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Review Article

The Partial Inner Product Space Method: A Quick Overview

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Many families of function spaces play a central role in analysis, in particular, in signal processing (e.g., wavelet or Gabor analysis). Typical are L^p spaces, Besov spaces, amalgam spaces, or modulation spaces. In all these cases, the parameter indexing the family measures the behavior (regularity, decay properties) of particular functions or operators. It turns out that all these space families are, or contain, scales or lattices of Banach spaces, which are special cases of *partial inner product spaces* (PIP-*spaces*). In this context, it is often said that such families should be taken as a whole and operators, bases, and frames on them should be defined globally, for the whole family, instead of individual spaces. In this paper, we will give an overview of PIP-spaces and operators on them, illustrating the results by space families of interest in mathematical physics and signal analysis. The interesting fact is that they allow a global definition of operators, and various operator classes on them have been defined.

1. Motivation

In the course of their curriculum, physics and mathematics students are usually taught the basics of Hilbert space, including operators of various types. The justification of this choice is twofold. On the mathematical side, Hilbert space is the example of an infinite-dimensional topological vector space that more closely resembles the familiar Euclidean space and thus it offers the student a smooth introduction into functional analysis. On the physics side, the fact is simply that Hilbert space is the daily language of quantum theory; therefore, mastering it is an essential tool for the quantum physicist.

However, the tool in question is actually insufficient. A pure Hilbert space formulation of quantum mechanics is both inconvenient and foreign to the daily behavior of most physicists, who stick to the more suggestive version of Dirac, although it lacks a rigorous formulation. On the other hand, the interesting solutions of most partial differential equations are seldom smooth or square integrable. Physically meaningful events correspond to changes of regime, which mean discontinuities and/or distributions. Shock waves are a typical example. Actually this state of affairs was recognized long ago by authors like Leray or Sobolev, whence they introduced the notion of *weak solution*. Thus it is no coincidence that many textbooks on PDEs begin with a thorough study of distribution theory [1–4].

All this naturally leads to the introduction of Rigged Hilbert Spaces (RHS) [5]. In a nutshell, a RHS is a triplet:

$$\Phi \hookrightarrow \mathcal{H} \hookrightarrow \Phi^{\times}, \tag{1.1}$$

where $\mathcal{H}$ is a Hilbert space, Φ is a dense subspace of the $\mathcal{H}$, equipped with a locally convex topology, finer than the norm topology inherited from $\mathcal{H}$, and $\Phi^{\times}$ is the space of continuous conjugate linear functionals on Φ, endowed with the strong dual topology. By duality, each space in (1.1) is dense in the next one and all embeddings are linear and continuous. In addition, the space Φ is in general required to be reflexive and nuclear. Standard examples of rigged Hilbert spaces are the Schwartz distribution spaces over $\mathbb{R}$ or $\mathbb{R}^n$, namely $\mathcal{S} \subset L^2 \subset \mathcal{S}^{\times}$ or $\mathfrak{D} \subset L^2 \subset \mathfrak{D}^{\times}$ [5–8].

The problem with the RHS (1.1) is that, besides the Hilbert space vectors, it contains only two types of elements: "very good" functions in Φ and "very bad" ones in $\Phi^{\times}$. If one wants a fine control on the behavior of individual elements, one has to interpolate somehow between the two extreme spaces. In the case of the Schwartz triplet, $\mathcal{S} \subset L^2 \subset \mathcal{S}^{\times}$, a well-known solution is given by a chain of Hilbert spaces, the so-called Hermite representation of tempered distributions [9].

In fact, this is not at all an isolated case. Indeed many function spaces that play a central role in analysis come in the form of families, indexed by one or several parameters that characterize the behavior of functions (smoothness, behavior at infinity, ...). The typical structure is a chain or a scale of Hilbert spaces, or a chain of (reflexive) Banach spaces (a discrete chain of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ is called a *scale* if there exists a self-adjoint operator $B \geqslant 1$ such that $\mathcal{H}_n = D(B^n)$, for all $n \in \mathbb{Z}$, with the graph norm $\|f\|_n = \|B^n f\|$. A similar definition holds for a continuous chain $\{\mathcal{H}_\alpha\}_{\alpha\in\mathbb{R}}$.). Let us give two familiar examples.

(i) First, consider the Lebesgue the Lebesgue L^p spaces on a finite interval, for example, $\mathcal{I} = \{L^p([0,1], dx), 1 \leqslant p \leqslant \infty\}$:

$$L^{\infty} \subset \cdots \subset L^{\overline{q}} \subset L^{\overline{r}} \subset \cdots \subset L^2 \subset \cdots \subset L^r \subset L^q \subset \cdots \subset L^1, \tag{1.2}$$

where $1 < q < r < 2$. Here L^q and $L^{\overline{q}}$ are dual to each other $(1/q + 1/\overline{q} = 1)$, and similarly are $L^r, L^{\overline{r}}$ $(1/r + 1/\overline{r} = 1)$. By the Hölder inequality, the (L^2) inner product

$$\langle f \mid g \rangle = \int_0^1 \overline{f(x)} g(x) dx \tag{1.3}$$

is well defined if $f \in L^q, g \in L^{\overline{q}}$. However, it is *not* well defined for two arbitrary functions $f, g \in L^1$. Take, for instance, $f(x) = g(x) = x^{-1/2} : f \in L^1$, but $fg = f^2 \notin L^1$.

Thus, on L^1, (1.3) defines only a *partial* inner product. The same result holds for any compact subset of $\mathbb{R}$ instead of [0,1].

(ii) As a second example, take the scale of Hilbert spaces built on the powers of a positive self-adjoint operator $A \geqslant 1$ in a Hilbert space $\mathcal{H}_0$. Let $\mathcal{H}_n$ be $D(A^n)$, the domain of A^n, equipped with the graph norm $\|f\|_n = \|A^n f\|,\ f \in D(A^n)$, for $n \in \mathbb{N}$ or $n \in \mathbb{R}^+$, and $\mathcal{H}_{\overline{n}} := \mathcal{H}_{-n} = \mathcal{H}_n^{\times}$ (conjugate dual)

$$\mathfrak{D}^{\infty}(A) := \bigcap_n \mathcal{H}_n \subset \ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_0 \subset \mathcal{H}_{\overline{1}} \subset \mathcal{H}_{\overline{2}} \cdots \subset \mathfrak{D}_{\overline{\infty}}(A) := \bigcup_n \mathcal{H}_n. \tag{1.4}$$

Note that, in the second example (ii), the index n could also be taken as real, the link between the two cases being established by the spectral theorem for self-adjoint operators. Here again the inner product of $\mathcal{H}_0$ extends to each pair $\mathcal{H}_n, \mathcal{H}_{-n}$, but on $\mathfrak{D}_{\overline{\infty}}(A)$ it yields only a partial inner product. The following examples are standard:

(i) $(A_{\mathrm{p}} f)(x) = (1 + x^2) f(x)$ in $L^2(\mathbb{R}, dx)$,

(ii) $(A_{\mathrm{m}} f)(x) = (1 - d^2/dx^2) f(x)$ in $L^2(\mathbb{R}, dx)$,

(iii) $(A_{\mathrm{osc}} f)(x) = (1 + x^2 - d^2/dx^2) f(x)$ in $L^2(\mathbb{R}, dx)$.

(The notation is suggested by the operators of position, momentum and harmonic oscillator energy in quantum mechanics, resp.). Note that both $\mathfrak{D}^{\infty}(A_{\mathrm{p}}) \cap \mathfrak{D}^{\infty}(A_{\mathrm{m}})$ and $\mathfrak{D}^{\infty}(A_{\mathrm{osc}})$ coincide with the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth functions of fast decay, and $\mathfrak{D}_{\overline{\infty}}(A_{\mathrm{osc}})$ with the space $\mathcal{S}^{\times}(\mathbb{R})$ of tempered distributions (considered here as continuous *conjugate linear* functionals on $\mathcal{S}$). As for the operator A_{m}, it generates the scale of Sobolev spaces $H^s(\mathbb{R}),\ s \in \mathbb{Z}$ or $\mathbb{R}$.

However, a moment's reflection shows that the total-order relation inherent in a chain is in fact an unnecessary restriction; partially ordered structures are sufficient, and indeed necessary in practice. For instance, in order to get a better control on the behavior of individual functions, one may consider the lattice built on the powers of A_{p} and A_{m} simultaneously. Then the extreme spaces are still $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}^{\times}(\mathbb{R})$. Similarly, in the case of several variables, controlling the behavior of a function in each variable separately requires a nonordered set of spaces. This is in fact a statement about tensor products (remember that $L^2(X \times Y) \simeq L^2(X) \otimes L^2(Y)$). Indeed the tensor product of two chains of Hilbert spaces, $\{\mathcal{H}_n\} \otimes \{\mathcal{K}_m\}$, is naturally a lattice $\{\mathcal{H}_n \otimes \mathcal{K}_m\}$ of Hilbert spaces. For instance, in the example above, for two variables x, y, that would mean considering intermediate Hilbert spaces corresponding to the product of two operators, $(A_{\mathrm{m}}(x))^n (A_{\mathrm{m}}(y))^m$.

Thus the structure to analyze is that of *lattices of Hilbert or Banach spaces*, interpolating between the extreme spaces of an RHS, as in (1.1). Many examples can be given, for instance, the lattice generated by the spaces $L^p(\mathbb{R}, dx)$, the amalgam spaces $W(L^p, \ell^q)$, the mixed-norm spaces $L^{p,q}_m(\mathbb{R}, dx)$, and many more. In all these cases, which contain most families of function spaces of interest in analysis and in signal processing, a common structure emerges for the "large" space V, defined as the union of all individual spaces. There is a lattice of Hilbert or reflexive Banach spaces V_r, with an (order-reversing) involution $V_r \leftrightarrow V_{\overline{r}}$, where $V_{\overline{r}} = V_r^{\times}$ (the space of continuous conjugate linear functionals on V_r), a central Hilbert space $V_o \simeq V_{\overline{o}}$, and a partial inner product on V that extends the inner product of V_o to pairs of dual spaces $V_r,\ V_{\overline{r}}$.

Moreover, many operators should be considered globally, for the whole scale or lattice, instead of on individual spaces. In the case of the spaces $L^p(\mathbb{R})$, such are, for instance, operators implementing translations ($x \mapsto x - y$) or dilations ($x \mapsto x/a$), convolution operators, Fourier transform, and so forth. In the same spirit, it is often useful to have a *common* basis for the whole family of spaces, such as the Haar basis for the spaces $L^p(\mathbb{R})$, $1 < p < \infty$. Thus we need a notion of operator and basis defined globally for the scale or lattice itself.

This state of affairs prompted A. Grossmann and one of us (the first author) to systematize this approach, and this led to the concept of *partial inner product space* or PIP-space [10–13]. After many years and various developments, we devoted a full monograph [14] to a detailed survey of the theory. The aim of this paper is to present the formalism of PIP-spaces, which indeed answers these questions. In a first part, the structure of PIP-space is derived systematically from the abstract notion of compatibility and then particularized to the examples listed above. In a second part, operators on PIP-spaces are introduced and illustrated by several operators commonly used in Gabor or wavelet analysis. Finally we describe a number of applications of PIP-spaces in mathematical physics and in signal processing. Of course, the treatment is sketchy, for lack of space. For a complete information, we refer the reader to our monograph [14].

2. Partial Inner Product Spaces

2.1. Basic Definitions

The basic question is how to generate PIP-spaces in a systematic fashion. In order to answer, we may reformulate it as follows: given a vector space V and two vectors $f, g \in V$, when does their inner product make sense? A way of formalizing the answer is given by the idea of *compatibility*.

Definition 2.1. A *linear compatibility relation* on a vector space V is a symmetric binary relation $f\#g$ which preserves linearity:

$$\begin{gathered} f\#g \Longleftrightarrow g\#f, \quad \forall f, g \in V, \\ f\#g,\, f\#h \Longrightarrow f\#(\alpha g + \beta h), \quad \forall f, g, h \in V,\ \forall \alpha, \beta \in \mathbb{C}. \end{gathered} \tag{2.1}$$

As a consequence, for every subset $S \subset V$, the set $S^\# = \{g \in V : g\#f, \text{for all } f \in S\}$ is a vector subspace of V and one has

$$S^{\#\#} = \left(S^\#\right)^\# \supseteq S, \qquad S^{\#\#\#} = S^\#. \tag{2.2}$$

Thus one gets the following equivalences:

$$\begin{aligned} f\#g &\Longleftrightarrow f \in \{g\}^\# \Longleftrightarrow \{f\}^{\#\#} \subseteq \{g\}^\# \\ &\Longleftrightarrow g \in \{f\}^\# \Longleftrightarrow \{g\}^{\#\#} \subseteq \{f\}^\#. \end{aligned} \tag{2.3}$$

From now on, we will call *assaying subspace* of V a subspace S such that $S^{\#\#} = S$ and denote by $\mathcal{F}(V, \#)$ the family of all assaying subsets of V, ordered by inclusion. Let F be the isomorphy class of $\mathcal{F}$, that is, $\mathcal{F}$ is considered as an abstract partially ordered set. Elements of F will be denoted by $r, q, \ldots$, and the corresponding assaying subsets by $V_r, V_q, \ldots$. By definition, $q \leqslant r$ if and only if $V_q \subseteqq V_r$. We also write $V_{\overline{r}} = V_r^{\#},\ r \in F$. Thus the relations (2.3) mean that $f\#g$ if and only if there is an index $r \in F$ such that $f \in V_r,\ g \in V_{\overline{r}}$. In other words, vectors should not be considered individually, but only in terms of assaying subspaces, which are the building blocks of the whole structure.

It is easy to see that the map $S \mapsto S^{\#\#}$ is a closure, in the sense of universal algebra, so that the assaying subspaces are precisely the "closed" subsets. Therefore one has the following standard result.

Theorem 2.2. *The family $\mathcal{F}(V,\#) \equiv \{V_r, r \in F\}$, ordered by inclusion, is a complete involutive lattice, that is, it is stable under the following operations, arbitrarily iterated:*

(i) *involution:* $V_r \leftrightarrow V_{\overline{r}} = (V_r)^{\#}$,

(ii) *infimum:* $V_{p\wedge q} \equiv V_p \wedge V_q = V_p \cap V_q,\ (p, q, r \in F)$,

(iii) *supremum:* $V_{p\vee q} \equiv V_p \vee V_q = (V_p + V_q)^{\#\#}$.

The smallest element of $\mathcal{F}(V,\#)$ is $V^{\#} = \bigcap_r V_r$ and the greatest element is $V = \bigcup_r V_r$. By definition, the index set F is also a complete involutive lattice; for instance,

$$\left(V_{p\wedge q}\right)^{\#} = V_{\overline{p\wedge q}} = V_{\overline{p}\vee\overline{q}} = V_{\overline{p}} \vee V_{\overline{q}}. \tag{2.4}$$

Definition 2.3. A *partial inner product* on $(V,\#)$ is a Hermitian form $\langle \cdot \mid \cdot \rangle$ defined exactly on compatible pairs of vectors. A *partial inner product space* (PIP-space) is a vector space V equipped with a linear compatibility and a partial inner product.

