = |\widetilde{\omega}(a_+) - \widetilde{\omega}(a_-)| \leq \widetilde{\omega}(a_+) + \widetilde{\omega}(a_-) = \widetilde{\omega}(|a|). \qquad \square$$

Proposition 2.18. *If ω admits an absolutely convergent slight extension $\breve{\omega}$ which is maximal positive, then the domain of $\breve{\omega}$ is* span $\{\mathcal{K}_\omega^\ddagger\}$, *the linear span of $\mathcal{K}_\omega^\ddagger$.*

Proof. Let $\breve{\omega}$ be a maximal positive extension and assume that $a_+ \in D(\breve{\omega})$, for every $a \in D(\breve{\omega})$. Since $\breve{\omega}(a_+) \geq 0$, it follows that $a_+ \in \mathcal{K}_\omega^\ddagger$. Similarly, $a_- \in \mathcal{K}_\omega^\ddagger$. This implies that a belongs to span $\{\mathcal{K}_\omega^\ddagger\}$, the linear span of $\mathcal{K}_\omega^\ddagger$. Since the converse inclusion is obvious, we conclude that $D(\breve{\omega}) = \text{span}\,\{\mathcal{K}_\omega^\ddagger\}$. $\square$

Proposition 2.19. *Every ω has a maximal absolutely convergent slight extension.*

Proof. Let $\mathcal{P}_\omega$ be as in subsection 2.2.4, and let AC_ω be the subfamily of $\mathcal{P}_\omega$, whose elements K satisfy the additional requirement

$$(\mathrm{p_{ac}}) \qquad (a, \ell) \in K \implies (a_+, \ell_+) \in K,$$

where, as usual, $\ell_+ = \max\{0, \ell\}$. Then AC_ω satisfies the assumptions of Zorn's lemma, hence it has a maximal element. $\square$

2.3. Positively regular extensions. In this section we will construct a particular slight extension of a nonclosable hermitian positive linear functional ω, following essentially the model of the construction of the Lebesgue integral. In this general framework, however, some further conditions must be imposed.

Definition 2.20. A slight extension $\breve{\omega}$ of a hermitian positive linear functional ω is said to be *positively regular* if

$$\breve{\omega}(a) = \sup\{\omega(b) : 0 \leq b \leq a;\, b \in \mathfrak{A}_0\}, \\ \text{for all } a \in D(\breve{\omega}) \cap \mathcal{P}(\mathfrak{A}).$$

We now give a condition for a positively regular extension of ω to exist. To this aim, for every $a \in \mathcal{P}(\mathfrak{A})$, we define

$$m(a) = \{b \in \mathfrak{A}_0 : 0 \leq b \leq a\},$$
$$\dot{\omega}(a) = \sup\{\omega(b) : b \in m(a)\}.$$

Proposition 2.21. *Assume that $\mathfrak{A}$ has property* (D) *and that the following conditions are fulfilled.*

(AP_1) *If (a_α) is an increasing net of elements of $\mathcal{P}(\mathfrak{A}_0)$, converging to $a \in \mathcal{P}(\mathfrak{A})$, then $\omega(a_\alpha) \to \dot{\omega}(a)$.*

(AP_2) *For every $a \in \mathcal{P}(\mathfrak{A})$, there exists a net $\{a_\alpha\} \subset m(a)$ such that $a_\alpha \xrightarrow{\tau} a$ increasingly and $\omega(a_\alpha) \to \dot{\omega}(a)$.*

Then, $\dot{\omega}$ defines an absolutely convergent positively regular slight extension of ω.

Proof. Assume that $\mathfrak{A}$ has property (D), and let ω be a nonclosable hermitian positive linear functional on $\mathfrak{A}_0 \times \mathfrak{A}_0$.

Now, put

$$\mathcal{P}(\dot{\omega}) = \{a \in \mathcal{P}(\mathfrak{A}) : \dot{\omega}(a) < \infty\}.$$

Conditions (AP_1) and (AP_2) imply that

$$a, b \in \mathcal{P}(\dot{\omega}) \Longrightarrow a + b \in \mathcal{P}(\dot{\omega}) \text{ and } \dot{\omega}(a + b) = \dot{\omega}(a) + \dot{\omega}(b) \tag{2}$$
$$a \in \mathcal{P}(\dot{\omega}), \lambda \geq 0 \Longrightarrow \lambda a \in \mathcal{P}(\dot{\omega}) \text{ and } \dot{\omega}(\lambda a) = \lambda \dot{\omega}(a). \tag{3}$$

Put

$$H(\dot{\omega}) = \{a \in \mathfrak{A} : a = a^* \text{ and } a_+, a_- \in \mathcal{P}(\dot{\omega})\}$$
$$D(\dot{\omega}) = \{a + ib : \ a, b \in H(\dot{\omega})\}.$$

If $a = a^*$, then a can be written, in a unique way, as $a = a_+ - a_-$. Thus, if $a \in D(\dot{\omega})$, we define

$$\dot{\omega}(a) := \dot{\omega}(a_+) - \dot{\omega}(a_-).$$

We need to prove that $\dot{\omega}$ is additive on hermitian elements. Let $a, b \in H(\dot{\omega})$. By (D2) of Definition 2.13, it follows that $(a + b)_+, (a + b)_- \in \mathcal{P}(\dot{\omega})$. It remains to prove that

$$\dot{\omega}(a + b) = \dot{\omega}(a) + \dot{\omega}(b), \quad \text{for all } a, b \in H(\dot{\omega}).$$

Put $c = a + b$. Then,

$$c_+ - c_- = a_+ - a_- + b_+ - b_-.$$

Whence

$$c_+ + a_- + b_- = c_- + a_+ + b_+.$$

Therefore, by (2),

$$\dot{\omega}(c_+) + \dot{\omega}(a_-) + \dot{\omega}(b_-) = \dot{\omega}(c_-) + \dot{\omega}(a_+) + \dot{\omega}(b_+).$$

From this equality the statement follows easily. Finally, we extend $\dot{\omega}$, by linearity, to $D(\dot{\omega})$. We finally remark that $\dot{\omega}$ is a slight extension because of (D) and (AP_2), and it is absolutely convergent since if $a \in D(\dot{\omega})$, $a = a^*$, then $a_+, a_- \in D(\dot{\omega})$ by definition. □

Remark 2.22. We notice that the construction itself of $\dot{\omega}$ implies that if $a \in D(\dot{\omega})$, then $a^* \in D(\dot{\omega})$ and $\dot{\omega}(a^*) = \overline{\dot{\omega}(a)}$, for every $a \in D(\dot{\omega})$.

Problem. Is $\dot{\omega}$ maximal? Or under which conditions is it maximal?

