L&PS – Logic & Philosophy of Science
Vol. VIII, No. 1, 2010, pp. 1-29

Logic and the Myth of the Perfect Language*

Marco Carapezza
Department FIERI-AGLAIA
University of Palermo
e-mail: caramak@unipa.it

Marcello D'Agostino
Department of Human Sciences
University of Ferrara
e-mail: marcello.dagostino@unife.it

1. The myth of the logically perfect language
2. Wittgenstein's 'adequate notation' and the futility of logic
3. The perfect language and mathematical logic
4. 'Unfeasible' tautologies
5. Perfect language and computational complexity
6. Cook's theorem and the inevitable imperfection of logical languages
7. Logic in a network of imperfect languages

ABSTRACT. We argue that the dream of a 'perfect language' – namely, a universal, unambiguous and semantically transparent medium of expression –, whose intriguing story has been told by Umberto Eco (1993), is deeply intertwined with the myth of *instant rationality*: the idea that a perfect language is one in which all logical relations become immediatly *visible*, so that the language itself "does the thinking for us" (Frege 1884). In the first part of this paper we trace this version of the dream in the works of Leibniz, Frege, Russell and Wittgenstein. In the second part we re-examine it in the light of more recent negative results in logic and theoretical computer science.

KEYWORDS: perfect language; instant rationality; logic; computational complexity.

* A preliminary version of this paper was presented at the workshop "Linguaggio, verità e storia in matematica", Mussomeli, 9 febbraio 2008. We wish to thank Gabriele Lolli, Franco Lo Piparo and Gianluigi Olivieri for interesting questions and suggestions.

Published by L&PS – Logic and Philosophy of Science
http://www2.units.it/~episteme – ISSN: 1826-1043

1. The myth of the logically perfect language

"It [operating with figures mechanically] only became possible at all after the mathematical notation has, as a result of genuine thought, been so developed that it does the thinking for us so to speak" (Frege 1884, p. IV). These words by Gottlob Frege clearly and concisely express the myth of the 'logically perfect language' that constitutes the theme of the present essay.

Reflection on the perfect (or original) language has been a recurrent theme in scientific-philosophical thought beginning from Greek antiquity.[1] However, this phrase has been used to mean several things that are quite different from one another. So, let us immediately clarify that here we will not deal with projects based on the idea that names express the profound essence of things or give information on lexical semantics, thus proving to be 'semantically transparent'. We will deal, instead, with the 'logically perfect language', meaning a language that can guarantee the correctness of processes of reasoning and one in which we can immediately recognize the relationships between propositions simply by means of *sensory perception.* Hence, this is a meaning of perfection in which semantic transparency concerns 'logical words' and not lexical terms. All that is required of the latter is referential univocity: to each conceptual content there must correspond only one term – a *real character*, to use the happy expression coined by Francis Bacon.

Like many other inventors of philosophical languages, Gottfried Wilhelm Leibniz believed that the possibility of a perfect language was founded on identification of the primitive notions that make up the knowable. A number should be attributed to every primitive notion and these numbers should combine together, producing all possible notions. A system of translation of these numbers into consonants and vowels would then allow us to assign to each notion a *character*[2] that refers univocally to the notion designated by it. Lastly, on the combinatory rules of these characters there should be founded the *ars iudicandi*:

> I necessarily arrived at this remarkable thought, namely that a kind of alphabet of human thoughts can be worked out and that everything can

---

[1] Among the numerous works that could be mentioned, two that after many years have aroused interest in this theme are Couturat (1903a) and Eco (1993).

[2] For an explanation of how to construct the 'characters', see Leibniz (1679a). More in general, on pp. 42-92 there is the first systematic attempt by Leibniz to develop a logical calculus. For an English translation of many of his writings, see Leibniz (2000). On the similarity and differences between Leibniz's ideas and those of other seventeenth-century authors engaged in projects for creating calculation languages, like George Dalgarno and John Wilkins, see Rossi (2000).

> be discovered and judged by a comparison of the letters of this alphabet and an analysis of the words made from them (Leibniz 1679c, p. 222).

