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The mechanically-based approach to 3D non-local linear elasticity theory: Long-range central interactions

Mario Di Paola [a], Giuseppe Failla [b], Massimiliano Zingales [a,*]

[a] University of Palermo, Dipartimento di Ingegneria Strutturale, Aerospaziale e Geotecnica (DISAG), Viale delle Scienze, 90128 Palermo, Italy
[b] University of Reggio Calabria, Dipartimento di Meccanica e Materiali (MECMAT), Via Graziella, 89122 Reggio Calabria, Italy

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ABSTRACT

This paper presents the generalization to a three-dimensional (3D) case of a mechanically-based approach to non-local elasticity theory, recently proposed by the authors in a one-dimensional (1D) case. The proposed model assumes that the equilibrium of a volume element is attained by contact forces between adjacent elements and by long-range forces exerted by non-adjacent elements. Specifically, the long-range forces are modelled as central body forces depending on the relative displacement between the centroids of the volume elements, measured along the line connecting the centroids. Further, the long-range forces are assumed to be proportional to a proper, material-dependent, distance-decaying function and to the products of the interacting volumes. Consistently with the modelling of the long-range forces as central body forces, the static boundary conditions enforced on the free surface of the solid involve only local stress due to contact forces.

The proposed 3D formulation is developed both in a mechanical and in a variational context. For this the elastic energy functionals of the solid with long-range interactions are introduced, based on the principle of virtual work to set the proper correspondence between the mechanical and the kinematic variables of the model. Numerical applications are reported for 2D solids under plane stress conditions.

1. Introduction

The classical rational continuum mechanics, based upon a convenient set of axioms, proves to be intrinsically size independent, as reported by several authors in the last decades (see Truesdell and Noll, 1992). This fact is evident as the same boundary value problems are involved ranging from quantum dots scales to macro-scale engineering problems. This represents, at the same time, the best and the worst feature of the continuum mechanics. On one hand, in fact, it has provided some beautiful solutions for mechanical and physical problems, used nowadays. On the other hand, however, effects observed in experimental tests such as size dependency and scaling of mechanical phenomena, which have attracted a considerable attention in recent times for problems involving thin films, quantum dots, damage and cracks propagations, nanowires and nanotubes, remain unexplained within classical theories. These considerations are well-known in the scientific community and several physical reasons have been found to explain such a breakdown of the continuum field theories. Among them we may enlist the increasing importance of surface energy at small scales, the inherent discrete nature of the solid matter, the presence of an inner microstructure that may play a significant role, with an increasing importance of internal motions, in enriched lattice models as those of liquid crystals, polymers and granular materials.

The most obvious route to investigate phenomena at a nanometric length scale is a molecular (atomistic) dynamics approach. However the atomistic theories, often involving tera-orders of degrees of freedom, are still not amenable for many engineering and physical problems at nano- and mesoscales even with the most powerful computing facilities. For this reason, continuum theories capable of describing the aforementioned effects are welcome, since they are computationally feasible and, very often, some beautiful and illustrative analytical solutions may be obtained.

In such a context non-local continuum theories involving scale effects to some extent have been proposed. They may be roughly divided into two main classes: (i) the gradient elasticity theories; (ii) the integral non-local elasticity theories.

The leading idea of the gradient elasticity theories is that the inner microstructure of a solid possesses additional degrees of freedom that correspond to additional microstresses and microstrains (Toupin, 1963; Mindlin, 1963; Krumhanls, 1967; Kunin, 1967, 1982; Mindlin and Eshel, 1968; Eringen, 1972a; Eringen and Edelen, 1972). A different formulation, based upon the introduction of gradients of the strain field in the elastic constitutive equations, have been also introduced at the beginning of the nineties (Aifantis, 1994, 2003) and in more recent times (Zhang and

* Corresponding author. Tel.: +39 091 6568458; fax: +39 0916568407.
E-mail address: massimiliano.zingales@unipa.it (M. Zingales).

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Sharma, 2005; Zhang et al., 2006). Some studies concerning the evaluation of elastic parameters involved in the gradient elasticity theories by quantum scattering approaches have been also recently conducted by Maranganti and Sharma (2007a,b) or by Di Vincenzo (1986) for a simplified microstructure topology.

Applications of the gradient elasticity theory may be found in the description of material defects (Kroner, 1970; Gutkin and Aifantis, 1999; Gutkin, 2000; Drugan, 2000; Frantziskonis and Aifantis, 2002), in damage analysis (Ganghoffer and De Borst, 2000), or in the context of plastic strains (Polizzotto and Borino, 1998). However, there are main difficulties in a systematic use of gradient elasticity theories due to the lack of mechanical consistency of the boundary conditions (B.C.), as stated by Polizzotto (2001).

As an alternative to gradient elasticity theories a competing approach, which does not involve the presence of gradients and hence does not run into higher-order differential equations, has been developed, almost independently, by Krumhanls (1963), Lax (1963), Kroner (1967) and Kunin (1967). In these studies an enriched version of the elastic continuum mechanics has been proposed as a generalization of lattice mechanics, thus formulating the so-called quasi-continuum theories (Kunin, 1982). In general, quasi-continuum theories involve a constitutive relation where a weighted integral of the strain field is introduced (Eringen, 1972a; Eringen and Edelen, 1972) and, for this reason, it is generally stated that at a given point inside the solid non-local effects due to the strain tensor field in the whole solid domain are accounted for. Such approach has been used by several authors to address dislocations, fluid mechanics, damage and plasticity (Eringen, 1978; Eringen, 1979; Borino et al., 1999; Bažant and Jirásek, 2002) or in an elasto-dynamics context (Eringen, 1972b; Artan and Altan, 2002; Chakraborty, 2007). However the integral non-local elasticity theory, albeit of simple use, involves the presence of an intrinsically non-convex elastic potential energy. Non-convexity may be a desirable feature of the elastic potential energy in the analysis of phase transitions of some kinds of alloys (see Puglisi and Truskinovsky, 2000; Del Piero and Truskinovsky, 2001), since such analysis involves non-linear stability of meta-stable portions of a non-linear elastic solid. However, non-convexity is certainly not desirable when only linear stress–strain relations are involved, as in the well-known Kroner–Eringen (KE) model (Kroner, 1967; Eringen, 1972b), where non-convex energies in bounded domains are not mechanically consistent. To overcome this major drawback, in fact, some particular classes of the weighting functions have been proposed (Marotti De Sciarra, 2008).

The introduction of a mechanically-based model of non-local elasticity may be then considered, in the authors' opinion, a necessary step to investigate, in the context of continuum field theories, those effects at meso- or nanolevel that are not predicted by the classical elasticity theory. A step in this direction has been recently made by Silling and co-workers (Silling, 2000; Silling et al., 2003; Silling and Lehoucq, 2008). They have assumed that the elastic potential energy stored in the body is a quadratic function of the relative displacements between the centroids of elementary volumes. Such an assumption leads to an integral formulation of the governing equations in terms of the displacement field. However, the enforcement of pertinent B.C. is still a pending problem since external surface loads must be turned into body forces applied to a thin boundary layer, whose dimensions shall be properly selected depending on the shape of the solid, on the applied load and on the material.

A mechanically-based model of non-local elasticity, including contact forces between adjacent volumes and long-range central forces between non-adjacent volumes, has been recently presented for a 1D linearly-elastic, isotropic and homogeneous bar (Di Paola and Zingales, 2008; Di Paola et al., 2009). Specifically, it has been assumed that non-adjacent volumes exchange a mutual force linearly depending on the product of the two volumes, on the relative displacement between the centroids of the two volumes and on a proper distance-decaying function. The latter describes the effects of distance-decaying interactions observed in next to the nearest next (NNN) lattice models (Born and Huang, 1954). The choice of the distance-decaying function ruling the long-range interactions is a crucial step in such a theory, since upon this choice several properties of the studied elastic domains may be highlighted (Kresse and Truskinovsky, 2003; Puglisi, 2007). In previous studies different forms of distance-decaying functions (both smooth and fractional functions, see Di Paola and Zingales, 2008; Di Paola et al., 2009) have been considered. It has been also shown that the proposed model can be cast within a consistent variational approach (Di Paola et al., 2010) and that reverts to the KE model, under some restrictions on the distance-decaying function and the weighting function of the KE model. However such an equivalence holds only for a 1D unbounded domain. Instead a different scenario appears for 1D bounded domains, since the elastic potential energy of the KE model is in general a non-convex function, whereas it is a convex, quadratic function of the state variables in the proposed model (Di Paola et al., 2010). In an elastodynamics context, similar considerations have been made about the dispersion of elastic waves travelling in a 1D elastic continuum with long-range interactions of smooth and fractional type (Cottone et al., 2009). Further, it has been shown that the proposed model is equivalent to a point-spring model with springs connecting non-adjacent elements featuring a distance-decaying stiffness. Due to such equivalence, the proposed model of non-local mechanics has been named mechanically-based model of non-local continuum.

In this paper the non-local elasticity model presented for a 1D bar (Di Paola and Zingales, 2008; Di Paola et al., 2009) will be generalized to a 3D linearly-elastic, isotropic and homogeneous continuum. It will be shown that the governing equations and the pertinent boundary conditions, first introduced on a purely-mechanical basis, can be cast within a consistent variational framework. For this, the elastic energy functionals are formulated based on the principle of virtual work, used to set the proper correspondence between the mechanical and the kinematic variables.

The paper is structured as follows: the proposed formulation of 3D non-local elasticity theory is reported in Section 2. The principle of virtual work and the static–kinematic duality pertinent to the proposed non-local model is then presented in Section 3, followed in Section 4 by the energy theorems. Numerical examples are reported in Section 5.