Proposition 2.23. *Let $\mathfrak{A}$ have property* (D) *and ω satisfy* $(\mathrm{AP})_1$ *and* $(\mathrm{AP})_2$. *Consider the statements*

(i) $\dot{\omega}$ *is maximal positive*;

(ii) $\mathcal{P}(\dot{\omega}) = \mathcal{K}^{\ddagger}_{\omega}$;

(iii) *Every $a \in \mathcal{P}(\mathfrak{A})$ such that there exists a net $\{a_\alpha\} \subset \mathcal{P}(\mathfrak{A}_0)$ with $a_\alpha \xrightarrow{\tau} a$ and $\lim \omega(a_\alpha) < \infty$ belongs to $\mathcal{P}(\dot{\omega})$.*

Then

$$\text{(i)} \iff \text{(ii)} \implies \text{(iii)}.$$

Proof. (i) $\Leftrightarrow$ (ii) is clear. We prove that (ii) $\Rightarrow$ (iii). Assume that $a \in \mathcal{P}(\mathfrak{A})$, and let $\{a_\alpha\} \subset \mathcal{P}(\mathfrak{A}_0)$ with $a_\alpha \xrightarrow{\tau} a$ and $\lim \omega(a_\alpha) = \ell < \infty$. Then $(a, \ell) \in \overline{G_\omega}$, with $\ell \geq 0$. This implies that $a \in \mathcal{K}^{\ddagger}_{\omega} = \mathcal{P}(\dot{\omega})$. □

Proposition 2.24. *If ω admits an absolutely convergent positively regular extension $\breve{\omega}$ which is maximal positive, then this extension is unique.*

Proof. Let ω' be another absolutely convergent positively regular and maximal positive extension. If $a \in D(\omega')$, then $a_+, a_- \in D(\breve{\omega}) \cap \mathcal{P}(\mathfrak{A})$ (because maximal positive extension are defined on the same set of positive elements). Moreover, if $a \in D(\omega') \cap \mathcal{P}(\mathfrak{A})$,

$$\omega'(a) = \sup\{\omega'(b); b \leq a\} = \sup\{\omega(b); b \leq a\} = \breve{\omega}(a).$$

Hence, if $a \in D(\omega')$, we have

$$\omega'(a) = \omega'(a_+) - \omega'(a) = \breve{\omega}(a_+) - \breve{\omega}(a) = \breve{\omega}(a),$$

i.e., $\omega' = \breve{\omega}$. □

Proposition 2.25. *Let $b \in D(\dot{\omega}) \cap \mathcal{P}(\mathfrak{A})$ and $a \in \mathfrak{A}$ such that $0 \leq a \leq b$. Then $a \in D(\dot{\omega}) \cap \mathcal{P}(\mathfrak{A})$.*

Proof. Let $c \in \mathfrak{A}_0$, with $0 \leq c \leq a$, then $c \leq b$. Hence,

$$\begin{aligned}\dot{\omega}(a) &= \sup\{\omega(c);\ c \in \mathfrak{A}_0,\ 0 \leq c \leq a\} \\ &\leq \sup\{\omega(c);\ c \in \mathfrak{A}_0,\ 0 \leq c \leq b\} \\ &= \dot{\omega}(b) < \infty.\end{aligned}$$ □

2.3.1. Further extensions. In this section we will strengthen the assumptions on $\mathfrak{A}[\tau]$ and on ω in order to construct extensions that behave more regularly.

Assume that $\mathfrak{A}_0$ contains an *approximate identity* with respect to τ, i.e., a net $\{e_\alpha\}$ of elements of $\mathcal{P}(\mathfrak{A}_0)$ with the properties

(i) $\alpha \leq \beta \Rightarrow e_\alpha \leq e_\beta$;

(ii) $e_\alpha a \xrightarrow{\tau} a$, for every $a \in \mathfrak{A}_0$.

Remark 2.26. Since * is τ-continuous, one also has $a e_\alpha \xrightarrow{\tau} a$, for every $a \in \mathfrak{A}_0$.

From now forth, with the symbol $\dot{\omega}$ we will always denote the extension of ω constructed in the proof of Proposition 2.21.

Let us now consider the vector space

$$D(^{\diamond}\omega) = \{a \in \mathfrak{A} : e_\alpha a \in D(\dot{\omega}), \text{ for all } \alpha, \text{ and } \{\dot{\omega}(e_\alpha a)\} \text{ is convergent}\}.$$

Then we define

$$^{\diamond}\omega(a) = \lim_\alpha \dot{\omega}(e_\alpha a), \quad a \in D(\diamond\omega).$$

It is easily seen that $^{\diamond}\omega$ is a linear functional on $D(^{\diamond}\omega)$, but, in general, $\mathfrak{A}_0 \not\subset D(^{\diamond}\omega)$ and, *a fortiori* $D(\dot{\omega}) \not\subset D(^{\diamond}\omega)$. So we cannot conclude that $^{\diamond}\omega$ is an extension of $\dot{\omega}$. Sufficient conditions for this to hold will be given in Proposition 2.28.

Remark 2.27. In general, the domain $D(^{\diamond}\omega)$ is not stable under involution. More precisely, if we put

$$D(\omega^{\diamond}) = \{a \in \mathfrak{A} : ae_\alpha \in D(\dot{\omega}), \text{ for all } \alpha, \text{ and } \{\dot{\omega}(ae_\alpha)\} \text{ is convergent}\},$$

then another functional $\omega^{\diamond}$ can be defined by

$$\omega^{\diamond}(a) = \lim_\alpha \dot{\omega}(ae_\alpha), \quad a \in D(\omega^{\diamond}).$$

In general, $\omega^{\diamond} \neq {}^{\diamond}\omega$ and both these possible extensions depend upon the net $\{e_\alpha\}$. Taking into account Remark 2.26, it is easily seen that $a \in D(^{\diamond}\omega) \Leftrightarrow a^* \in D(\omega^{\diamond})$.

In what follows we will suppose that $\mathfrak{A}_0$ is a pre C*-algebra (i.e., a normed *-algebra whose norm $\|\cdot\|_0$ satisfies the C*-property $\|a^*a\|_0 = \|a\|_0^2$, for every $a \in \mathfrak{A}_0$). Then its completion $\widetilde{\mathfrak{A}_0}$ is a C*-algebra. For every $a, b \in \mathfrak{A}_0$ the following inequality holds

$$b^*a^*ab \leq \|a\|_0^2\, b^*b.$$

Hence, since for any pair of elements x, y of a C*-algebra such that $0 \leq x \leq y$, one has $x^{1/2} \leq y^{1/2}$, we also have

$$(b^*a^*ab)^{1/2} \leq \|a\|_0\, (b^*b)^{1/2}, \tag{4}$$

i.e.,

$$|ab| \leq \|a\|_0\, |b|. \tag{5}$$

We remind that by (D4) of Definition 2.13, if $a, b \in \mathfrak{A}_0$ then $|ab|, |b| \in \mathfrak{A}_0$.

