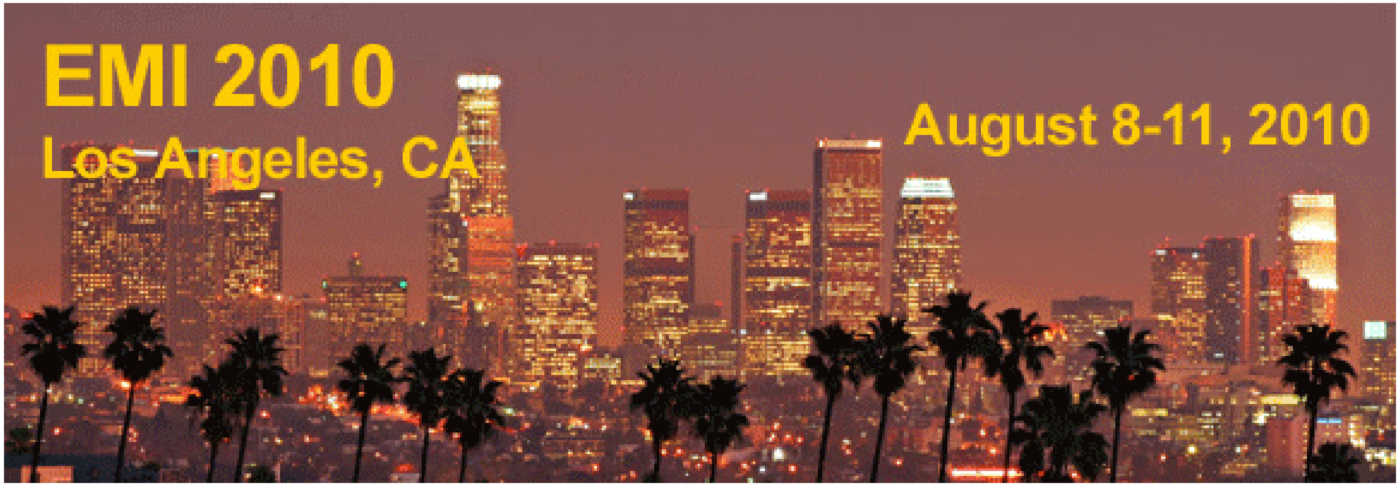

THE MECHANICAL MODEL OF FRACTIONAL VISCOELASTICITY

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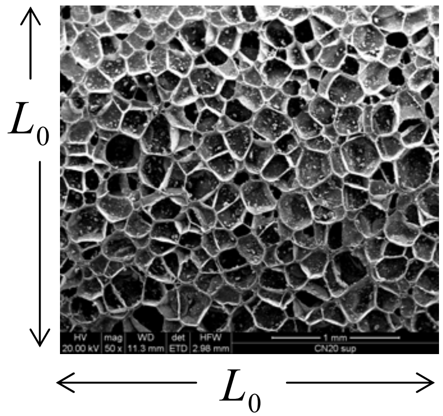
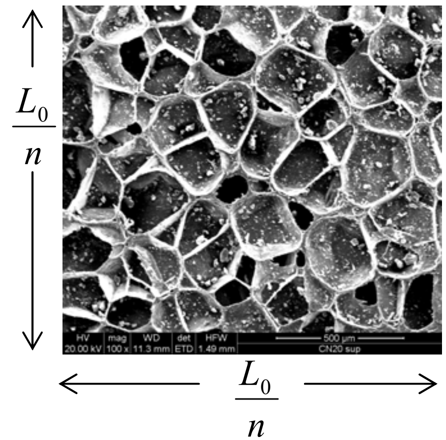
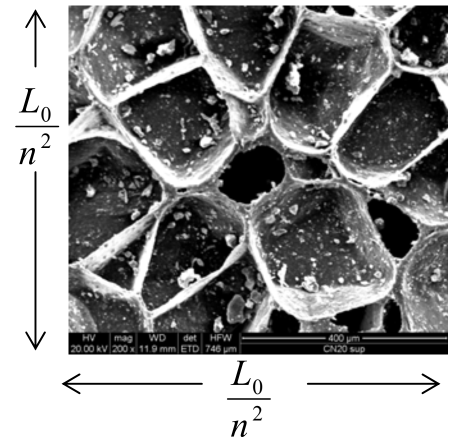
OUTLINE

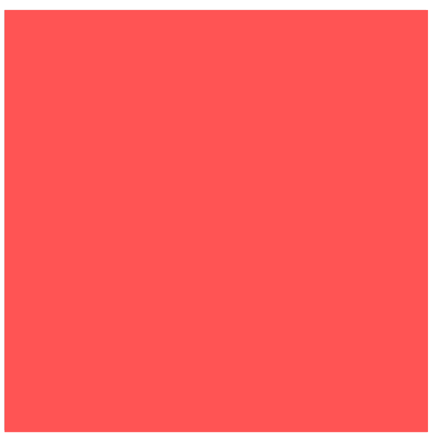
- Fractional derivative for linear viscoelastic model
- Schmidt-Gaul transformation
- Modification of Schmidt-Gaul transformation
- Mechanical model with the modified Schmidt-Gaul transformation
- Fractal model of linear viscoelastic material
- Conclusions

Polymers at different scales

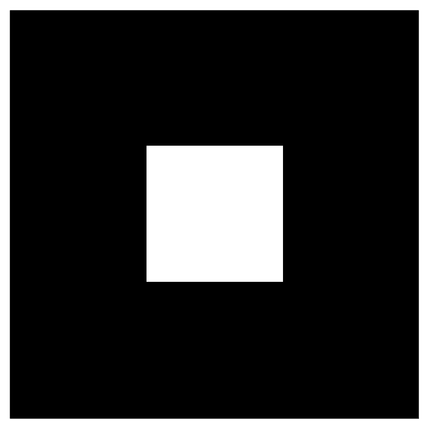




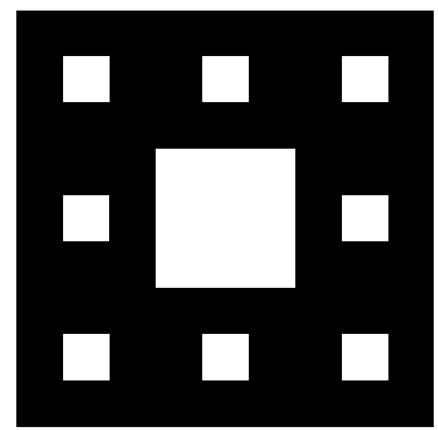




Parent



Generator

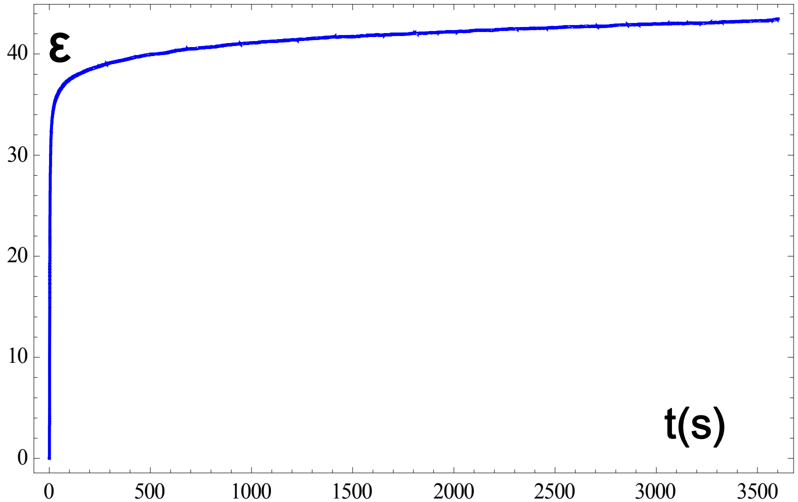


Prefractal

MECHANICAL BEHAVIOR

CREEP TEST $\sigma(t) = U(t)\sigma_0$

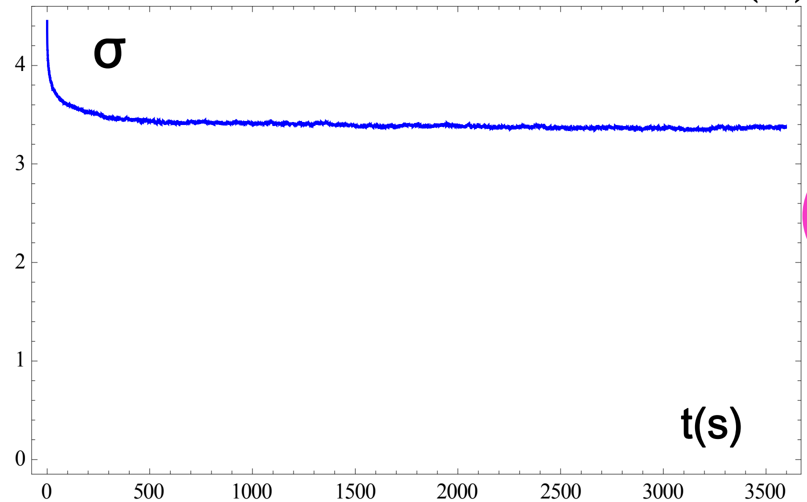
$$\varepsilon(t) = G(t)\sigma_0 U(t)$$



$G(t)$ Creep compliance

RELAXATION TEST $\varepsilon(t) = U(t)\varepsilon_0$

$$\sigma(t) = J(t)\varepsilon_0 U(t)$$



$$J(s)G(s) = \frac{1}{s^2}$$

$J(t)$ Relaxation function

$$J(s) = \mathscr{L}\{J(t);s\}$$

$$G(s) = \mathscr{L}\{G(t);s\}$$

If J(t) or (G(t)) is known, then G(t) or (J(t)) is obtained

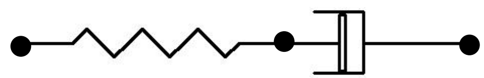
BOLTZMAN SUPERPOSITION INTEGRAL (Linear viscoelastic material)

$$\sigma(t) = \int_{-\infty}^{t} J(t-\tau)\dot{\varepsilon}(\tau)\,d\tau$$

W. Flugge, *Viscoelasticity*. Blaisdell Publishing Company, Massachusetts (1967).

exponential decay $J(t) = \exp(-\alpha t) \qquad \alpha > 0$

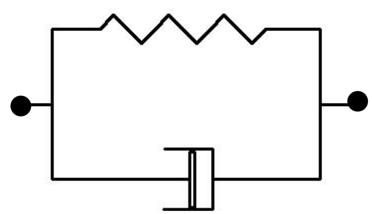
$$\sigma(t) = \int_{-\infty}^{t} e^{-\alpha(t-\tau)}\dot{\varepsilon}(\tau)\,d\tau \quad \Rightarrow \quad \dot{\sigma} + \alpha\,\sigma = \dot{\varepsilon}(t)$$