Note that the partial inner product is not required to be positive definite.

The partial inner product clearly defines a notion of *orthogonality:* $f \perp g$ if and only if $f\#g$ and $\langle f \mid g \rangle = 0$.

Definition 2.4. The PIP-space $(V, \#, \langle \cdot \mid \cdot \rangle)$ is *nondegenerate* if $(V^{\#})^{\perp} = \{0\}$, that is, if $\langle f \mid g \rangle = 0$ for all $f \in V^{\#}$ implies that $g = 0$.

We will assume henceforth that our PIP-space $(V, \#, \langle \cdot \mid \cdot \rangle)$ is nondegenerate. As a consequence, $(V^{\#}, V)$ and every couple $(V_r, V_{\overline{r}})$, $r \in F$, are dual pairs in the sense of topological vector spaces [15]. We also assume that the partial inner product is positive definite.

Now one wants the topological structure to match the algebraic structure, in particular, the topology τ_r on V_r should be such that its conjugate dual be $V_{\overline{r}}$: $(V_r[\tau_r])^{\times} = V_{\overline{r}},$ for all $r \in F$. This implies that the topology τ_r must be finer than the weak topology $\sigma(V_r, V_{\overline{r}})$ and coarser than the Mackey topology $\tau(V_r, V_{\overline{r}})$:

$$\sigma(V_r, V_{\overline{r}}) \preceq \tau_r \preceq \tau(V_r, V_{\overline{r}}). \tag{2.5}$$

From here on, we will assume that every V_r carries its Mackey topology $\tau(V_r, V_{\overline{r}})$. This choice has two interesting consequences. First, if $V_r[\tau_r]$ is a Hilbert space or a reflexive Banach space, then $\tau(V_r, V_{\overline{r}})$ coincides with the norm topology. Next, $r < s$ implies that $V_r \subset V_s$, and the embedding operator $E_{sr} : V_r \rightarrow V_s$ is continuous and has dense range. In particular, $V^{\#}$ is dense in every V_r.

2.2. Examples

2.2.1. Sequence Spaces

Let V be the space ω of *all* complex sequences $x = (x_n)$ and define on it (i) a compatibility relation by $x\#y \Leftrightarrow \sum_{n=1}^{\infty} |x_n y_n| < \infty$ and (ii) a partial inner product $\langle x \mid y \rangle = \sum_{n=1}^{\infty} \overline{x_n} y_n$.

Then $\omega^{\#} = \varphi$, the space of finite sequences, and the complete lattice $\mathcal{F}(\omega, \#)$ consists of Köthe's perfect sequence spaces [15, § 30]. Among these, typical assaying subspaces are the weighted Hilbert spaces

$$\ell^2(r) = \left\{ (x_n) : \sum_{n=1}^{\infty} |x_n|^2 r_n^{-2} < \infty \right\}, \tag{2.6}$$

where $r = (r_n), r_n > 0$, is a sequence of positive numbers. The involution is $\ell^2(r) \leftrightarrow \ell^2(\overline{r}) = \ell^2(r)^{\times}$, where $\overline{r}_n = 1/r_n$. In addition, there is a central, self-dual Hilbert space, namely, $\ell^2(1) = \ell^2(\overline{1}) = \ell^2$, where $1 = (1)$.

2.2.2. Spaces of Locally Integrable Functions

Let now V be $L^1_{\text{loc}}(\mathbb{R}, dx)$, the space of Lebesgue measurable functions, integrable over compact subsets, and define a compatibility relation on it by $f\#g \Leftrightarrow \int_{\mathbb{R}} |f(x)g(x)| dx < \infty$ and a partial inner product $\langle f \mid g \rangle = \int_{\mathbb{R}} \overline{f(x)} g(x) dx$.

Then $V^{\#} = L^{\infty}_c(\mathbb{R})$, the space of bounded measurable functions of compact support. The complete lattice $\mathcal{F}(L^1_{\text{loc}}, \#)$ consists of Köthe function spaces [16, 17]. Here again, typical assaying subspaces are weighted Hilbert spaces

$$L^2(r) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}, dx) : \int_{\mathbb{R}} |f(x)|^2 r(x)^{-2} dx < \infty \right\}, \tag{2.7}$$

with $r, r^{-1} \in L^2_{\text{loc}}(\mathbb{R}, dx),\ r(x) > 0$ a.e. The involution is $L^2(r) \leftrightarrow L^2(\overline{r})$, with $\overline{r} = r^{-1}$, and the central, self-dual Hilbert space is $L^2(\mathbb{R}, dx)$.

2.2.3. Nested Hilbert Spaces

This is the original construction of Grossmann [18] for finding an "easy" substitute to distributions, and actually one of the motivations for introducing PIP-spaces. And indeed the two are closely related; see [14, Section 2.4.1]

2.2.4. Rigged Hilbert Spaces

This is the simplest example of PIP-space, but it is a rather poor one. Indeed, in the RHS (1.1), two elements are compatible if both belong to $\mathscr{H}$, or one of them belongs to Φ. Thus the three defining spaces are the only assaying subspaces. The partial inner product is, of course, simply that of $\mathscr{H}$, provided the sesquilinear form that puts Φ and $\Phi^\times$ in duality has been correctly normalized.

3. Lattices of Hilbert or Banach Spaces

From the previous examples, we learn that $\mathcal{F}(V,\#)$ is a huge lattice (it is complete!) and that assaying subspaces may be complicated, such as Fréchet spaces, nonmetrizable spaces, and so forth. This situation suggests to choose an involutive sublattice $\mathcal{I} \subset \mathcal{F}$, indexed by I, such that

(i) $\mathcal{I}$ is *generating*:

$$f\#g \Longleftrightarrow \exists\, r \in I \text{ such that } f \in V_r,\ g \in V_{\overline{r}}, \tag{3.1}$$

(ii) every $V_r,\ r \in I$, is a Hilbert space or a reflexive Banach space,

(iii) there is a unique self-dual assaying subspace $V_o = V_{\overline{o}}$, which is a Hilbert space.

In that case, the structure $V_I := (V, \mathcal{I}, \langle \cdot \mid \cdot \rangle)$ is called, respectively, a *lattice of Hilbert spaces* (LHS) or a *lattice of Banach spaces* (LBS). Both types are particular cases of the so-called indexed PIP-spaces [14]. Note that $V^\#$, V themselves usually do *not* belong to the family $\{V_r, r \in I\}$, but they can be recovered as

$$V^\# = \bigcap_{r\in I} V_r, \quad V = \sum_{r\in I} V_r. \tag{3.2}$$

In the LBS case, the lattice structure takes the following forms:

(i) $V_{p\wedge q} = V_p \cap V_q$, with the *projective* norm

$$\|f\|_{p\wedge q} = \|f\|_p + \|f\|_q, \tag{3.3}$$

(ii) $V_{p\vee q} = V_p + V_q$, with the *inductive* norm

$$\|f\|_{p\vee q} = \inf_{f=g+h}\left(\|g\|_p + \|h\|_q\right), \quad g \in V_p,\ h \in V_q. \tag{3.4}$$

These norms are usual in interpolation theory [19]. In the LHS case, one takes similar definitions with squared norms, in order to get Hilbert norms throughout.

In the rest of this section, we will list a series of concrete examples of LHS/LBSs. Some more examples, which are of particular interest in signal processing, will be given in Section 6.2. For simplicity, we will restrict ourselves to one dimension, although most spaces may be defined on $\mathbb{R}^n,\ n > 1$, as well.

3.1. Chains of Hilbert or Banach Spaces

Typical are the two examples described in Section 1.

(1) The chain of Lebesgue spaces on a finite interval $\mathcal{I} = \{L^p([0,1], dx),\ 1 < p < \infty\}$. The chain (1.2) is a (totally ordered) lattice. The corresponding lattice completion is obtained by adding "nonstandard" spaces such as

$$L^{p-} = \bigcap_{1<q<p} L^q \quad \text{(non-normable Fréchet)}, \qquad L^{p+} = \bigcup_{p<q<\infty} L^q \quad \text{(nonmetrizable)}. \tag{3.5}$$

(2) The scale (1.4) of Hilbert spaces $\{\mathcal{H}_n, n \in \mathbb{Z}\}$ built on powers of $A = A^* \geqslant 1$. The lattice completion is similar to the previous one, introducing analogous "nonstandard" spaces [14, Section 5.1].

3.2. Sequence Spaces

3.2.1. A LHS of Weighted ℓ^2 Spaces

In ω, with the compatibility # and the partial inner product defined in Section 2.2.1, we may take the lattice $\mathcal{I} = \{\ell^2(r)\}$ of the weighted Hilbert spaces defined in (2.6), with lattice operations:

(i) infimum: $\ell^2(p \wedge q) = \ell^2(p) \wedge \ell^2(q) = \ell^2(r),\ r_n = \min(p_n, q_n)$,

(ii) supremum: $\ell^2(p \vee q) = \ell^2(p) \vee \ell^2(q) = \ell^2(s),\ s_n = \max(p_n, q_n)$,

(iii) duality: $\ell^2(p \wedge q) \leftrightarrow \ell^2(\overline{p} \vee \overline{q}), \ell^2(p \vee q) \leftrightarrow \ell^2(\overline{p} \wedge \overline{q})$.

As a matter of fact, the norms above are equivalent to the projective and inductive norms, respectively. Then, it is easy to show that the lattice $\mathcal{I} = \{\ell^2(r)\}$ is generating in $\mathcal{F}(\omega, \#)$.

3.2.2. Köthe Perfect Sequence Spaces

We have already noticed that the complete lattice $\mathcal{F}(\omega, \#)$ consists precisely of all Köthe perfect sequence spaces. Indeed, these are defined as the assaying subspaces corresponding to the compatibility #, which is called *α-duality* [15]. Among these, there is an interesting class, the so-called ℓ_ϕ spaces associated to symmetric norming functions.

Definition 3.1. A real-valued function ϕ defined on the space φ of finite sequences is said to be a *norming function* if

(n_1) $\phi(x) > 0$ for every sequence $x \in \varphi$, $x \neq 0$,

(n_2) $\phi(\alpha x) = |\alpha|\phi(x)$, for all $x \in \varphi$, for all $\alpha \in \mathbb{C}$,

(n_3) $\phi(x + y) \leqslant \phi(x) + \phi(y)$, for all $x, y \in \varphi$,

(n_4) $\phi(1, 0, 0, 0, \ldots) = 1$.

A norming function ϕ is *symmetric* if

(n_5) $\phi(x_1, x_2, \ldots, x_n, 0, 0, \ldots) = \phi(|x_{j_1}|, |x_{j_2}|, \ldots, |x_{j_n}|, 0, 0, \ldots)$,

where $j_1, j_2, \ldots, j_n$ is an arbitrary permutation of $1, 2, \ldots, n$.

From property (n_5), it is clear that a symmetric norming function ϕ is entirely determined by its values on the set $[\varphi]$ of finite, positive, nonincreasing sequences. Hence, from conditions (n_2) and (n_4), we deduce that

$$\phi_\infty(x) \leqslant \phi(x) \leqslant \phi_1(x), \quad \forall\, x \in \varphi, \tag{3.6}$$

where $\phi_\infty(x) = \max_{i=1,\ldots,n}|x_i|$ and $\phi_1(x) = \sum_{i=1}^n |x_i|$.

To every symmetric norming function ϕ, one can associate a Banach space ℓ_ϕ as follows. Given a sequence $x \in \omega$, define its nth section as $x^{(n)} = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. Then the sequence $(\phi(x^{(n)}))$ is nondecreasing, so that one can define

$$\ell_\phi = \left\{ x \in \omega : \sup_n \phi\left(x^{(n)}\right) < \infty \right\} \tag{3.7}$$

and then extend the norming function ϕ to the whole of ℓ_ϕ by putting $\phi(x) = \lim_n \phi(x^{(n)})$. This relation defines a norm ϕ on ℓ_ϕ, for which it is complete, hence, a Banach space. In other words, we can also say that $\ell_\phi = \{x \in \omega : \phi(x) < \infty\}$ is the natural domain of definition of the extended norming function ϕ. Clearly, one has $\ell_{\phi_\infty} = \ell^\infty$ and $\ell_{\phi_1} = \ell^1$. Similarly, $\ell^p = \ell_{\phi_p}$, where $\phi_p(x) = \left(\sum_n |x_n|^p\right)^{1/p}$. Thus every space ℓ_ϕ contains ℓ^1 and is contained in ℓ^∞.

In addition, the set of Banach spaces ℓ_ϕ constitutes a lattice. Given two symmetric norming functions ϕ and ψ, one defines their infimum and supremum, exactly as for the general case:

(i) $\phi \wedge \psi := \max\{\phi, \psi\}$, which defines on the space $\ell_{\phi\wedge\psi} := \ell_\phi \cap \ell_\psi$ a norm equivalent to $\phi(x) + \psi(x)$,

(ii) $\phi \vee \psi := \min\{\phi, \psi\}$, which defines on the space $\ell_{\phi\vee\psi} := \ell_\phi + \ell_\psi$ a norm equivalent to $\inf_{x=y+z}\{\phi(y) + \psi(z)\},\ x \in \ell_\phi + \ell_\psi,\ y \in \ell_\phi,\ z \in \ell_\psi$.

It remains to analyze the relationship of the spaces ℓ_ϕ with the PIP-space structure of ω. Define, for any finite, positive, nonincreasing sequence $y \in [\varphi]$,

$$\overline{\phi}(y) := \max_{x\in[\varphi]} \frac{\langle x \mid y\rangle}{\phi(x)}. \tag{3.8}$$

The function $\overline{\phi}$ thus defined is a symmetric norming function; hence, it can be extended to the corresponding Banach space $\ell_{\overline{\phi}}$. The function $\overline{\phi}$ is said to be *conjugate* to ϕ and the space $\ell_{\overline{\phi}}$ is the conjugate dual of ℓ_ϕ with respect to the partial inner product, that is, $\ell_{\overline{\phi}} = (\ell_\phi)^{\#}$. Clearly one has $\overline{\overline{\phi}} = \phi$; hence, $\ell_{\overline{\overline{\phi}}} = (\ell_\phi)^{\#\#} = \ell_\phi$.

In addition, it is easy to show that $\ell_{\overline{\phi\wedge\psi}} = \ell_{\overline{\phi}\vee\overline{\psi}}$ and $\ell_{\overline{\phi\vee\psi}} = \ell_{\overline{\phi}\wedge\overline{\psi}}$. In other words, one gets the following result.

Proposition 3.2. *The family of Banach spaces ℓ_ϕ, where ϕ is a symmetric norming function, is an involutive sublattice of the lattice $\mathcal{F}(\omega,\#)$ and a LBS.*

Actually, since every ϕ satisfies the inclusions $\ell^1 \subset \ell_\phi \subset \ell^\infty$, the family $\{\ell_\phi\}$ is also an involutive sublattice of the lattice $\mathcal{F}(\ell^\infty,\#)$ obtained by restricting to ℓ^∞ the PIP-space structure of ω.