Let us also suppose that the norm topology of $\mathfrak{A}_0$ is finer than the topology induced on $\mathfrak{A}_0$ by τ. Then $\widetilde{\mathfrak{A}_0}$ contains a bounded approximate identity, that is, a net $\{e_\alpha\}$ of elements of $\widetilde{\mathfrak{A}_0}$ with the properties

(i) $\|e_\alpha\|_0 \leq 1$, for every α;

(ii) $\alpha \leq \beta \Rightarrow e_\alpha \leq e_\beta$;

(iii) $\lim_\alpha \|e_\alpha a - a\|_0 = 0$, for all $a \in \mathfrak{A}_0$.

However, in general, $\{e_\alpha\}$ is not contained in $\mathfrak{A}_0$, unless $\mathfrak{A}_0$ is an ideal of $\widetilde{\mathfrak{A}_0}$. If $\{e_\alpha\} \subset \mathfrak{A}_0$, then $\{e_\alpha\}$ is an approximate identity also with respect to τ, since τ is coarser than $\|\cdot\|$.

Moreover, by (5), it follows that

$$|\omega(ab)| \leq \omega(|ab|) \leq \|a\|_0 \omega(|b|), \quad \text{for all } a, b \in \mathfrak{A}_0. \tag{6}$$

In the next proposition we will assume that (6) extends to $D(\dot{\omega})$.

Proposition 2.28. *Let $\mathfrak{A}[\tau]$ and ω satisfy (AP_1) and (AP_2). Assume, in addition, that*

(i) *$\mathfrak{A}_0$ is a pre C^*-algebra with a norm $\|\cdot\|_0$ such that the norm topology of $\mathfrak{A}_0$ is finer than the topology induced on $\mathfrak{A}_0$ by τ and $\mathcal{P}(\mathfrak{A}_0) = \mathfrak{A}_0 \cap \mathcal{P}(\widetilde{\mathfrak{A}_0})$;*

(ii) *$\mathfrak{A}_0$ admits an approximate identity $\{e_\alpha\}$, with the property $\|e_\alpha\|_0 \leq 1$, for every α;*

(iii) *if $a \in \mathfrak{A}_0$ and $x \in D(\dot{\omega})$, then $ax \in D(\dot{\omega})$ and $|\dot{\omega}(ax)| \leq \|a\|_0 \dot{\omega}(|x|)$.*

Then $D(\dot{\omega}) \subseteq D(^\diamond\omega)$ and $^\diamond\omega(a) = \dot{\omega}(a)$, for every $a \in D(\dot{\omega})$.

Proof. It is sufficient to prove the statement for $a \in \mathcal{P}(\mathfrak{A})$. By (AP_2), if $a \in \mathcal{P}(\mathfrak{A})$ there exists a net $\{a_\gamma\} \subset \mathcal{P}(\mathfrak{A}_0)$ such that $a_\gamma \leq a$, for every γ, $a_\gamma \xrightarrow{\tau} a$ and $\omega(a_\gamma) \to \dot{\omega}(a)$. The τ-continuity of the multiplication implies that, for each fixed α, $e_\alpha a_\gamma \xrightarrow{\tau} e_\alpha a$. Now, by (iii) it follows that $e_\alpha a \in D(\dot{\omega})$. It remains to prove that

$$\lim_\alpha \dot{\omega}(e_\alpha a) = \dot{\omega}(a).$$

First observe that the assumption $\mathcal{P}(\mathfrak{A}_0) = \mathfrak{A}_0 \cap \mathcal{P}(\widetilde{\mathfrak{A}_0})$ implies [**7**, Theorem 4.3.2] that ω is continuous on $\mathfrak{A}_0$. Then, for each fixed γ, we have

$$|\omega(e_\alpha a_\gamma - a_\gamma)| \leq \|\omega\| \|e_\alpha a_\gamma - a_\gamma\|_0 \longrightarrow 0.$$

Hence, for each fixed γ and for every $\varepsilon > 0$, there exists an $\alpha_1(\gamma, \varepsilon)$ such that

$$|\omega(e_\alpha a_\gamma - a_\gamma)| \leq \frac{\varepsilon}{3}, \quad \text{for all } \alpha > \alpha_1(\gamma, \varepsilon).$$

Finally, from $\omega(a_\gamma) \to \dot{\omega}(a)$, it follows that there exists γ_ε such that

$$|\omega(a_\gamma) - \dot{\omega}(a)| < \frac{\varepsilon}{3}, \quad \text{for all } \gamma > \gamma_\varepsilon.$$

Moreover, also taking into account that $a - a_\gamma \geq 0$, we obtain from (iii),

$$|\omega(e_\alpha a_\gamma) - \dot{\omega}(e_\alpha a)| \leq \dot{\omega}(|a_\gamma - a|) = \dot{\omega}(a - a_\gamma) < \frac{\varepsilon}{3}, \quad \forall \gamma > \gamma_\varepsilon.$$

Hence, for $\gamma > \gamma_\varepsilon$ and $\alpha > \alpha_1(\gamma_\varepsilon, \varepsilon)$ we get

$$\begin{aligned} |\dot{\omega}(a) - \dot{\omega}(e_\alpha a)| &\leq |\dot{\omega}(a) - \omega(a_\gamma)| + |\omega(a_\gamma) - \omega(e_\alpha a_\gamma)| \\ &\quad + |\omega(e_\alpha a_\gamma) - \dot{\omega}(e_\alpha a|) < \varepsilon. \qquad \square \end{aligned}$$

We show now a situation where the assumptions of Proposition 2.28 are fulfilled.

Proposition 2.29. *Let $\mathfrak{A}[\tau]$ satisfy the following conditions*

(a) *The map $a \to |a|$ is τ-continuous*;

(b) *$\mathcal{P}(\mathfrak{A})$ is τ-closed.*

Assume that $\mathfrak{A}_0$ is a pre C^*-algebra. Then the following statements hold.