2. The linear elastic problem with long-range central forces

Let us consider a linearly-elastic body embedded in a region V of an Euclidean space and be S its boundary surface. The body is referred to an orthogonal reference system $O(x_1, x_2, x_3)$, as shown in Fig. 1a. Denote by S_c the constrained part of S, in which the displacements are prescribed, and by S_f the unconstrained part of S, in which the external loads are prescribed, so that $S = S_c \cup S_f$. Denote by $\mathbf{u}(\mathbf{x})$ the displacement vector field $\left(\mathbf{x} = [x_1 \quad x_2 \quad x_3]^T\right)$. Be $\bar{\mathbf{u}}(\mathbf{x})$ the displacement vector field on S_c and be $\bar{\mathbf{p}}_n(\mathbf{x})$ the prescribed external surface loads (per unit surface) applied to S_f, being $\mathbf{n}$ the outward normal to S_f. The prescribed external body forces (per unit volume) are denoted by $\bar{\mathbf{b}}(\mathbf{x})$; in addition to such external body forces, it is assumed that the unit volume at point $\mathbf{x}$ is subjected to other *internal* body forces, denoted by $\mathbf{f}(\mathbf{x}, \xi)$, due to the volume elements $dV(\xi)$ at all points ξ in the body. That is, such body forces are long-range forces exchanged between non-adjacent volume elements. In the mechanically-based model of non-local elasticity proposed for a 1D bar by Di Paola et al. (2009), they have been

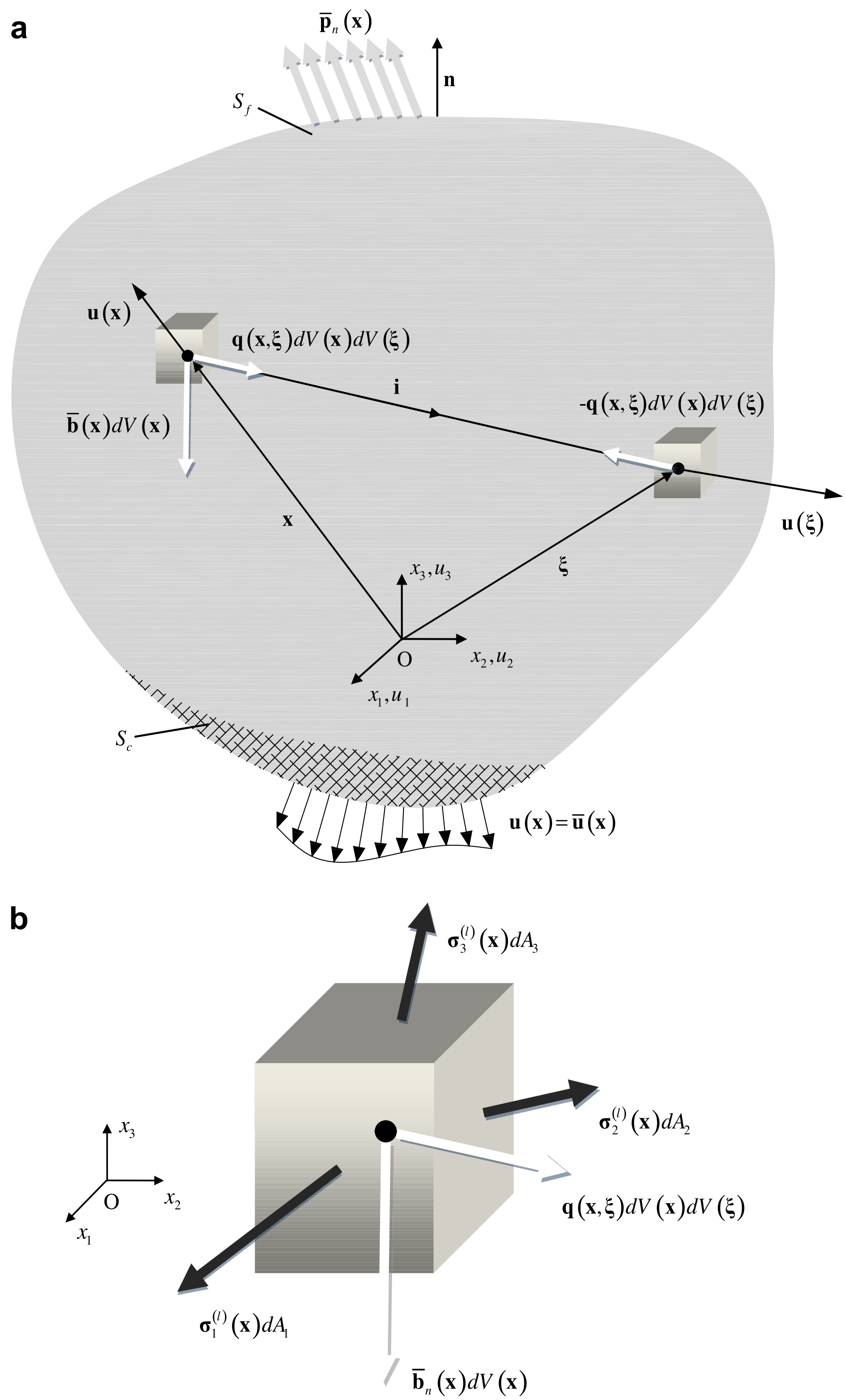

Fig. 1. 3D continuum with long-range interactions.

modelled as central forces depending on the products of the interacting volume elements located at $\mathbf{x}$ and ξ. The formulation proposed for a 1D bar may be readily generalized to a 3D continuum as follows.

Denote by $\mathbf{q}(\mathbf{x}, \xi)$ the (specific) long-range force exerted on a unit volume at $\mathbf{x}$ by a unit volume at ξ, as shown in Fig. 1a. The long-range force applied on the volume $dV(\mathbf{x})$ is modelled, in analogy to the 1D case, as proportional to the product of the interacting volumes, i.e. $q_k(\mathbf{x}, \xi)dV(\xi)dV(\mathbf{x})$. Therefore, the kth component of the long-range force on the unit volume at $\mathbf{x}$, due to a volume element $dV(\xi)$ at ξ, is given by

$$df_k(\mathbf{x}, \xi) = q_k(\mathbf{x}, \xi)dV(\xi) \tag{1}$$

where $q_k(\mathbf{x}, \xi)(k = 1, 2, 3)$ is the kth component of $\mathbf{q}(\mathbf{x}, \xi)$. The specific long-range force $q_k(\mathbf{x}, \xi)$ is modelled as a central force and is represented as

$$q_k(\mathbf{x}, \xi) = g_{kj}(\mathbf{x}, \xi)\eta_j(\mathbf{x}, \xi) \tag{2}$$

where $\eta_k(\mathbf{x}, \xi)(k = 1, 2, 3)$ are the components of the relative displacement vector $\boldsymbol{\eta}(\mathbf{x}, \xi)$ between the centroids of the volume elements located at $\mathbf{x}$ and ξ, that reads

$$\eta_k(\mathbf{x}, \xi) = u_k(\xi) - u_k(\mathbf{x}) \tag{3}$$

being $u_k(\mathbf{x})(k = 1, 2, 3)$ the kth component of $\mathbf{u}(\mathbf{x})$. Also, in Eq. (2), under the assumption of isotropic elastic continuum $g_{kj}(\mathbf{x}, \xi)$ is the double, symmetric and material-dependent tensor

$$g_{kj}(\mathbf{x},\xi) = g(\mathbf{x},\xi)J_{kj}(\mathbf{x},\xi) \quad (j,k=1,2,3) \tag{4}$$

where $J_{kj}(\mathbf{x},\xi)$ is the geometric Jacoby directional tensor given by $J_{kj}(\mathbf{x},\xi) = i_k(\mathbf{x},\xi)i_j(\mathbf{x},\xi)$, being i_k the kth component of the unit vector associated with the direction $\mathbf{x}-\xi$ and defined as:

$$i_k = \frac{(\xi_k - x_k)}{\sqrt{(\xi_j - x_j)(\xi_j - x_j)}} \tag{5}$$

In Eq. (4) $g(\mathbf{x},\xi)$ is a scalar, real-valued and symmetric function, i.e. $g(\mathbf{x},\xi) = g(\xi,\mathbf{x})$; also, it is material-dependent and distance decaying, that is it belongs to the functional class for which $g(\mathbf{x}_1,\xi_1) < g(\mathbf{x}_2,\xi_2)\ \forall|\mathbf{x}_1-\xi_1| > |\mathbf{x}_2-\xi_2|$. Therefore, Eq. (2) is the constitutive equation for the long-range forces.

It is worth remarking that, under the assumption of small displacements, the proposed model of 3D non-local interactions is invariant with respect to rigid-body displacements. It can be readily seen, in fact, that the long-range forces (2) vanish for any rigid-body displacement field, including rigid rotations.

In the context of the proposed model of 3D continuum with long-range central forces, the volume element $dV(\mathbf{x})$ is in equilibrium under the external body forces $b_k(\mathbf{x})dV(\mathbf{x})$, the long-range internal body forces $f_k(\mathbf{x})dV(\mathbf{x})$ and under contact stresses exerted by the adjacent volume elements, denoted by $\boldsymbol{\sigma}_k^{(l)}(\mathbf{x})$, where the apex in parenthesis means *local* since they are the classical Cauchy stresses (see Fig. 1b). Specifically, the long-range internal body forces $f_k(\mathbf{x})$ are the overall resultants of the long-range forces exerted on the volume element $dV(\mathbf{x})$ by all the volume elements at ξ, that is

$$f_k(\mathbf{x}) = \int_V q_k(\mathbf{x},\xi)dV(\xi) \quad (k=1,2,3) \tag{6}$$

Since $f_k(\mathbf{x})dV(\mathbf{x})$ are infinitesimals of third order, as $b_k(\mathbf{x})dV(\mathbf{x})$, the stress vector $\boldsymbol{\sigma}_n^{(l)}(\mathbf{x})$ acting on an elementary plane in $\mathbf{x}$, defined by its normal $\mathbf{n}$, is still given by the Cauchy stress relation as

$$\sigma_{nk}^{(l)} = \sigma_{kj}^{(l)} n_j \quad (j,k=1,2,3) \tag{7}$$

where n_j is the jth component of $\mathbf{n}$. Based on Eq. (7), the following remarkable feature of the proposed 3D non-local continuum may be derived: the static B.C. do not involve long-range interactions, that is

$$\bar{p}_{nk} = \sigma_{kj}^{(l)} n_j \quad (j,k=1,2,3) \text{ on } S_f \tag{8}$$

where n_j is the jth component of the outward normal $\mathbf{n}$ to the boundary surface. Such a result is not surprising since the long-range non-local interactions are modelled as (internal) body forces and then they do not affect the mechanical boundary conditions. Also, it has to be noted that the result in Eq. (8) cannot be obtained by defining the stress as in the KE non-local continuum. A rigorous proof of Eq. (8) has been also provided for a 1D bar in some previous works by Di Paola and Zingales (2008) and by Di Paola et al. (2009).