Although for Leibniz all knowledge was potentially translatable into the *Characteristica*, this instrument could also have worked *in specific domains*, and therefore could also have been used *before* the whole alphabet of human thought was identified. In this connection, the grammar of this language would be unique and indifferent to the sphere of application. Such grammar would not be different from logic, seen as "the art of using the understanding not only to judge proposed truth, but also to discover hidden truth" (Leibniz 1696, p. 475). Hence, it would be, at one and the same time, a method of discovery (*ars inveniendi*) and a method of decision (*ars iudicandi*). Thus, once the fundamental notions have been identified, there could be a *calculus ratiocinator* in the light of which all controversies become vain:

> Obviously, once this is performed, every paralogism is nothing but a *calculation mistake*, and [...] every sophism, expressed in this kind of new writing, is nothing but a solecism or a barbarism, to be resolved easily through the laws of this philosophical grammar itself.
> Henceforth, when controversies arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, sit down with their counting-tables (having called in a friend, if they like) and say to each other: *let us calculate* (Leibniz 1684, p. 200, our translation).

Such a system would not be applicable to historical-natural language,[3] but to ideographic language, the *characteristica universalis*, in which there is a rigorous one-to-one correspondence between simple signs and simple ideas and, consequently, between compound signs and compound ideas. Only a similar artificial language would ensure this possibility, and not ordinary languages. The fact is that the latter:

> although they serve for reasoning, nevertheless they are subject to innumerable misunderstandings, nor can they be employed for calculation, that is to say in such a way that the errors of reasoning can be discovered (Leibniz 1684, p. 205).

---

[3] Awareness of the historical character of languages is at the heart of Leibniz's theoretical reflection, and does not contradict logico-combinatory works like *Dissertatio de arte combinatoria*. See Mugnai (1976) and Gensini (1991).

Hence *characteristica* will not make all men equally able to attain prodigious results in discovery, but it will make them all able not to commit errors or, possibly, to recognize errors by themselves, whether committed by them or by others. On one side, therefore, we have a system of discovery, linked to individual ability, and on the other a system of control that would become mechanical, and even *immediate*:

> And this is the advantage of our method – we can judge *at once*, through numbers, whether proposed proposition are proved, and so we accomplish, solely with the guidance of characters and the use of a definite method which is truly analytic, what others have scarcely achieved with the greatest mental effort and by accident. And therefore we can succeed in presenting conclusion within our own century which would scarcely be provided for mortals in many thousands of years otherwise (Leibniz 1679b, p. 236, our emphasis).

Thus the *characteristica universalis* should make us *immediately* able to judge the correctness of reasoning and this immediateness would be made possible by the reduction of control to simple sensory perception. As for Leibniz, in later authors too this strong idea of *instant rationality* will characterize the projects of logically perfect languages.

"Leibniz's dream", as Giuseppe Peano was to call this grandiose programme of the German philosopher,[4] was taken up in the late nineteenth century and the early twentieth, interweaving with projects for refounding logic. Explicit references to Leibniz's programme can be found in Boole and, even more markedly, in Frege, who at the beginning of his *Begriffsschrift* harks back precisely to Leibniz's "*calculus philosophicus or ratiocinator*" (1879, p. 6). In this work, Frege, after indicating in the symbolic systems of arithmetic, geometry and chemistry partial realizations of Leibniz's project, though relating to particular fields of knowledge, claims for his own ideography the merit of having added a new field and "indeed the central one, which borders on all the others" (1879, p. 7). In an article devoted to clarifying the difference between Peano's ideography and his own, Frege actually writes: "it [the *Begriffsschrift*] is, to use an expression coined by Leibniz, a *lingua characterica*" (Frege 1897).[5]

[4] See Peano (1908). The expression is taken up by Bertrand Russell (1901). On this topic see also Mondadori (1986).