Maxwell model (for creep)

Creep compliance

$$J(s)G(s) = \frac{1}{s^2} \quad \Rightarrow \quad G(s) = \frac{1}{\alpha s^2} \quad \Rightarrow \quad G(t) = \frac{t}{\alpha}$$



Kelvin Voight model (for relaxation)

Two different models for creep and relaxation (physically meaningless)

Power law decay

$$J(t)=\frac{E}{\Gamma(1-\beta)}\left(\frac{t}{\eta}\right)^{-\beta}$$

$0<\beta<1$

$E\eta^{\beta},\beta \rightarrow$ to be determined from experimental data

$$\sigma(t)=\int_0^t J(t-\tau)\dot{\varepsilon}(\tau)d\tau=\frac{E\eta^{\beta}}{\Gamma(1-\beta)}\int_0^t (t-\tau)^{-\beta}\,\dot{\varepsilon}(\tau)d\tau$$

or

$$\sigma(t)=E\eta^{\beta}\ {}_{C}D_t^{\beta}\varepsilon(t)$$

${}_{C}D_t^{\beta}$ Caputo's FD

1) Caputo's FD coalesces with Riemann Liouville FD for quiescent system at t=0 or for systems that operates from t= - ∞

2) Riemann Liouville FD is an intermediate operator between zero-th and first ordinary derivative $0<\beta<1$

3) for β=0 restitutes $\sigma(t)=E\varepsilon(t)$ elastic behavior (pure solid)
   for β=1 restitutes $\sigma(t)=E\eta\dot{\varepsilon}(t)$ purely viscous (pure fluid)

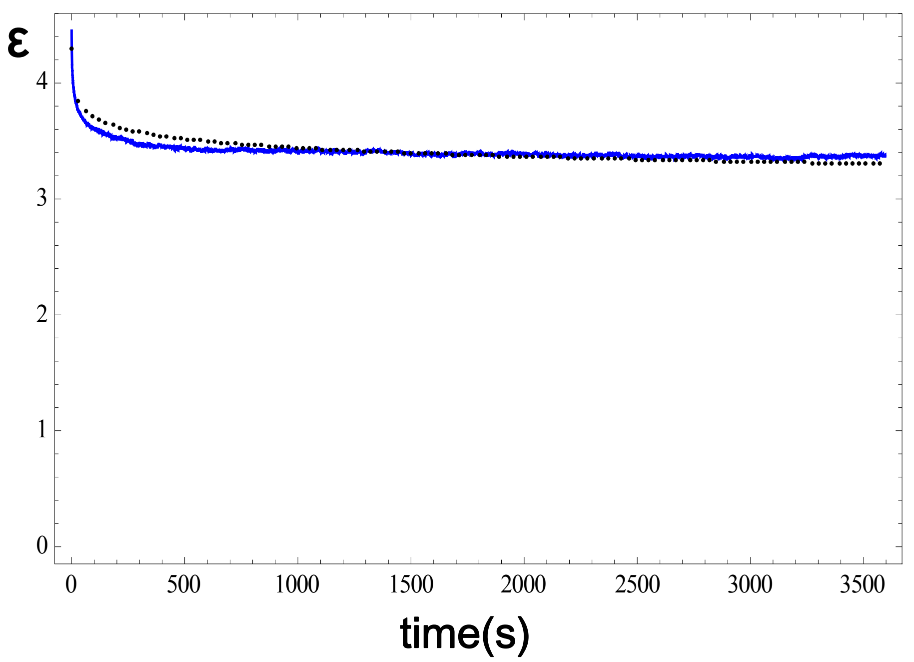
4) J(t) and G(t) are now physically consistent since they descend from the same fractional differential equation

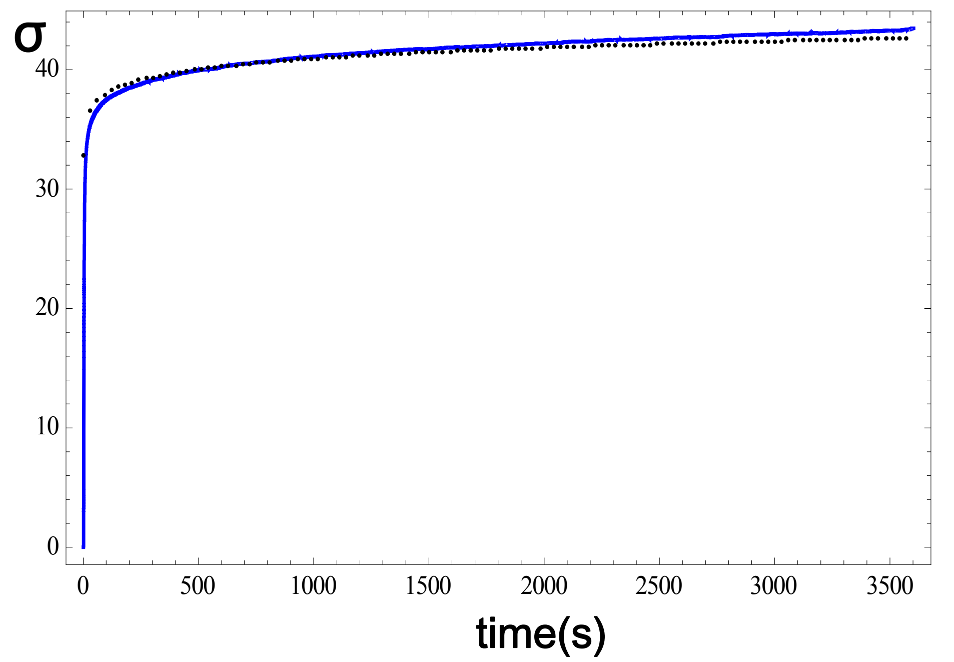
Relaxation and Creep function

$$J(t)=\frac{E}{\Gamma(1-\beta)}\left(\frac{t}{\eta}\right)^{-\beta} \Rightarrow J(s)=E\eta^{\beta}s^{\beta-1}$$

$$J(s)G(s)=\frac{1}{s^2} \Rightarrow G(s)=\frac{\eta^{-\beta}s^{-\beta-1}}{E}$$

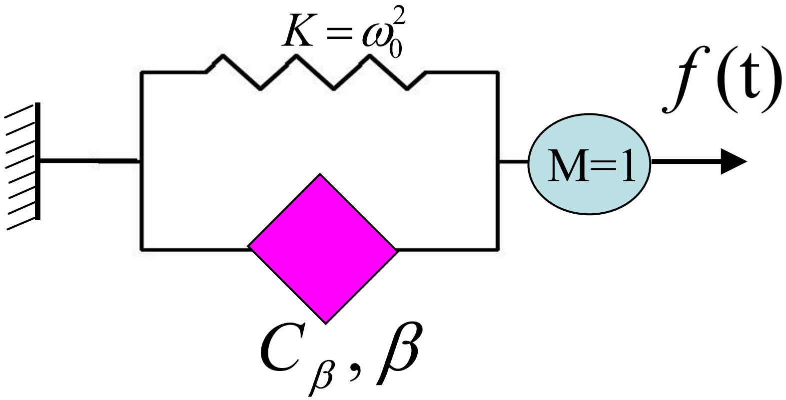
$$\mathscr{L}^{-1}\left[\frac{\eta^{-\beta}s^{-\beta-1}}{E}\right]=\frac{1}{E\,\Gamma(1+\beta)}\left(\frac{t}{\eta}\right)^{\beta}=G(t)$$





FRACTIONAL VISCOELASTIC SYSTEM

$$\left[C_\beta\right] = \sec^{\beta-2}$$



$$\ddot{X}(t) + C_\beta \left({}_C\mathcal{D}_t^\beta X\right)(t) + \omega_0^2 X(t) = f(t)$$

$$\left({}_C\mathcal{D}_t^\beta X\right)(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\dot{X}(\tau)}{(t-\tau)^\beta} d\tau$$

$$\frac{1}{\Gamma(1-\beta)} = \Gamma(\beta)\frac{\sin(\beta\pi)}{\pi} \quad ; \quad \Gamma(\beta) = \int_0^\infty e^{-z} z^{\beta-1} dz \qquad 0<\beta<1$$

$$z = (t-\tau) y^2 \quad ; \quad dz = 2(t-\tau)\, y\, dy$$

A.Schmidt, L. Gaul
Mech. Res. Com. 33 (2006) 99–107

$$\left({}_C D_t^\beta X\right)\ (t) = \left(2\sin(\beta\pi)/\pi\right) \int_0^\infty y^{2\beta-1} \left(\int_0^t e^{-y^2(t-\tau)} \dot{X}(\tau)\, d\tau \right) dy$$

- Results with this transformation are available leading to a set of ordinary differential equation
- Mechanical model of the differential equation does not appear (unless β=1/2) and then the corresponding fractal model does not comes out