These spaces $\{\ell_\phi\}$ may be generalized further to what is called the theory of Banach ideals of sequences. See [14, Section 4.3] for more details.

3.3. *Spaces of Locally Integrable Functions*

3.3.1. A LHS of Weighted L^2 Spaces

In $L^1_{\rm loc}(\mathbb{R},dx)$, we may take the lattice $\mathcal{I} = \{L^2(r)\}$ of the weighted Hilbert spaces defined in (2.7), with

(i) infimum: $L^2(p\wedge q) = L^2(p)\wedge L^2(q) = L^2(r),\ r(x) = \min(p(x),q(x))$,

(ii) supremum: $L^2(p\vee q) = L^2(p)\vee L^2(q) = L^2(s),\ s(x) = \max(p(x),q(x))$,

(iii) duality: $L^2(p\wedge q) \leftrightarrow L^2(\overline{p}\vee\overline{q}),\ L^2(p\vee q) \leftrightarrow L^2(\overline{p}\wedge\overline{q})$.

Here too, these norms are equivalent to the projective and inductive norms, respectively.

3.3.2. The Spaces $L^p(\mathbb{R},dx),\ 1<p<\infty$

The spaces $L^p(\mathbb{R},dx),\ 1<p<\infty$ do not constitute a scale, since one has only the inclusions $L^p\cap L^q \subset L^s, p<s<q$. Thus one has to consider the lattice they generate, with the following lattice operations:

(i) $L^p\wedge L^q = L^p\cap L^q$, with projective norm,

(ii) $L^p\vee L^q = L^p + L^q$, with inductive norm.

For $1<p,q<\infty$, both spaces $L^p\wedge L^q$ and $L^p\vee L^q$ are reflexive Banach spaces and their conjugate duals are, respectively, $(L^p\wedge L^q)^\times = L^{\overline{p}}\vee L^{\overline{q}}$ and $(L^p\vee L^q)^\times = L^{\overline{p}}\wedge L^{\overline{q}}$.

It is convenient to introduce the following unified notation:

$$L^{(p,q)} = \begin{cases} L^p\wedge L^q, & \text{if } p \geqslant q, \\ L^p\vee L^q, & \text{if } p\leqslant q. \end{cases} \tag{3.9}$$

Then, for $1<p,q<\infty$, $L^{(p,q)}$ is a reflexive Banach space, with conjugate dual $L^{(\overline{p},\overline{q})}$.

Next, if we represent (p,q) by the point of coordinates $(1/p,1/q)$, we may associate all the spaces $L^{(p,q)}$ $(1\leqslant p,q\leqslant\infty)$ in a one-to-one fashion with the points of a unit square $\mathrm{J} = [0,1]\times[0,1]$ (see Figure 1). Thus, in this picture, the spaces L^p are on the main diagonal, intersections $L^p\cap L^q$ above it and sums L^p+L^q below.

The space $L^{(p,q)}$ is contained in $L^{(p',q')}$ if (p,q) is on the left and/or above (p',q'). Thus the smallest space is

$$V_{\mathrm{J}}^{\#} = L^{(\infty,1)} = L^\infty\cap L^1 \tag{3.10}$$

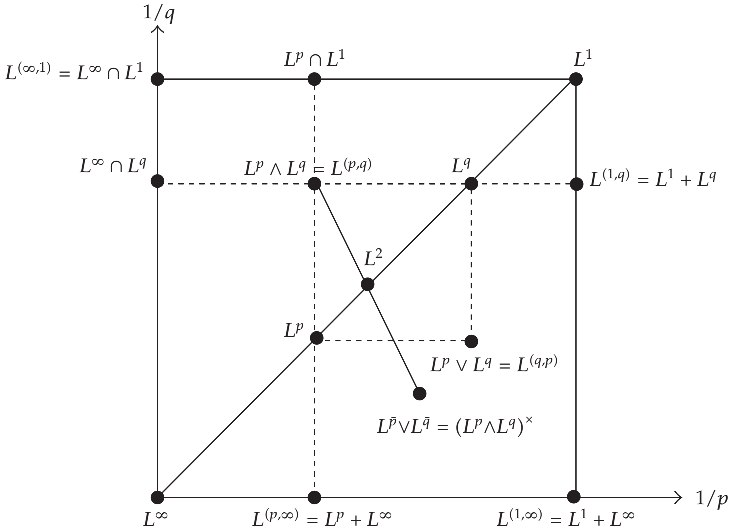

Figure 1: The unit square describing the lattice J.

and it corresponds to the upper-left corner, while the largest one is

$$V_{\mathrm{J}} = L^{(1,\infty)} = L^1 + L^\infty, \tag{3.11}$$

corresponding to the lower-right corner. Inside the square, duality corresponds to (geometrical) symmetry with respect to the center (1/2,1/2) of the square, which represents the space L^2. The ordering of the spaces corresponds to the following rule:

$$L^{(p,q)} \subset L^{(p',q')} \Longleftrightarrow (p,q) \leqslant (p',q') \Longleftrightarrow p \geqslant p', \quad q \leqslant q'. \tag{3.12}$$

With respect to this ordering, J is an involutive lattice with the operations

$$\begin{aligned}
(p,q) \wedge (p',q') &= (p \vee p', q \wedge q'),\\
(p,q) \vee (p',q') &= (p \wedge p', q \vee q'),\\
\overline{(p,q)} &= (\overline{p}, \overline{q}),
\end{aligned} \tag{3.13}$$

where $p \wedge p' = \min\{p,p'\}$, $p \vee p' = \max\{p,p'\}$. It is remarkable that the lattice $\mathcal{J}$ generated by $\mathcal{I} = \{L^p\}$ is obtained at the first "generation". One has, for instance, $L^{(r,s)} \wedge L^{(a,b)} = L^{(r\vee a, s\wedge b)}$, both as sets and as topological vector spaces.

3.3.3. *Mixed-Norm Lebesgue Spaces* $L_m^{p,q}$

An interesting class of function spaces, close relatives to the Lebesgue L^p spaces, consists of the so-called L^P spaces with mixed norm. Let (X,μ) and (Y,ν) be two σ-finite measure spaces and $1 \leqslant p,q \leqslant \infty$ (in the general case, one considers n such spaces and n-tuples

$P := (p_1, p_2, \ldots, p_n)$). Then, a function $f(x, y)$ measurable on the product space $X \times Y$ is said to belong to $L^{(p,q)}(X \times Y)$ if the number obtained by taking successively the p-norm in x and the q-norm in y, in that order, is finite (exchanging the order of the two norms leads in general to a different space). If $p, q < \infty$, the norm reads

$$\|f\|_{(p,q)} = \left(\int_Y \left(\int_X |f(x,y)|^p d\mu(x) \right)^{q/p} d\nu(y) \right)^{1/q}. \tag{3.14}$$

The analogous norm for p or $q = \infty$ is obvious. For $p = q$, one gets the usual space $L^p(X \times Y)$.

These spaces enjoy a number of properties similar to those of the L^p spaces: (i) each space $L^{(p,q)}$ is a Banach space and it is reflexive if and only if $1 < p, q < \infty$; (ii) the conjugate dual of $L^{(p,q)}$ is $L^{(\overline{p},\overline{q})}$, where, as usual, $p^{-1} + \overline{p}^{-1} = 1$, $q^{-1} + \overline{q}^{-1} = 1$; thus the topological conjugate dual coincides with the Köthe dual; (iii) the mixed-norm spaces satisfy a generalized Hölder inequality and have nice interpolation properties.

The case $X = Y = \mathbb{R}^d$ with Lebesgue measure is the important one for signal processing [20, Section 11.1]. More generally, one can add a weight function m and obtain the spaces $L_m^{p,q}(\mathbb{R}^d)$ (we switch to a notation more suitable for the applications):

$$\|f\|_m^{p,q} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x,\omega)|^p \; m(x,\omega)^p dx \right)^{q/p} d\omega \right)^{1/q}. \tag{3.15}$$

Here the weight function m is a nonnegative locally integrable function on $\mathbb{R}^{2d}$, assumed to be v-moderate, that is, $m(z_1 + z_2) \leqslant v(z_1) m(z_2)$, for all $z_1, z_2 \in \mathbb{R}^{2d}$, with v a submultiplicative weight function, that is, $v(z_1 + z_2) \leqslant v(z_1) v(z_2)$, for all $z_1, z_2 \in \mathbb{R}^{2d}$. The typical weights are of polynomial growth: $v_s(z) = (1 + |z|)^s, \; s \geqslant 0$.

The space $L_m^{p,q}(\mathbb{R}^{2d})$ is a Banach space for the norm $\|\cdot\|_m^{p,q}$. The duality property is, as expected, $(L_m^{p,q})^\times = L_{1/m}^{\overline{p},\overline{q}}$. Of course, things simplify when $p = q$: $L_m^{p,p}(\mathbb{R}^{2d}) = L_m^p(\mathbb{R}^{2d})$, a weighted L^p space.

Concerning lattice properties of the family of $L_m^{p,q}$ spaces, we cannot expect more than for the L^p spaces. Two $L_m^{p,q}$ spaces are never comparable, even for the same weight m, so one has to take the lattice generated by intersection and duality.

A different type of mixed-norm spaces is obtained if one takes $X = Y = \mathbb{Z}^d$, with the counting measure. Thus one gets the space $\ell_m^{p,q}(\mathbb{Z}^{2d})$, which consists of all sequences $a = (a_{kn}), \; k, n \in \mathbb{Z}^d$, for which the following norm is finite:

$$\|a\|_{\ell_m^{p,q}} := \left(\sum_{n \in \mathbb{Z}^d} \left(\sum_{k \in \mathbb{Z}^d} |a_{kn}|^p \; m(k,n)^p \right)^{q/p} \right)^{1/q}. \tag{3.16}$$

Contrary to the continuous case, here we do have inclusion relations: if $p_1 \leqslant p_2, q_1 \leqslant q_2$ and $m_2 \leqslant C m_1$, then $\ell_{m_1}^{p_1,q_1} \subseteq \ell_{m_2}^{p_2,q_2}$.

Discrete mixed-norm spaces have been used extensively in functional analysis and signal processing. For instance, they are key to the proof that certain operators are bounded between two given function spaces, such as modulation spaces (see below) or ℓ^p spaces. In general, a mixed-norm space will prove useful whenever one has a signal consisting of sequences labeled by two indices that play different roles. An obvious example is time-frequency or time-scale analysis: a Gabor or wavelet basis (or frame) is written as $\{\psi_{j,k}, j,k \in \mathbb{Z}\}$, where j indexes the scale or frequency and k the time. More generally, this applies whenever signals are expanded with respect to a dictionary with two indices.

3.3.4. Köthe Function Spaces

The mixed-norm Lebesgue spaces $L_m^{p,q}$ are special cases of a very general class, the so-called *Köthe function spaces*. These have been introduced (and given that name) by Dieudonné [16] and further studied by Luxemburg-Zaanen [21]. The procedure here is entirely parallel to that used in Section 3.2.2 above for introducing the sequence spaces ℓ_ϕ.

Let (X,μ) be a σ-finite measure space and M^+ the set of all measurable, non-negative functions on X, where two functions are identified if they differ at most on a μ-null set. A *function norm* is a mapping $\rho : M^+ \to \overline{\mathbb{R}}$ such that

(i) $0 \leqslant \rho(f) \leqslant \infty$, for all $f \in M^+$ and $\rho(f) = 0$ if and only if $f = 0$,

(ii) $\rho(f_1 + f_2) \leqslant \rho(f_1) + \rho(f_2)$, for all $f_1, f_2 \in M^+$,

(iii) $\rho(af) = a\rho(f)$, for all $f \in M^+$, for all $a \geqslant 0$,

(iv) $f_1 \leqslant f_2 \Rightarrow \rho(f_1) \leqslant \rho(f_2)$, for all $f_1, f_2 \in M^+$.

A function norm ρ is said to have the *Fatou property* if and only if $0 \leqslant f_1 \leqslant f_2 \leqslant \dots,\ f_n \in M^+$ and $f_n \to f$ pointwise implies that $\rho(f_n) \to \rho(f)$.

Given a function norm ρ, it can be extended to all complex measurable functions on X by defining $\rho(f) = \rho(|f|)$. Denote by L_ρ the set of all measurable f such that $\rho(f) < \infty$. With the norm $\|f\| = \rho(f), L_\rho$ is a normed space and a subspace of the vector space V of all measurable μ-a.e. finite, functions on X. Furthermore, if ρ has the Fatou property, L_ρ is complete, that is, a Banach space.

A function norm ρ is said to be *saturated* if, for any measurable set $E \subset X$ of positive measure, there exists a measurable subset $F \subset E$ such that $\mu(F) > 0$ and $\rho(\chi_F) < \infty$ (χ_F is the characteristic function of F).

Let ρ be a saturated function norm with the Fatou property. Define

$$\rho'(f) = \sup\left\{ \int_X |fg| d\mu : \rho(g) \leqslant 1 \right\}. \tag{3.17}$$

Then ρ' is a saturated function norm with the Fatou property and $\rho'' \equiv (\rho')' = \rho$. Hence, $L_{\rho'}$ is a Banach space. Moreover, one has also

$$\rho'(f) = \sup\left\{ \left| \int_X fg\, d\mu \right| : \rho(g) \leqslant 1 \right\}. \tag{3.18}$$

For each ρ as above, L_ρ is a Banach space and $L_{\rho'} = (L_\rho)^\#$, that is, each L_ρ is assaying. The pair $\langle L_\rho, L_{\rho'} \rangle$ is actually a dual pair, although $\langle V^\#, V \rangle$ is not. The space $L_{\rho'}$ is called the *Köthe dual* or *α-dual* of L_ρ and denoted by $(L_\rho)^\alpha$.

However, $L_{\rho'}$ is in general only a closed subspace of the Banach conjugate dual $(L_\rho)^\times$; thus, the Mackey topology $\tau(L_\rho, L_{\rho'})$ is coarser than the ρ-norm topology, which is

$\tau(L_\rho, (L_\rho)^\times)$. This defect can be remedied by further restricting ρ. A function norm ρ is called *absolutely continuous* if $\rho(f_n) \searrow 0$ for every sequence $f_n \in L_\rho$ such that $f_1 \geqslant f_2 \geqslant \ldots \searrow 0$ pointwise a.e. on X. For instance, the Lebesgue L^p-norm is absolutely continuous for $1 \leqslant p < \infty$, but the L^∞-norm is *not*! Also, even if ρ is absolutely continuous, ρ' need not be. Yet, this is the appropriate concept, in view of the following results:

(i) $L_{\rho'} = (L_\rho)^\alpha = (L_\rho)^\times$ if and only if ρ is absolutely continuous;

(ii) L_ρ is reflexive if and only if ρ *and* ρ' are absolutely continuous and ρ has the Fatou property.