[5] An analogous concept is found in Frege (1880-81). This reference is important for understanding that what Frege really meant to obtain was not a calculus – and so a variant of Boole's logic. Thanks to the possibilities afforded by "predicate letters, variables and quantifiers", the

An important implication of Leibniz's programme is that if one succeeded in taking it to its conclusion, there would *be no more need for logic as such*: correct syntactic formulation would in itself constitute a guarantee of correct logical reasoning. That the 'end of logic' is the inevitable result of the fulfilment of Leibniz's dream is particularly clear in the words of Frege:

> if we had a logically perfect language we would perhaps further need no logic, or we could read it off language. But we are at a vast distance from being in this condition (Frege 1915, p. 252).

2. Wittgenstein's 'adequate notation' and the futility of logic

Although in Wittgenstein's work there are no explicit references to Leibniz, and it is a very arduous undertaking to reconstruct his library, it appears very likely that the work of the German philosopher, possibly through the mediation of Frege and Russell, was very much present in the mind of the author of the *Tractatus.* In this connection, there are manifold (though little investigated) echoes of Leibniz in his writings.[6]

In the *Tractatus*,[7] Wittgenstein raises the question of an 'adequate notation', meaning a notation in which the grammatical structure and the logical structure of sentences coincide, and one through which each sentence *shows* its sense. For Wittgenstein, the sense of a proposition is to be identified with the possibility of its being true or false: "The sense of a proposition is its agreement and disagreement with the possibilities of the existence and non-existence of the states of affairs" (T. 4.2).[8] He distinguishes two types of propositions, elementary propositions [*Elementarsätze*] and propositions that have a

proposition becomes articulated and a meaning can be expressed by means of a language that can be considered a *lingua characteristica*, as already remarked by Van Heijenoort (1967b). A similar view is expressed in Hintikka (1997). Recently Korte has maintained that the decision to consider his *Begriffsschrift* a *lingua characteristica* "is related to his logistic program, which was to show that judgements of arithmetic are not synthetic, as Kant had claimed, but analytic" (Korte 2010).

[6] Among the few authors that relate the works of the first analytic philosophers with earlier artificial language projects, there are Soren Stenlund (2002) and Nicolay Milkov (2006).

[7] The reference is to Wittgenstein (1921). Here we quote the English translation, sometimes with slight modifications.

[8] Here we will not consider the consequences of an adequate notation in the framework of the distinction between senselessness [*Sinnloss*] of logical propositions, which at all events represent the framework of the world (cf. T. 6.124), and the nonsense [*Unsinn*] of non-logical propositions (cf. T. 4.003). See the now classic Diamond (1991), and Conant (2000).

complex structure, being formed by elementary propositions. While the truth of elementary propositions consists in the existence or non-existence of a certain fact about the world, the truth of the other propositions depends on the relations that link the elementary propositions contained in them: complex propositions are truth functions of the elementary propositions. As Wittgenstein writes: "A proposition is the expression of agreement and disagreement with the truth-possibilities of the elementary propositions" (T. 4.4).

Thus if the sense of a proposition consists in the conditions in which it is true or false, an adequate notation should be able to show these conditions explicitly. Wittgenstein has no doubt about the fact that this should be possible. In the *Tractatus* there is even a certain tension in relation to the idea that the proposition already shows its meaning in ordinary language – "a proposition *shows* its sense" (T. 4.022) – and not only in the presence of adequate symbolism. Nevertheless, Wittgenstein shares with Frege and Russell, however different their positions on common language may be, the idea that: "[in common language] it is humanly impossible to deduce the logic of language" (T. 4.002), because the grammatical structure does not mirror the logical structure of the sentence itself. The logic underlying linguistic utterances could instead be made evident by a more appropriate symbolism, one capable of making it immediately *visible*[9] without resorting to any 'deductive process'. This seems also to be lurking in the back of Russell's mind, when he writes:

> In a logically perfect language, there will be one word and no more for every simple object, and anything that is not simple will be expressed by a combination of words. [...] A language of that sort will be *completely analytic*, and will show *at a glance* the logical structure of the facts asserted or denied (Russell 1918, p. 176, our emphasis).