The equilibrium equations of the elementary volume $dV(\mathbf{x})$ singled out from the solid (Fig. 1b) is simply written in the form

$$\sigma_{kj,j}^{(l)} = -\bar{b}_k - f_k \tag{9}$$

where, according to Eqs. (2) and (6), the kth component of f_k reads

$$f_k(\mathbf{x}) = \int_V q_k(\mathbf{x},\xi)dV(\xi) = \int_V g_{kj}(\mathbf{x},\xi)\eta_j(\mathbf{x},\xi)dV(\xi), \tag{10}$$

Then, the equations governing the 3D non-local continuum include the strain–displacement equations

$$\varepsilon_{kj} = \frac{1}{2}(u_{k,j} + u_{j,k}) \tag{11}$$

and the constitutive equations relating the local, contact stress field to the strain field that are provided by

$$\sigma_{kj}^{(l)} = 2\mu^*\varepsilon_{kj} + \delta_{kj}\lambda^*\varepsilon_{hh} \tag{12}$$

where $\mu^* = \beta_1\mu$ and $\lambda^* = \beta_1\lambda$, being μ and λ the Lamè elastic constants and β_1 a dimensionless real coefficient, $0 \leqslant \beta_1 \leqslant 1$, weighing the amount of local interactions (Polizzotto, 2001) in analogy to those non-local theories where the non-local elastic material is conceived as a two-phase elastic material (Altan, 1989). Also, in Eq. (12) δ_{kj} is the Kronecker delta.

In summary, the governing equations of the linearly-elastic, isotropic and homogeneous 3D solid in presence of long-range interactions are

$$\begin{array}{ll} \text{Equilibrium} & \left\{ \begin{array}{ll} \sigma_{kj,j}^{(l)}(\mathbf{x}) = -\bar{b}_k(\mathbf{x}) - f_k(\mathbf{x}) & \forall \mathbf{x}\in V \\ \\ \varepsilon_{kj}(\mathbf{x}) = \frac{1}{2}(u_{k,j}(\mathbf{x}) + u_{j,k}(\mathbf{x})) & \forall \mathbf{x}\in V \\ \eta_k(\mathbf{x},\xi) = u_k(\xi) - u_k(\mathbf{x}) & \forall \mathbf{x},\xi\in V \\ \\ \sigma_{kj}^{(l)}(\mathbf{x}) = 2\mu^*\varepsilon_{kj}(\mathbf{x}) + \delta_{kj}\lambda^*\varepsilon_{hh}(\mathbf{x}) & \forall \mathbf{x}\in V \\ q_k(\mathbf{x},\xi) = g_{kj}(\mathbf{x},\xi)\eta_j(\mathbf{x},\xi) & \forall \mathbf{x},\xi\in V \end{array} \right. \\ \text{Constitutive} & \end{array} \tag{13a-e}$$

with the kinematic and static B.C.

$$\begin{array}{l} \text{Kinematic B.C.} \\ \text{Static B.C.} \end{array} \left\{ \begin{array}{ll} u_k(\mathbf{x}) = \bar{u}_k(\mathbf{x}) & \forall \mathbf{x}\in S_c \\ \sigma_{kj}^{(l)}(\mathbf{x})n_j = \bar{p}_{nk}(\mathbf{x}) & \forall \mathbf{x}\in S_f \end{array} \right. \tag{14a, b}$$

The reason for which Eq. (13c) shall be consistently considered among the equations governing the proposed non-local continuum will appear more evident in Section 4, where Eq. (13c) will be derived based on the principle of virtual forces.

In a previous paper (Di Paola et al., 2009) it has been shown that the proposed model reverts, in the simple 1D case, to an equivalent point-spring network, fully analogous to NNN lattice models. This motivates the definition of the proposed non-local continuum as a "mechanically-based" model. Such an equivalence holds true also for a 3D continuum; in this context, in analogy to the 1D point-spring model non-adjacent volumes are connected by linear springs of distance-decaying stiffness, applied to the centroids of interacting volumes.

Eqs. (13a) along with the static B.C. Eq. (14b) define a statically-admissible field of Cauchy stresses $\sigma_{kj}^{(l)}(\mathbf{x})$ and long-range forces $f_k(\mathbf{x})$; Eqs. (13b-c) along with Eqs. (14a) define a kinematically-admissible field of displacements $u_k(\mathbf{x})$ and strains $\varepsilon_{kj}(\mathbf{x})$. The solution to the full set of governing Eqs. (13) and (14) may be built, for instance, by the stiffness method, i.e. upon replacing Eq. (13b) for $\varepsilon_{kj}(\mathbf{x})$ in Eq. (13d) and then by replacing Eq. (13d) for $\sigma_{kj}^{(l)}(\mathbf{x})$ into Eq. (13a). This yields the following solving equations in an integro-differential form

$$\mu^*\nabla^2 u_k(\mathbf{x}) + (\lambda^*+\mu^*)u_{i,ik}(\mathbf{x}) + \int_V g_{kj}(\mathbf{x},\xi)\eta_j(\mathbf{x},\xi)dV(\xi) = -\bar{b}_k(\mathbf{x}) \quad \mathbf{x}\in V, \tag{15}$$

where $\nabla^2[\cdot] = [\cdot]_{,jj}$ is the Laplace operator. Correspondingly, the B.C. read

$$\left\{ \begin{array}{ll} \mu^*(u_{k,j}+u_{j,k})n_j + \lambda^* u_{j,j} = \bar{p}_{nk} & \text{on } S_f \\ u_k(\mathbf{x}) = \bar{u}_k(\mathbf{x}) & \text{on } S_c \end{array} \right. \tag{16a, b}$$

Eqs. (15) and (16) represent the generalization of the Navier equations for the proposed mechanically-based model of non-local continuum. Specifically, the first two terms in the l.h.s. of Eq. (15) are the classical, local terms and the third term accounts for non-local effects due to the long-range, internal body forces.

3. The static–kinematic duality in presence of long-range interactions

In this Section equations (13a) and (13b,c) will be derived based on the principle of virtual work.

To this aim, first let us consider an arbitrary displacement field $\tilde{u}_k(\mathbf{x})$ that satisfies the strain–displacement equations (13b) and the kinematic B.C. Eq. (14a) on S_c. Be $\hat{\sigma}^{(l)}_{kj}(\mathbf{x})$ the local stress field satisfying the equilibrium equations (13a) under arbitrary body forces $\hat{b}_k(\mathbf{x})$, $\hat{f}_k(\mathbf{x})$ and the static B.C. Eq. (14b) under external surface loads $\hat{p}_{nk}(\mathbf{x})$ on S_f. In the context of the proposed model of 3D continuum with long-range interactions, the principle of virtual work states that the work done by the surface loads $\hat{p}_{kn}(\mathbf{x})$ and the body forces $\hat{b}_k(\mathbf{x})$, $\hat{f}_k(\mathbf{x})$, due to the displacements of the application points $\tilde{u}_k(\mathbf{x})$, is equal to the work done by the local stresses $\hat{\sigma}^{(l)}_{kj}(\mathbf{x})$ due to the strain field $\tilde{\varepsilon}_{kj}(\mathbf{x})$, related to $\tilde{u}_k(\mathbf{x})$ by means of the strain-displacement equations (13b). That is,

$$\int_S \hat{p}_{nk}(\mathbf{x})\tilde{u}_k(\mathbf{x})dS + \int_V \hat{b}_k(\mathbf{x})\tilde{u}_k(\mathbf{x})dV + \int_V \hat{f}_k(\mathbf{x})\tilde{u}_k(\mathbf{x})dV = \int_V \hat{\sigma}^{(l)}_{kj}(\mathbf{x})\tilde{\varepsilon}_{kj}(\mathbf{x})dV \tag{17}$$

where the extra-term $\int_V \hat{f}_k(\mathbf{x})\tilde{u}_k(\mathbf{x})dV$ in the l.h.s. is the work done by the body forces $\hat{f}_k(\mathbf{x})$. The identity in Eq. (17) may be easily proved by applying the Green's theorem on the last two integrals in the l.h.s. of Eq. (17), upon taking into account the equilibrium equations (13a).