Let ρ be a saturated, absolutely continuous function norm on X, with the Fatou property and such that ρ' is also absolutely continuous. Then $\langle L_\rho, L_{\rho'} \rangle$ is a reflexive dual pair of Banach spaces. In addition, the set J of all function norms with these properties is an involutive lattice with respect to the following partial order: $\rho_1 \leqslant \rho_2$ if and only if $\rho_1(f) \leqslant \rho_2(f)$, for every measurable f. The lattice operations are the following:

(i) $(\rho_1 \vee \rho_2)(f) = \max\{\rho_1(f), \rho_2(f)\}$,

(ii) $(\rho_1 \wedge \rho_2)(f) = \inf\{\rho_1(f_1) + \rho_2(f_2); f_1, f_2 \in M^+, f_1 + f_2 = |f|\}$,

(iii) involution : $\rho \leftrightarrow \rho'$.

For the corresponding Banach spaces, we have the relations

$$L_{(\rho_1 \vee \rho_2)} = \left(L_{\rho_1} \cap L_{\rho_2}\right)_{\text{proj}}, \quad L_{(\rho_1 \wedge \rho_2)} = \left(L_{\rho_1} + L_{\rho_2}\right)_{\text{ind}}. \tag{3.19}$$

Consider now the usual space $V = L^1_{\text{loc}}(X, d\mu)$, with the compatibility and partial inner product defined in Section 2.2.2, so that $V^\# = L^\infty_c(X, d\mu)$. Then the construction outlined above provides $L^1_{\text{loc}}(X, d\mu)$ with the structure of a LBS. Indeed, one has the following result.

Proposition 3.3. *Let J be the set of saturated, absolutely continuous function norms ρ on X, with the Fatou property and such that ρ' is also absolutely continuous. Let $\mathcal{J}$ denote the set $\mathcal{J} := \{L_\rho : \rho \in J \text{ and } L_\rho \subset L^1_{\text{loc}}\}$. Then $\mathcal{J}$ is a LBS, with the lattice operations defined above.*

More general situations may be considered, for which we refer to [14, Section 4.4].

4. Comparing PIP-Spaces

The definition of LBS/LHS given in Section 3 leads to the proper notion of comparison between two linear compatibilities on the same vector space. Namely, we shall say that a compatibility $\#_1$ is *finer* than $\#_2$, or that $\#_2$ is *coarser* than $\#_1$, if $\mathcal{F}(V, \#_2)$ is an involutive cofinal sublattice of $\mathcal{F}(V, \#_1)$ (given a partially ordered set F, a subset $K \subset F$ is *cofinal* to F if, for any element $x \in F$, there is an element $k \in K$ such that $x \leqslant k$).

Now, suppose that a linear compatiblity $\#$ is given on V. Then, every involutive cofinal sublattice of $\mathcal{F}(V, \#)$ defines a coarser PIP-space, and *vice versa*. Thus coarsening is always possible, and will ultimately lead to a minimal PIP-space, consisting of $V^\#$ and V only, that is, the situation of distribution spaces. However, the operation of refining is not always possible; in particular there is no canonical solution, *a fortiori* no unique maximal solution. There are exceptions, however, for instance, when one is given explicitly a larger set of assaying subspaces that also form, or generate, a larger involutive sublattice. To give an example, the weighted L^2 spaces of Section 3.3.1 form an involutive sublattice of the involutive lattice $\mathcal{I}$ of Köthe function spaces of Section 3.3.4; thus, $\mathcal{I}$ is a genuine refinement of the original LHS.

In the case of a LHS, refining is possible, with infinitely many solutions, by use of interpolation methods or the spectral theorem for self-adjoint operators, which are essentially equivalent in this case. In particular, one may always refine a *discrete* scale of Hilbert spaces into a (nonunique) *continuous* one. Indeed, for the scale described in Section 1, Example (ii), one has, by definition, $\mathcal{H}_n = D(A^n)$, the domain of A^n, equipped with the graph norm $\|f\|_n = \|A^n f\|,\ f \in D(A^n)$, for $n \in \mathbb{N}$. Then, for each $0 \leqslant \alpha \leqslant 1$, one may define

$$\mathcal{H}_{n+\alpha} := \left\{ f \in \mathcal{H}_0 : \int_1^\infty s^{2n+2\alpha}\, d\langle f \mid E(s) f\rangle < \infty \right\}, \tag{4.1}$$

where $\{E(s), 1 \leq s < \infty\}$ is the spectral family of A. With the inner product

$$\langle f \mid g\rangle_{n+\alpha} = \langle A^{n+\alpha} f \mid A^{n+\alpha} g\rangle, \quad f, g \in \mathcal{H}_{n+\alpha}, \tag{4.2}$$

$\mathcal{H}_{n+\alpha}$ is a Hilbert space and one has the continuous embeddings

$$\mathcal{H}_{n+1} \hookrightarrow \mathcal{H}_{n+\beta} \hookrightarrow \mathcal{H}_{n+\alpha} \hookrightarrow \mathcal{H}_n, \quad 0 \leqslant \alpha \leqslant \beta \leqslant 1. \tag{4.3}$$

One may go further, as follows. Let φ be any continuous, positive function on $[1, \infty)$ such that $\varphi(t)$ is unbounded for $t \to \infty$, but increases slower than any power t^α $(0 < \alpha 1)$. An example is $\varphi(t) = \log\, t$ $(t \geqslant 1)$. Then $\varphi(A)$ is a well-defined self-adjoint operator, with domain

$$D(\varphi(A)) = \left\{ f \in \mathcal{H}_0 : \int_1^\infty (1 + \varphi(s))^2\, d\langle f \mid E(s) f\rangle < \infty \right\}. \tag{4.4}$$

With the corresponding inner product

$$\langle f \mid g\rangle_\varphi = \langle f \mid g\rangle + \langle \varphi(A) f \mid \varphi(A) g\rangle, \tag{4.5}$$

$D(\varphi(A))$ becomes a Hilbert space $\mathcal{H}_\varphi$. For every α, $0 < \alpha \leqslant 1$, one has, with proper inclusions and continuous embeddings,

$$\mathcal{H}_\alpha \hookrightarrow \mathcal{H}_\varphi \hookrightarrow \mathcal{H}_0. \tag{4.6}$$

This can be continued as far as one wants, with the result that every scale of Hilbert spaces possesses infinitely many proper refinements which are themselves chains of Hilbert spaces [14, Chapter 5].

Another type of refinement consists in refining a RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$, by inserting a number of intermediate spaces, called *interspaces*, namely, spaces $\mathcal{E}$ such that $\Phi \hookrightarrow \mathcal{E} \hookrightarrow \Phi^\times$ (which implies that the conjugate dual $\mathcal{E}^\times$ is also an interspace). Upon some additional conditions, the most important of which being that Φ be dense in $\mathcal{E} \cap \mathcal{F}$ with its projective topology, for any pair $\mathcal{E}, \mathcal{F}$ of interspaces, one obtains in that way a proper refining of the original RHS. With this construction, which goes under the name of *multiplication framework*, one succeeds, for instance, in defining a valid (partial) multiplication between distributions. A thorough analysis may be found in [14, Section 6.3].

5. Operators on PIP-Spaces

5.1. General Definitions

As already mentioned, the basic idea of (indexed) PIP-spaces is that vectors should not be considered individually, but only in terms of the subspaces V_r ($r \in F$ or $r \in I$), the building blocks of the structure; see (3.1). Correspondingly, an operator on a PIP-space should be defined in terms of assaying subspaces only, with the proviso that only bounded operators between Hilbert or Banach spaces are allowed. Thus an operator is a *coherent collection* of bounded operators. More precisely, one has the following.

Definition 5.1. Given a LHS or LBS $V_I = \{V_r, r \in I\}$, an *operator* on V_I is a map $A : \mathfrak{D}(A) \to V$, such that

(i) $\mathfrak{D}(A) = \bigcup_{q\in \mathrm{d}(A)} V_q$, where $\mathrm{d}(A)$ is a nonempty subset of I,

(ii) for every $q \in \mathrm{d}(A)$, there exists a $p \in I$ such that the restriction of A to V_q is linear and continuous into V_p (we denote this restriction by A_{pq}),

(iii) A has no proper extension satisfying (i) and (ii).

The linear bounded operator $A_{pq} : V_q \to V_p$ is called a *representative* of A. In terms of the latter, the operator A may be characterized by the set $\mathrm{j}(A) = \{(q,p) \in I \times I : A_{pq} \text{ exists}\}$. Thus the operator A may be identified with the collection of its representatives:

$$A \simeq \{A_{pq} : V_q \longrightarrow V_p : (q,p) \in \mathrm{j}(A)\}. \tag{5.1}$$

By condition (ii), the set $\mathrm{d}(A)$ is obtained by projecting $\mathrm{j}(A)$ on the "first coordinate" axis. The projection $\mathrm{i}(A)$ on the "second coordinate" axis plays, in a sense, the role of the range of A. More precisely,

$$\begin{aligned} \mathrm{d}(A) &= \{q \in I : \text{there is a } p \text{ such that } A_{pq} \text{ exists}\}, \\ \mathrm{i}(A) &= \{p \in I : \text{there is a } q \text{ such that } A_{pq} \text{ exists}\}. \end{aligned} \tag{5.2}$$

The following properties are immediate see the (see Figure 2):

(i) $\mathrm{d}(A)$ is an initial subset of I: if $q \in \mathrm{d}(A)$ and $q' < q$, then $q' \in \mathrm{d}(A)$, and $A_{pq'} = A_{pq}E_{qq'}$, where $E_{qq'}$ is a representative of the unit operator (this is what we mean by a 'coherent' collection),

(ii) $\mathrm{i}(A)$ is a final subset of I: if $p \in \mathrm{i}(A)$ and $p' > p$, then $p' \in \mathrm{i}(A)$ and $A_{p'q} = E_{p'p}A_{pq}$.

(iii) $\mathrm{j}(A) \subset \mathrm{d}(A) \times \mathrm{i}(A)$, with strict inclusion in general.

We denote by $\mathrm{Op}(V_I)$ the set of all operators on V_I. Of course, a similar definition may be given for operators $A : V_I \to Y_K$ between two LHSs or LBSs.

Since $V^{\#}$ is dense in V_r, for every $r \in I$, an operator may be identified with a separately continuous sesquilinear form on $V^{\#} \times V^{\#}$. Indeed, the restriction of any representative A_{pq}

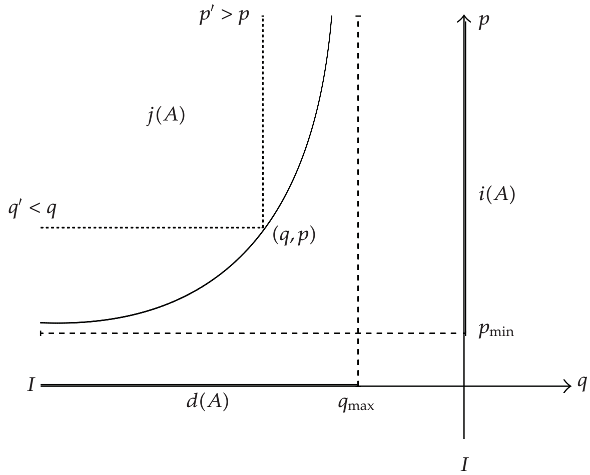

Figure 2: Characterization of the operator A, in the case of a scale.

to $V^{\#} \times V^{\#}$ is such a form, and all these restrictions coincide. Equivalently, an operator may be identified with a continuous linear map from $V^{\#}$ into V (continuity with respect to the respective Mackey topologies).

But the idea behind the notion of operator is to keep also the *algebraic operations* on operators; namely, we define the following operations:

(i) *Adjoint*: Every $A \in \mathrm{Op}(V_I)$ has a unique adjoint $A^\times \in \mathrm{Op}(V_I)$, defined by the relation

$$\langle A^\times x \mid y \rangle = \langle x \mid Ay \rangle, \quad \text{for } y \in V_r,\ r \in \mathrm{d}(A),\ x \in V_{\overline{s}},\ s \in \mathrm{i}(A), \tag{5.3}$$

that is, $(A^\times)_{\overline{rs}} = (A_{sr})^*$ (usual Hilbert/Banach space adjoint). ¡list-item¿¡label/¿

It follows that $A^{\times\times} = A$, for every $A \in \mathrm{Op}(V_I)$: no extension is allowed, by the maximality condition (iii) of Definition 5.1.

(ii) *Partial Multiplication*: The product AB is defined if and only if there is a $q \in \mathrm{i}(B) \cap \mathrm{d}(A)$, that is, if and only if there is a continuous factorization through some V_q:

$$V_r \xrightarrow{B} V_q \xrightarrow{A} V_s, \quad \text{that is,}\ (AB)_{sr} = A_{sq}B_{qr}. \tag{5.4}$$

It is worth noting that, for a LHS/LBS, the domain $\mathfrak{D}(A)$ is always a vector subspace of V (this is not true for a general PIP-space). Therefore, $\mathrm{Op}(V_I)$ is a vector space and a *partial *-algebra* [22].

The concept of PIP-space operator is very simple, yet it is a far-reaching generalization of bounded operators. It allows indeed to treat on the same footing all kinds of operators, from bounded ones to very singular ones. By this, we mean the following, loosely speaking. Take

$$V_r \subset V_o \simeq V_{\overline{o}} \subset V_s \quad (V_o = \text{Hilbert space}). \tag{5.5}$$

Three cases may arise:

(i) if A_{oo} exists, then A corresponds to a bounded operator $V_o \to V_o$,

(ii) if A_{oo} does not exist, but only $A_{or} : V_r \to V_o$, with $r < o$, then A corresponds to an unbounded operator, with domain $D(A) \supset V_r$,

(iii) if no A_{or} exists, but only $A_{sr} : V_r \to V_s$, with $r < o < s$, then A corresponds to a singular operator, with Hilbert space domain possibly reduced to $\{0\}$.

5.2. Special Classes of Operators on PIP-Spaces

Exactly as for Hilbert or Banach spaces, one may define various types of operators between PIP-spaces, in particular LBS/LHSs. We discuss briefly the most important classes, namely, regular operators, homomorphisms and isomorphisms, unitary operators, symmetric operators, and orthogonal projections. Further details may be found in the monograph [14].

5.2.1. Regular and Totally Regular Operators

An operator A on a nondegenerate PIP-space V_I, with positive-definite partial inner product, in particular, a LBS/LHS, is called *regular* if $\mathtt{d}(A) = \mathtt{i}(A) = I$ or, equivalently, if $A : V^\# \to V^\#$ and $A : V \to V$ continuously for the respective Mackey topologies. This notion depends only on the pair $(V^\#, V)$, not on the particular compatibility #. The set of all regular operators $V_I \to V_I$ is denoted by $\mathrm{Reg}(V_I)$. Thus a regular operator may be multiplied both on the left and on the right by an arbitrary operator. Clearly, the set $\mathrm{Reg}(V_I)$ is a $*$-algebra and can often be identified with an O^*-algebra [22, 23].

We give two examples.