(i) *$D(\dot{\omega})$ is a left module over $\mathfrak{A}_0$ (i.e., $ax \in D(\dot{\omega})$, for every $x \in D(\dot{\omega})$, $a \in \mathfrak{A}_0$);*

(ii) *$|\dot{\omega}(ax)| \leq \|a\|_0 \dot{\omega}(|x|)$, for all $x \in D(\dot{\omega})$, $a \in \mathfrak{A}_0$.*

Proof. Let $x \in D(\dot{\omega})$. Then there exists a net $\{x_\alpha\} \subset \mathfrak{A}_0$ such that $x_\alpha \xrightarrow{\tau} x$ and $\omega(x_\alpha) \to \dot{\omega}(x)$. For $a \in \mathfrak{A}_0$ we get, by (5),

$$|ax_\alpha| \leq \|a\|_0 |x_\alpha|.$$

Since $x \to |x|$ is τ-continuous, taking the limit over α, we obtain

$$|ax| \leq \|a\|_0 |x|.$$

By Proposition 2.25, it follows that $|ax| \in D(\dot{\omega})$ and then $ax \in D(\dot{\omega})$. Since $\dot{\omega}$ is positive on $D(\dot{\omega}) \cap \mathcal{P}(\mathfrak{A})$, by Proposition 2.17, we get

$$|\dot{\omega}(ax)| \leq \|a\|_0 \dot{\omega}(|x|), \quad \text{for all } x \in D(\dot{\omega}), a \in \mathfrak{A}_0. \qquad \square$$

Remark 2.30. Assume that ω is a *trace*, i.e., $\omega(ab) = \omega(ba)$, for every $a, b \in \mathfrak{A}_0$. If the assumptions of Proposition 2.29 are satisfied, then one easily proves that $\dot{\omega}(ax) = \dot{\omega}(xa)$, for every $x \in D(\dot{\omega})$ and $a \in \mathfrak{A}_0$. This, in turn, implies that ${}^\diamond\omega = \omega^\diamond$.

Assume that $\dot{\omega}$ satisfies the following condition of *lower semicontinuity*:

(LS) *For every net $\{a_\alpha\}$ of elements of $\mathcal{P}(\mathfrak{A})$ such that $a_\alpha \to a \in \mathfrak{A}$, one has*

$$\dot{\omega}(a) \leq \liminf_\alpha \dot{\omega}(a_\alpha).$$

Of course, $a \in \mathcal{P}(\mathfrak{A})$, by (b). Under these assumptions we can prove the following abstract version of the Lebesgue dominated convergence theorem, whose proof is similar to the classical one.

Proposition 2.31. *Let $\mathfrak{A}[\tau]$ satisfy* (a) *and* (b) *of Proposition* 2.29 *and $\dot{\omega}$ satisfy* (LS). *Let $\{a_\alpha\}$ be a net in $\mathfrak{A}$, τ-converging to $a \in \mathfrak{A}$. Suppose that there exists $b \in D(\dot{\omega})$ such that $|a_\alpha| \leq b$, for every α. Then $a \in D(\dot{\omega})$ and $\lim_\alpha \dot{\omega}(|a_\alpha - a|) = 0$.*

Proof. Since $|a_\alpha| \leq b$, then $|a| \leq b$, $\mathcal{P}(\mathfrak{A})$ being τ-closed. By Proposition 2.25 it follows that $|a| \in D(\dot{\omega})$. This implies that $a \in D(\dot{\omega})$.

Clearly, $|a_\alpha - a| \leq 2b$. Let $\{c_\alpha\}$ be the net defined by $c_\alpha = 2b - |a_\alpha - a|$. Then, $\lim_\alpha c_\alpha = 2b$. Consequently, by (LS), we have

$$\dot{\omega}(2b) \leq \liminf_\alpha \dot{\omega}(2b - |a_\alpha - a|) = \dot{\omega}(2b) - \limsup_\alpha \dot{\omega}(|a_\alpha - a|).$$

This implies that

$$\limsup_\alpha \dot{\omega}(|a_\alpha - a|) \leq 0.$$

Hence,

$$\lim_\alpha \dot{\omega}(|a_\alpha - a|) = 0. \qquad \square$$

Remark 2.32. If $\dot{\omega}$ satisfies (LS), then the three conditions of Proposition 2.23 are equivalent.

Proposition 2.33. *Let* $\mathfrak{A}[\tau]$ *satisfy* (a) *of Proposition* 2.29 *and* $\dot{\omega}$ *satisfy* (LS). *If* $a \in \mathfrak{A}$ *and there exists a net* $\{a_\alpha\} \subset \mathfrak{A}_0$ *such that* $a_\alpha \xrightarrow{\tau} a$ *with* $\lim_\alpha \omega(|a_\alpha|) < \infty$*, then* $a \in D(\dot{\omega})$.

Proof. From $a_\alpha \xrightarrow{\tau} a$, it follows that $|a_\alpha| \xrightarrow{\tau} |a|$. Then by (LS),

$$\dot{\omega}(|a|) \leq \liminf_\alpha \omega(|a_\alpha|) < \infty.$$

Hence, $|a| \in D(\dot{\omega})$. This, in turn, implies that $a \in D(\dot{\omega})$. $\square$

Remark 2.34. The previous proposition says, in other words, that $\dot{\omega}$ is a maximal absolutely convergent slight extension of ω. Moreover it is positively regular and positive. Then, by Proposition 2.24, it is the unique extension of ω with these properties.

3. Applications to commutative integration. Let us consider the situation of Example 2.2. In this case, the positively regular slight extension $\dot{\omega}$ of ω exists and $\dot{\omega}$ is nothing but the Lebesgue integral on $[0, 1]$. This extension is not maximal. There exist, in fact, many possible extensions of the Lebesgue integral. We consider in what follows the Henstock-Kurzweil (HK) integral and we show that it actually is a slight extension of ω.

In what follows, we denote by $\mathsf{m}(E)$ the Lebesgue measure of a Lebesgue measurable set E.

Proposition 3.1. *If f is HK-integrable on $[a,b]$, then there exists a sequence $\{f_n\}$ of continuous functions such that $f_n \to f$ in measure and $\int_a^b f_n \to (HK)\int_a^b f$.*

Proof. The claim is well known for Lebesgue integrable functions [**17**, Theorem 3.14].

In [**12**] it was proved that if f is HK-integrable on $[a,b]$, then there exists a sequence of measurable sets $\{E_n\}$ such that $\cup_n E_n = [a,b]$, f is Lebesgue integrable on E_n, for each n, and $(L)\int_{E_n} f \to (HK)\int_a^b f$.

Now, for each n, let $\{g_{n,m}\}$ be a sequence of continuous functions such that $g_{n,m} \to f\chi_{E_n}$ in measure and $\int_a^b g_{n,m} \to (L)\int_a^b f\chi_{E_n}$. We define $f_n = g_{n,m_n}$, where m_n is the first m satisfying the conditions:

$$\left|\int_a^b g_{n,m_n} - (L)\int_a^b f\chi_{E_n}\right| < \frac{1}{n}; \tag{7}$$

$$\mathsf{m}\left(\left\{x \in [a,b] : |g_{n,m_n}(x) - f\chi_{E_n}(x)| > \frac{1}{n}\right\}\right) < \frac{1}{n}. \tag{8}$$

Remark that

$$\begin{aligned}\{x \in [a,b] : |f_n(x) - f(x)| > \varepsilon\} \\ \subset \left\{x \in [a,b] : |f_n(x) - f\chi_{E_n}(x)| > \frac{\varepsilon}{2}\right\} \\ \cup \left\{x \in [a,b] : |f(x) - f\chi_{E_n}(x)| > \frac{\varepsilon}{2}\right\},\end{aligned}$$

and that $f\chi_{E_n} \to f$ in measure.