Proposed Transormation

$$\left({}_{C}\mathcal{D}_{t}^{\beta}X\right)(t)=\frac{1}{\Gamma(1-\beta)}\int_{0}^{t}\frac{\dot{X}(\tau)}{(t-\tau)^{\beta}}d\tau \qquad \Gamma(\beta)=\int_{0}^{\infty}e^{-z}z^{\beta-1}dz$$

$$z=(t-\tau)y^{2\gamma} \quad ; \quad dz=2\gamma(t-\tau)y^{2\gamma-1}dy \qquad \gamma>0$$

$$\left({}_{C}\mathcal{D}_{t}^{\beta}X\right)(t)=\left(4\gamma\sin(\beta\pi)/\pi\right)\int_{0}^{\infty}y^{2\gamma\beta-1}\left\{\int_{0}^{t}e^{-y^{2\gamma}(t-\tau)}\dot{X}(\tau)d\tau\right\}dy$$

By selecting $2\gamma\beta-1=0$ ⇒ $\gamma=1/2\beta$ $\qquad U_{y}(t)$

$$\left({}_{C}\mathcal{D}_{t}^{\beta}X\right)(t)=\left(2\sin(\beta\pi)/\beta\pi\right)\int_{0}^{\infty}U_{y}(t)dy$$

Proposed Transormation

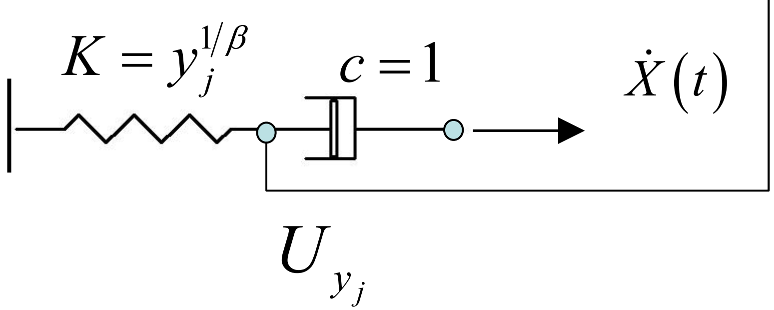
$$z = (t-\tau)\, y^{2\gamma} \quad ; \quad dz = 2\gamma (t-\tau)\, y^{2\gamma-1}\, dy$$

By selecting $2\gamma\beta - 1 = 0$ ➡ $\gamma = 1/2\beta$

$$\left({}_C\mathcal{D}_t^{\beta} X\right)(t) = \left(2\sin(\beta\pi)/\beta\pi\right)\int_0^{\infty} U_y(t)\, dy \approx \left(2\sin(\beta\pi)/\beta\pi\right)\Delta y \sum_{j=1}^{\infty} U_{y_j}(t)$$

$$U_{y_j}(t) = \int_0^t e^{-y_j^{1/\beta}(t-\tau)}\, \dot{X}(\tau)\, d\tau$$

$$\begin{cases} \dot{U}_{y_j}(t) + y_j^{1/\beta} U_{y_j}(t) = \dot{X}(t) \\ U_{y_j}(0) = 0 \end{cases}$$



Maxwell unit

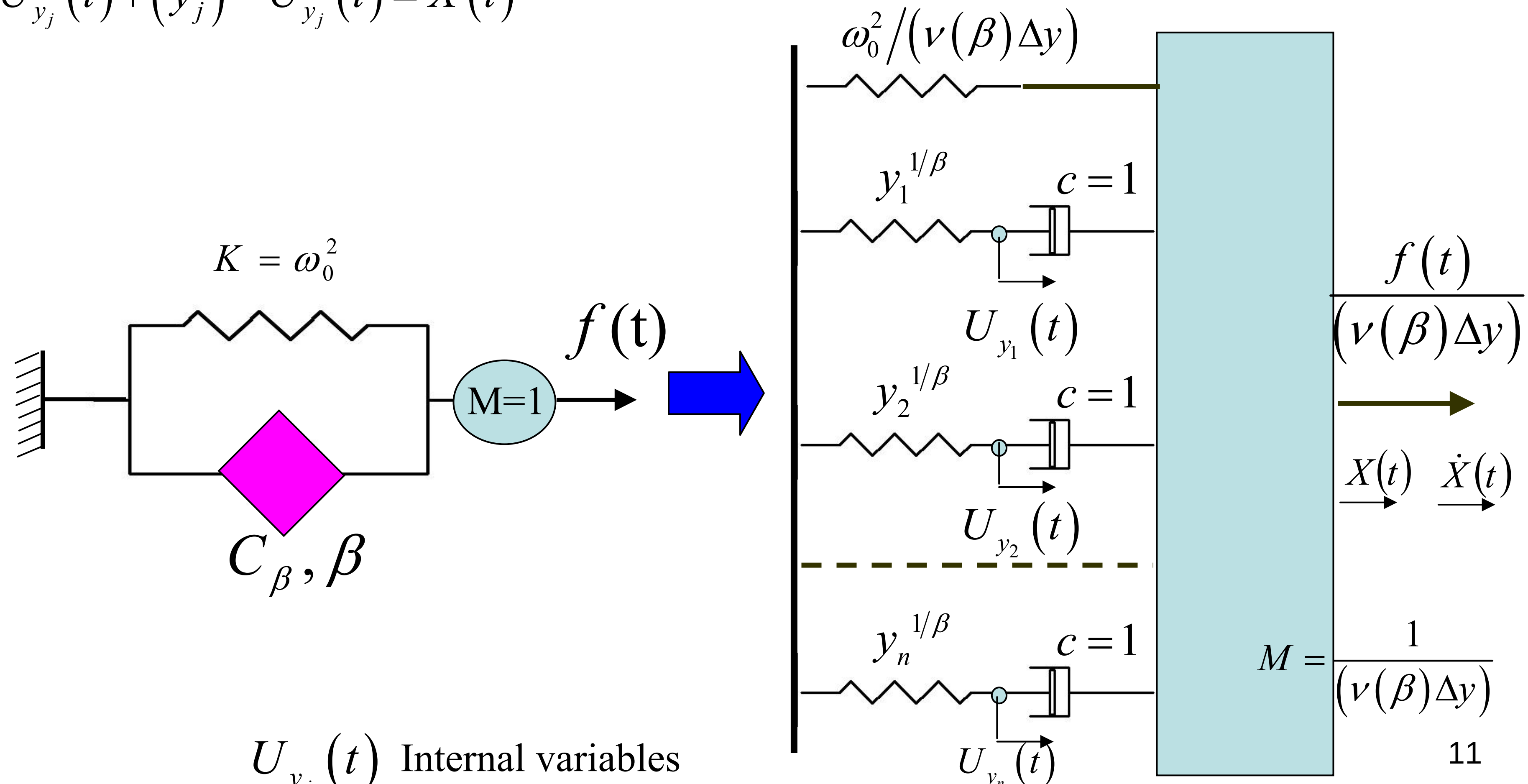
MECHANICAL INTERPRETATION

$$\ddot{X}(t)+C_\beta\left({}_C\mathcal{D}_t^\beta X\right)(t)+\omega_0^2X(t)=f(t)$$

$$\begin{cases}\dfrac{\ddot{X}(t)}{\nu(\beta)\Delta y}+\displaystyle\sum_{j=1}^{\infty}U_{y_j}(t)\Delta y_j+\dfrac{\omega_0^2X(t)}{\nu(\beta)\Delta y}=\dfrac{f(t)}{\nu(\beta)\Delta y}\\ \dot{U}_{y_j}(t)+\left(y_j\right)^{1/\beta}U_{y_j}(t)=\dot{X}(t)\end{cases}$$

$$\nu(\beta)=C_\beta\left(2\sin(\beta\pi)/\beta\pi\right)$$

$$y_j=j\Delta y$$



$U_{y_j}(t)$ Internal variables

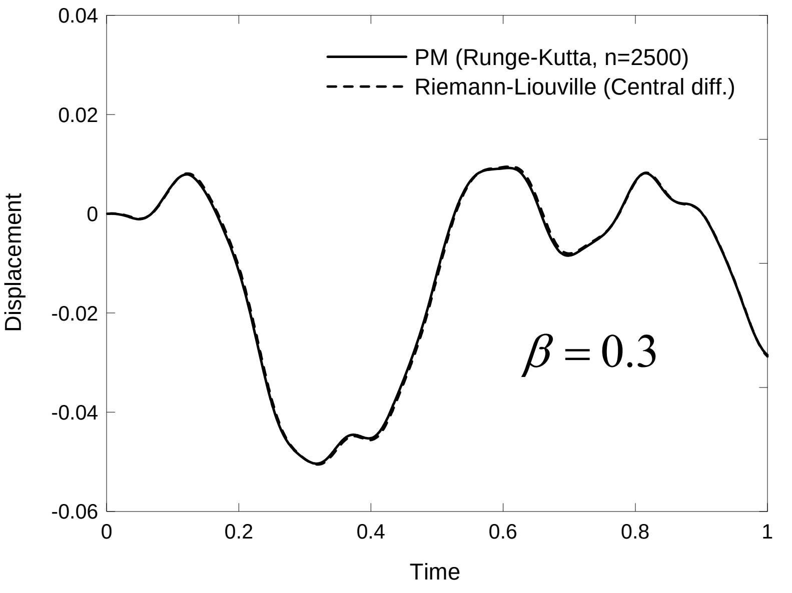
NUMERICAL APPLICATIONS

$$\ddot{X}(t)+C_\beta\left(\mathcal{D}_+^\beta X\right)(t)+\omega_0^2 X(t)=W(t)$$