(1) If $V = \omega,\ V^\# = \varphi$, then $\mathrm{Op}(\omega)$ consists of arbitrary infinite matrices and $\mathrm{Reg}(\omega)$ of infinite matrices with finite rows and finite columns.

(2) If $V = \mathcal{S}^\times$, $V^\# = \mathcal{S}$, then $\mathrm{Op}(\mathcal{S}^\times)$ consists of arbitrary tempered kernels, while $\mathrm{Reg}(\mathcal{S}^\times)$ contains those kernels that can be extended to $\mathcal{S}^\times$ and map $\mathcal{S}$ into itself. A nice example is the Fourier transform.

An operator A is called *totally regular* if $\mathtt{j}(A)$ contains the diagonal of $I \times I$, that is, A_{rr} exists for every $r \in I$ or A maps every V_r into itself continuously.

5.2.2. Homomorphisms

Let V_I, Y_K be two LHSs or LBSs. An operator $A \in \mathrm{Op}(V_I, Y_K)$ is called a *homomorphism* if

(i) $\mathtt{j}(A) = I \times K$ and $\mathtt{j}(A^\times) = K \times I$,

(ii) $f \ \#_I \ g$ implies that $Af \ \#_K Ag$.

We denote by $\mathrm{Hom}(V_I, Y_K)$ the set of all homomorphisms from V_I into Y_K. The following properties are easy to prove:

(i) $A \in \mathrm{Hom}(V_I, Y_K)$ if and only if $A^\times \in \mathrm{Hom}(Y_K, V_I)$,

(ii) if $A \in \mathrm{Hom}(V_I, Y_K)$, then $\mathsf{j}(A^\times A)$ contains the diagonal of $I \times I$ and $\mathsf{j}(AA^\times)$ contains the diagonal of $K \times K$.

The homomorphism $M \in \mathrm{Hom}(W_I, Y_K)$ is a *monomorphism* if $MA = MB$ implies that $A = B$, for any two elements of $A, B \in \mathrm{Hom}(V_I, W_L)$, where V_I is any PIP-space. Typical examples of monomorphisms are the inclusion maps resulting from the restriction of a support. Take for instance, $L^1_{\mathrm{loc}}(X, d\mu)$, the space of locally integrable functions on a measure space (X, μ). Let Ω be a measurable subset of X and Ω' its complement, both of nonzero measure, and construct the space $L^1_{\mathrm{loc}}(\Omega, d\mu)$, which is a PIP-subspace of $L^1_{\mathrm{loc}}(X, d\mu)$ (see Section 5.2.5). Given $f \in L^1_{\mathrm{loc}}(X, d\mu)$, define $f^{(\Omega)} = f\chi_\Omega$, where χ_Ω is the characteristic function of χ_Ω. Then we obtain an injection monomorphism $M^{(\Omega)} : L^1_{\mathrm{loc}}(\Omega, d\mu) \to L^1_{\mathrm{loc}}(X, d\mu)$ as follows:

$$\left(M^{(\Omega)} f^{(\Omega)}\right)(x) = \begin{cases} f^{(\Omega)}(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \notin \Omega, \end{cases} \qquad f^{(\Omega)} \in L^1_{\mathrm{loc}}(\Omega, d\mu). \tag{5.6}$$

If we consider the lattice of weighted Hilbert spaces $\{L^2(r)\}$ in this PIP-space, then the correspondence $r \leftrightarrow r^{(\Omega)} = r\chi_\Omega$ is a bijection between the corresponding involutive lattices.

The homomorphism $A \in \mathrm{Hom}(V_I, Y_K)$ is an *isomorphism* if there exists a homomorphism $B \in \mathrm{Hom}(Y_K, V_I)$ such that $BA = 1_V$, $AB = 1_Y$, where 1_V, 1_Y denote the identity operators on V_I, Y_K, respectively.

For instance, the Fourier transform is an isomorphism of the Schwartz RHS $\mathcal{S} \subset L^2 \subset \mathcal{S}^\times$ onto itself and, similarly, of the Feichtinger triplet (6.16) onto itself. In both cases, the property extends to several scales of lattices interpolating between the two extreme spaces, for instance, the Hilbert chain of the Hermite representation of tempered distributions.

5.2.3. Unitary Operators and Group Representations

The operator $U \in \mathrm{Op}(V_I, Y_K)$ is *unitary* if $U^\times U$ and $UU^\times$ are defined and $U^\times U = 1_V, UU^\times = 1_Y$. We emphasize that unitary operators need *not* be homomorphisms ! In fact, unitarity is a rather weak property and it is insufficient for group representations.

Thus a unitary representation of a group G into a PIP-space V_I is defined as a homomorphism of G into the *unitary isomorphisms* of V_I. Given such a unitary representation U of G into V_I, where the latter has the central Hilbert space $\mathscr{H}_0$, consider the representative $U_{00}(g)$ of $U(g)$ in $\mathscr{H}_0$. Then $g \mapsto U_{00}(g)$ is a unitary representation of G into $\mathscr{H}_0$, in the usual sense.

To give an example, let V_I be the scale built on the powers of the operator (Hamiltonian) $H = -\Delta + v(\mathbf{r})$ on $L^2(\mathbb{R}^3, d\mathbf{x})$, where Δ is the Laplacian on $\mathbb{R}^3$ and v is a (nice) rotation invariant potential. The system admits as symmetry group $G = \mathrm{SO}(3)$ (the full symmetry group might be larger; for instance, the Coulomb potential admits SO(4) as symmetry group for its bound states.) and the representation U_{00} is the natural representation of SO(3) in $L^2(\mathbb{R}^3)$:

$$[U_{00}(\rho)\psi](\mathbf{x}) = \psi\left(\rho^{-1}\mathbf{x}\right), \quad \rho \in \mathrm{SO}(3). \tag{5.7}$$

Then U_{00} extends to a unitary representation U by totally regular isomorphisms of V_I. Angular momentum decompositions, corresponding to irreducible representations of SO(3), extend to V_I as well. In addition, this is a good setting also for representations of the Lie algebra $\mathfrak{so}(3)$. Notice that the representation U is totally regular, but this need not be the case. For instance, if the potential v is not rotation invariant, U will no longer be totally regular, although it is still an isomorphism.

5.2.4. Symmetric Operators

An operator $A \in \mathrm{Op}(V_I)$ is *symmetric* if $A^\times = A$. Since one has $A^{\times\times} = A$, no extension is allowed, by the maximality condition. Thus symmetric operators behave essentially like bounded self-adjoint operators in a Hilbert space. Yet, they can be very singular, as indicated above, for a chain

$$V^\# \subset \cdots \subset V_r \subset V_o = V_{\overline{o}} \subset V_s \subset \cdots \subset V. \tag{5.8}$$

In this case, the question is whether a symmetric operator $A \in \mathrm{Op}(V_I)$ has a self-adjoint restriction to the central Hilbert space V_o. In a Hilbert space context, an answer is given by the celebrated KLMN theorem (KLMN stands for Kato, Lax, Lions, Milgram, Nelson). Actually, this classical result already has a distinct PIP-space flavor. Thus is not surprising that the KLMN theorem has a natural generalization to a LHS or a PIP-space with positive-definite partial inner product and central Hilbert space $V_o = V_{\overline{o}}$, and a quadratic form version as well [14, Section 3.3.5].

An interesting application is a correct description of very singular Hamiltonians in quantum mechanics, typically with δ or δ' interactions. For instance, one can treat in this way the Kronig-Penney crystal model, which consists of a δ potential at each node of a lattice, in one, two, or three dimensions [24, 25].

5.2.5. Orthogonal Projections, Bases, Frames

An operator P on a nondegenerate PIP-space V, respectively, a LBS/LHS V_I, is an *orthogonal projection* if $P \in \mathrm{Hom}(V_I)$ and $P^2 = P = P^\times$. It follows immediately from the definition that an orthogonal projection is totally regular, that is, $\mathsf{j}(P)$ contains the diagonal $I \times I$, or still that P leaves every assaying subspace invariant. Equivalently, P is an orthogonal projection if P is an idempotent operator (that is, $P^2 = P$) such that $\{Pf\}^\# \supseteq \{f\}^\#$ for every $f \in V$ and $\langle g \mid Pf \rangle = \langle Pg \mid f \rangle$ whenever $f\#g$. We denote by $\mathrm{Proj}(V)$ the set of all orthogonal projections in V and similarly for $\mathrm{Proj}(V_I)$.

These projection operators enjoy several properties similar to those of Hilbert space projectors. Two of them are of special interest in the present context.

(i) Given a nondegenerate PIP-space V, there is a natural notion of subspace, called *orthocomplemented*, which guarantees that such a subspace W of V is again a nondegenerate PIP-space with the induced compatibility relation and the restriction of the partial inner product. There are equivalent topological conditions, so that orthocomplemented subspaces are the proper PIP-subspaces [26]. Then the basic theorem about projections states that a subspace W of V is orthocomplemented if and only if W is the range of an orthogonal projection $P \in \mathrm{Proj}(V)$, that is, $W = PV$. Then $V = W \oplus Z$, where Z is another orthocomplemented subspace.

(ii) An orthogonal projection P is of finite rank if and only if $W = \mathrm{Ran}\, P \subset V^{\#}$ and $W \cap W^{\perp} = \{0\}$ (this condition is, of course, superfluous when the partial inner product is positive definite).

There is a natural partial order on the set of projections:

$$P_W \leq P_Y \quad \text{if and only if} \quad W \subseteq Y, \tag{5.9}$$

but the lattice properties of $\mathrm{Proj}(V)$ are unknown. Thus we expect that quantum logic may be reformulated in a PIP-space language only under additional restrictions on V.

Property (ii) has important consequences for the structure of bases. First we recall that a sequence $\{e_n, n = 1, 2, \dots\}$ of vectors in a Banach space E is a *Schauder basis* if, for every $f \in E$, there exists a unique sequence of scalar coefficients $\{c_k, k = 1, 2, \dots\}$ such that $\lim_{m \to \infty} \| f - \sum_{k=1}^{m} c_k e_k \| = 0$. Then one may write

$$f = \sum_{k=1}^{\infty} c_k e_k. \tag{5.10}$$

The basis is *unconditional* if the series (5.10) converges unconditionally in E (i.e., it keeps converging after an arbitrary permutation of its terms).

A standard problem is to find, for instance, a sequence of functions that is an unconditional basis for *all* the spaces $L^p(\mathbb{R}), 1 < p < \infty$. In the PIP-space language, this statement means that the basis vectors must belong to $V^{\#} = \bigcap_{1<p<\infty} L^p(\mathbb{R})$. Also, since (5.10) means that f may be approximated by finite sums, the property (ii) of orthogonal projections implies that all the basis vectors must belong to $V^{\#}$. Some examples are given in Section 6.2.5.

Actually, in the context of signal processing, orthogonal (in the Hilbert sense) bases are not enough; one needs also biorthogonal bases and, more generally, *frames*. We recall that a countable family of vectors $\{\psi_n\}$ in a Hilbert space $\mathscr{H}$ is called a *frame* if there are two positive constants $\mathtt{m}, \mathtt{M}$, with $0 < \mathtt{m} \leqslant \mathtt{M} < \infty$, such that

$$\mathrm{m} \| f \|^2 \leqslant \sum_{n=1}^{\infty} |\langle \psi_n \mid f \rangle|^2 \leqslant \mathrm{M} \| f \|^2, \quad \forall f \in \mathscr{H}. \tag{5.11}$$

The two constants $\mathtt{m}, \mathtt{M}$ are called *frame bounds*. If $\mathtt{m} = \mathtt{M}$, the frame is said to be *tight*. Consider the analysis operator $C : \mathscr{H} \to \ell^2$ defined by $C : f \mapsto \{ |\langle \psi_n \mid f \rangle| \}$ and the frame operator

$S = C^*C$. Then the vectors $\widetilde{\psi}_n = S^{-1}\psi_n$ also constitute a frame, called the *canonical dual frame*, and one has the (strongly converging) expansions

$$f = \sum_{n=1}^{\infty} \left| \langle \psi_n \mid f \rangle \right| \widetilde{\psi}_n = \sum_{n=1}^{\infty} \left| \langle \widetilde{\psi}_n \mid f \rangle \right| \psi_n. \tag{5.12}$$

Then the considerations made above for bases should apply to frame vectors as well, that is, the vectors $\psi_n, \ \widetilde{\psi}_n$ should also belong to $V^{\#}$.

6. Applications of PIP-Spaces

6.1. Applications in Mathematical Physics

PIP-spaces have found many applications in mathematical physics, mostly in quantum physics. We will sketch a few of them in this section. Most of what follows is described in detail in [14].

6.1.1. Dirac Formalism in Quantum Mechanics

The mathematical description of a quantum system rests on three basic principles: (i) The *superposition principle*, which implies that the set of states of the system has a linear structure; (ii) The notion of *transition amplitude*, given by an inner product: $A(\psi_1 \rightarrow \psi_2) = \langle \psi_2 \mid \psi_1 \rangle$, which yields transition probabilities by $P(\psi_1 \rightarrow \psi_2) = |\langle \psi_2 \mid \psi_1 \rangle|^2$; and (iii) The *probabilistic interpretation*, which requires that $\langle \psi \mid \psi \rangle = \|\psi\|^2 > 0$, whenever $\psi \neq 0$.

Combining these basic principles implies that the set of states of the system is a positive-definite inner product space Φ, that is, a pre-Hilbert space. On this basis, Dirac developed a formalism for quantum physics with great computational capacity and broad predictive power. The essential features of Dirac's formalism are the following.

(i) Physical observables are represented by linear operators in the space Φ and these operators form an algebra. Therefore, it makes sense to arbitrarily add and multiply operators to form new operators.

(ii) For a given quantum physical system, there exist complete systems of commuting observables (CSCO) in the algebra of observables. The system of eigenvectors for a chosen CSCO provides a basis for the space Φ, that is, every vector $\phi \in \Phi$ can be expanded into the eigenvectors of the CSCO.

In the simplest case, a CSCO consists of only one observable A, with a mixed spectrum consisting of discrete eigenvalues $\{\lambda_n\} = \sigma_p(A)$ and a continuous part $\{\lambda\} = \sigma_c(A)$. The corresponding eigenvectors, written as $|\lambda_n\rangle, \ |\lambda\rangle$, respectively, obey "orthogonality" relations

$$\langle \lambda_m \mid \lambda_n \rangle = \delta_{mn}, \quad \langle \lambda \mid \lambda_n \rangle = 0, \quad \langle \lambda \mid \lambda' \rangle = \delta(\lambda - \lambda'). \tag{6.1}$$

Then every $\phi \in \Phi$ can be expanded as

$$\phi = \sum_n |\lambda_n\rangle \langle \lambda_n \mid \phi \rangle + \int_{\sigma_c(A)} d\lambda |\lambda\rangle \langle \lambda \mid \phi \rangle. \tag{6.2}$$

Clearly the eigenvectors $|\lambda\rangle$ cannot belong to the pre-Hilbert space Φ, nor to its completion $\mathscr{H}$. Thus Dirac's formalism, while extremely practical and used by physicists on a daily basis, is not mathematically well defined.