The work done by the force field $\hat{f}_k(\mathbf{x})$ can be rewritten in the form

$$\int_V \hat{f}_k(\mathbf{x})\tilde{u}_k(\mathbf{x})dV(\mathbf{x}) = -\frac{1}{2}\int_V\int_V \hat{q}_k(\mathbf{x},\xi)\tilde{\eta}_k(\mathbf{x},\xi)dV(\mathbf{x})dV(\xi) \tag{18}$$

This identity may be easily proved by considering that $\hat{f}_k(\mathbf{x})$ is the resultant of the central forces acting on the elementary volume located at $\mathbf{x}$, as reported in Eq. (6), so that the internal work $\int_V \hat{f}_k(\mathbf{x})\tilde{u}_k(\mathbf{x})dV$ reads

$$\int_V \hat{f}_k(\mathbf{x})\tilde{u}_k(\mathbf{x})dV(\mathbf{x}) = \int_V\int_V \hat{q}_k(\mathbf{x},\xi)\tilde{u}_k(\mathbf{x})dV(\mathbf{x})dV(\xi) = \int_V\int_V g_{kj}(\mathbf{x},\xi)\hat{\eta}_j(\mathbf{x},\xi)\tilde{u}_k(\mathbf{x})dV(\mathbf{x})dV(\xi) \tag{19}$$

Being $g_{kj}(\mathbf{x},\xi) = g_{kj}(\xi,\mathbf{x})$ due to Eqs. (4) and (5), the role of the arguments $\mathbf{x}$, ξ can be exchanged in the integral at the r.h.s. of Eq. (19). Therefore, taking into account that $\hat{\eta}_j(\mathbf{x},\xi) = \hat{u}_j(\xi) - \hat{u}_j(\mathbf{x})$ (see Eq. (3)), it can be seen that

$$\int_V\int_V g_{kj}(\mathbf{x},\xi)\hat{\eta}_j(\mathbf{x},\xi)\tilde{u}_k(\mathbf{x})dV(\mathbf{x})dV(\xi) = -\int_V\int_V g_{kj}(\mathbf{x},\xi)\hat{\eta}_j(\mathbf{x},\xi)\tilde{u}_k(\xi)dV(\mathbf{x})dV(\xi) \tag{20}$$

Then from Eqs. (19) and (20) it yields

$$\int_V \hat{f}_k(\mathbf{x})\tilde{u}_k(\mathbf{x})dV(\mathbf{x}) = \frac{1}{2}\int_V\int_V g_{kj}(\mathbf{x},\xi)\hat{\eta}_j(\mathbf{x},\xi)[\tilde{u}_k(\mathbf{x}) - \tilde{u}_k(\xi)]dV(\mathbf{x})dV(\xi) = -\frac{1}{2}\int_V\int_V \hat{q}_k(\mathbf{x},\xi)\tilde{\eta}_k(\mathbf{x},\xi)dV(\mathbf{x})dV(\xi) \tag{21}$$

that proves Eq. (18). Based on Eq. (18) the principle of virtual work (17) can be then rewritten as

$$\int_S \hat{p}_{nk}(\mathbf{x})\tilde{u}_k(\mathbf{x})dS + \int_V \hat{b}_k(\mathbf{x})\tilde{u}_k(\mathbf{x})dV(\mathbf{x}) = \int_V \hat{\sigma}^{(l)}_{kj}(\mathbf{x})\tilde{\varepsilon}_{kj}(\mathbf{x})dV(\mathbf{x}) + \frac{1}{2}\int_V\int_V \hat{q}_k(\mathbf{x},\xi)\tilde{\eta}_k(\mathbf{x},\xi)dV(\mathbf{x})dV(\xi) \tag{22}$$

It will be shown next that Eq. (22) sets a static–kinematic duality between the equations (13a) and (13b,c), which will be derived respectively by the principle of virtual displacements and the principle of virtual forces. Also, it will be shown that the energy balance (22) leads to appropriate elastic energy functionals, based on which the proposed 3D non-local elastic continuum will be given a consistent variational framework.

3.1. The principle of virtual displacements

Let us assume that in Eq. (22) the Cauchy stresses $\sigma^{(l)}_{kj}(\mathbf{x})$, the external body forces $\bar{b}_k(\mathbf{x})$, the long-range forces $f_k(\mathbf{x})$ and the external surface loads $\bar{p}_{nk}(\mathbf{x})$ are the true, equilibrated, mechanical quantities. Also, be $\tilde{u}_k(\mathbf{x}) = \delta u_k(\mathbf{x})$, $\tilde{\varepsilon}_{kj}(\mathbf{x}) = \delta\varepsilon_{kj}(\mathbf{x})$ and $\delta\eta_k(\mathbf{x},\xi) = \delta u_k(\mathbf{x}) - \delta u_k(\xi)$ arbitrary variations in the class of kinematically-admissible solutions, so that $\delta u_k = 0$ on S_c. In this case Eq. (22) takes the form

$$\int_{S_f} \bar{p}_{nk}(\mathbf{x})\delta u_k(\mathbf{x})dS + \int_V \bar{b}_k(\mathbf{x})\delta u_k(\mathbf{x})dV(\mathbf{x}) = \int_V \sigma^{(l)}_{kj}(\mathbf{x})\delta\varepsilon_{kj}(\mathbf{x})dV(\mathbf{x}) + \frac{1}{2}\int_V\int_V q_k(\mathbf{x},\xi)\delta\eta_k(\mathbf{x},\xi)dV(\mathbf{x})dV(\xi) \tag{23}$$

Bearing in mind that $\delta\varepsilon_{kj}(\mathbf{x})$ and $\delta\eta_k(\mathbf{x},\xi)$ satisfy equations (13b,c), i.e. $\delta\varepsilon_{kj}(\mathbf{x}) = [(\delta u_k)_{,j} + (\delta u_j)_{,k}]/2$ and $\delta\eta_k(\mathbf{x},\xi) = \delta u_k(\xi) - \delta u_k(\mathbf{x})$ in V, the principle of virtual displacements (23) yields the equilibrium equations (13a) involving $\sigma^{(l)}_{kj}(\mathbf{x})$, $\bar{b}_k(\mathbf{x})$ and $f_k(\mathbf{x})$, along with the static B.C. Eq. (14b) involving $\sigma^{(l)}_{kj}(\mathbf{x})$ and $\bar{p}_{nk}(\mathbf{x})$. This is shown by considering that the r.h.s. of Eq. (23) can be cast in the equivalent form

$$\int_V \sigma^{(l)}_{kj}(\mathbf{x})\delta\varepsilon_{kj}(\mathbf{x})dV(\mathbf{x}) + \frac{1}{2}\int_V\int_V q_k(\mathbf{x},\xi)\delta\eta_k(\mathbf{x},\xi)dV(\mathbf{x})dV(\xi) = -\int_V \sigma^{(l)}_{kj,j}(\mathbf{x})\delta u_k(\mathbf{x})dV(\mathbf{x}) + \int_{S_f} \sigma^{(l)}_{kj}(\mathbf{x})n_j\delta u_k(\mathbf{x})dS_f(\mathbf{x}) - \int_V f_k(\mathbf{x})\delta u_k(\mathbf{x})dV(\mathbf{x}) \tag{24}$$

where the latter term in the r.h.s. is obtained by from Eq. (18). Based on Eq. (24) and taking into account that Eq. (23) holds true for any virtual displacement field, the equilibrium equations (13a) along with the static B.C. Eq. (14a) are retrieved.

3.2. The principle of virtual forces

Let us assume that in Eq. (22) the displacements $u_k(\mathbf{x})$, the relative displacements $\eta_k(\mathbf{x},\xi)$ and the strains $\varepsilon_{kj} = (u_{k,j} + u_{j,k})/2$ are the real displacement and strain fields, respectively. Further, we assume that $\hat{\sigma}^{(l)}_{kj}(\mathbf{x}) = \delta\sigma^{(l)}_{kj}(\mathbf{x})$ and $\hat{q}_k(\mathbf{x},\xi) = \delta q_k(\mathbf{x},\xi)$ are arbitrary variations in the class of statically-admissible Cauchy stresses and long-range forces. Therefore, $\delta\sigma^{(l)}_{ij}(\mathbf{x})$ and $\delta q_k(\mathbf{x},\xi)$ are in equilibrium with vanishing external body forces $b_k(\mathbf{x})$ and surface loads $p_{nk}(\mathbf{x})$ on S_f so that the equilibrium equations read

$$\begin{aligned} &\left(\delta\sigma^{(l)}_{kj}(\mathbf{x})\right)_{,j} = -\delta f_k(\mathbf{x}) && \forall\mathbf{x}\in V \\ &\delta\sigma^{(l)}_{kj}(\mathbf{x})n_j = 0 && \forall\mathbf{x}\in S_f \\ &\delta\sigma^{(l)}_{kj}(\mathbf{x})n_j = \delta p_{nk}(\mathbf{x}) && \forall\mathbf{x}\in S_c \end{aligned} \tag{25a, b, c}$$

where $\delta\sigma^{(l)}_{kj}(\mathbf{x})$ are arbitrary on S_c.

The fundamental identity (22), rewritten in terms of $\hat{b}_k(\mathbf{x}) = \delta b_k(\mathbf{x}) = 0$, $\hat{p}_{nk}(\mathbf{x}) = \delta p_{nk}(\mathbf{x}) = 0$ on S_f, $\hat{\sigma}^{(l)}_{ij}(\mathbf{x}) = \delta\sigma^{(l)}_{ij}(\mathbf{x})$ and $\hat{q}_k(\mathbf{x},\xi) = \delta q_k(\mathbf{x},\xi)$, may be then recast as follows

$$\int_{S_u} u_k(\mathbf{x})\delta p_{nk}(\mathbf{x})dS(\mathbf{x}) = \int_V \varepsilon_{kj}(\mathbf{x})\delta\sigma^{(l)}_{kj}(\mathbf{x})dV(\mathbf{x}) + \frac{1}{2}\int_V\int_V \eta_k(\mathbf{x},\xi)\delta q_k(\mathbf{x},\xi)dV(\mathbf{x})dV(\xi) \tag{26}$$

Based on Eqs. (25a), the following relation holds

$$\int_V \left(\delta\sigma_{kj}^{(l)}(\mathbf{x})\right)_{,j} u_k(\mathbf{x})dV(\mathbf{x}) = -\int_V \delta\sigma_{kj}^{(l)}(\mathbf{x})u_{k,j}(\mathbf{x})dV(\mathbf{x}) + \int_{S_c} \delta\sigma_{kj}^{(l)}(\mathbf{x})n_j(\mathbf{x})u_k(\mathbf{x})dS(\mathbf{x}) + \int_V \delta f_k(\mathbf{x})u_k(\mathbf{x})dV(\mathbf{x}) = 0 \quad (27)$$

If Eq. (27) is added to the r.h.s. of Eq. (26), taking into account Eq. (21) and due to the symmetry of the double tensor $\delta\sigma_{kj}^{(l)}(\mathbf{x})$, it yields

$$\int_{S_c} [u_k(\mathbf{x}) - \bar{u}_k(\mathbf{x})]\delta p_{nk}(\mathbf{x})dS(\mathbf{x}) = \int_V \left\{\varepsilon_{kj}(\mathbf{x}) - \left[u_{k,j}(\mathbf{x}) + u_{j,k}(\mathbf{x})\right]/2\right\}\delta\sigma_{kj}^{(l)}(\mathbf{x})dV(\mathbf{x}) + \frac{1}{2}\int_V\int_V \{\eta_k(\mathbf{x},\xi) - [u_k(\xi) - u_k(\mathbf{x})]\}\delta q_k(\mathbf{x},\xi)dV(\mathbf{x})dV(\xi) \quad (28)$$

Since Eq. (28) holds true for any set of virtual Cauchy stresses $\delta\sigma_{ij}^{(l)}(\mathbf{x})$ and long-range forces $\delta q_k(\mathbf{x},\xi)$, Eqs. (13b,c) along with the kinematic B.C. Eq. (14a) are retrieved. This motivates the need to include Eqs. (13c) among the equations governing the proposed 3D non-local continuum.