For Wittgenstein the logically perfect language is the practicable result of his proposal of a new kind of symbolism in which recognition of tautologies can be *immediate.* Since the deducibility of q from p (where p indicates the conjunction of the premises and q the conclusion of a deductive process) is equivalent to the tautologyhood of $p \rightarrow q$, the correctness of the inference of q from p would prove, in a symbolism of the kind, to be immediately visible.

Frege and Russell had demonstrated that it was possible, although extremely complicated, to show in a purely formal way the tautological character of a proposition through the creation of a proof system founded on self-

[9] Wittgenstein made explicit use of visual metaphors probably more than any other philosopher and the centrality of vision in cognitive processes is one of the main themes of the *Tractatus*.

evident logical axioms (propositions whose tautological character is immediately recognizable) starting from which it was possible to derive all the other tautologies through valid rules of inference (which preserve tautologyhood).

> We prove a logical proposition by creating it out of other logical propositions by applying in succession certain operations, which again generate tautologies out of the first. (And from a tautology only tautologies *follow*.) (T. 6.126).

In Wittgenstein's view, however, the systems of Frege and Russell were not as perspicuous as they should. In the first place, their approach made logical relationships opaque, causing propositions that are in fact identical to appear as distinct. For example, $p \rightarrow q$ (if p then q) and $\sim p \vee q$ (not p or q), though different signs, for Wittgenstein are *the same* proposition because both are true for all values of p and q, except in the case in which p is true (T) and q *is* false (F). Secondly, the privileged position of the tautologies that play the role of axioms is entirely arbitrary in such systems. Furthermore, recognizing that a proposition is a tautology, through a formal derivation, may generally be an extremely complicated process, and this appears in sharp contrast with the idea that "every tautology itself shows that it is a tautology" (T. 6.127).[10] Hence for Wittgenstein, the systems of Frege and Russell do not realize an 'adequate notation' for expressing logical connections. Indeed, in such a notation

> That the truth of one proposition follows from the truth of other propositions, we perceive from the structure of the propositions (T. 5.13, our emphasis)

and

> we can recognize in an adequate notation the formal properties of the propositions by mere inspection (T. 6.122).

Hence:

> 'Laws of inference', which – as in Frege and Russell – are to justify the conclusion are senseless and would be superfluous (T. 5.132).

[10] For a discussion of the effective decidability of the propositions of logic within the structure of the *Tractatus* see Marconi (2005).

Thus the outcome of this position is the thesis according to which in an *adequate notation logical deduction would be wholly superfluous!*[11]

It seems plausible that, according to Wittgenstein, an adequate notation could be provided by the famous method of the truth-tables introduced by himself in the *Tractatus.* The truth-table of a proposition explicitly shows its truth conditions in terms of the truth and falsehood of the elementary propositions contained in it. Further, for Wittgenstein every truth-table constitutes a propositional sign (T. 4.442):

For example the following is a propositional sign:

«

p	q	
T	T	T
F	T	T
T	F	
F	F	T

».

The table in the example illustrates the conditions of truth (and of falsehood) of the implication *if p then q* (but also of the disjunction *not-p or q*) in terms of the truth and falsehood of the elementary propositions *p* and *q*. It follows that *if p then q* is false only in the case in which *p* is true and *q* false, while it is instead true in all other cases. An abbreviation of this scheme could be (*TT-T*) (*p*, *q*) or, more explicitly, (*TTFT*) (*p*, *q*). Other logical connections between the same elementary propositions would evidently give rise to other configurations.

Thus, the ideal of an adequate notation, a symbolic system in which propositions explicitly show their own truth conditions, seems to be fully accomplished. The theoretical move that allows for this result consists in considering the truth-table itself as a 'propositional sign', that is to say, a configuration of signs able to serve as a proposition. The difficulty about recognising tautologies that afflicted Frege's and Russell's systems seems to dissolve into a symbolism in which logical relations become immediately (and *instantly*) visible. In this vein, Wittgenstein writes: "if two propositions contradict one another, then their structure shows it; the same is true if one of them follows from the other. And so on" (T. 4.1211). Hence we could say that we *see* from the propositional signs (*TTFT*) (*p*, *q*) and (*FFTF*) (*p*, *q*) – that is to say from a comparison between the arrangements of the signs *T* and *F* appearing in them – that

[11] On this point see also D'Agostino and Floridi (2009).

the propositions in question contradict one another.[12] Therefore we do not need any logical demonstrations.