For that reason, von Neumann formulated a rigorous version of quantum mechanics, in a pure Hilbert space language. His formulation consists in the following two axioms: (i) *Pure states* are represented by rays (i.e., one-dimensional subspaces) in a Hilbert space $\mathscr{H}$; and (ii) *Observables* are represented by self-adjoint operators in $\mathscr{H}$. This formulation is well defined mathematically, but too restrictive. Nonnormalizable eigenvectors, corresponding to points of a continuous spectrum, cannot belong to $\mathscr{H}$, yet they are extremely useful and often have a clear physical meaning (plane waves, for instance). Observables may be unbounded, so that domain considerations must be taken into account. In particular, unbounded operators may not always be multiplied. Thus it is understandable that the large majority of physicists stay with Dirac's formalism.

This had prompted several authors [27–31] to propose a rigorous version in terms of a RHS $\Phi \subset \mathscr{H} \subset \Phi^\times$. In this scheme, the space Φ is constructed from the basic observables (*labeled* observables) of the system at hand and is interpreted as the space of all physically preparable states. The conjugate dual $\Phi^\times$ contains idealized states (probes), identified with measurement devices. In that context, let A be an observable, represented by a self-adjoint operator in $\mathscr{H}$ such that $A : \Phi \to \Phi$ continuously. Then A may be transposed by duality to a linear operator $A^\times : \Phi^\times \to \Phi^\times$, which is an extension of $A^\dagger := A^* \upharpoonright \Phi$, where A^* is the usual Hilbert space adjoint operator, namely,

$$\langle \phi \mid A^\times F\rangle = \langle A\phi \mid F\rangle, \quad \forall \phi \in \Phi,\ F \in \Phi^\times. \tag{6.3}$$

For such an operator, the vector $\xi_\lambda \in \Phi^\times$ is called a *generalized eigenvector* of A, with eigenvalue $\lambda \in \mathbb{C}$, if it satisfies

$$\langle \phi \mid A^\times \xi_\lambda\rangle := A^\times \xi_\lambda(\phi) = \overline{\lambda}\ \xi_\lambda(\phi) \equiv \overline{\lambda}\langle \phi \mid \xi_\lambda\rangle, \quad \forall\, \phi \in \Phi. \tag{6.4}$$

Then the spectral theorem of Gel'fand-Maurin [5, 6] states that A possesses in $\Phi^\times$ a complete orthonormal set of generalized eigenvectors $\xi_\lambda \in \Phi^\times,\ \lambda \in \mathbb{R}$. This means that, for any two $\phi, \psi \in \Phi$, one has (we split again into the discrete and the continuous part of the spectrum of A)

$$\langle \phi \mid \psi\rangle = \sum_n \langle \phi \mid \lambda_n\rangle\langle \lambda_n \mid \psi\rangle + \int \langle \phi \mid \lambda\rangle\langle \lambda \mid \psi\rangle \rho(\lambda) d\lambda, \tag{6.5}$$

where ρ is a non-negative integrable function. In that way one recovers essentially Dirac's bra-and-ket formalism. This approach is based on a RHS, but the construction is such that a PIP-space version is easily obtained—and is in fact closer to Dirac's spirit. For more details, see [14, Section 7.1.1].

6.1.2. Symmetries, Singular Interactions in Quantum Mechanics

Several other topics in quantum mechanics can be advantageously formulated in a RHS or PIP-space language, for instance, the implementation of symmetries, with the two dual points of view, the active one and the passive one [32]. A symmetry group is represented by a unitary representation in $\mathscr{H}$ that extends to a unitary representation U in the enveloping PIP-space, in the sense defined in Section 5.2.3. Then, in accordance with the physical interpretation given above, the active point of view corresponds to the action of U in $V^{\#} \equiv \Phi$, the passive one to the action on $V \equiv \Phi^{\times}$.

Another problem is a correct definition of a Hamiltonian with a singular interaction, already mentioned in Section 5.2.4. In the simplest case, the standard definition is $H = -(\Delta/2m) + V$, where the interaction V is given by some reasonable function (potential). However, there are cases where a singular interaction is needed, for instance when V is replaced formally by a δ function or a δ' function, with support in a point (or several) or a manifold of lower dimension. Then the usual formulation is based on von Neumann's theory of self-adjoint extensions of symmetric operators, sometimes coupled with Krein's formula [33]. But here the PIP-space approach is a convenient substitute to that approach, as shown in [24, 25] and [14, Section 7.1.3].

6.1.3. Quantum Scattering Theory

In scattering theory, it is common to use scales of Hilbert spaces built on the powers of $A_1 := (1 + |\mathbf{x}|^2)$ or $A_2 := (1 + |\mathbf{p}|^2)$, and the LHS obtained by combinations of both. This example contains the Sobolev spaces (the scale built on A_2), the weighted spaces L^2_s (the scale built on A_1), and spaces of mixed type. In particular, operators of the form $f(\mathbf{x})g(\mathbf{p})$, for suitable functions f, g, play an essential role in the so-called phase-space approach to scattering theory and they may be controlled by this LHS. For instance, their trace ideal properties may be derived in this way and they are used for proving the absence of singular continuous spectrum by the limiting absorption principle.

On the other hand, the Weinberg-van Winter (WVW) formulation of scattering theory [34–36] has a very natural interpretation in terms of a LHS, whose components, including the extreme ones, are Hilbert spaces consisting of functions analytic in a sector; thus the indexing parameter is the opening angle of that sector. This technique has allowed to show that the WVW formalism is a particular case of the Complex Scaling Method [14, Section 7.2.3], a result hitherto unknown.

6.1.4. Quantum Field Theory

Mathematically rigorous formulations of QFT rely heavily on a RHS or a PIP-space approach, primarily Wightman's axiomatic formulation. There, indeed, a quantum field is defined as an operator-valued distribution, which is customarily written in terms of an unsmeared field (field at a point) $A(x)$, as

$$A(f) = \int_{\mathbb{R}^4} A(x) f(x) dx, \quad f \in \mathcal{S}(\mathbb{R}^4). \tag{6.6}$$

Under quite reasonable assumptions, the unsmeared field can be defined as a map from $\mathcal{S}(\mathbb{R}^4)$ into $\mathrm{Op}(V)$, where V is a conveniently chosen PIP-space. This allows to give the previous formula a rigorous mathematical meaning [14, Section 7.3.1].

Another PIP-space version of QFT is the Fock construction (tensor algebra) over the RHS

$$\mathcal{S}(\mathcal{V}_m^+) \hookrightarrow L^2(\mathcal{V}_m^+, d\mu) \hookrightarrow \mathcal{S}^\times(\mathcal{V}_m^+), \tag{6.7}$$

where $\mathcal{V}_m^+$ denotes the forward mass shell $\mathcal{V}_m^+ = \{p \in \mathbb{R}^4 : p^2 = m^2,\ p_0 > 0\}$ and $d\mu$ the Lorentz invariant measure d^3p/p_0 on it. Write $\Phi_1 = \mathcal{S}(\mathcal{V}_m^+)$, the space of "good" one-particle states. Then define

$$\Phi_n = \Phi_1^{\bigotimes_s n}, \tag{6.8}$$

where the right-hand side denotes the symmetrized tensor product of n copies of Φ_1, corresponding to n-boson states. Again Φ_n is reflexive, complete, and nuclear with respect to its natural topology, and it can be described as the end space of a scale of Hilbert spaces. Finally, define

$$\Phi = \bigoplus_{n=0}^{\infty} \Phi_n, \tag{6.9}$$

that is, the topological direct sum of the component spaces. Elements of Φ are finite sequences $f = \{f_0, f_1, \ldots, f_n, \ldots\}$, $f_0 \in \mathbb{C}$, $f_n \in \Phi_n$, that is, totally symmetric functions of Schwartz type. The space Φ is reflexive, complete, and nuclear with respect to the direct sum topology. Its dual is the topological product

$$\Phi^\times = \prod_{n=0}^{\infty} \Phi_n^\times. \tag{6.10}$$

Thus we get a suitable RHS, in which the central Hilbert space $\mathcal{H}$ is Fock space, that is, the tensor algebra $\Gamma(\Phi_1)$ over Φ_1.

Other examples are the construction of QFT via the Borchers algebra, Nelson's Euclidean field theory, or the precise treatment of unsmeared fields (fields at a point). See [14, Section 7.3] for a detailed presentation.

6.1.5. Representations of Lie Groups and Lie Algebras

Let us return to the situation described in Section 5.2.3. We start with a strongly continuous unitary representation U_{00} of a Lie group G in a Hilbert space $\mathcal{H}_0$ and seek to build a PIP-space V_I, with $\mathcal{H}_0$ being its central Hilbert space, such that U_{00} extends to a unitary representation U into V_I.

The solution of this problem is well known from Nelson's theory of analytic vectors. Let $\overline{\Delta}$ be the closure of the Nelson operator $\Delta := \sum_{j=1}^n X_j^2$, where $\{X_j, j = 1, \ldots, n\}$ are the representatives under U_{00} of the elements of a basis of the Lie algebra $\mathfrak{g}$ of G. Δ is essentially self-adjoint on the so-called Gårding domain $\mathcal{H}_0^G$, $\overline{\Delta}$ is self-adjoint, and $\overline{\Delta} \geqslant 0$. Define $V_I := \{\mathcal{H}_n, n \in \mathbb{Z}\}$ as the canonical scale of Hilbert spaces generated by the operator $(\overline{\Delta} + 1)$:

$$V^{\#} := \mathfrak{D}^{\infty}\left(\overline{\Delta}\right) \hookrightarrow \mathcal{H}_0 \hookrightarrow V := \mathfrak{D}_{\overline{\infty}}\left(\overline{\Delta}\right). \tag{6.11}$$

First, one has $\mathfrak{D}^{\infty}(\overline{\Delta}) = \bigcap_{n=1}^{\infty} D(\overline{\Delta}^n) = \mathcal{H}_0^{\infty}$, the space of C^{∞}-vectors of U_{00}. Next, for every $g \in G$, $U_{00}(g)$ leaves each $\mathcal{H}_n$, $n \in \mathbb{N}$, invariant and its restriction $U_{nn}(g) : \mathcal{H}_n \to \mathcal{H}_n$ is continuous; thus it can be transposed to a continuous map $U_{nn}(g^{-1}) : \mathcal{H}_{\overline{n}} \to \mathcal{H}_{\overline{n}}$. It follows that U_{00} extends to a unitary representation U in the LHS V_I. Corresponding to the triplet (6.11), we have three representations of G, namely, U_{00}, its restriction $U_{\infty\infty}$, and the dual $U_{\overline{\infty\infty}}$ of the latter, which is an extension of the first two. All three are continuous. Moreover, if one of the three is topologically irreducible (i.e., there is no proper invariant closed subspace), so are the other two.

In addition to the representations of the group G, the scale V_I is the natural tool for studying the properties of the operators representing elements of the Lie algebra $\mathfrak{g}$ or the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of G. For every element $x \in \mathfrak{g}$ or $L \in \mathfrak{U}(\mathfrak{g})$, the representative $U(x)$, respectively $U(L)$, originally defined on $\mathcal{H}_0^G$, extends to a regular operator on V_I. These regular operators have in general no $\{0, 0\}$-representative, since x and L are represented in $\mathcal{H}_0$ by unbounded operators. As in the case of the group G, one gets three *-representations of the enveloping algebra $\mathfrak{U}(\mathfrak{g})$, and in particular of the Lie algebra $\mathfrak{g}$, in the three spaces of the triplet (6.11). Namely, one has, for every $L, L_1, L_2 \in \mathfrak{U}(\mathfrak{g})$,

$$\begin{aligned} U(L_1)U(L_2) &= U(L_1 L_2), \\ U(L^+) &= U(L)^{\times}, \end{aligned} \tag{6.12}$$

where $L \leftrightarrow L^+$ is the involution on $\mathfrak{U}(\mathfrak{g})$. These representations have the same irreducibility properties as the corresponding ones of the group. See [14, Section 7] for further details.

6.2. Applications in Analysis and Signal Processing

Many families of function spaces of interest in analysis or signal processing are, or contain, lattices of Banach spaces. To quote a few: amalgam spaces, modulation spaces, Besov spaces, α-modulation spaces, coorbit spaces, which contain many of the previous cases. We shall describe them briefly in succession. For further information about those spaces, we refer to our monograph [14, Chapters 4 and 8].

6.2.1. Amalgam Spaces

A situation intermediate between the mixed-norm spaces $L^{p,q}(\mathbb{R}^{2d})$ (for $m \equiv 1$) and the spaces $\ell^{p,q}(\mathbb{Z}^{2d})$ is that of the so-called *amalgam spaces*. They were introduced specifically to overcome the inability of the L^p norms to distinguish between the local and the global behavior of functions. The simplest ones are the spaces $W(L^p, \ell^q)$ of Wiener [37], which consist of functions on $\mathbb{R}$ which are locally in L^p with ℓ^q behavior at infinity, in the sense that the L^p norms over the intervals $(n, n+1)$ form an ℓ^q sequence. It is a Banach space for the norm

$$\|f\|_{p,q} = \left\{ \sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p dx \right]^{q/p} \right\}^{1/q}, \quad 1 \leqslant p, q < \infty. \tag{6.13}$$

The following inclusion relations, with all embeddings continuous, derive immediately from those of the L^p and the ℓ^q spaces.

(i) If $q_1 \leqslant q_2$, then $W(L^p, \ell^{q_1}) \subset W(L^p, \ell^{q_2})$.

(ii) If $p_1 \leqslant p_2$, then $W(L^{p_2}, \ell^q) \subset W(L^{p_1}, \ell^q)$.

Thus the smallest space is $W(L^\infty, \ell^1)$ and the largest space is $W(L^1, \ell^\infty)$. As for duality, one has $W(L^p, \ell^q)^\times = W(L^{\overline{p}}, \ell^{\overline{q}})$, for $1 \leqslant p, q < \infty$.

The interesting fact is that, for $1 \leqslant p, q \leqslant \infty$, the set $\mathcal{Q}$ of all amalgam spaces $\{W(L^p, \ell^q)\}$ may be represented by the points (p, q) of the *same* unit square J as in the example of the L^p spaces, with the *same* order structure. However, $\mathcal{Q}$ is *not* a lattice with respect to the order (3.12). One has indeed

$$\begin{gathered} W(L^p, \ell^q) \wedge W\left(L^{p'}, \ell^{q'}\right) \supset W\left(L^{p \vee p'}, \ell^{q \wedge q'}\right), \\ W(L^p, \ell^q) \vee W\left(L^{p'}, \ell^{q'}\right) \subset W\left(L^{p \wedge p'}, \ell^{q \vee q'}\right), \end{gathered} \tag{6.14}$$

where again $\wedge$ means intersection with projective norm and $\vee$ means vector sum with inductive norm, but equality is not obtained. Thus, as in the previous case, one gets chains by varying either p or q, but not both.