3.3. Energy balance and variational principles

The energy balance between the work done by the applied loads and the corresponding elastic potential energy stored in the solid is obtained from the assumption that, in the fundamental identity (22), the kinematic variables $u_k(\mathbf{x}),\ \eta_k(\mathbf{x},\xi),\ \varepsilon_{kj}(\mathbf{x})$ and the static variables $\sigma_{kj}^{(l)}(\mathbf{x}),\ q_k(\mathbf{x},\xi)$ are the real solutions of the elastic problem. Under this assumption Eq. (22) leads to

$$L_{ext} = \frac{1}{2}\left[\int_V \bar{b}_k(\mathbf{x})u_k(\mathbf{x})dV(\mathbf{x}) + \int_{S_f}\bar{p}_{nk}(\mathbf{x})u_k(\mathbf{x})dS_f(\mathbf{x}) + \int_{S_c}\bar{p}_{nk}(\mathbf{x})\bar{u}_k(\mathbf{x})dS_c(\mathbf{x})\right] = \frac{1}{2}\left[\int_V \sigma_{kj}^{(l)}(\mathbf{x})\varepsilon_{kj}(\mathbf{x})dV(\mathbf{x}) + \frac{1}{2}\int_V\int_V q_k(\mathbf{x},\xi)\eta_k(\mathbf{x},\xi)dV(\mathbf{x})dV(\xi)\right] = L_{int} \quad (29)$$

The l.h.s. of Eq. (29) is the external work done by the body forces $\bar{b}_k(\mathbf{x})$ in V, the surface loads $\bar{p}_{nk}(\mathbf{x})$ on S_f and the corresponding reactions $\bar{p}_{nk}(\mathbf{x})$ on S_c. The r.h.s. of Eq. (29) is the internal work done by the Cauchy stresses $\sigma_{kj}^{(l)}(\mathbf{x})$ and the long-range forces $f_k(\mathbf{x})$. Based on Eq. (29) the proposed non-local model can be given a variational formulation, where the elastic potential energy $\Phi(\varepsilon_{kj},\eta_k)$ and the complementary elastic energy $\Psi(\sigma_{kj}^{(l)},q_k)$ are given in the form

$$\Phi(\varepsilon_{kj},\eta_k) = \Phi^{(l)}(\varepsilon_{kj}) + \Phi^{(nl)}(\eta_k) = \int_V \phi^{(l)}(\varepsilon_{kj}(\mathbf{x}))dV(\mathbf{x}) + \frac{1}{2}\int_V\int_V \phi^{(nl)}(\eta_k(\mathbf{x},\xi))dV(\mathbf{x})dV(\xi) \quad (30a)$$

$$\Psi(\sigma_{kj}^{(l)},q_k) = \Psi^{(l)}(\sigma_{kj}^{(l)}) + \Psi^{(nl)}(q_k) = \int_V \psi^{(l)}(\sigma_{kj}^{(l)}(\mathbf{x}))dV(\mathbf{x}) + \frac{1}{2}\int_V\int_V \psi^{(nl)}(q_k(\mathbf{x},\epsilon))dV(\mathbf{x})dV(\xi) \quad (30b)$$

In Eqs. (30) the kernels of the volume integrals $\phi^{(l)}(\varepsilon_{kj}(\mathbf{x}))$ and $\phi^{(nl)}(\eta_k(\mathbf{x},\xi))$ are specific elastic potential energies associated with the strain field and to the relative displacement field, respectively defined as

$$\phi^{(l)}(\varepsilon_{kj}(\mathbf{x})) = \mu^*\varepsilon_{kj}(\mathbf{x})\varepsilon_{kj}(\mathbf{x}) + \frac{\lambda^*}{2}\varepsilon_{kk}(\mathbf{x})\varepsilon_{kk}(\mathbf{x}),$$
$$\phi^{(nl)}(\eta_k(\mathbf{x},\xi)) = \frac{1}{2}g_{kj}(\mathbf{x},\xi)\eta_j(\mathbf{x},\xi)\eta_k(\mathbf{x},\xi). \quad (31a,b)$$

The counterparts of $\phi^{(l)}(\varepsilon_{kj}(\mathbf{x}))$ and $\phi^{(nl)}(\eta_k(\mathbf{x},\xi))$, namely the specific complementary elastic energies $\psi^{(l)}(\sigma_{kj}^{(l)}(\mathbf{x}))$ and $\psi^{(nl)}(q_k(\mathbf{x},\xi))$ in Eq. (30b), respectively associated with the local Cauchy stress and with the long-range forces, are given as

$$\psi^{(l)}\left(\sigma_{kj}^{(l)}(\mathbf{x})\right) = \frac{1}{2}\left(\frac{\sigma_{kj}^{(l)}(\mathbf{x})\sigma_{kj}^{(l)}(\mathbf{x})}{2\mu^*} - \frac{3\lambda\sigma_{kk}^{(l)}(\mathbf{x})\sigma_{kk}^{(l)}(\mathbf{x})}{2\mu^*(2\mu^*+3\lambda^*)}\right),$$
$$\psi^{(nl)}(q_k(\mathbf{x},\xi)) = \frac{1}{2}(g_{kj}(\mathbf{x},\xi))^{-1}q_j(\mathbf{x},\xi)q_k(\mathbf{x},\xi). \quad (32a,b)$$

It may be verified that the constitutive relations used in Section 2 may be derived directly from Eqs. (31a,b) as $\sigma_{kj}^{(l)}(\mathbf{x}) = \partial\phi^{(l)}(\varepsilon_{kj}(\mathbf{x}))/\partial\varepsilon_{kj}$ and $q_k(\mathbf{x},\boldsymbol{\epsilon}) = \partial\phi^{(nl)}(\eta_k(\mathbf{x},\boldsymbol{\epsilon}))/\partial\eta_k = g_{kj}(\mathbf{x},\xi)\eta_j(\mathbf{x},\boldsymbol{\epsilon})$. Similar considerations hold true also for the inverses forms obtained from Eq. (32a,b) as $\varepsilon_{kj}(\mathbf{x}) = \partial\psi^{(l)}(\sigma_{kj}^{(l)}(\mathbf{x}))/\partial\sigma_{kj}^{(l)}$ and $\eta_k(\mathbf{x},\boldsymbol{\epsilon}) = \partial\psi^{(nl)}(q_k(\mathbf{x},\boldsymbol{\epsilon}))/\partial q_k = (g_{kj}(\mathbf{x},\xi))^{-1}q_j$. At this stage it is necessary to make some further comments about the functional class of the distance-decaying function $g(\mathbf{x},\xi)$. Such a function has been introduced, on a mechanical basis, assuming that the long-range central forces counteract the relative displacements between non-adjacent elements. On this basis, therefore, $g(\mathbf{x},\xi)$ has been taken as a symmetric and strictly positive function of the distance between two interacting volumes. Further, if the material is isotropic, $g(\mathbf{x},\xi)$ is chosen as a function depending only on the distance between the interacting volumes $d = \|\mathbf{x}-\xi\| = [(x_i-\xi_i)(x_i-\xi_i)]^{1/2}$, i.e. $g(\mathbf{x},\xi) = g(\|\mathbf{x}-\xi\|)$, and not on the volume locations. The requirement $g(\mathbf{x},\xi) \geqslant 0$ in the whole solid domain is mandatory as it is related to the material stability criterion in presence of long-range interactions as it has been assessed in previous studies (Di Paola et al., 2009).

The total potential energy stored in the 3D solid, defined as $\Pi(u_k,\varepsilon_{kj},\eta_k) = \Phi(\eta_k,\varepsilon_{kj}) + P(u_k)$ where $P(u_k)$ is the potential energy associated with the conservative fields $\bar{b}_k(\mathbf{x})$ and $\bar{p}_{nk}(\mathbf{x})$, attains its minimum at the solution of the elastic equilibrium problem (see Di Paola et al., 2010 for details). Further the Euler–Lagrange equations as well as the mechanical boundary conditions associated with the total potential energy $\Pi(u_k,\varepsilon_{kj},\eta_k)$ coincide with Eq. (15) in V and with Eq. (16a) on S_f, thus proving the mathematical consistency of the proposed model of non-local interactions. Similar considerations hold true also for the total complementary energy functional $\Xi\left(\bar{b}_k,\bar{p}_k,\sigma_{kj}^{(l)},q_k\right)$ as reported in previous papers (Di Paola et al., 2009c; Failla et al., 2010) and they have not been included for brevity's sake.

In the next section some numerical applications will be reported for 2D cases of plane stress.

4. Numerical applications

Two study cases will be considered, (i) a square plate under in-plane loads and (ii) a circular plate under in-plane and symmetrically-distributed radial loads. Solutions will be built by (a) a standard finite difference method and (b) a Galerkin method, where the basis functions used to express the sought displacement response are also used as weighting functions.