By way of example, remembering that "Every proposition of logic is a *modus ponens* present in signs" (T. 6.1264), consider this simple inference:

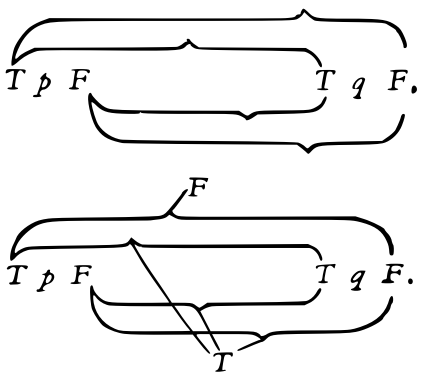
If it rains I put on my hat
It rains
Then I put on my hat

Let us now consider the truth-table that shows that this inference – which exemplifies *modus ponens* – is tautological (p and q stand for arbitrary propositions):

p	q	$p \to q$	$(p \to q) \wedge p$	$((p \to q) \wedge p) \to q)$
T	T	T	T	T
F	T	T	F	T
T	F	F	F	T
F	F	T	F	T

Recognition of the tautological character of the proposition in the last column – which guarantees the correctness of the inference in question – does not here require any 'logical demonstration', but the simple inspection of the propositional sign representing the proposition in the tabular notation.

In the *Tractatus* (6.1203), Wittgenstein also provides another example of adequate notation. It is not clear whether this intended to be an alternative method to the tabular one, serving for visual recognition of the tautologies or, as it would seem, amounts to essentially the same method:



[12] On this point see Piana (1973, p. 54).

Let *p* and *q* be the connected elementary propositions in a complex proposition. Each proposition may have two truth values T and F that will be connected to those of the other proposition, yielding the same values as obtained with the truth-tables.[13]

As we will see more clearly in the next sections, the central problem in both methods is that the number of possible assignments of truth-values increases exponentially with the number of elementary propositions occurring in the complex proposition, making both the tabular notation and the graphic one outlined above entirely unfeasible. Wittgenstein does not consider this problem,[14] and does not provide examples of how the most complex tautologies should be written in one of these symbolisms.

We will argue, in the next sections, that this is by no means an empirical problem that falls outside the austere perspective of *Tractatus*, and that the unfeasibility of the symbolisms proposed by Wittgenstein is only a special case of a general logical problem that (with all probability) does not admit of any solution.

3. The perfect language and mathematical logic

Through the interpretation of the truth-table as a propositional sign, the myth of the logically perfect language seems to find partial realization in Wittgenstein's *Tractatus*, though in the narrow domain of propositional logic. In this connection, the truth-table of a proposition fully expresses the sense of the logical words that occur in it and it would therefore provide an example of 'adequate notation' that, according to Wittgenstein, would make the process of logical deduction entirely superfluous. It is true that Church-Turing's undecidability theorem (1936)[15] excludes the possibility of finding a similar perfect language for more powerful logics, such as the logic of quantifiers (with non-monadic predicates). But it is also true that this negative result does not imply that there cannot be, even for the logic of quantifiers, an 'almost-perfect' language, which works reasonably well in all practical contexts. After all, the impact of the undecidability theorem on the massive attempts set going in the

[13] Max Black (1964, comment on T 6.1203) defines this system "coarse and unusable", while claiming that its tabular equivalent is by far better.

[14] As Pasquale Frascolla puts it, the speaker able to see all meta-logical relations is not the empirical speaker, because he would be a logically omniscient being; and Frascolla observes that this raises a new the problem: "in what sense, if any, can formal relations be said to exist even when the speaker is not in fact able to see them?" (2006, p. 180).