A very useful class of amalgam spaces is the family $W(\mathcal{F}L^p, \ell^q), 1 \leqslant p, q \leqslant \infty$, where $\mathcal{F}L^p$ denotes the set of Fourier transforms of L^p functions (one may even add weights on both spaces). These spaces have, for instance, nice inclusion and convolution properties.

Among these, the most interesting one is $\mathcal{S}_0 = W(\mathcal{F}L^1, \ell^1)$, called the *Feichtinger algebra*. The space $\mathcal{S}_0$ has many interesting properties; for instance, one has the following.

(i) $\mathcal{S}_0$ is a Banach space for the norm $\|f\|_{\mathcal{S}_0} = \|V_{g_0} f\|_1$, and $\mathcal{S} \hookrightarrow \mathcal{S}_0 \hookrightarrow L^2$, with all embeddings continuous with dense range. Here g_0 is the Gaussian and $V_g f$ denotes the Short-Time Fourier (or Gabor) Transform of $f \in L^2(\mathbb{R}^d)$, given in (6.17) below.

(ii) $\mathcal{S}_0$ is a Banach algebra with respect to pointwise multiplication and convolution.

(iii) Time-frequency shifts $T_x M_\omega$ are isometric on $\mathcal{S}_0$: $\|T_x M_\omega f\|_{\mathcal{S}_0} = \|f\|_{\mathcal{S}_0}$, where $(T_x g)(y) = g(y - x)$ (translation) and $(M_\omega h)(y) = e^{2\pi i y \omega} h(y)$ (modulation). $\mathcal{S}_0$ is continuously embedded in any Banach space with the same property and containing g_0; thus it is the smallest Banach space with this property.

(iv) The Fourier transform is an isometry on $\mathcal{S}_0$: $\|\mathcal{F}f\|_{\mathcal{S}_0} = \|f\|_{\mathcal{S}_0}$.

As for the (conjugate) dual $\mathcal{S}_0^\times$ of $\mathcal{S}_0$, it is a Banach space with norm $\|f\|_{\mathcal{S}_0^\times} = \|V_g f\|_\infty$. The space $\mathcal{S}_0^\times$ contains both the δ function and the pure frequency $\chi_\omega(x) = e^{-2\pi i x \omega}$.

In virtue of (i) above, we have

$$\mathcal{S} \hookrightarrow \mathcal{S}_0 \hookrightarrow L^2 \hookrightarrow \mathcal{S}_0^\times \hookrightarrow \mathcal{S}^\times, \tag{6.15}$$

where all embeddings are continuous and have dense range. It turns out that the triplet

$$\mathcal{S}_0(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}_0^{\times}(\mathbb{R}^d) \tag{6.16}$$

is the prototype of a *Banach Gel'fand triple*, that is, a RHS (or LBS) in which the extreme spaces are (nonreflexive) Banach spaces. This is often a very convenient substitute for Schwartz' RHS and it is widely used in signal processing.

6.2.2. Modulation Spaces and Gabor Analysis (Time-Frequency Analysis)

Modulation spaces are closely linked to, and in fact defined in terms of, the Short-Time Fourier (or Gabor) Transform. Given a C^∞ window function $g \neq 0$, the *Short-Time Fourier Transform* (STFT) of $f \in L^2(\mathbb{R}^d)$ is defined by

$$(V_g f)(x,\omega) = \langle M_\omega T_x g \mid f \rangle := \int_{\mathbb{R}^d} \overline{g(y-x)} f(y) e^{-2\pi i y \omega} dy, \quad x,\omega \in \mathbb{R}^d, \tag{6.17}$$

where, as usual, $(T_x g)(y) = g(y-x)$ (translation) and $(M_\omega h)(y) = e^{2\pi i y \omega} h(y)$ (modulation).

Then, given a v-moderate weight function $m(x,\omega)$, (see Section 3.3.3) the modulation space $M_m^{p,q}$ is defined in terms of a mixed norm of an STFT:

$$M_m^{p,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}^{\times}(\mathbb{R}^d) : V_g f \in L_m^{p,q}(\mathbb{R}^{2d}) \right\}, \quad 1 \leqslant p,q \leqslant \infty. \tag{6.18}$$

For $p = q$, one writes $M_m^p \equiv M_m^{p,p}$. The space $M_m^{p,q}$ is a Banach space for the norm

$$\|f\|_{M_m^{p,q}} := \|V_g f\|_{L_m^{p,q}}. \tag{6.19}$$

Actually, there is a slightly more restrictive definition, which uses the weight function $m_s(x,\omega) \equiv w_s(\omega) = (1+|\omega|)^s, s \geqslant 0$, (or, equivalently, $\widetilde{m}_s(x,\omega) = (1+|\omega|^2)^{s/2}$), so that the norm reads

$$\|f\|_{M_{w_s}^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\langle M_\omega T_x g \mid f \rangle|^p dx \right)^{q/p} (1+|\omega|)^{sq} \, d\omega \right)^{1/q}. \tag{6.20}$$

Equivalently, one may define a modulation space as the inverse Fourier transform of a Wiener amalgam space:

$$M_{w_s}^{p,q} = \mathcal{F}^{-1}\left(W\left(L^p, \ell_{w_s}^q \right) \right). \tag{6.21}$$

This space is independent of the choice of window g, in the sense that different window functions define equivalent norms.

The lattice properties of the family $\{M_m^{p,q}, 1 \leqslant p,q \leqslant \infty\}$ are, of course, the same as those of the mixed-norm spaces $L_m^{p,q}$. As for duality, one has $(M_m^{p,q})^{\times} = M_{1/m}^{\overline{p},\overline{q}}$. Inclusion relations hold, leading again to a lattice structure: if $p_1 \leqslant p_2$, $q_1 \leqslant q_2$, and $m_2 \leqslant Cm_1$, for some constant $C > 0$, then $M^{p_1,q_1}_{m_1} \subseteq M^{p_2,q_2}_{m_2}$. In particular, one has

$$M^1_v \subseteq M^{p,q}_m \subseteq M^\infty_{1/v}. \tag{6.22}$$

The class of modulation spaces $M^{p,q}_{w_s}$ contains several well-known spaces, such as the following:

(i) The Bessel potential spaces or fractional Sobolev spaces $H^s = M^2_{\widetilde{m}_s}$:

$$H^s(\mathbb{R}^d) = M^2_{\widetilde{m}_s}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}^\times : \int_{\mathbb{R}^d} \left|\widehat{f}(t)\right|^2 \left(1 + |t|^2\right)^s dt < \infty \right\}, \quad s \in \mathbb{R}. \tag{6.23}$$

(ii) $L^2(\mathbb{R}^d) = M^2(\mathbb{R}^d)$.

(iii) The Feichtinger algebra $\mathcal{S}_0 = M^1$ and its dual $\mathcal{S}_0^\times = M^\infty$.

By construction, modulation spaces are function spaces well adapted to *Gabor analysis*, although they can often be replaced by amalgam spaces. A wealth of information about the spaces and their application in Gabor analysis may be found in the monograph of Gröchenig [20]. Here we just indicate a few relevant points, especially those that are of a PIP-space nature. We consider in particular the action of several types of operators on such spaces.

(i) *Translation and Modulation Operators*

(a) Every amalgam space $W(L^p, \ell^q)$ and every mixed-norm space $L^{p,q}_m$ are invariant under translation, that is, T_y is a totally regular operator in the corresponding PIP-space.

(b) Every modulation space $M^{p,q}_m$ is invariant under time-frequency shifts (translation and modulation), that is, T_y and M_ξ are totally regular operators.

(ii) *Fourier Transform*

(a) For $1 \leqslant p, q \leqslant 2$, $\mathcal{F}$ maps $W(L^p, \ell^q)$ into $(WL^{\overline{q}}, \ell^{\overline{p}})$ continuously, that is, $J(\mathcal{F})$ contains every pair (p, q), $(\overline{q}, \overline{p})$.

(b) If $m(\xi, -x) \leqslant Cm(x, \xi)$, then every space M^p_m is invariant under Fourier transform.

(iii) *Gabor Frame Operators*

Given a nonzero window function $g \in L^2(\mathbb{R}^d)$ and lattice parameters $\alpha, \beta > 0$, the set of vectors

$$\mathcal{G}(g, \alpha, \beta) = \left\{ M_{n\beta} T_{k\alpha} g,\ k, n \in \mathbb{Z}^d \right\} \tag{6.24}$$

is called a *Gabor system*. The system $\mathcal{G}(g, \alpha, \beta)$ is a *Gabor frame* if there exist two constants $\mathtt{m} > 0$ and $\mathtt{M} < \infty$ such that

$$\mathrm{m}\|f\|^2 \leqslant \sum_{k,n \in \mathbb{Z}^d} \left|\langle \mathrm{M}_{n\beta} T_{k\alpha} g \mid f \rangle\right|^2 \leqslant \mathrm{M}\|f\|^2, \quad \forall f \in L^2(\mathbb{R}^d). \tag{6.25}$$

The associated Gabor frame operator $S_{g,g}$ is given by

$$S_{g,g}f := \sum_{k,n\in\mathbb{Z}^d} \langle M_{n\beta}T_{k\alpha}g \mid f\rangle M_{n\beta}T_{k\alpha}g. \tag{6.26}$$

The main results of the Gabor time-frequency analysis stem from the following proposition.

Proposition 6.1. *If $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame, there exists a dual window $\breve{g} = S_{gg}^{-1}g$ such that $\mathcal{G}(\breve{g},\alpha,\beta)$ is a frame, called the dual frame. Then one has, for every $f \in L^2(\mathbb{R}^d)$,*

$$f = \sum_{k,n\in\mathbb{Z}^d} \langle M_{n\beta}T_{k\alpha}g \mid f\rangle M_{n\beta}T_{k\alpha}\breve{g} = \sum_{k,n\in\mathbb{Z}^d} \langle M_{n\beta}T_{k\alpha}\breve{g} \mid f\rangle M_{n\beta}T_{k\alpha}g, \tag{6.27}$$

with unconditional convergence in $L^2(\mathbb{R}^d)$.

The outcome of the theory is that the modulation spaces $M_m^{p,q}$ turn out to be the natural class of function spaces for Gabor analysis [20]. Define indeed the following operator, generalizing (6.26) slightly:

$$S_{g,g'}f := \sum_{k,n\in\mathbb{Z}^d} \langle M_{n\beta}T_{k\alpha}g \mid f\rangle M_{n\beta}T_{k\alpha}g', \quad g,g' \in L^2(\mathbb{R}^{2d}). \tag{6.28}$$

Then one has the following results (they are highly nontrivial and their proof requires deep analysis)

(i) If $g,g' \in W(L^\infty,\ell^1)$, then the Gabor frame operator $S_{g,g'}$ is bounded on every $L^p(\mathbb{R}^{2d})$, $1 \leqslant p \leqslant \infty$.

(ii) If $g,g' \in M_v^1$, then $S_{g,g'}$ is bounded on $M_m^{p,q}$ for all $1 \leqslant p,q \leqslant \infty$, all v-moderate weights m, and all α, β.

(iii) If $\breve{g}$ is a dual window of g, that is, $S_{g,\breve{g}} = 1$ on L^2, then the two expansions in (6.27) converge unconditionally in $M_m^{p,q}$ if $p,q < \infty$.

Clearly statements (i) and (ii) can be translated into PIP-space language, by saying that $S_{g,g'}$ is a totally regular operator in the chain $\{L^p, 1 \leqslant p \leqslant \infty\}$, respectively, any PIP-space built from modulation spaces.

These results should suffice to convince the reader that the modulation spaces $M_m^{p,q}$ are the "natural" spaces for Gabor analysis. Actually, most of this remains true if one replaces modulation spaces by amalgam spaces $W(L^p,\ell_m^q)$. Second, it is obvious that most of the statements have a distinctly PIP-space flavor: it is not some individual space $M_m^{p,q}$ or $W(L^p,\ell_m^q)$ that counts, but the whole family, with many operators being regular in the sense of PIP-spaces.

6.2.3. Besov Spaces and Wavelet Analysis (Time-Scale Analysis)

Besov spaces were introduced around 1960 for providing a precise control on the smoothness of solutions of certain partial differential equations. Later on, it was discovered that they are closely linked to wavelet analysis, exactly as the (much more recent) modulation spaces are structurally adapted to Gabor analysis. In fact, there are many equivalent definitions of Besov spaces. We restrict ourselves to a "discrete" formulation, based on a dyadic partition of unity. Other definitions may be found in the literature quoted in [14], in particular [19, 38–40].

Let us consider a weight function $\varphi \in \mathcal{S}(\mathbb{R})$ with the following properties:

(i) $\operatorname{supp} \varphi = \{\xi : 2^{-1} \leqslant |\xi| \leqslant 2\}$,

(ii) $\varphi(\xi) > 0$ for $2^{-1} < |\xi| < 2$,

(iii) $\sum_{j=-\infty}^{\infty} \varphi(2^{-j}\xi) = 1 \ (\xi \neq 0)$.

Then one defines the following functions by their Fourier transform:

(i) $\widehat{\varphi_j}(\xi) = \varphi(2^{-j}\xi),\ j \in \mathbb{Z}$: high "frequency" for $j > 0$, low "frequency" for $j < 0$,

(ii) $\widehat{\varphi}(\xi) = 1 - \sum_{j=1}^{\infty} \varphi(2^{-j}\xi)$: low "frequency" part.

Given the weight function φ, the inhomogeneous Besov space B_{pq}^s is defined as

$$B_{pq}^s = \left\{ f \in \mathcal{S}^\times : \|f\|_{pq}^s < \infty \right\}, \tag{6.29}$$

where $\|\cdot\|_{pq}^s$ denotes the norm

$$\|f\|_{pq}^s := \|\varphi * f\|_p + \left(\sum_{j=1}^{\infty} \left(2^{sj} \|\varphi_j * f\|_p \right)^q \right)^{1/q}, \quad s \in \mathbb{R},\ 1 \leqslant p, q \leqslant \infty. \tag{6.30}$$

The space B_{pq}^s is a Banach space and it does not depend on the choice of the weight function φ, since a different choice defines an equivalent norm. Note that $B_{22}^s = H^s$, the (fractional) Sobolev space or Bessel potential space.

For $f \in B_{pq}^s$, one may write the following (weakly converging) expansion, known as a *dyadic Littlewood-Paley decomposition*:

$$f = \varphi * f + \sum_{j=1}^{\infty} \varphi_j * f. \tag{6.31}$$

Clearly the first term represents the (relatively uninteresting) low-"frequency" part of the function, whereas the second term analyzes in detail the high-"frequency" component.

Besov spaces enjoy many familiar properties (for more details, we refer to the literature, e.g., [19, Section 6.2] or [39, Chapter 2]).