In both cases (i) and (ii), an exponential decay is assumed for the long-range forces, that is

$$g(\mathbf{x},\xi) = C\exp(-\|\mathbf{x}-\xi\|/l) \quad (33)$$

where C is a constant parameter and l is the internal length (Marotti De Sciarra, 2008). Clearly both C and $l$depend on the material under study and shall be selected based on experimental evidence. However, the full set of material parameters used in the following, including the non-local parameters C and l, will be given suitable values chosen to enhance non-local effects in the response, case in which it appears more meaningful to assess the matching between the two solution methods used.

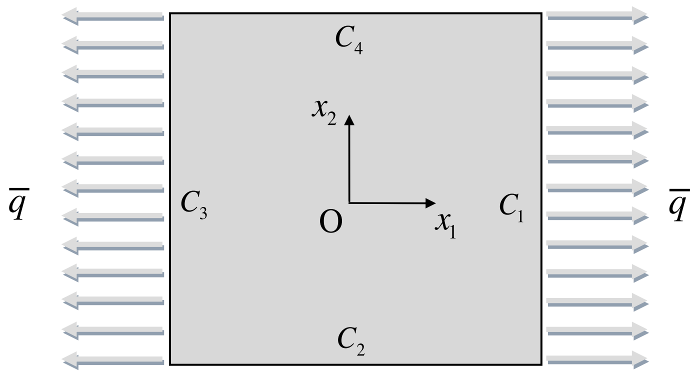

Fig. 2. Square plate acted upon by in-plane self-equilibrated surface loads.

4.1. Square plate

Consider the square plate shown in Fig. 2. Be $O(x_1, x_2)$ a Cartesian coordinate system with origin O located at the center of the plate. Therefore, under plane stress conditions the only non-vanishing local stress components $\sigma_{ij}^{(l)}$ correspond to indexes $i, j = 1, 2$ and $q_3 = 0$ for the non-local forces.

The governing equations as well as the pertinent boundary conditions can be obtained as a particular case of Eqs. (16)–(17),

$$\mu'\nabla^2 u_k(\mathbf{x}) + (\lambda' + \mu')u_{j,jk} + s\int_A g_{kj}(\mathbf{x}, \xi)\eta_j(\mathbf{x}, \xi)dA(\xi) = 0 \quad \forall \mathbf{x} \in A \tag{34a}$$

$$\mu'(u_{k,j} + u_{j,k})\eta_j + \lambda' u_{j,j} = \bar{p}_{nk} \quad \forall \mathbf{x} \in C_1, C_3 \tag{34b}$$

$$\mu'(u_{k,j} + u_{j,k})\eta_j + \lambda' u_{j,j} = 0 \quad \forall \mathbf{x} \in C_2, C_4 \tag{34c}$$

for $j, k = 1, 2$, where s is the thickness and A is the surface area. In Eqs. (34) the elastic moduli μ' and λ' are the reduced Lamé elastic constants for plane stress analysis, related to μ and λ by

$$\lambda' = \frac{2\lambda\mu}{(\lambda + \mu)}\beta_1 \quad \mu' = \mu\beta_1 \tag{35}$$

where, as explained in Section 2 (see Eq. (12)), β_1 weighs the amount of local interactions (Polizzotto, 2001).

Numerical results are presented for a steel square plate ($E = 230$ GPa, $\nu = 0.3$) of side $a = 1$ m, $s = 2 \times 10^{-2}$ m, and loaded along the sides $x_1 = \pm a/2$ by a set of self-equilibrated loads per unit length equal to $\bar{q} = 10^4$ Nm^{-1} (that is, $\bar{p}_{nk} = \bar{q}/s$ in Eq. (34b)). Also, $\beta_1 = 0.9$ and $C = 5 \times 10^{16}$ Nm^{-7} and $l = 0.1$ m are selected in Eq. (33) for the exponential decay. The finite difference solution to the integro-differential system in Eqs. (34) is obtained based on uniform $n \times n$ grid, with $\Delta x_i = 1/(n-1)$. The Galerkin solution is built by expanding the sought displacement functions $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ in the forms

$$u_1(x_1, x_2) = \sum_{p=1}^{m_1}\sum_{q=1}^{n_1} c_{pq} J_{2p-1}(x_1) J_{2q}(x_2) \tag{36a}$$

$$u_2(x_1, x_2) = \sum_{p=1}^{m_2}\sum_{q=1}^{n_2} d_{pq} J_{2p}(x_1) J_{2q-1}(x_2) \tag{36b}$$

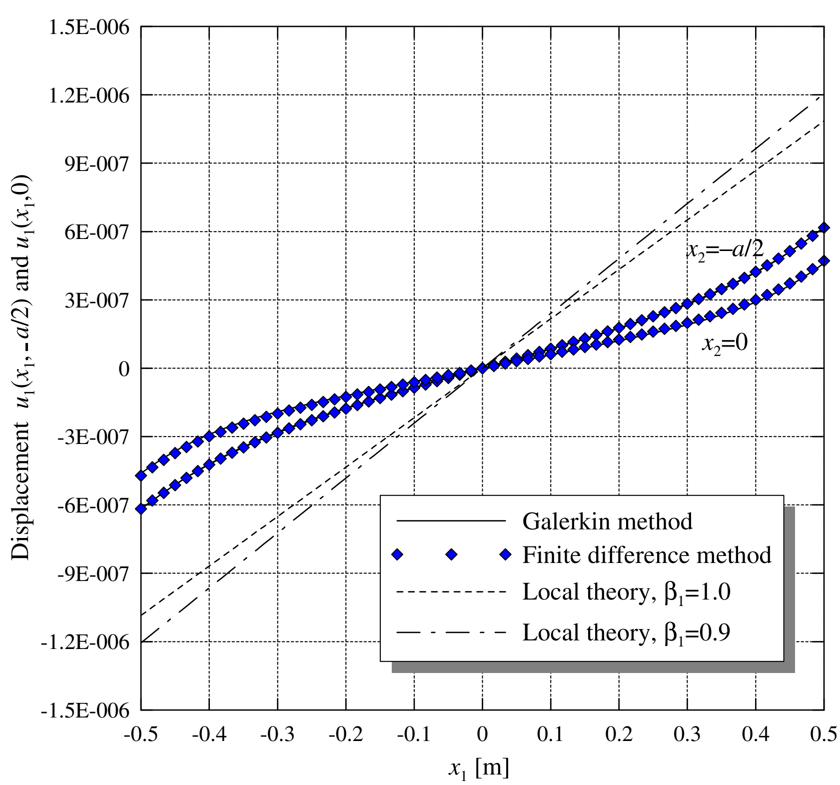

Fig. 3. Displacement $u_1(x_1, x_2)$ at $x_2 = 0$ and $x_2 = -a/2$.

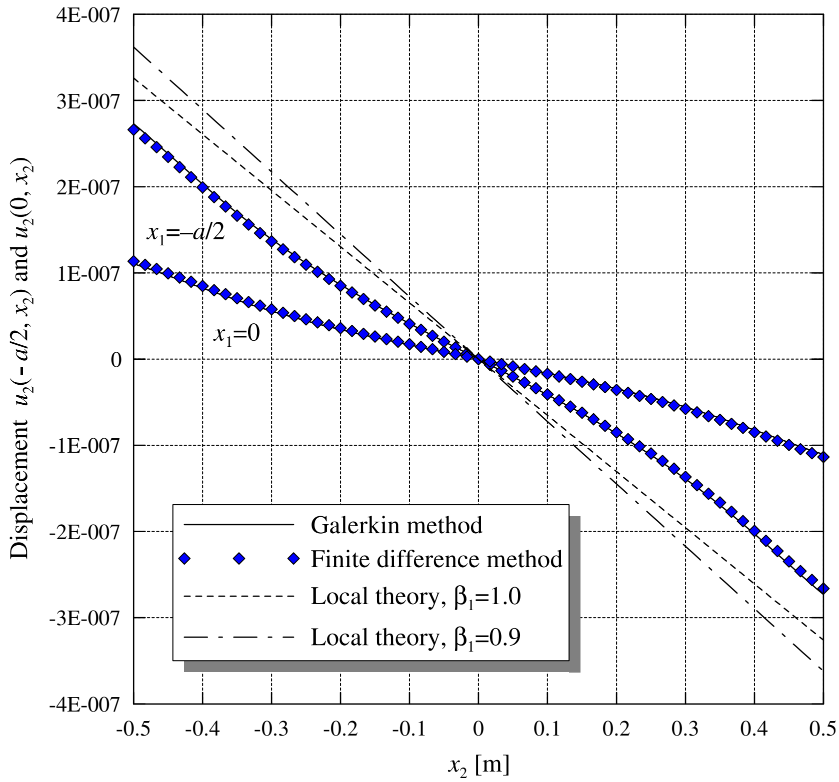

Fig. 4. Displacement $u_2(x_1, x_2)$ at $x_1 = 0$ and $x_1 = -a/2$.

where c_{pq} and d_{pq} are unknown coefficients, while $J_k(\cdot)$ is the kth order Jacobi polynomial. Figs. 3–6 show the displacement and the strain fields obtained by the finite difference method, when $n = 61$ is selected, and by the Galerkin method, when $m_1 = n_1 = m_2 = n_2 = 5$ is set in Eqs. (36). At different sections along the plate domain, the two methods appear in a satisfactory agreement. Note that no significant differences are encountered if the finite difference grid is refined ($n > 61$) or if the number of terms in