[15] For a historical reconstruction of the decision problem and of the negative results of Church and Turing, see Kneale and Kneale (1962).

1950s to realize Leibniz' dream through so-called 'automated deduction' was practically none. If anything, there prevailed the enormous impression aroused by the *positive* results, also made possible by theoretical contributions such as Herbrand's theorem, that provided methods of great practical interest for reducing quantificational reasoning to Boolean reasoning.

Analogous considerations also hold for other famous negative results of modern logic, first of all Gödel's theorems. These meta-mathematical exploits struck at the heart the hard-core of Hilbert's philosophical programme on the foundations of mathematics, that is to say, the assumption that an adequate formalization of mathematics is possible (Gödel's first theorem), and the assumption that it should be possible to show the consistency of this formalization with reliable means (Gödel's second theorem).[16] From the point of view of deductive *practice* and the possibility of a logically perfect language that would make it trivial or even superfluous, the first theorem is clearly the important one, but it is not at all obvious what its real scope is. As Solomon Feferman has observed:

> A common complaint about this result is that it just uses the diagonal method to "cook up" an example of an undecidable statement. What one would really like to show undecidable by PA or some other formal system is a natural number-theoretical or combinatorial statement of prior interest. The situation is analogous to Cantor's use of the diagonal method to infer the existence of transcendental numbers from the denumerability of the set of algebraic numbers; however, that did not provide any natural example. The existence of transcendentals had previously been established by an explicit but artificial example by Liouville. Neither argument helped to show that e and π, among other reals, are transcendental, but they did at least show that questions of transcendence are non-vacuous. Similarly, Gödel's first incompleteness theorem shows that the question of decidability of sentences by PA or any one of its consistent extensions is non-vacuous. That suggests looking for natural arithmetical statements which have resisted attack so far to try to see whether that is because they are not decided by systems that formalize a significant part of mathematical practice (2006, pp. 436-437).

But so far this search has produced no results.[17] Indeed, one of the most accredited candidates, the statement of Fermat's conjecture, has recently been proved,

[16] For an entertaining discussion of Hilbert's programme within the framework of the search for the perfect language, see Chaitin (2009), inspired by Eco (1997).

[17] Paris and Harrington's result, according to which a certain modification of a famous theorem of Ramsey is independent of Peano arithmetic, in a sense constitutes an exception. Nev-

and according to some logicians its proof could be formalized in Peano arithmetic. For none of the other candidates that survive, including Goldbach's conjecture and Riemann's hypothesis, has anyone ever succeeded in showing their independence of Peano arithmetic or of any of its coherent extensions.

Hence the famous negative results of 20th-century meta-mathematics, obviously without denying their importance on the philosophical side, have not had such a practical impact as to establish once and for all that the very idea of a logically perfect language is void of content, that is to say, cannot have any real instantiation that corresponds, *more or less*, to what Leibniz, Frege and Wittgenstein had in mind. From this point of view, perhaps it is not the negative results that are surprising. What is truly surprising, as George Kreisel once remarked, is the discovery that certain branches of mathematics – such as pure logical reasoning on truth functions and quantifiers or elementary Euclidean geometry – are, after all, mechanizable.[18]

Therefore in this perspective the emphasis must be, if anything, on the numerous *positive* results, first of all the decidability of elementary Euclidean geometry, proved by Tarski in 1951,[19] and one may wonder what Leibniz's reaction would have been to a result of the kind! From this perspective, the dream of a logically perfect language would perhaps be impossible 'in principle' – that is to say, there would be no foundation to the grand theoretical pretension of finding a *universal* adequate notation, in which the solution of *all* logical problems can be 'read' off their statement itself – but it may well be realizable 'in practice', in the vast majority of the interesting cases that come to our attention and that, in the imperfect language of ordinary reasoning, can only be solved through tiresome (and therefore highly fallible) deductive processes. Leibniz's dream, and the myth of the perfect language that is indissolubly associated with it, would die as a *philosophical* research programme, but would survive as the metaphysical hard-core of a *scientific* research programme that, in spite of the negative results, appears to be highly progressive.[20]

ertheless, as Feferman himself observes, even in this case it was not a matter of an undemonstrable proposition whose truth had previously been the object of conjecture. For Ramsey's theorem and Paris and Harrington's result see Mangione and Bozzi (1993, in particular p. 516 and pp. 856-857).