O. P. Agraval (2001)

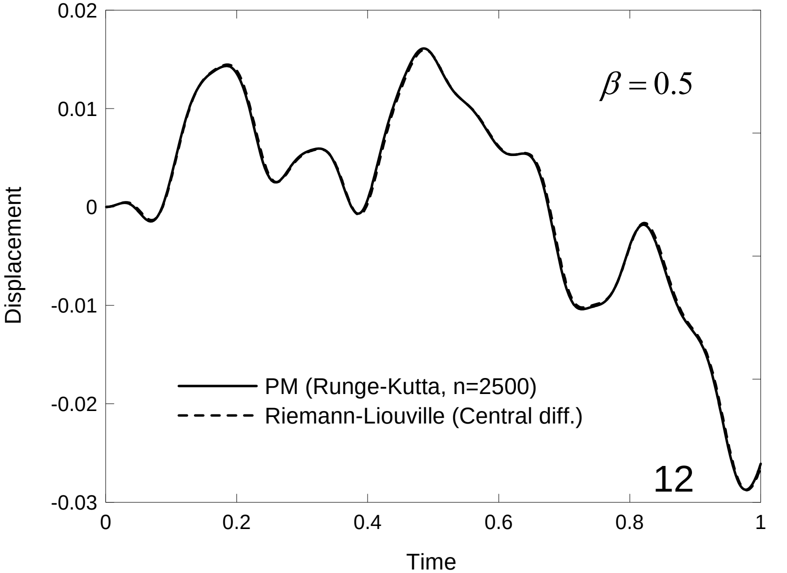
$$\omega_0=10\,;\qquad C_\beta=3$$

$$q=1.0$$



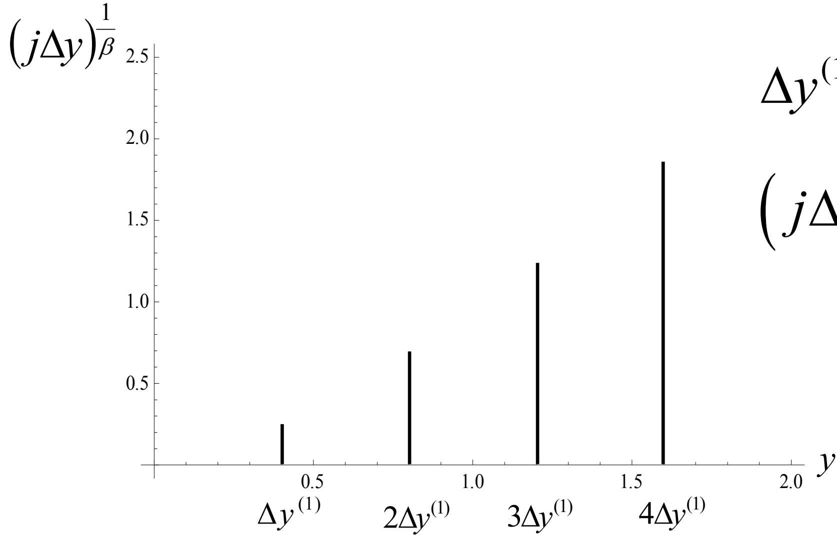
proposed method:

$$\begin{cases}\dfrac{\ddot{X}(t)}{\nu(\beta)\Delta y}+\displaystyle\sum_{j=1}^{n}U_{y_j}(t)\Delta y_j+\dfrac{\omega_0^2X(t)}{\nu(\beta)\Delta y}=\dfrac{W(t)}{\nu(\beta)\Delta y}\\ \dot{U}_{y_j}(t)+(y\Delta y)^{1/\beta}U_{y_j}(t)=\dot{X}(t)\qquad j=1,2,\ldots n\end{cases}$$



$$\nu(\beta)=C_\beta\left(2\sin(\beta\pi)/\beta\pi\right)$$

FRACTAL INTERPRETATION



$$\Delta y^{(1)}$$

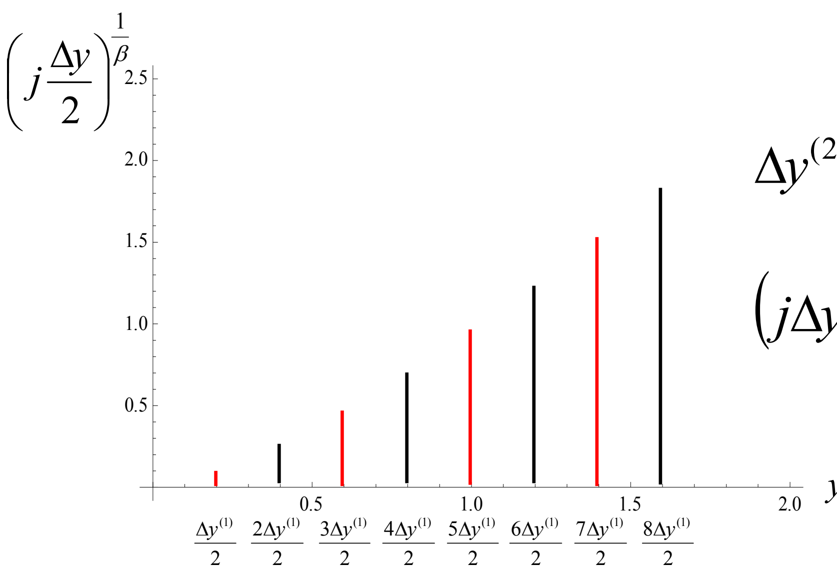
$$\left(j\Delta y^{(1)}\right)^{\frac{1}{\beta}}$$