Given $\varepsilon, \eta > 0$ let $N > \sup\{2/\varepsilon, 2/\eta\}$ be such that for $n > N$ we have

$$\mathsf{m}\left(\left\{x \in [a,b] : |f\chi_{E_n}(x) - f(x)| > \frac{\varepsilon}{2}\right\}\right) < \frac{\eta}{2}.$$

Then, by (8), taking into account that $f_n = g_{n,m_n}$, for $n > N$, one has

$$\begin{aligned} &\mathsf{m}\left(\{x \in [a,b] : |f_n(x) - f(x)| > \varepsilon\}\right) \\ &\qquad \leq \mathsf{m}\left(\left\{x : |f_n(x) - f\chi_{E_n}(x)| > \frac{\varepsilon}{2}\right\}\right) \\ &\qquad\quad + \mathsf{m}\left(\left\{x : |f(x) - f\chi_{E_n}(x)| > \frac{\varepsilon}{2}\right\}\right) \\ &\qquad \leq \mathsf{m}\left(\left\{x : |f_n(x) - f\chi_{E_n}(x)| > \frac{1}{n}\right\}\right) + \frac{\eta}{2} \\ &\qquad < \frac{1}{n} + \frac{\eta}{2} < \frac{1}{N} + \frac{\eta}{2} < \frac{\eta}{2} + \frac{\eta}{2} = \eta. \end{aligned}$$

This implies $f_n \to f$ in measure.

The convergence $\int_a^b f_n \to (HK) \int_a^b f$ follows immediately by (7) and by the convergence $(L) \int_{E_n} f \to (HK) \int_a^b f$. □

The next proposition shows that the HK-integral is not even a maximal slight extension.

Proposition 3.2. *There exists a measurable function f which is not HK-integrable on $[0,1]$, and a sequence $\{f_n\}$ of continuous functions such that $f_n \to f$ in measure and the sequence $\{\int_0^1 f_n\}$ is convergent.*

Proof. Let C be the Cantor ternary set. Then

$$C = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{2^n} J_{in},$$

where with $J_{1n}, J_{2n}, \ldots, J_{2^n n}$ we denote the intervals involved at the step n of the construction of the Cantor set. Let $\{p_k\}$ be a fixed sequence of natural numbers, with $0 = p_1 < p_2 < \cdots < p_n < \cdots$ such that

$$2^{p_{n+1} - p_n - 1} - 1 > n \cdot 3^{p_n}.$$

We define $h(n) = p_k$ whenever $p_k \leq n < p_{k+1}$.

It is known that each point $x \in C$ can be uniquely represented as $x = \sum_n c_n/3^n$, where $c_n = 0$ or $c_n = 2$, for every n. Let F be the function defined on $[0,1]$ by $F(x) = \sum_n c_{h(n)}/3^n$, if $C \ni x = \sum_n c_n/3^n$, and linear on each interval contiguous to C.

Foran [**3**] proved that the function f defined as $f(x) = F'(x)$ on each interval contiguous to C and as $f(x) = 0$ on C, is not HK-integrable on $[0,1]$.

By an easy adaptation of the algorithm used in the proof of Proposition 3.1, to complete the proof it is enough to prove that there exists a sequence $\{g_n\}$ of Lebesgue integrable functions such that $g_n \to f$ almost everywhere and the sequence $\{(L)\int_0^1 g_n\}$ is convergent.

For each natural n, let F_n be defined by

$$F_n\left(\sum_{i=1}^{\infty} \frac{c_i}{3^i}\right) = \sum_{i=1}^{n} \frac{c_{h(i)}}{3^i} \text{ on } C$$

and linear on each interval contiguous to C. Then, for each $x \in [0,1]$ we have

$$F_n(x) \longrightarrow F(x). \tag{9}$$

Now let J be one of the intervals J_{in}. Then there exist $d_1, \dots, d_n$, with $d_i = 0$ or $d_i = 2$ for $i = 1, \dots, n$, such that

$$J = \left[\sum_{i=0}^{n} \frac{d_i}{3^i}, \sum_{i=1}^{n} \frac{d_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{2}{3^i}\right].$$

Therefore, each $x \in J$ can be represented by

$$x = \sum_{i=1}^{n} \frac{d_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{c_i}{3^i},$$

with $c_i \in \{0,1,2\}$, for $i = n+1, \dots$. Consequently, for $x \in J$ we have

$$F_n(x) = \sum_{i=1}^{n} \frac{d_{h(i)}}{3^i}.$$

Thus the function F_n is absolutely continuous on $[0,1]$, being constant on the intervals J_{in}, and linear on each interval contiguous to $\cup_{i=1}^{2^n} J_{in}$.

Let g_n be defined as follows:

$$g_n(x) = \begin{cases} 0 & \text{if } x \in \cup_{i=1}^{2^n} J_{in} \\ F'_n(x) & \text{on } [0,1] \setminus \cup_{i=1}^{2^n} J_{in}. \end{cases}$$

It is clear that g_n is Lebesgue integrable and that $(L)\int_0^1 g_n = F_n(1)$. Thus, by (9),

$$\lim_n (L)\int_0^1 g_n = F(1) = 1.$$

Now let $I = (a,b)$ be an interval contiguous to C. Then, for every $x \in I$ we have, by the definitions of F and F_n, $f(x) = (F(b) - F(a))/(b-a)$ and $g_n(x) = (F_n(b) - F_n(a))/(b-a)$, for every $n \in \mathbf{N}$. Therefore $g_n \to f$ pointwise in $[0,1]$. □

Lemma 3.3. *Let f be a measurable function on $[0,1]$ such that there exists a sequence $\{f_n\}$ of continuous functions converging in measure to f, with $\lim_n \int_0^1 |f_n| < +\infty$. Then, f is Lebesgue integrable.*

Proof. Indeed, if $f_n \to f$ in measure, then $\{f_n\}$ contains a subsequence $\{f_{n_k}\}$ converging to f almost everywhere. By Fatou's lemma, we then obtain

$$\int_0^1 |f| \leq \liminf_{k\to\infty} \int_0^1 |f_{n_k}| < \infty. \qquad \square$$

Remark 3.4. The previous lemma cannot be deduced directly from Proposition 2.33, since the condition (LS) is not necessarily satisfied if τ is taken as the topology of convergence in measure.