[18] On this point see Kreisel and Krivine (1971, p. 165-166).

[19] Tarski (1951).

[20] As Gregory Chaitin puts it: "There's a wonderful intellectual tension between incompleteness and the fact that people still believe in formal proof and absolute truth. People still want to go ahead and carry out Hilbert's program and actually formalize everything, just as if Gödel and Turing had never happened!" (2009, p. 19).

However, this hope for the realizability of a 'practically perfect' logical language is also, with all probability, entirely unfounded. We will see that this pessimistic conclusion is the consequence of a profound result obtained in a field of research, that of computational complexity, whose origins are apparently very distant from typical philosophical problems, such as that of the perfect language, and are instead linked to a vast constellation of practical problems raised by the development of computer technology.

4. 'Unfeasible' tautologies

In his *Games of Arithmetic and Interesting Problems*, published for the first time in 1925, Giuseppe Peano introduces a series of problems serving to make "arithmetic more pleasant and less boring." Among the "captious problems", in which "the true answer is not the one that first presents itself to the mind", there was the following one:

> A party of 7 travellers go to a hotel and ask for a bed for each traveller. The hotelier answers: "I only have six beds, distinguished with the letters A, B, C, D, E and F. But I will try to accommodate you anyway." So he tells two travellers to sleep in bed A, then one in bed B, and that is three; then one in C, and that is four; then one in D, and that is five; then one in E, and that is six; then he takes one of those that were in A and transfers him to F, and that is seven. Thus the 7 travellers sleep in 6 beds, one per bed. How did he do this? Anyone who plays the game represents the beds with six cards, and proceeds fast, so that the listener does not realize that a traveller has been counted twice (Peano 1925).

The impossibility of an 'honest' solution to the problem of the hotel keeper is ensured by a fundamental combinatorial principle, also known as 'pigeonhole principle' or 'Dirichlet's principle', according to which $n+1$ objects cannot be placed in n boxes unless one of the boxes contains more than one object. Despite its obviousness from the intuitive point of view, this principle is surprisingly useful for showing an enormous variety of mathematical facts, from simple curiosities, for example that in London there are at least two people that have the same number of hairs, to numerical matters that are anything but obvious, for example that among N integers chosen at random there are always two whose difference is divisible by $N-1$, or that for any irrational number a there exist infinite rational numbers $r = p/q$ such that $|a-r| < q^{-2}$.

It is well known that this important combinatorial principle can be expressed in a simple and natural way through a class of tautologies of Boolean logic. Indicating with $p_{i.j}$ the sentence according to which the object *i* occupies box *j*, the proposition

$$(*) \quad (p_{1.1} \vee p_{1.2}) \wedge (p_{2.1} \vee p_{2.2}) \wedge (p_{3.1} \vee p_{3.2}) \wedge \neg(p_{1.1} \wedge p_{2.1}) \wedge \wedge \neg(p_{1.2} \wedge p_{2.2}) \wedge \neg(p_{1.1} \wedge p_{3.1}) \wedge \neg(p_{1.2} \wedge p_{3.2}) \wedge \neg(p_{2.1} \wedge p_{3.1}) \wedge \wedge \neg(p_{2.2} \wedge p_{3.2})$$

asserts, for example, that three objects can be placed in two boxes in such a way that each box contains at most one object. The impossibility of the situation described by this proposition is therefore a special case of the pigeonhole principle, with $n = 2$. The interesting thing is that this proposition is *logically* inconsistent (even simply at the level of Boolean logic) and therefore its negation is a *tautology.* Hence the pigeonhole principle is expressed by the class of all the tautologies obtained by negating the propositions constructed on the model of (*) for each positive whole number $n > 1$.