$$\dot{U}_{y_j}(t) + \left(y_j\right)^{1/\beta} U_{y_j}(t) = \dot{X}(t)$$

$$\dot{U}_{y_j}(t) + \left(j\Delta y\right)^{1/\beta} U_{y_j}(t) = \dot{X}(t)$$

$$y_j = j\Delta y$$

Δy Is an arbitrary small value

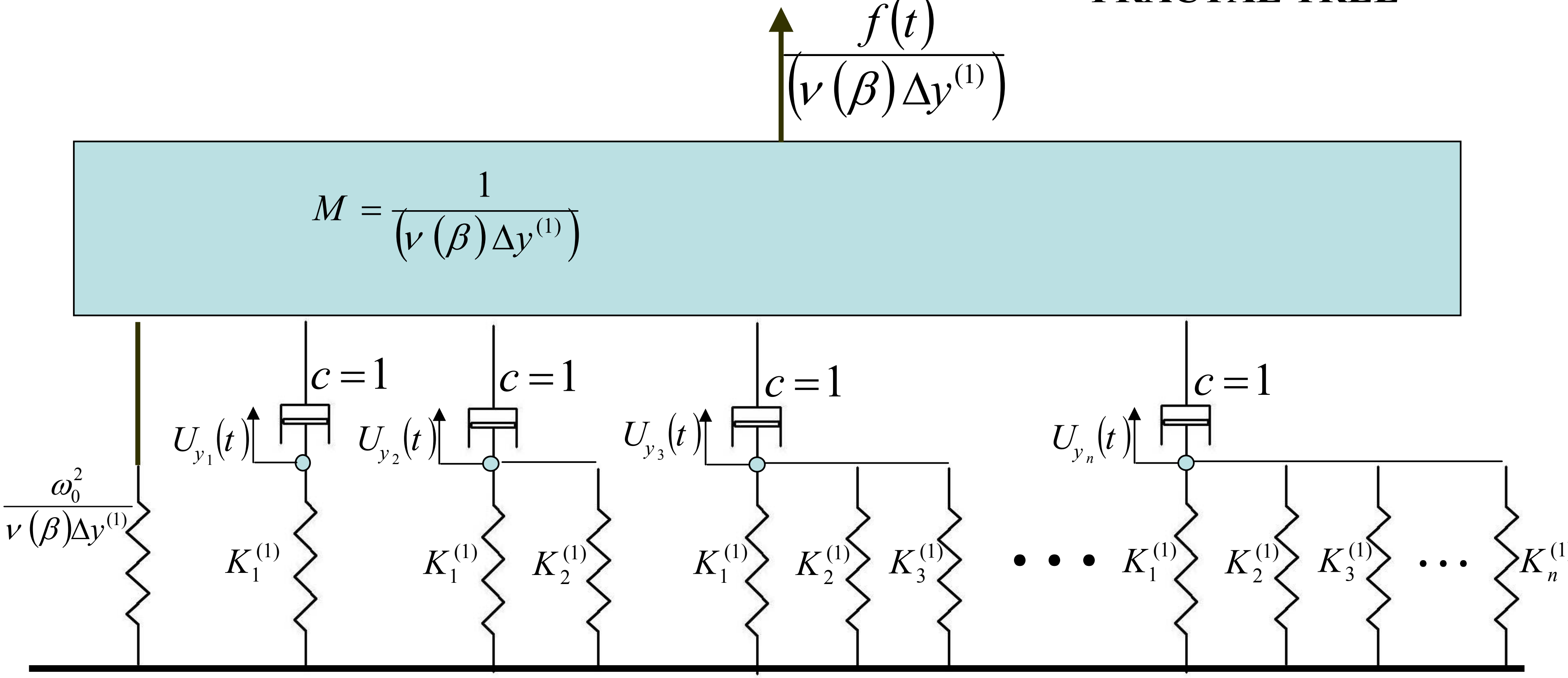


$$\Delta y^{(2)} = \frac{\Delta y^{(1)}}{2}$$

$$\left(j\Delta y^{(2)}\right)^{\frac{1}{\beta}}$$

⋮

FRACTAL TREE



$$\Delta y = \Delta y^{(1)}$$

$$K_1^{(1)} = \left(\Delta y^{(1)}\right)^{1/\beta}$$

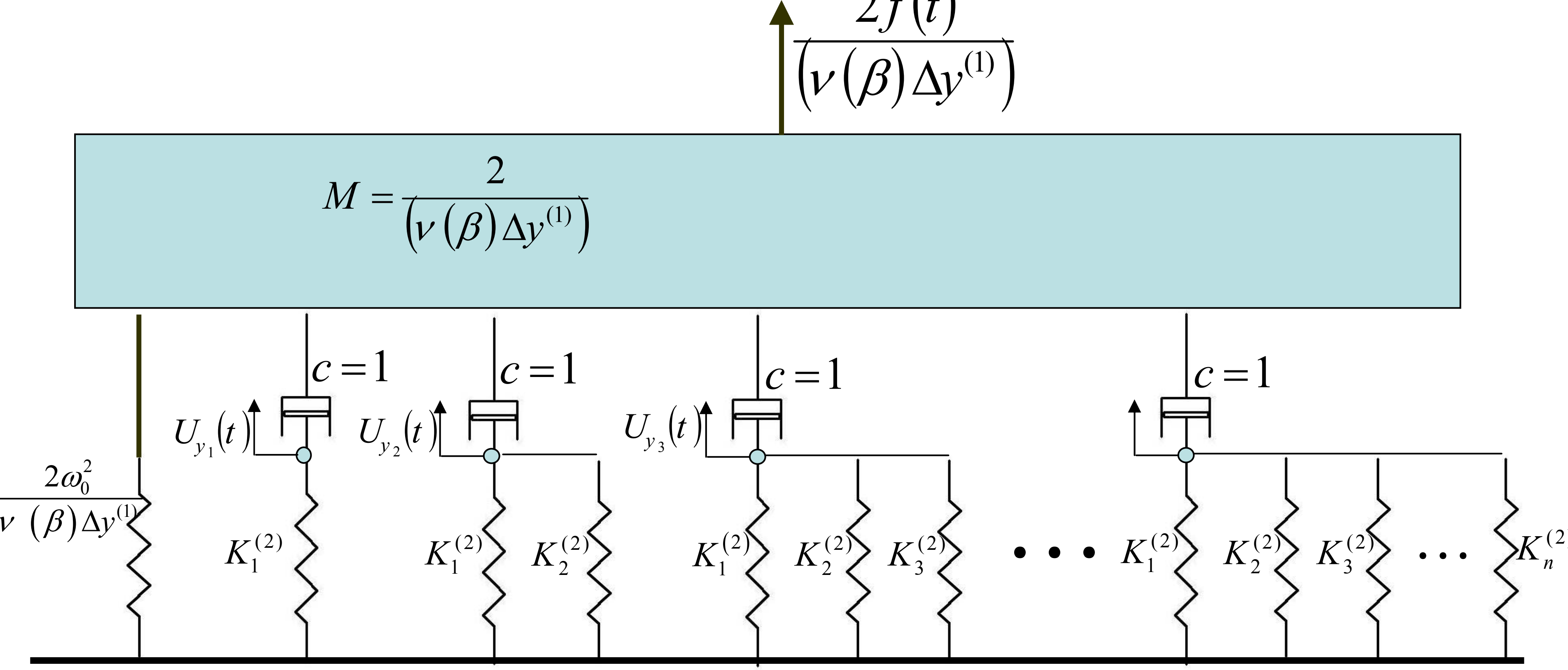
$$K_2^{(1)} = \left(\Delta y^{(1)}\right)^{1/\beta}\left(2^{1/\beta} - 1^{1/\beta}\right)$$

$$K_3^{(1)} = \left(\Delta y^{(1)}\right)^{1/\beta}\left(3^{1/\beta} - 2^{1/\beta}\right)$$

$$\vdots$$

$$K_n^{(1)} = \left(\Delta y^{(1)}\right)^{1/\beta}\left(n^{1/\beta} - (n-1)^{1/\beta}\right)$$

FRACTAL TREE



$$\Delta y = \Delta y^{(2)} = \frac{\Delta y^{(1)}}{2}$$

$$K_1^{(2)} = \left(\frac{1}{2}\right)^{1/\beta} K_1^{(1)}$$

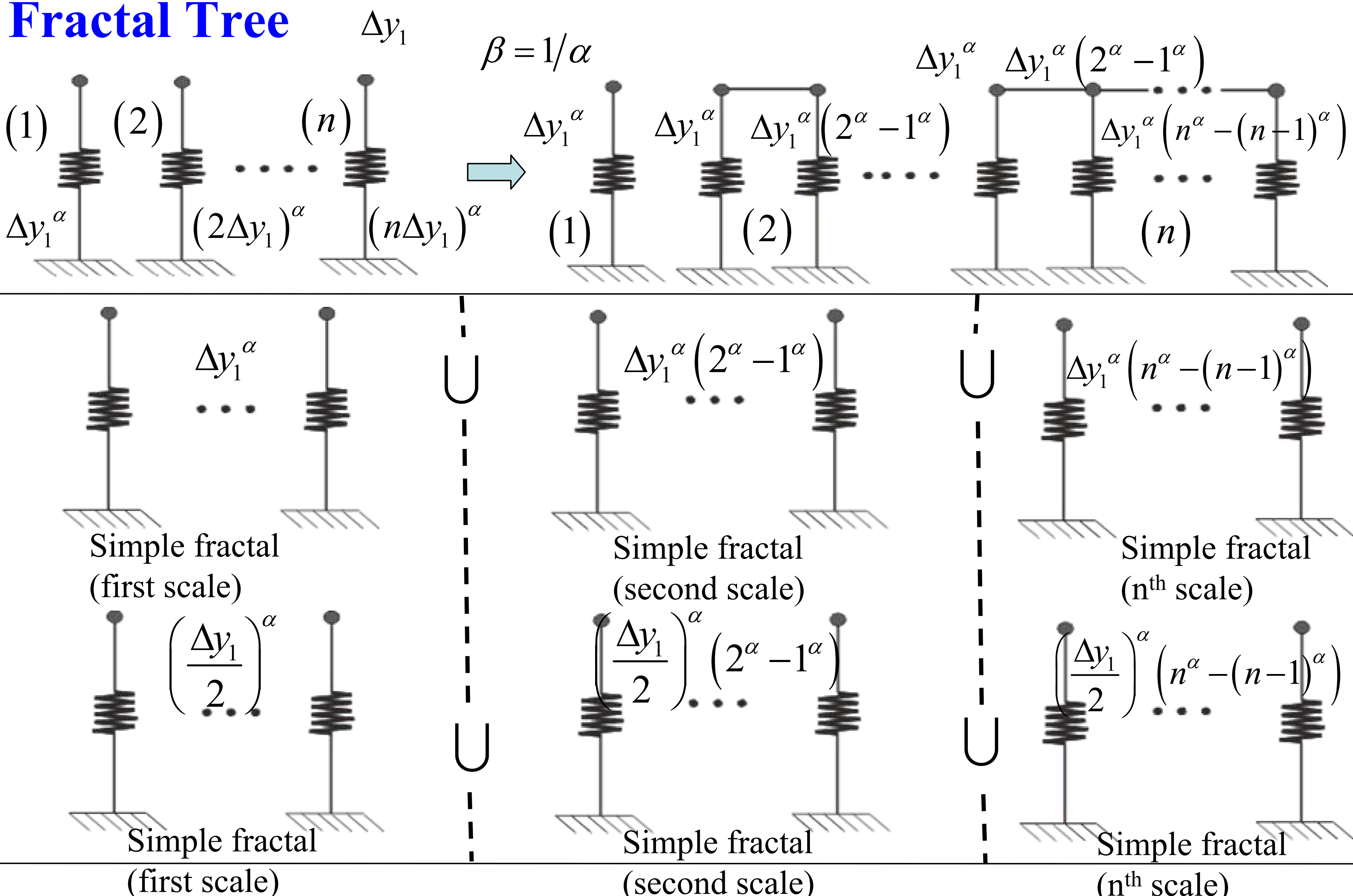
$$K_2^{(2)} = \left(\frac{1}{2}\right)^{1/\beta} K_2^{(1)}$$

$$K_3^{(2)} = \left(\frac{1}{2}\right)^{1/\beta} K_3^{(1)}$$

$$\vdots$$

$$K_n^{(2)} = \left(\frac{1}{2}\right)^{1/\beta} K_n^{(1)}$$

Fractal Tree



$$D = 1/\alpha = \beta$$

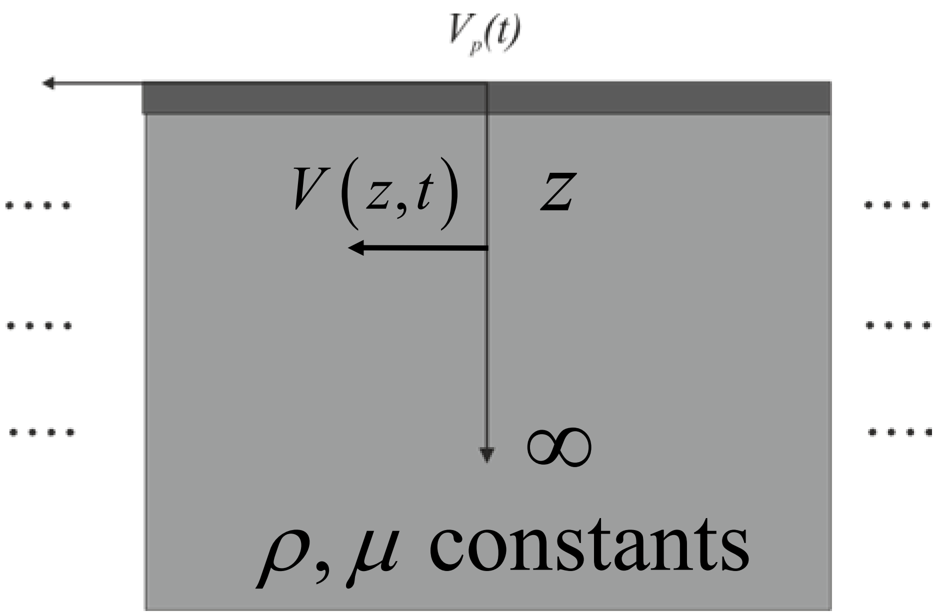
→ **Housdorff-Besitckovich dimension**

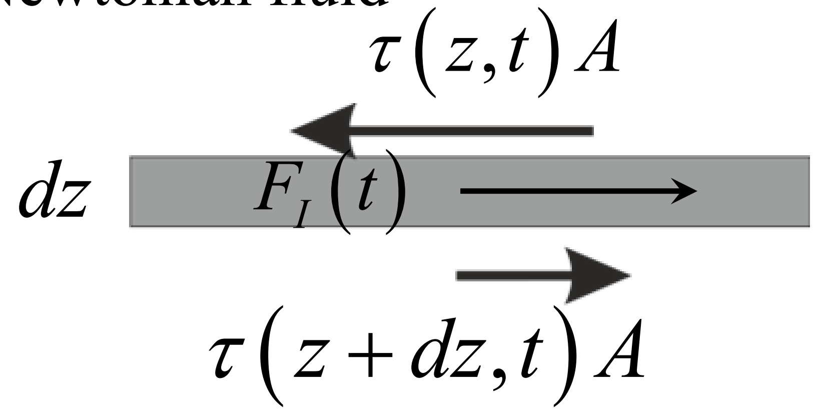
CONCLUSIONS

- Fractional derivative for linear Viscoelastic model
- Schmidt-Gaul transformation
- Viscoelastic behavior (creep and relaxation)are well fitted with a power law
- This corresponds to a Caputo's fractional derivative
- By using a modified Schmidt-Gaul transformation a mechanical interpretation of the viscoelastic model appears
- This mechanical model is interpreted as a mechanical fractal