Proposition 3.5. *Let f be a non negative measurable function. Then $f \in \mathcal{K}_\omega$ if, and only if, f is Lebesgue integrable.*

Proof. It is well known that if f is Lebesgue integrable, then $f \in \mathcal{K}_\omega$. Vice versa, if $f \in \mathcal{K}_\omega$ then f is Lebesgue integrable, by Lemma 3.3. □

The previous proposition shows that the Lebesgue integral is a maximal positive extension of the integral on $C(X)$ where $X = [a,b]$. It is also absolutely continuous and positively regular. Hence, by Proposition 2.24, there are no other absolutely convergent extensions of the Riemann integral.

4. Applications to noncommutative integration. Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and let σ be a normal faithful semifinite trace defined on $\mathfrak{M}_+$.

Segal, who begun the studies on non-commutative integration, introduced the notion of *measurable operator*. For the basic definitions and properties on noncommutative integration we refer to [**14, 15, 18**].

We first remind how the topology of convergence *in measure* on $\mathfrak{M}$ is defined.

For $\epsilon, \delta > 0$, let $N(\epsilon, \delta) = \{A \in \mathfrak{M} :$ for some projection $P \in \mathfrak{M}, \|AP\| \leq \varepsilon$ and $\sigma(P^\perp) \leq \delta\}$. We endow $\mathfrak{M}$ with the translation-invariant topology τ generated by the system $\{N(\epsilon, \delta); \epsilon, \delta > 0\}$ of neighborhoods of 0. The τ-completion $\widetilde{\mathfrak{M}}$ of $\mathfrak{M}$ is a *-algebra and it is called the *-algebra of *measurable operators*. We remind that the mappings

(i) $A \mapsto A^*$ of $\mathfrak{M} \to \mathfrak{M}$;

(ii) $(A, B) \mapsto A + B$ of $\mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$;

(iii) $(A, B) \mapsto AB$ of $\mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$,

have unique continuous extensions as mappings of $\widetilde{\mathfrak{M}} \to \widetilde{\mathfrak{M}}$ and $\widetilde{\mathfrak{M}} \times \widetilde{\mathfrak{M}} \to \widetilde{\mathfrak{M}}$. The mappings (i), (ii) and their extensions, are uniformly continuous. The map (iii) is uniformly continuous on products of sets which are bounded in measure [**18**].

Let $1 \leq p < +\infty$, and put

$$\mathcal{J}_p = \{X \in \mathfrak{M} : \sigma(|X|^p) < \infty\}.$$

Then, $\mathcal{J}_p$ is a *-ideal of $\mathfrak{M}$. As in [**14**], we denote with $L^p(\sigma)$ the Banach space completion of $\mathcal{J}_p$ with respect to the norm

$$\|X\|_p := \sigma(|X|^p)^{1/p}, \quad X \in \mathcal{J}_p.$$

As usual, we put $L^\infty(\sigma) = \mathfrak{M}$.

If X is measurable and $X = X^*$, then the operators X_+, X_- and $|X|$ are measurable operators.

If A is a *measurable* operator and $A \geq 0$, one defines

$$\mu(A) = \sup\{\sigma(X);\ 0 \leq X \leq A,\ X \in \mathcal{J}_1\}.$$

Then the space $L^1(\sigma)$ can also be defined [**14**] as the space of all measurable operators A such that $\mu(|A|) < \infty$.

The *integral* of an element $A \in L^1(\sigma)$ can be defined, in an obvious way, taking into account that any measurable operator A can be decomposed as $A = B_+ - B_- + iC_+ - iC_-$, where $B = (A + A^*)/2$ and $C = (A - A^*)/2i$.

Proposition 4.1. *The following statements hold.*

(i) *For all $A \in \widetilde{\mathfrak{M}}$, A is a closed, densely defined operator affiliated with $\mathfrak{M}$.*

(ii) *If $A \in \widetilde{\mathfrak{M}}$ then for all $\epsilon > 0$ there is a projection P in $\mathfrak{M}$ with $AP \in \mathfrak{M}$ and $\sigma(P^\perp) \leq \epsilon$.*

From the previous discussion, it follows that μ is exactly the positively regular extension $\dot{\sigma}$ of σ, as defined in Section 2. More precisely,

Proposition 4.2. *μ is a positively regular, absolutely convergent extension of σ.*

Moreover, the following generalization of (ii) of Proposition 2.29 holds [**15**].

Lemma 4.3. *Let $A \in L^1(\sigma)$ and $X \in \mathfrak{M}$. Then $XA \in L^1(\sigma)$ and*

$$|\mu(XA)| \leq \|X\|\mu(|A|). \tag{10}$$

Also, in this case, one can pose the problem of extending σ to other *measurable* operators. As far as we know, extensions of σ to other classes of measurable operators have not been explicitly defined. This will be done in what follows simply by applying the results of subsection 2.3.1 (in particular, Proposition 2.28). As noticed in

Remark 2.26, since σ is a trace, $\mu^\diamond = {}^\diamond\mu$ and the results do not depend on the order in which the operators are multiplied.

Since $\mathcal{J}_1$ is a *-ideal of the von Neumann algebra $\mathfrak{M}$, $\mathcal{J}_1$ contains an approximate identity of $\mathfrak{M}$ [**2**]; in this concrete case, this means that there exists a net $\{E_\alpha\}_{\alpha\in I}$ of positive operators in $\mathcal{H}$ with the properties

(i) $E_\alpha \in \mathcal{J}_1$, for every $\alpha \in I$;

(ii) $\|E_\alpha\| \leq 1$, for every $\alpha \in I$;

(iii) $\alpha \leq \beta$ implies $E_\alpha \leq E_\beta$;

(iv) $\|E_\alpha A - A\| \to 0$ for every $A \in \mathcal{J}_1$.

Since $\mathcal{J}_1$ is a *-ideal, one also has $\|AE_\alpha - A\| \to 0$, for all $A \in \mathcal{J}_1$.

Let

$$D({}^\diamond\mu) = \{A \in \widetilde{\mathfrak{M}} : E_\alpha A \in L^1(\sigma),$$
$$\text{for all } \alpha \in I \text{ and } \lim_\alpha \mu(E_\alpha A) \text{ exists finite}\}.$$

For $A \in D({}^\diamond\mu)$, we put

$${}^\diamond\mu(A) = \lim_\alpha \mu(E_\alpha A).$$

From Lemma 4.3, it follows that Proposition 2.28 can be applied to state that ${}^\diamond\mu$ is an extension of μ once we have checked that the condition $\mathcal{P}(\mathcal{J}_1) = \mathcal{J}_1 \cap \mathcal{P}(\overline{\mathcal{J}_1})$ holds. But, by the definition itself, we have

$$\mathcal{P}(\mathcal{J}_1) = \mathfrak{M}^+ \cap \mathcal{J}_1 \supseteq \mathcal{P}(\overline{\mathcal{J}_1}) \cap \mathcal{J}_1.$$

The converse inclusion is clear.