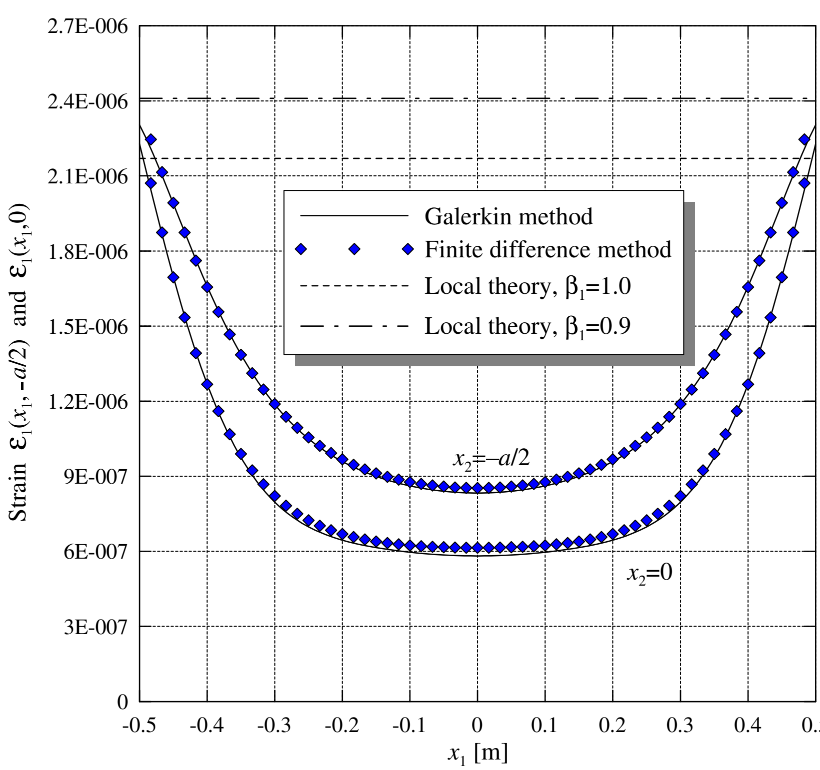

Fig. 5. Strain $\epsilon_1(x_1, x_2)$ at $x_2 = 0$ and $x_2 = -a/2$.

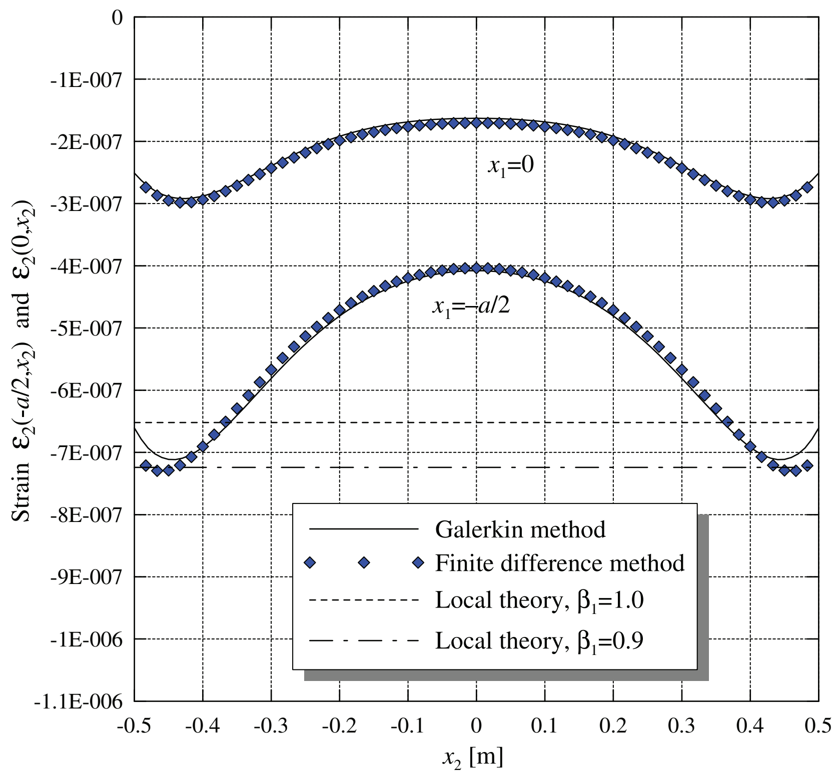

Fig. 6. Strain $\epsilon_2(x_1, x_2)$ at $x_1 = 0$ and $x_1 = -a/2$.

Eqs. (36) is increased; pertinent results have then been omitted for brevity. For comparison the local solution also is considered in Figs. 3–6; specifically, the local solution obtained when the non-local terms are neglected in Eq. (34) and, also, the classical local solution obtained for standard material parameters, i.e. for $\beta_1 = 1.0$. It is then seen that, if compared to the corresponding displacements predicted by local elasticity, the displacements are smaller in the inner part of the plate and they increase towards the boundary.

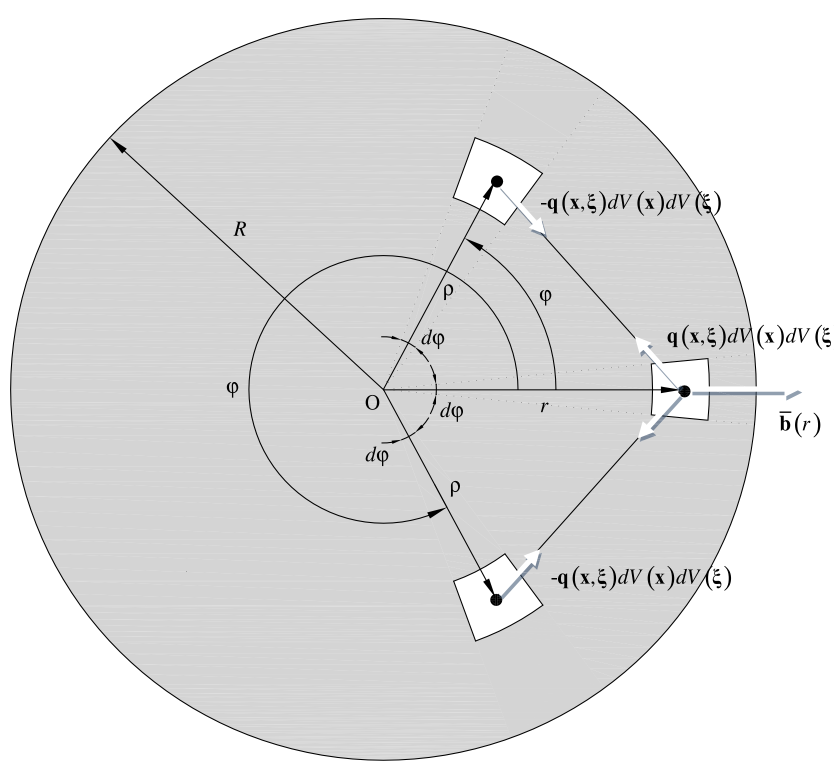

Fig. 7. Circular plate with long-range interactions.

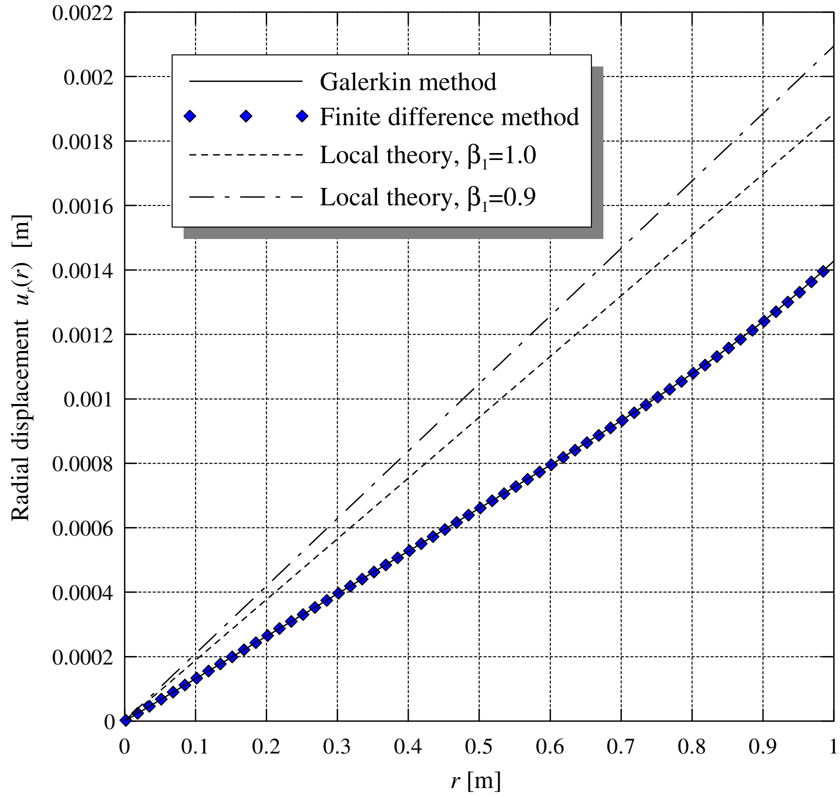

Fig. 8. Radial displacement $u_r(r)$.

Therefore, both the displacements $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ do exhibit the same qualitative behaviour as the displacement found for a 1D bar acted upon by self-equilibrated end forces (Di Paola et al., 2009; Failla et al., 2010).

4.2. Circular plate

Let us consider a circular plate as in Fig. 7. Let $O(r, \theta)$ be a polar coordinate system whose origin O is located at the center of the plate; r is the radial distance from the origin and θ is the azimuth, taken to be positive if counter-clockwise; R is the radius. The circular plate can be subjected to symmetrically-distributed in-plane body forces, denoted by $\bar{b}(r)$, and boundary forces at $r = R$, denoted by $\bar{p}_R$. In this case, the response variables depend on the radial distance only. Therefore, be $u(r)$ the radial displacement, $\varepsilon_r(r) = u_{r,r}$ the radial strain and $\varepsilon_\theta(r) = u/r$ the circumferential strain. The stress state is a plane stress and, therefore, the only non-vanishing components are the radial components $\sigma_r(r)$ (local stress) and $f_r(r)$ (non-local body force) and the circumferential component $\sigma_\theta(r)$ (local stress). The governing equations of the plane elasticity problem, formulated in polar coordinates, are reported in the Appendix A.