A mechanical model of fractional decay (Bagley-Torvik, 1984)

- A rigid plate moving on a semi-infinite Newtonian fluid





Constitutive Equations

$$\tau = \mu \frac{\partial V}{\partial z} \quad ; \quad F_I = \rho A dz \frac{\partial V}{\partial t}$$

Governing equation

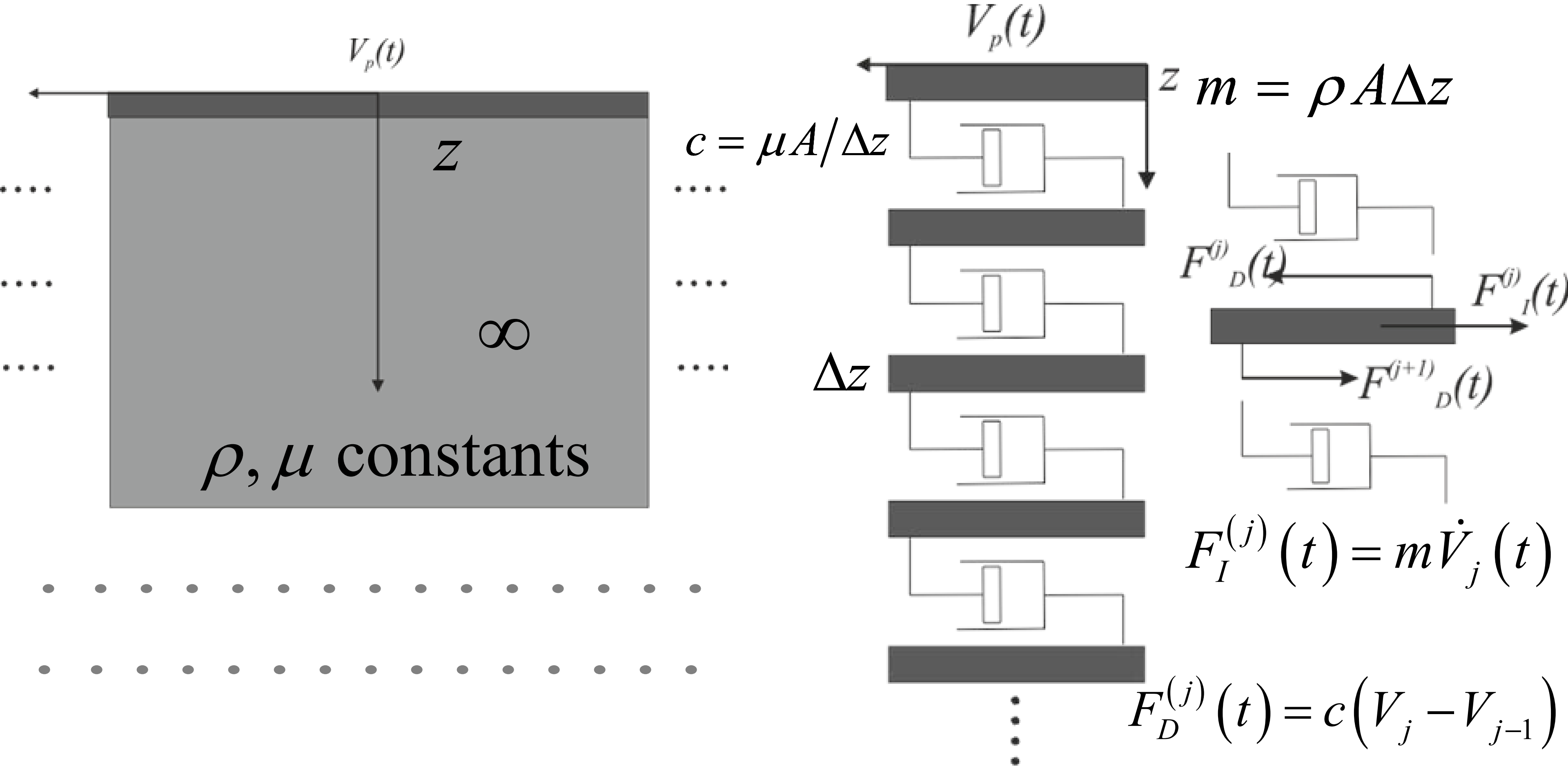
$$\mu \frac{\partial^2 V}{\partial z^2} = \rho \frac{\partial V}{\partial t}$$

- The relation between stress and velocity of the <u>overall model: Fractional derivative of order 1/2</u>

$$\tau(z,t) = \sqrt{\mu\rho}\left(D_0^{1/2} V\right)(t)$$

The discrete Bagley-Torvik model

- The Discrete Representation of the Bagley-Torvik Model



- Equilibrium equation: $$c\left(V_{j+1} - 2V_j + V_{j-1}\right) = m\dot{V}_j(t)$$

- Bagley-Torvik introducing scaling: $\Delta z \rightarrow dz$

The Discrete Rheological Model

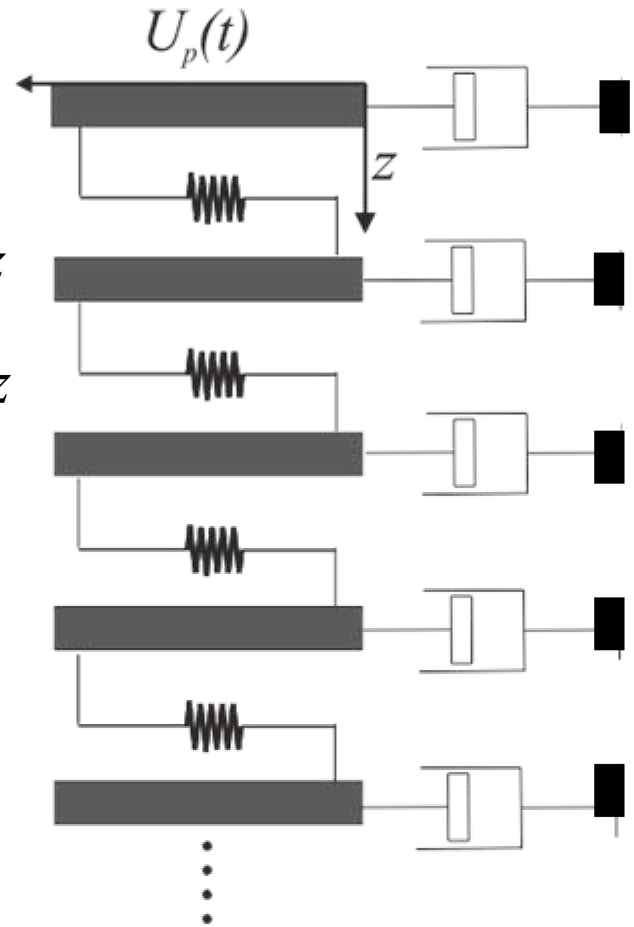
- Introducing the mechanical analogy: Velocity vs Displacement