Finally we remark that if $A \in L^1(\sigma)$, then $E_\alpha A \in L^1(\sigma)$ for every $\alpha \in \mathcal{I}$, by Lemma 4.3. In conclusion, we have shown that

$$L^1(\sigma) \subseteq D({}^\diamond\mu).$$

In general, the equality does not hold, as we will see from the following example.

Example 4.4. Let $\mathcal{H}$ be a separable Hilbert space and $\{\varphi_k\}_{k=1}^{+\infty}$ an orthonormal basis of $\mathcal{H}$. For any positive operator $A \in \mathcal{B}(\mathcal{H}) = \mathfrak{M}$, the *natural* trace tr on $\mathcal{B}(\mathcal{H})$ is defined as

$$tr\,(A) = \sum_{k=1}^{\infty} (\varphi_k, A\varphi_k).$$

As is well known, $tr\,(A)$ is independent of the orthonormal basis and tr is a normal faithful semifinite trace on the von Neumann algebra $\mathfrak{M} = \mathcal{B}(\mathcal{H})$.

If $p \geq 1$, we denote as usual, by $\mathcal{T}_p$ the following *-ideal of $\mathcal{B}(\mathcal{H})$.

$$\mathcal{T}_p = \{X \in \mathfrak{M} : tr\,(|X|^p) < \infty\}.$$

Then, as it is well known, $\mathcal{T}_p$ is a Banach space with respect to the norm

$$\|X\|_p = (tr\,(|X|^p))^{1/p}\,, \quad X \in \mathcal{T}_p.$$

In what follows we will only consider the cases $p = 1$ and $p = 2$. With the notations introduced in the previous sections, we have, in this case, $L^1(tr) = \mathcal{T}_1$. We remind that

$$\mathcal{T}_1 \subset \mathcal{T}_2 \quad \text{and} \quad \|A\| \leq \|A\|_2 \leq \|A\|_1, \text{ for all } A \in \mathcal{T}_1.$$

Let us now define, for each $\ell \geq 1$, the following closed subspaces of $\mathcal{H}$,

$$\mathcal{H}_\ell := \text{linear span}\, \overline{\{\varphi_n,\ n \in \{1, 2, \dots \ell\}\}}^{\|\cdot\|},$$

and let P_ℓ denote the projection onto $\mathcal{H}_\ell$. The family $\{P_\ell\}_{\ell \in \mathbf{N}}$ is an approximate identity of $\mathfrak{M}$.

Indeed, for every $A \in \mathcal{T}_1 \subseteq \mathcal{T}_2$,

$$\|P_\ell A - A\|^2 \leq \|P_\ell A - A\|_2^2 = \sum_{k=1}^{\infty} ((P_\ell A - A)\varphi_k, (P_\ell A - A)\varphi_k).$$

Now, $P_\ell \varphi_k = \varphi_k$, if $k \leq \ell$, and $P_\ell \varphi_k = 0$, if $k > \ell$. Then, a simple computation shows that, if $A\varphi_k = \sum_{r=1}^{\infty} c_{kr}\varphi_r$, then

$$\sum_{k=1}^{\infty} ((P_\ell A - A)\varphi_k, (P_\ell A - A)\varphi_k) = \sum_{k=\ell+1}^{\infty} |c_{kr}|^2 \longrightarrow 0, \text{ as } \ell \to \infty.$$

For every

$$f = \sum_{k=1}^{\infty} d_k \varphi_k \in \mathcal{H},$$

we put

$$Af = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} d_k \varphi_k.$$

Then A is a well-defined bounded operator on $\mathcal{H}$. Indeed,

$$\begin{aligned}\|Af\|^2 &= \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\, d_k \varphi_k,\ \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} d_j \varphi_j \right) \\ &\leq \sum_{k=1}^{\infty} \frac{1}{k^2} |d_k|^2 \leq \|f\|^2.\end{aligned}$$

Moreover,

$$tr\,(|A|) = \sum_{k=1}^{\infty} (\varphi_k, |A|\varphi_k) = \sum_{k=1}^{\infty} \frac{1}{k} (\varphi_k, \varphi_k) = +\infty.$$

Hence, $A \notin \mathcal{T}_1$.

For every $\ell \in \mathbf{N}$, we have $AP_\ell \in \mathcal{T}_1$. Moreover, $A \in D(tr^\diamond)$, since

$$\begin{aligned}{}^\diamond tr\,(A) &= \lim_{\ell \to +\infty} tr\,(P_\ell A) = \lim_{\ell \to +\infty} \sum_{k=1}^{\infty} (\varphi_k, P_\ell A \varphi_k) \\ &= \lim_{\ell \to +\infty} \sum_{k=1}^{\ell} (\varphi_k, A\varphi_k) \\ &= \lim_{\ell \to +\infty} \sum_{k=1}^{\ell} \frac{(-1)^{k-1}}{k} (\varphi_k, \varphi_k) = \ln 2.\end{aligned}$$

This example shows that, in general,

$$D({}^\diamond \mu) \setminus L^1(\sigma) \neq \varnothing.$$

By the definition itself, every $A \in D({}^\diamond \mu)$ can be approximated by a net $\{A_\alpha\}$ of operators of $\mathcal{J}_1$ such that $A_\alpha \to A$ in measure and

$\mu(A_\alpha) \to {}^\diamond\mu\,(A)$. The next proposition shows that, in fact, a sequence $\{A_n\}$ enjoying the same properties can be found.

Proposition 4.5. *If $A \in D({}^\diamond\mu)$, then there exists a sequence $\{A_n\}$ of bounded operators of $\mathcal{J}_1$ such that $A_n \to A$ in measure and $\mu\,(A_n) \to {}^\diamond\mu\,(A)$.*

Proof. If $A \in D({}^\diamond\mu)$, then $\lim_\alpha \mu(E_\alpha A)$ exists finite. For each $n \in \mathbf{N}$, let us choose an element $E_{\alpha_n}A$ of $L^1(\sigma)$ such that $|{}^\diamond\mu(A) - \mu(E_{\alpha_n}A)| < 1/2n$.