Numerical results are presented for a circular aluminium plate ($E = 78$ GPa, $\nu = 0.34$) of radius $R = 1$ m, $s = 1 \times 10^{-3}$ m, and loaded at $r = R$ by a symmetrically-distributed boundary load per unit length equal to $\bar{p}_R = 2 \times 10^5$ Nm^{-1}. Also, $\beta_1 = 0.9$ and $C = 5 \times 10^{16}$ Nm^{-7}, $l = 0.1$ m are selected in Eq. (33) for the exponential decay that, in polar coordinates, is rewritten as

$$g(r, \rho, \varphi) = C \exp\left(-\sqrt{(\rho\cos\varphi - r)^2 + (\rho\sin\varphi)^2} \Big/ l\right) \quad (37)$$

The finite difference solution to the integro-differential equation (A.8) is obtained based on a uniform $n_r \times n_\theta$ grid, with $\Delta r = 1/(n_r - 1)$ and $\Delta\theta = 1/(n_\theta - 1)$. The Galerkin solution is built by expanding the sought displacement function $u(r)$ in the following series expansion in terms of odd-order Jacobi polynomials

$$u(r) = \sum_{k=1}^{m} c_k J_{2k-1}(r) \quad (38)$$

Figs. 8 and 9 show a very good agreement between the solutions in terms of radial displacement $u(r)$ and radial strain $\varepsilon_r(r)$, obtained by the finite difference method for $n_r = 600$, $n_\theta = 30,000$ and by the Galerkin method for $m = 5$. As expected, as compared to the previous application on a square plate, the finite difference method requires a very fine grid to approximate the circular plate domain. Especially a coarse grid on the azimuth θ, in fact, determines an increasing distance between grid points as the radial distance increases. However, no substantial differences have been found in the results as $n_r > 600$ or $n_\theta > 30,000$ and, for this reason, additional results are omitted for brevity. The same can be stated for the Galerkin solution, where no significant differences are encountered as $m > 5$. For comparison the local solutions also are considered in Figs. 8 and 9: the local solution obtained when the non-local terms are neglected in Eq. (A.8) and the classical local solution obtained for standard material parameters (i.e., for $\beta_1 = 1.0$). As already pointed out for the square plate discussed in Section 4.1 it is seen that, if compared to the radial displacement predicted by local elasticity for a circular slab under the same radial loads, the non-local radial displacement is smaller in the inner part of the slab and increases as the distance r tends to the external radius R.

5. Conclusions

This paper presents the generalization of a previously proposed 1D model of non-local elasticity to a 3D linearly-elastic, isotropic and homogeneous solid under static load. The mathematical model

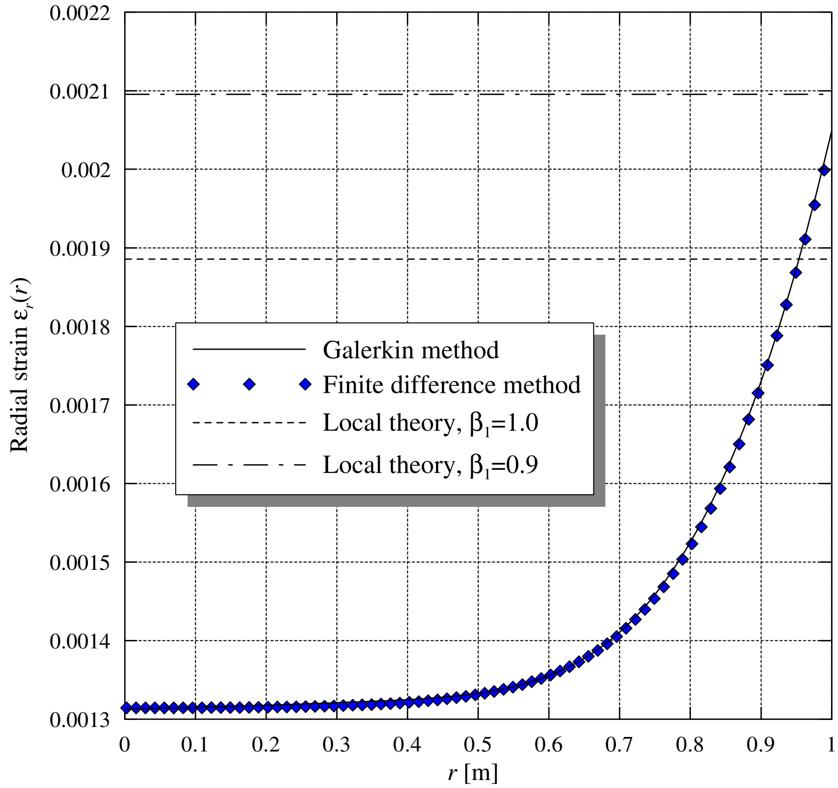

Fig. 9. Radial strain $\epsilon_r(r)$.

has been introduced upon the consideration that crystal lattices may exchange only central-type interactions, so that a continuum model accounting for such effects has been formulated. The long-range interactions between non-adjacent elements of the continuum model have been modelled as dependent on the product of the interacting volumes, on the relative displacement along the central direction and on a proper distance-decaying function. The specific choice of the functional class of the distance-decaying interactions as well as the parameters of the decay are strictly related to the material microstructure interactions.

The elastic problem requires the solution of an integro-differential system of equations involving either the gradients of the displacement field as well as the integrals of the relative displacement field extended to the volume of the solid. The integro-differential system must be supplemented by the kinematic and static B.C., where the latter are expressed as in classical, local mechanics since non-local interactions are represented by body forces and, therefore, they do not contribute at the boundary of the solid. The formulation of the elastic problem has been reported based on mechanical as well as on variational considerations; a proper correspondence between the static and kinematic variables has been preliminarily set based on the principle of virtual work. It has been also shown that the solution of the proposed mechanically-based model of non-local elasticity corresponds to a global minimum of the elastic potential energy function, involving the displacement, the strain and the relative displacement field as state variables.

Numerical applications showing the capabilities of the proposed method have been reported for different cases of plane elasticity where long-range interactions are ruled by an exponential distance-decaying function. Solutions built based on the finite difference method and the Galerkin method have been found in a very good agreement.

More generally, for 3D continua of complex geometry standard finite element solutions can be built. To this aim the principle of virtual displacements given in Section 3.1 or the total potential energy functional given in Section 3.3 can be easily used to derive the local and non-local stiffness matrices, upon replacing the displacement functions built based on a given mesh.

Acknowledgements

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Appendix A. Governing equations in polar coordinates

Let us consider the circular plate in Fig. 7. The in-plane equilibrium equation along the radial direction may be written in polar coordinates as

$$\sigma_{r,r} + \frac{\sigma_r - \sigma_\theta}{r} + \bar{b}_r + f_r = 0 \tag{A.1}$$

where f_r is the non-local force per unit-volume. Based on the Jacobian of the coordinate transformation

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta; \tag{A.2a}$$
$$\xi_1 = \rho\cos\varphi, \quad \xi_2 = \rho\sin\varphi; \tag{A.2b}$$

the Cartesian integral form (6) for the non-local force f_r reverts to the following integral form

$$f_r(r) = 2s\int_0^\pi \int_0^R g(r,\rho,\varphi)(\eta_1 i_1^2 + \eta_2 i_1 i_2)\rho\, d\rho\, d\varphi \tag{A.3}$$

where s is the thickness,

$$\eta_1(r,\rho,\varphi) = u_r(\rho)\cos\varphi - u_r(r), \quad \eta_2(\rho,\varphi) = u_r(\rho)\sin\varphi \tag{A4a, b}$$

$$i_1 = \frac{(\rho\cos\varphi - r)}{\sqrt{(\rho\cos\varphi - r)^2 + (\rho\sin\varphi)^2}},$$
$$i_2 = \frac{\rho\sin\varphi}{\sqrt{(\rho\cos\varphi - r)^2 + (\rho\sin\varphi)^2}} \quad \text{(A5a, b)}$$

Eq. (A.3) for f_r may be also built by computing the resultant force per unit-volume at $(r, 0)$, due to all the volume elements $dV(\rho,\varphi)$ at (ρ,φ); note that such resultant has only a radial component (i.e., f_r in Eq. (A.3)) since the circumferential component vanishes due to polar symmetry. Under plane-stress conditions, σ_r and σ_θ are given respectively by

$$\sigma_r(r) = \frac{E}{1-\nu^2}[\varepsilon_r + \nu\varepsilon_\theta] = \frac{E}{1-\nu^2}\left[u_{r,r} + \nu\frac{u}{r}\right] \quad \text{(A.6)}$$
$$\sigma_\theta(r) = \frac{E}{1-\nu^2}[\varepsilon_\theta + \nu\varepsilon_r] = \frac{E}{1-\nu^2}\left[\frac{u}{r} + \nu u_{r,r}\right] \quad \text{(A.7)}$$

By replacing Eq. (A.6) for σ_r and Eq. (A.7) for σ_θ, Eq. (A.1) can be rewritten in terms of the radial displacement $u(r)$ only in the form

$$\frac{E}{1-\nu^2}\left[u_{r,r} + \nu\frac{u}{r} + r u_{r,rr} + \nu(u_{r,r} - u_r/r) - \frac{u}{r} - \nu u_{r,r}\right] + 2rs\int_0^\pi\int_0^R g(r,\rho,\varphi)\left(\eta_1 i_1^2 + \eta_2 i_1 i_2\right)\rho\, d\rho\, d\varphi = 0 \quad \text{(A.8)}$$

Eq. (A.8) is the integro-differential equation governing the in-plane equilibrium of a symmetrically-loaded circular plate in presence of long-range central interactions. The pertinent B.C. are analogous to the classical elasticity theory, that is they read

$$u(0) = 0; \quad u(R) = \bar{u}_R \text{ or alternatively } \sigma_r(R) = \bar{p}_R \quad \text{(A.9)}$$

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