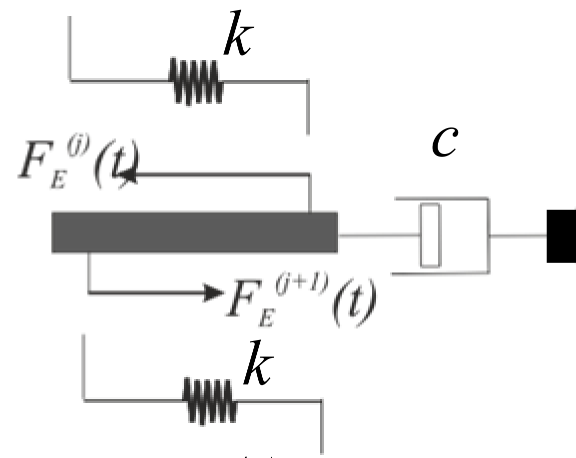
$$V_j(t) \equiv U_j(t)$$

$$m = \rho A \Delta z$$

$$c = \mu A / \Delta z$$

$$\Delta z$$





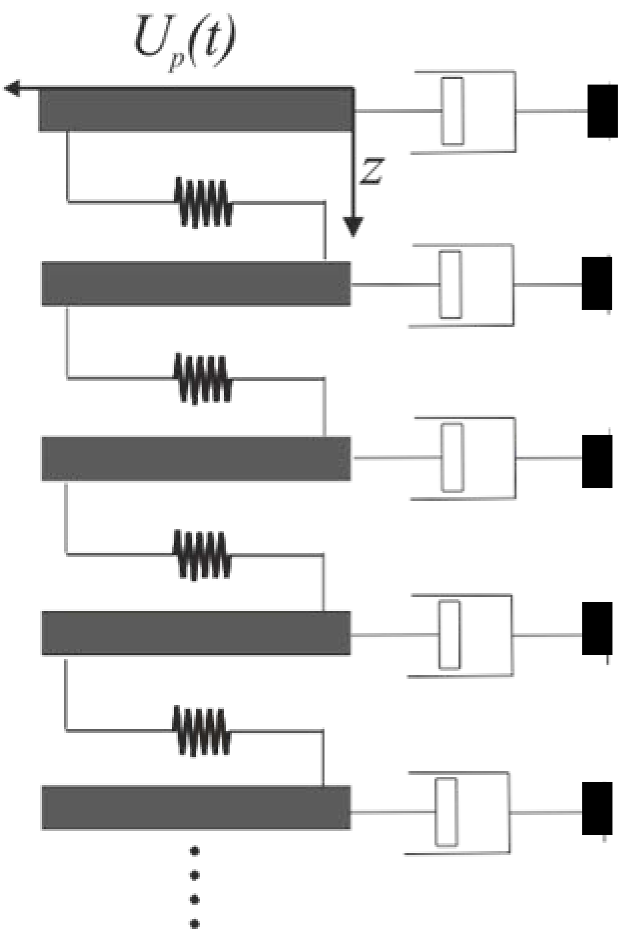
$$\bar{c} = \bar{\mu} A \Delta z$$

$$k = EA / \Delta z$$

$$c\left(V_{j+1} - 2V_j + V_{j-1}\right) = m\dot{V}_j(t)$$

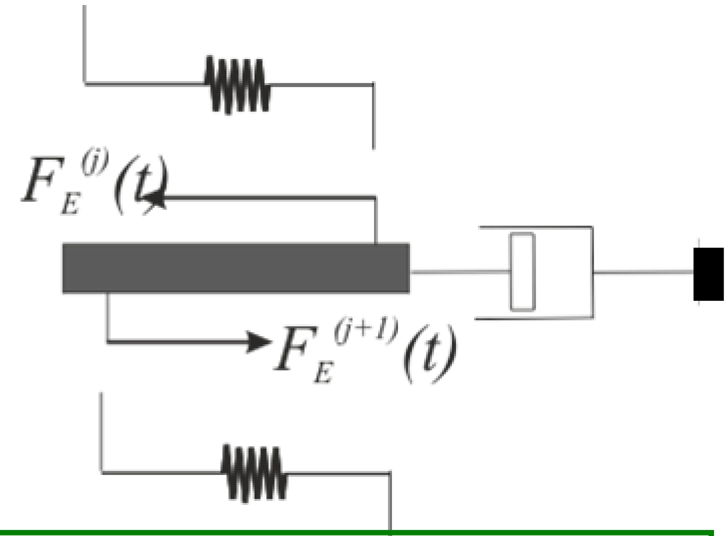
$$k\left(U_{j+1} - 2U_j + U_{j-1}\right) = \bar{c}\dot{U}_j(t)$$

The Fractional Rheological Model



$$k = EA/\Delta z$$

$$\overline{c} = \overline{\mu} A \Delta z$$



$$k\left(U_{j+1} - 2U_j + U_{j-1}\right) = \overline{c}\dot{U}_j(t)$$

Governing equation: $$E\frac{\partial^2 U}{\partial z^2} = \overline{\mu}\frac{\partial U}{\partial t}$$

$$\Delta z \rightarrow dz$$

$$\gamma(z,t) = \partial U / \partial z$$

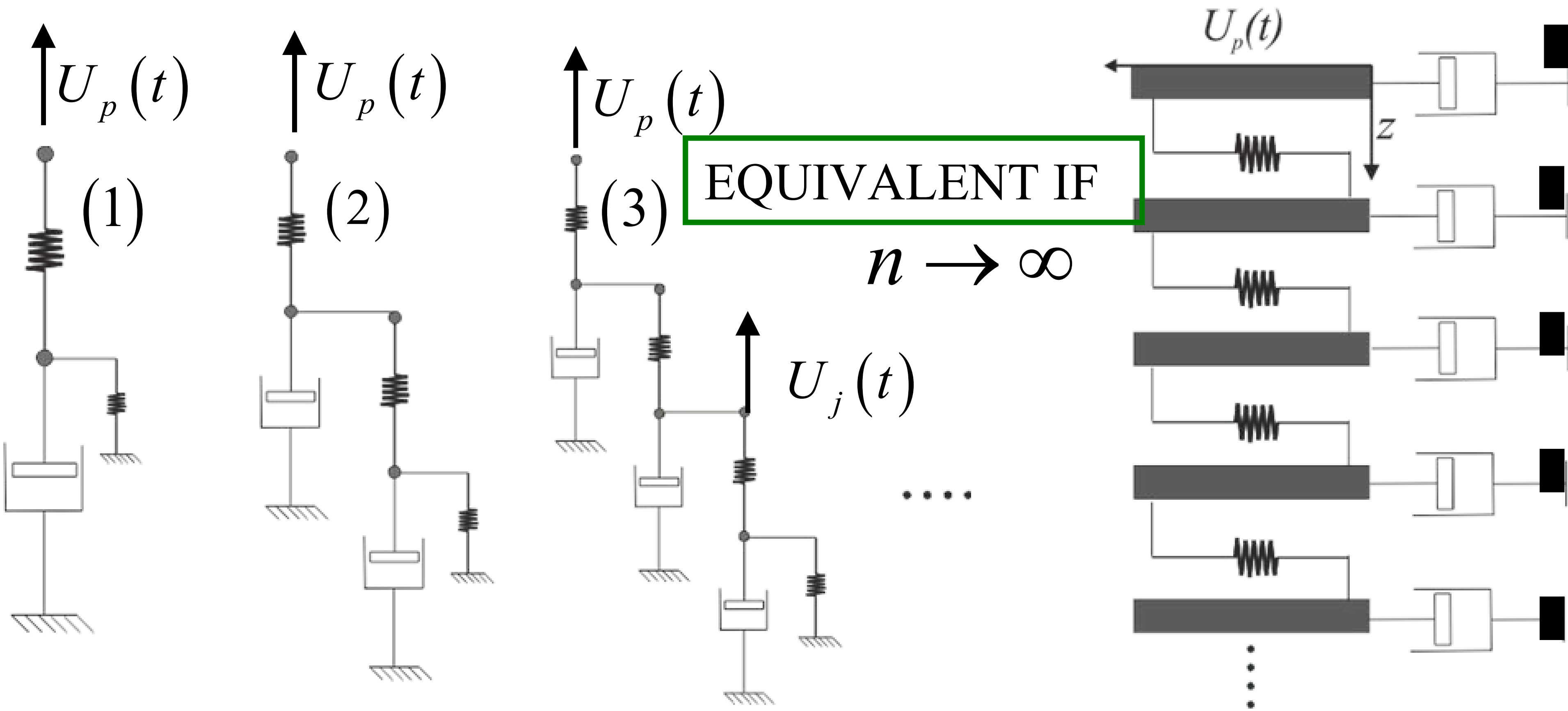
- A semi-infinite elastic shear layer equipped with external dahspot

- The relation between stress and velocity strain of the <u>overall model: Fractional derivative of order 1/2</u>

$$\tau(z,t) = \sqrt{E\overline{\mu}}\left(D_0^{1/2}\gamma\right)(t)$$

The fractal ladder for the rheological fractional behavior

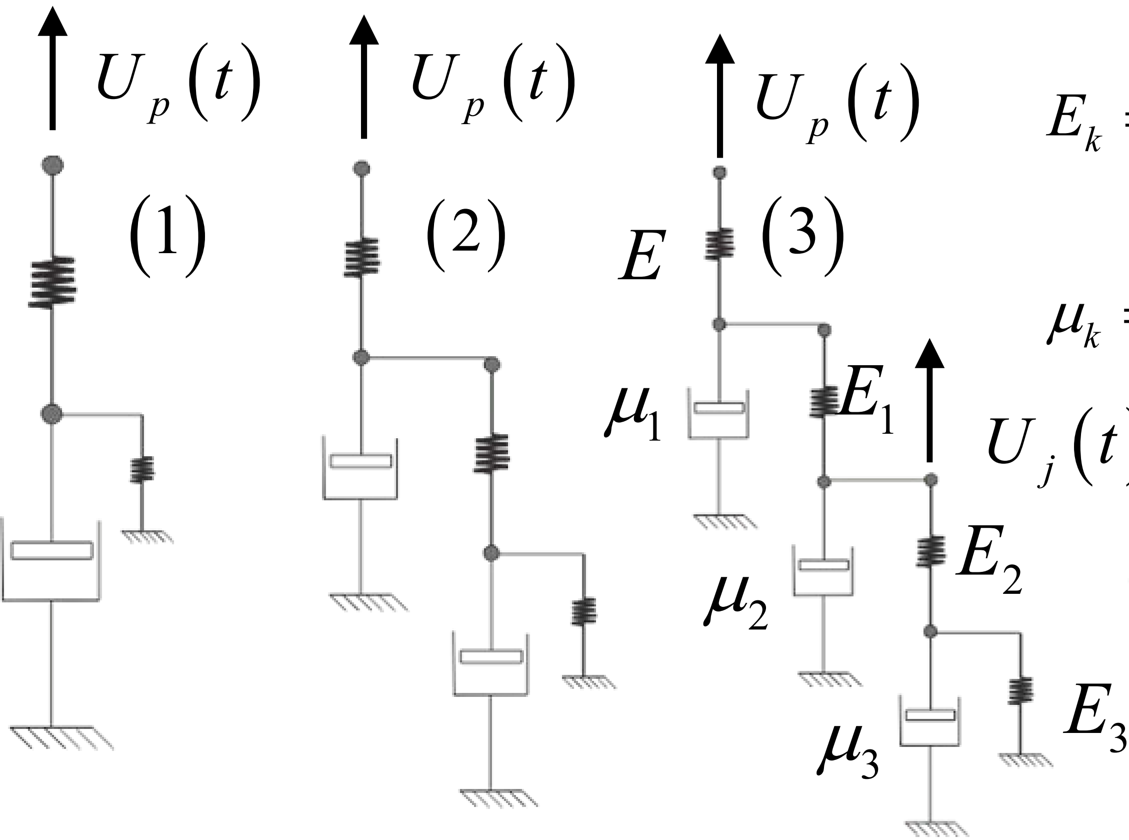
• A Different order of fractional exponent may be obtained introducing fractal ladders (Schiessel-Blumen, 1993).