(i) Inclusion Relations

The following relations hold, where all embeddings are continuous:

(a) $\mathcal{S} \hookrightarrow B_{pq}^s \hookrightarrow \mathcal{S}^\times$,

(b) $B_{pq}^s \hookrightarrow L^p$, if $1 \leqslant p, q \leqslant \infty$ and $s > 0$,

(c) for $s_1 < s_2$, $B_{pq}^{s_2} \hookrightarrow B_{pq}^{s_1}$ $(1 \leqslant q, p \leqslant \infty)$,

(d) for $1 \leqslant q_1 < q_2 \leqslant \infty$, $B_{pq_1}^{s} \hookrightarrow B_{pq_2}^{s}$ $(s \in \mathbb{R}, 1 \leqslant p \leqslant \infty)$,

(e) for $s - 1/p = s_1 - 1/p_1, B_{pq}^{s} \hookrightarrow B_{p_1 q_1}^{s_1}$ $(s, s_1 \in \mathbb{R},\ 1 \leqslant p \leqslant p_1 \leqslant \infty,\ 1 \leqslant q \leqslant q_1 \leqslant \infty)$.

In the terminology of Section 4, the first statement means that the spaces B_{pq}^{s} are interspaces for the RHS $\mathcal{S} \hookrightarrow L^2 \hookrightarrow \mathcal{S}^{\times}$. The inclusion relations above mean that the family of spaces B_{pq}^{s} contains again many chains of Banach spaces.

(ii) Interpolation

Besov spaces enjoy nice interpolation properties, in all three parameters $s,\ p,\ q$.

(iii) Duality

One has $(B_{pq}^{s})^{\times} = B_{\overline{p}\,\overline{q}}^{-s}$ $(s \in \mathbb{R})$.

(iv) Translation and Dilation Invariance

Every space B_{pq}^{s} is invariant under translation and dilation (the unitary dilation operator in L^2 reads $(D_a f)(x) = a^{-1/2} f(x/a)$.)

(v) Regularity Shift

Let $J^{\sigma} : \mathcal{S}^{\times} \to \mathcal{S}^{\times}$ denote the operator $J^{s} f = \mathcal{F}^{-1}\{(1 + |\cdot|^2)^{s/2} \mathcal{F} f\}$, $s \in \mathbb{R}$. Then J^{σ} is an isomorphism from B_{pq}^{s} onto $B_{pq}^{s-\sigma}$. Thus J^{σ} is totally regular for $\sigma \leqslant 0$, but not for $\sigma > 0$.

It is also useful to consider the homogeneous Besov space $\dot{B}_{pq}^{s}$, defined as the set of all $f \in \mathcal{S}^{\times}$ for which $\|f\dot{\|}_{pq}^{s} < \infty$, where the quasinorm $\| \cdot \dot{\|}_{pq}^{s}$ is defined by

$$\|f\dot{\|}_{pq}^{s} := \left(\sum_{j=-\infty}^{\infty} \left(2^{sj} \|\varphi_j * f\|_p \right)^q \right)^{1/q}. \tag{6.32}$$

(This is only a quasi-norm since $\|f\dot{\|}_{pq}^{s} = 0$ if and only if supp $\widehat{f} = \{0\}$, i.e., f is a polynomial.) Note that, if $0 \notin$ supp $\widehat{f}$, then $f \in \dot{B}_{pq}^{s}$ if and only if $f \in B_{pq}^{s}$.

The spaces $\dot{B}_{pq}^{s}$ have properties similar to the previous ones and, in addition, one has $B_{pq}^{s} = L^p \cap \dot{B}_{pq}^{s}$ for $s > 0,\ 1 \leqslant p, q \leqslant \infty$. In particular, every space $\dot{B}_{pq}^{s}$ is invariant under translation and dilation, which is not surprising, since these spaces are in fact based on the $ax + b$ group, consisting precisely of dilations and translations of the real line, via the coorbit space construction (see Section 6.2.4 below).

Besov spaces are well adapted to *wavelet analysis*, because the definition (6.29) essentially relies on a dyadic partition (powers of 2). Historically, the connection was made with the *discrete* wavelet analysis, for that reason. Indeed, there exists an equivalent definition given in terms of decay of wavelet coefficients. More precisely, if a function f is expanded in a wavelet basis, the decay properties of the wavelet coefficients allow to characterize precisely to which Besov space the function f belongs, as we shall see below. In addition, the Besov spaces may also be characterized in terms of the *continuous* wavelet transform (see [14, Section 8.4]).

In order to go into details, we have to recall some basic facts about the wavelet transform (for simplicity, we restrict ourselves to one dimension). Whereas the STFT is defined in terms of translation and modulation, the continuous wavelet transform is based on translations and dilations

$$(W_{\psi}s)(b,a) = a^{-1}\int_{-\infty}^{\infty} \overline{\psi\left(a^{-1}(x-b)\right)}s(x)dx, \quad a>0,\ b\in\mathbb{R},\ s\in L^2(\mathbb{R}). \tag{6.33}$$

Note that we use here the so-called L^1 normalization. It is more frequent to use the L^2 normalization, in which the prefactor is $a^{-1/2}$ instead of a^{-1}. In this relation, the wavelet ψ is assumed to satisfy the admissibility condition

$$c_{\psi} := \int_{-\infty}^{\infty} d\omega|\omega|^{-1}\left|\widehat{\psi}(\omega)\right|^2 < \infty, \tag{6.34}$$

which implies that $\int_{-\infty}^{\infty}\psi(x)dx = 0$. This condition is only necessary, but becomes sufficient under some mild restrictions, so that it is commonly used as admissibility condition in practice.

However, discretizing the two parameters a and b in (6.33) leads in general only to frames. In order to get orthogonal wavelet bases, one relies on the so-called *multiresolution analysis* of $L^2(\mathbb{R})$. This is defined as an increasing sequence of closed subspaces of $L^2(\mathbb{R})$:

$$\cdots \subset \mathcal{V}_{-2} \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \tag{6.35}$$

with $\bigcap_{j\in\mathbb{Z}}\mathcal{V}_j = \{0\}$ and $\bigcup_{j\in\mathbb{Z}}\mathcal{V}_j$ dense in $L^2(\mathbb{R})$ and such that

(1) $f(x)\in\mathcal{V}_j \Leftrightarrow f(2x)\in\mathcal{V}_{j+1}$,

(2) there exists a function $\phi\in\mathcal{V}_0$, called a *scaling function*, such that the family $\{\phi(\cdot-k),\ k\in\mathbb{Z}\}$ is an orthonormal basis of $\mathcal{V}_0$.

Combining conditions (1) and (2), one sees that $\{\phi_{jk}\equiv 2^{j/2}\phi(2^j\cdot-k),\ k\in\mathbb{Z}\}$ is an orthonormal basis of $\mathcal{V}_j$. The space $\mathcal{V}_j$ can be interpreted as an *approximation* space at resolution 2^j. Defining $\mathcal{W}_j$ as the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$, that is, $\mathcal{V}_j\oplus\mathcal{W}_j = \mathcal{V}_{j+1}$, we see that $\mathcal{W}_j$ contains the additional *details* needed to improve the resolution from 2^j to 2^{j+1}. Thus one gets the decomposition $L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}}\mathcal{W}_j$. The crucial theorem then asserts the existence of a function ψ, called the *mother wavelet*, explicitly computable from ϕ, such that $\{\psi_{jk}\equiv 2^{j/2}\psi(2^j\cdot-k),\ k\in\mathbb{Z}\}$ constitutes an orthonormal basis of $\mathcal{W}_j$ and thus $\{\psi_{jk}\equiv 2^{j/2}\psi(2^j\cdot-k),\ j,k\in\mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$: these are the *orthonormal wavelets*. Thus the expansion of an arbitrary function $f\in L^2$ into an orthogonal wavelet basis $\{\psi_{jk},\ j,k\in\mathbb{Z}\}$ reads

$$f = \sum_{j,k\in\mathbb{Z}} c_{jk}\psi_{jk}, \quad \text{with } c_{jk} = \langle\psi_{jk}\mid f\rangle. \tag{6.36}$$

Additional regularity conditions can be imposed to the scaling function ϕ. Given $r \in \mathbb{N}$, the multiresolution analysis corresponding to ϕ is called *r-regular* if

$$\left|\frac{d^n\phi}{dx^n}\right| \leqslant c_m\left(1+|x|^m\right), \quad \text{for all } n \leqslant r \text{ and all integers } m \in \mathbb{N}. \tag{6.37}$$

Well-known examples include the Haar wavelets, the B-splines, and the various Daubechies wavelets.

As a result of the "dyadic" definition (6.29)-(6.30), it is natural that Besov spaces can be characterized in terms of an r-regular multiresolution analysis $\{\mathcal{V}_j\}$. Let $E_j : L^2 \to \mathcal{V}_j$ be the orthogonal projection on $\mathcal{V}_j$ and $D_j = E_{j+1} - E_j$ that on $\mathcal{W}_j$. Let $0 < s < r$ and $f \in L^p(\mathbb{R})$. Then, $f \in B^s_{pq}(\mathbb{R})$ if and only if $\|D_j f\|_p = 2^{-js}\delta_j$, where $(\delta_j) \in \ell^q(\mathbb{N})$, and one has ($\asymp$ means equivalence of norms)

$$\|f\|^s_{pq} \asymp \|E_0 f\|_p + \left(\sum_{j\in\mathbb{Z}} 2^{jsq}\|D_j f\|^q_p\right)^{1/q}. \tag{6.38}$$

Specializing to $p = q = 2$, one gets a similar result for Sobolev spaces: given $f \in H^{-r}(\mathbb{R})$ and $|s| < r$, $f \in H^s(\mathbb{R})$ if, and only if, $E_0 f \in L^2(\mathbb{R})$ and $\|D_j f\|_2 = 2^{-js}\epsilon_j$, $j \in \mathbb{N}$, where $(\epsilon_j) \in \ell^2(\mathbb{N})$.

But there is more. Indeed, modulation spaces and Besov spaces admit decomposition of elements into wavelet bases and each space can be uniquely characterized by the decay properties of the wavelet coefficients. To be precise, let $\{\psi_{jk},\ j,k \in \mathbb{Z}\}$ be an orthogonal wavelet basis coming from an r-regular multiresolution analysis based on the scaling function ϕ. Then the following results are typical [41, Chapters II.9 and VI.10].

(i) *Inhomogeneous Besov Spaces:* $f \in B^s_{pq}(\mathbb{R})$ if it can be written as

$$f(x) = \sum_{k\in\mathbb{Z}} \beta_k \phi(x-k) + \sum_{j\geqslant 0, k\in\mathbb{Z}} c_{jk}\psi_{jk}, \tag{6.39}$$

where $(\beta_k) \in \ell^p$ and $\left(\sum_{k\in\mathbb{Z}} |c_{jk}|^p\right)^{1/p} = 2^{-j(s+1/2-1/p)}\gamma_j$, with $(\gamma_j) \in \ell^q(\mathbb{Z})$.

(ii) *Homogeneous Besov Spaces:* Let $|s| < r$. Then, if $f \in \dot{B}^s_{pq}(\mathbb{R})$, its wavelet coefficients c_{jk} verify $\left(\sum_{k\in\mathbb{Z}} |c_{jk}|^p\right)^{1/p} = 2^{-j(s+1/2-1/p)}\ \gamma_j$, where $(\gamma_j) \in \ell^q(\mathbb{Z})$. Conversely, if this condition is satisfied, then $f = g + P$, where $g \in \dot{B}^s_{pq}$ and P is a polynomial.

6.2.4. α-Modulation Spaces, Coorbit Spaces

The α-modulation spaces ($\alpha \in [0,1]$) are spaces intermediate between modulation and Besov spaces, to which they reduce for $\alpha = 0$ and $\alpha \to 1$, respectively. As for coorbit spaces, they are a far-reaching generalization, based on integrable group representations [42]. They contain most of the previous spaces, but we will refrain from describing them in detail, for lack of space.

6.2.5. Unconditional Bases

We conclude this section with some examples of unconditional wavelet bases, as announced in Section 5.2.5. Actually, the concept of wavelet basis can be further generalized to *biorthogonal bases*, obtained by considering two scales of the type (6.35) and imposing cross-orthogonality relations [43, Section 8.3]. For precise definitions, we refer to the literature.

(i) The Haar wavelet basis is defined by the scaling function $\phi_H = \chi_{[0,1]}$ and the mother wavelet $\psi_H = \chi_{[0,1/2]} - \chi_{[1/2,1]}$. It is a standard result that the Haar system is an unconditional basis for every $L^p(\mathbb{R}),\ 1 < p < \infty$.

(ii) The Lemarié-Meyer wavelet basis is an unconditional basis for all L^p spaces, Sobolev spaces, and homogeneous Besov spaces $\dot{B}^s_{pq}$ $(1 \leqslant p, q < \infty)$ [44].

(iii) There is a class of wavelet bases (Wilson bases of exponential decay) that are unconditional bases for every modulation space $M^{p,q}_m$, $1 \leqslant p, q < \infty$, but *not* for $L^p,\ 1 < p < \infty,\ \ p \neq 2$ [45].

7. Conclusion

Most families of function spaces used in analysis and in signal processing come in scales or lattices and in fact are, or contain, PIP-spaces. The (lattice) indices defining the (partial) order characterize the properties of the corresponding functions or distributions: smoothness, local integrability, decay at infinity. Thus it seems natural to formulate the properties of various operators globally, using the theory of PIP-space operators; in particular the set $\mathsf{j}(A)$ of an operator encodes its properties in a very convenient and visual fashion. In addition, it is often possible to determine uniquely whether a function belongs to one of those spaces simply by estimating the (asymptotic) behavior of its Gabor or wavelet coefficients, a real breakthrough in functional analysis [41].

A legitimate question is whether there are instances where a PIP-space is really *needed*, or a RHS could suffice. The answer is that there are plenty of examples, among the applications we have enumerated in Section 6 (without details, for lack of space). We may therefore expect that the PIP-space formalism will play a significant role in Gabor/wavelet analysis, as well as in mathematical physics.

Concerning the applications in mathematical physics, in almost all cases, the relevant structure is a scale or a chain of Hilbert spaces, which allows a finer control on the behavior of operators. For instance (all details may be found in our monograph [14]) the following are considered.

(i) For the singular interactions in quantum mechanics (δ or δ' potentials), the approach of Grossmann et al. [18, 24, 25] is definitely the most appropriate; a RHS would be irrelevant.

(ii) The very formulation of the WVW approach to quantum scattering theory [34–36] requires a LHS, whose end spaces are in fact Hilbert spaces.

(iii) For quantum field theory, the energy bounds of Fredenhagen and Hertel [46] rely in an essential way on the scale generated by the Hamiltonian, and so does Nelson's approach to Euclidean field theory [47].

As for the applications in signal processing, all families of spaces routinely used are, or contain, chains of Banach spaces, which are needed for a fine tuning of elements (usually, distributions) and operators on them. Such are, for instance, L^p spaces, amalgam spaces, modulation spaces, Besov spaces, or coorbit spaces, mentioned above. Here again, a RHS, even with Banach end spaces like the Feichtinger algebra and its conjugate dual, is clearly not sufficient. The whole Section 6.2 illustrates the statement.

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