Now, for each $n \in \mathbf{N}$, let $\{G_{n,m}\}$ be a sequence of bounded operators of $\mathcal{J}_1$ such that $G_{n,m} \to E_{\alpha_n}A$ in $\|\cdot\|_1$. We define $A_n = G_{n,m_n}$ where m_n is the first m satisfying the condition

$$|\mu(G_{n,m_n}) - \mu(E_{\alpha_n}A)| < \frac{1}{2n}. \tag{11}$$

Therefore,

$$|{}^\diamond\mu(A) - \mu(A_n)| \leq |\mu(G_{n,m_n}) - \mu(E_{\alpha_n}A)| + |{}^\diamond\mu(A) - \mu(E_{\alpha_n}A)| < \frac{1}{n}.$$

Finally, we notice that, since $E_{\alpha_n} \to I$ in measure, then also $A_n \to A$ in measure. □

The following statement is the analog to Lemma 3.3 in the non commutative case.

Proposition 4.6. *Let $\mathfrak{M}$ be a von Neumann algebra, and let A be a measurable operator with respect to the finite trace σ on $\mathfrak{M}$. If there exists a sequence of operators $\{A_n\}$ of $\mathfrak{M}$ converging in measure to A, with $\lim_n \sigma(|A_n|) < +\infty$, then $A \in L^1(\sigma)$.*

Proof. Without loss of generality we may assume that A and all the A_n's are hermitian. Moreover, since, by assumption, $\sigma(I) < \infty$, we may also assume that $\sigma(I) = 1$.

Since $A_n \to A$ in measure, for every $n \in \mathbf{N}$, we can choose a projection P_n, such that

$$\|(A - A_n)P_n\| < 1 \quad \text{and} \quad \sigma(P_n^\perp) < \frac{1}{2^n}.$$

Since $\sigma(P_n^\perp) \to 0$, we also get $\sigma(P_n^\perp \vee P_k^\perp) \to \sigma(P_k^\perp)$, for every $k \in \mathbf{N}$. Then, we can define inductively a sequence of natural numbers $N_1 < N_2 < \cdots < N_k < \cdots$ such that

(i) $N_1 = 1$;

(ii) $\sigma(P_{N_k}^\perp \vee P_{N_{k+1}}^\perp) < \sigma(P_{N_k}^\perp) + 1/2^k$, $k = 1, 2, \dots$.

For $k = 1, 2, \dots$, we set $Q_k = \wedge_{r=1}^\infty P_{N_{k+r}}$. Then

$$Q_1 \leq Q_2 \leq \cdots \leq Q_k \leq \cdots .$$

Moreover, by (ii), it follows that

$$\sigma(P_{N_k}^\perp \vee \cdots \vee P_{N_{k+r}}^\perp) \leq \sigma(P_{N_k}^\perp) + \frac{1}{2^k} + \cdots + \frac{1}{2^{k+r}}, \\ k = 1, 2, \dots ; r = 0, 1, 2, \dots .$$

The previous inequality implies that

$$\sigma(Q_k) \geq \sigma(P_{N_k}) - \frac{1}{2^{k-1}}, \quad k = 1, 2, \dots .$$

Hence, for $\sigma(P_{N_k}^\perp) \to 0$, we have

$$1 \geq \lim_{k\to\infty} \sigma(Q_k) \geq \lim_{k\to\infty} \sigma(P_{N_k}) = \sigma(I) = 1.$$

Now, by the definition of P_{N_k}, it follows that

$$\begin{aligned} \|P_{N_k}A - P_{N_k}A_{N_k}\|_1 &\leq \|P_{N_k}A - P_{N_k}A_{N_k}\| \, \|I\|_1 \\ &= \|AP_{N_k} - A_{N_k}P_{N_k}\| \, \|I\|_1 \leq 1. \end{aligned}$$

Hence, by the properties of the norm,

$$\|P_{N_k}A\|_1 \leq 1 + \|P_{N_k}A_{N_k}\|_1 \leq 1 + \|A_{N_k}\|_1,$$

since $|P_{N_k}A_{N_k}| \leq |A_{N_k}|$. Similarly, taking into account that $Q_k \leq P_{N_k}$, it is easily shown that $|Q_kA|$ is bounded and $|Q_kA| \leq |P_{N_k}A|$. Hence,

$$\sigma(|Q_kA|) \leq \sigma(|P_{N_k}A|) \leq 1 + \sigma(|A_{N_k}|).$$

Since $\lim_n \sigma(|A_n|) < +\infty$, for k large enough, we get

$$\sigma(|Q_k A|) \leq 2 + \lim_k \sigma(|A_{N_k}|).$$

The sequence $\{Q_k\}$ is increasing (using the faithfulness of σ, it can also be shown that $Q_k \uparrow I$) and, therefore, also $\{|Q_k A|\}$ is increasing. Moreover, as seen before, $\sup_k \|Q_k A\|_1 < \infty$. Therefore, by [**15**, Corollary 3.2] there exists $B = \sup |Q_k A| \in L^1(\sigma)$ and $\||Q_k A| - B\|_1 \to 0$. It remains to prove that $B = |A|$. For this, we first show that $Q_k A \to A$ in measure. Due to the continuity of the involution, this is equivalent to showing that $AQ_k \to A$ in measure. Let $\delta > 0$ and R be a projection in $\mathfrak{M}$ such that AR is bounded and $\sigma(R^\perp) < \delta$. Define $R_k = Q_k \wedge R$. Then we have $\|(AQ_k - A)R_k\| = 0$ for every $k \in \mathbf{N}$ and $\sigma(R_k^\perp) = \sigma(Q_k^\perp \vee R^\perp) < \delta$, for sufficiently large k, since $Q_k \uparrow I$. Thus, $Q_k A \to A$ in measure. This implies that $AQ_k A \to A^2$ in measure. Now, from $\||Q_k A| - B\|_1 \to 0$, it follows that $|Q_k A| \to B$ in measure, whence $AQ_k A \to B^2$ in measure. Therefore $A^2 = B^2$ and, in conclusion $|A| = B$. □

Proposition 4.7. *Let A be a positive measurable operator. Then $A \in \mathcal{K}_\sigma$ if, and only if, $A \in L^1(\sigma)$.*

Proof. By the definition itself, if $A \in L^1(\sigma)$ then $A \in \mathcal{K}_\sigma$. Vice versa, if $A \in \mathcal{K}_\sigma$, then $A \in L^1(\sigma)$, by Proposition 4.6. □

Concluding remark. In the case of non commutative integration, we have discussed the possibility of constructing an extension ${}^\diamond\mu$ of the *integral* μ, based on the choice of an approximating identity $\{E_\alpha\}$ in the von Neumann algebra $\mathfrak{M}$, and we have given sufficient conditions for ${}^\diamond\mu$ to enjoy basic reasonable properties. Nevertheless, it is clear that there are still too many possibilities of choosing $\{E_\alpha\}$ for making our approach fully satisfactory. A more restricted choice (possibly, uniqueness) could only be obtained by requiring that the corresponding extension ${}^\diamond\mu$ enjoys a series of properties making it closer and closer to the known extensions of the Lebesgue integral. We hope to discuss this aspect in a further paper.

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