• The solution of the Schiessel-Blumen with properly variable viscous coefficient and stiffness yields fractional rheological models ??????

The fractal ladder for the rheological fractional behavior II

- The solution of the Schiessel-Blumen with properly variable viscous coefficient and stiffness yields fractional rheological models ??????



Model Coefficients

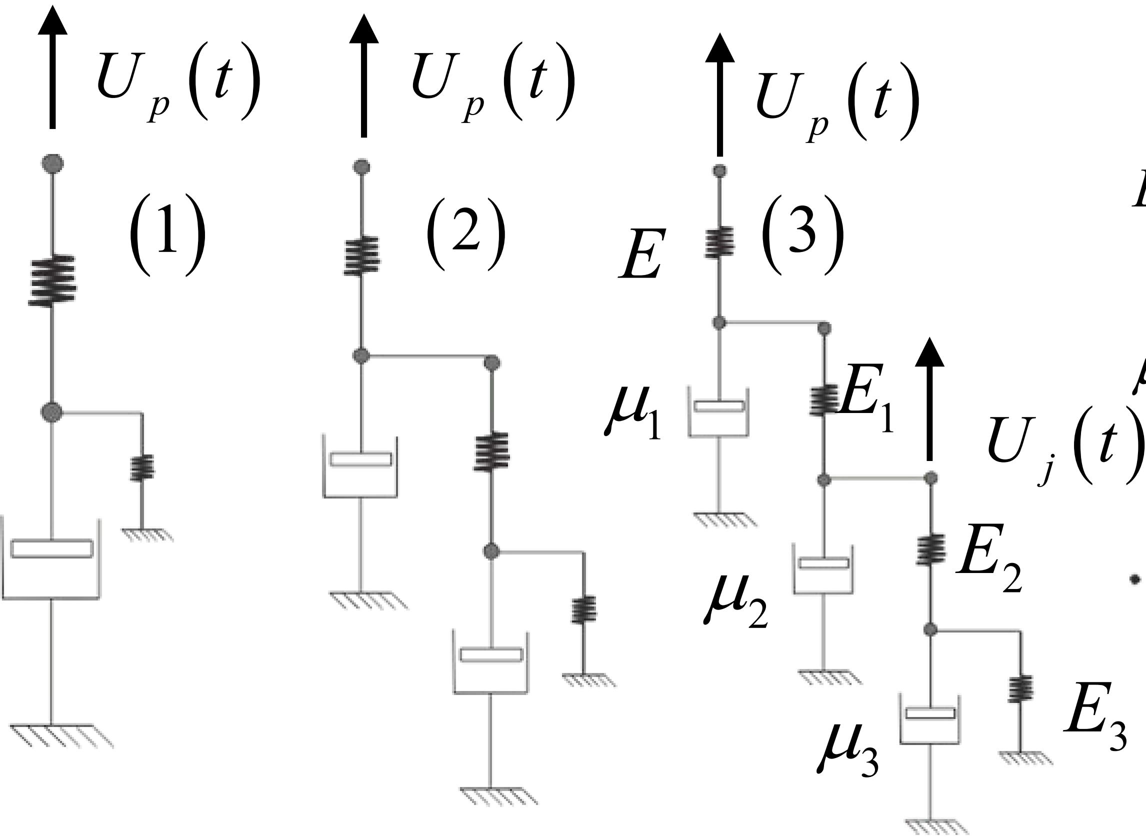
$$E_k = \frac{1}{2k-1}\frac{\Gamma(\alpha)}{\Gamma(1-\alpha)}\frac{\Gamma(k+1-\alpha)}{\Gamma(k-1-\alpha)}E$$

$$\mu_k = 2\frac{\Gamma(\alpha)}{\Gamma(1-\alpha)}\frac{\Gamma(k+1-\alpha)}{\Gamma(k+\alpha)}\bar{\mu}$$

$\cdots\cdots\ n \to \infty$

NO, ONLY AS A NON-PHYSICAL APPROXIMATION !!!!!

The Schiessel-Blumen fractional model



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Model Coefficients

$$E_k = \frac{1}{2k-1}\frac{\Gamma(\alpha)}{\Gamma(1-\alpha)}\frac{\Gamma(k+1-\alpha)}{\Gamma(k-1-\alpha)}E$$

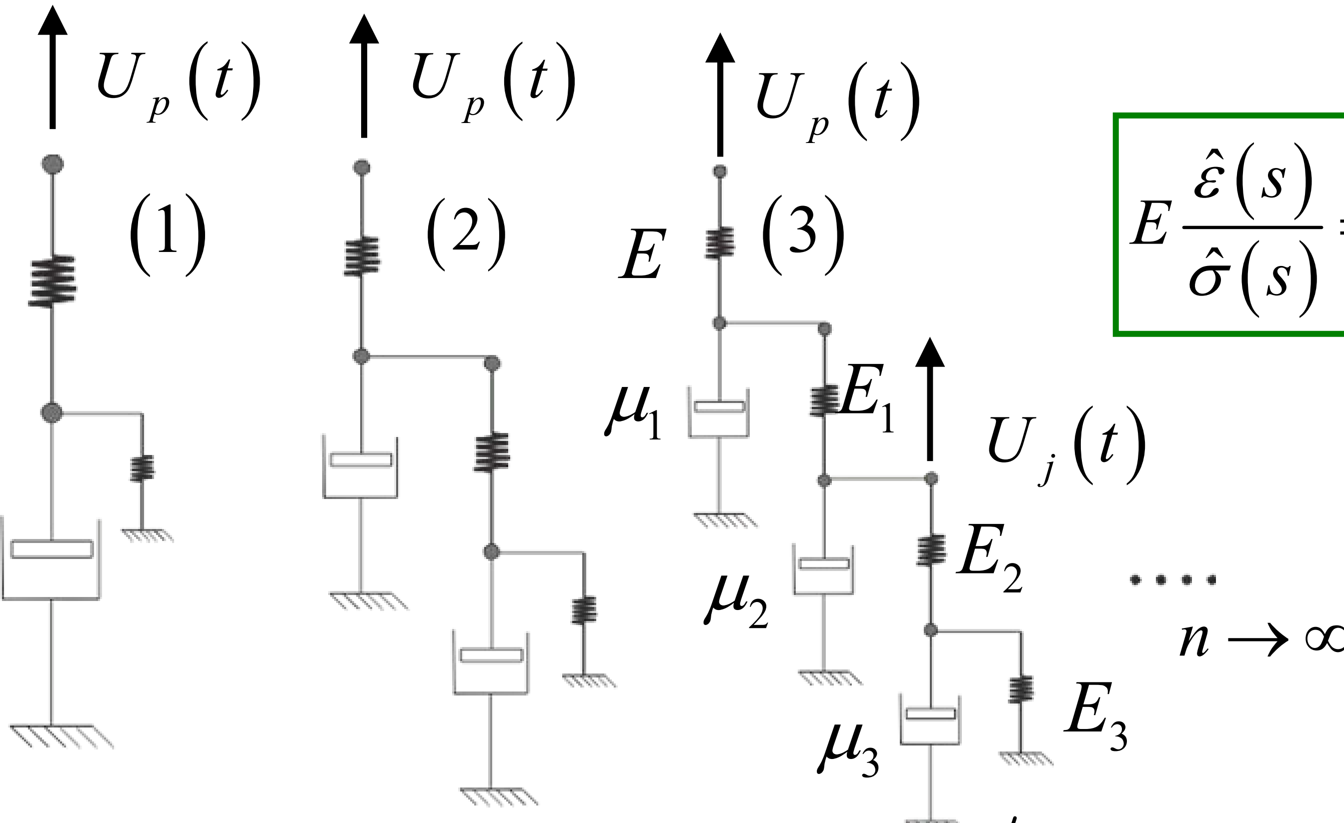
$$\mu_k = 2\frac{\Gamma(\alpha)}{\Gamma(1-\alpha)}\frac{\Gamma(k+1-\alpha)}{\Gamma(k+\alpha)}\bar{\mu}$$

$$n \to \infty$$

- A Continuous fraction expansion of the displacement of the model yields (Laplace Domain) $n \to \infty$

$$E\frac{\hat{\varepsilon}(s)}{\hat{\sigma}(s)} = 1 + \left(\frac{c}{s}\right)\left(1+\frac{c}{s}\right)^{\alpha-1} \qquad c = \frac{EA}{\bar{\mu}} \qquad (A=1)$$

The Schiessel-Blumen fractional model II



$$E\frac{\hat{\varepsilon}(s)}{\hat{\sigma}(s)} = 1 + \left(\frac{c}{s}\right)\left(1+\frac{c}{s}\right)^{\alpha-1}$$

$$E\frac{\hat{\varepsilon}(s)}{\hat{\sigma}(s)} \cong \left(\frac{c}{s}\right)^{\alpha}$$

- A first approximation holding for $c/s \gg 1$
- THE APPROXIMATE FRACTIONAL DERIVATIVE FOR THE REHOLOGICAL MODEL

$$\sigma(t) \cong \frac{E}{c^{\alpha}}\left(D_0^{\alpha}\varepsilon\right)(t)$$

- ONLY FOR STEADY-STATE: HOW LONG IS THE STEADY STATE ?