Well, the tautology that expresses the pigeonhole principle for a given n contains $f(n) = (n + 1) \times n$ distinct propositional letters and the length of the expression representing it in the still "imperfect" logical language of Frege and Russell contains (including the parentheses and counting each $p_{i.j}$ as a single symbol) an overall number of symbols equal to:

$$g(n) = 7/2\, n^3 + 11/2\, n^2 + 4n + 1.$$

On the other hand, the number of lines contained in the complete truth-table for this expression, that is to say in its translation into the 'adequate notation', is equal to $h(n) = 2^{f(n)}$. The problem is that with an increase in n, the length of the translation into the 'logically perfect' language of the truth-tables grows much faster than the length of the proposition expressed in a standard Boolean language. While $g(n)$ is a *polynomial* function of n (of degree 3), $h(n)$ is an *exponential* function of n, and it is well known that the speed of growth of exponential functions very soon takes the length of the translation beyond the limits of feasibility. For example, returning to the 'captious' problem of Peano, to express the fact that 7 travellers cannot be put in 6 beds, a string containing 979 symbols is sufficient in an ordinary logical language, but its translation into the perfect language would require the construction of a truth-table containing 4 398 046 511 104 lines (without calculating the length of each line)! It can easily be verified that, even for relatively low values of n, the overall number of symbols occurring in the truth-table required to 'see immediately' that the expression is a tautology would be higher than the number of atoms contained in the known universe, while the same tautology expressed in the imperfect standard language containts only a few thousand symbols. It follows that translating a proposition from the ordinary logical language of Frege and Russell into the logically perfect language of the truth-tables – in which it is possible to recognize that a certain expression is a tautology simply by examining the expression itself – is a task that is *practically impossible*. If a logically perfect language exists, and it must be possible to use it in practice, it cannot be that in which the propositional signs are constituted by the truth-tables.

This *reductio ad absurdum* of the idea that a logically perfect language could be that of the truth-tables is obviously not sufficient to render research in this direction entirely vain. There might be a more concise way to give a complete representation of the sense of a proposition – that is to say, to perform the same task as Wittgenstein assigned to the truth-tables – which however does not determine any combinatorial explosion. After all, in the truth-table method there are obvious inefficiencies that could easily be eliminated: (a) the same propositional letter can occur many times, so that a symbolic expression could be 'compressed' by resorting to a representation in the form of a graph, and (b) the enumeration of the truth conditions is manifestly redundant: for example, any 'state of affairs' that makes p true also makes true the disjunction $p \vee q$, and it is not necessary to consider *all* possible assignments to the atomic propositions that occur in q. It is legitimate to wonder whether the elimination of these inefficiencies would not in itself be sufficient to render truly perfect the language of the truth-tables; or, alternatively, whether it is possible to contrive some other representation of propositions, maybe wholly different from that proposed by Wittgenstein, that really is able to satisfy his request for an 'adequate notation' in which the tautological character of a proposition proves to be 'immediately' perceivable. At this precise point the theory of computational complexity enters the scene. Its impact on the scientific research programmes descending from Leibniz' dream and from the myth of the perfect language, which survived the limitative theorems of the 1930s without big traumas, has been very important and has involved a profound change of perspective whose philosophical consequences are not yet fully understood.

5. Perfect language and computational complexity

The theory of computational complexity can be considered a refinement of the traditional theory of computability taking into account the *resources* (time and space) used by algorithms. Its principal innovation consists in having replaced the concept of 'effective procedure' with that of 'feasible procedure'.[21] An *effective procedure* or *algorithm* by and large consists in a 'mechanical method', i.e. one executable *in principle* by a machine, to solve a given class of problems. An effective procedure is *feasible* when it can also be carried out by a machine *in practice*, and not only in principle. The expression 'in practice' involves a certain degree of vagueness that computational complexity researchers have removed by agreeing to consider as feasible, executable in practice, algorithms that can be performed in *polynomial* space and time. To clarify the meaning of this convention, let us observe first of all that a *problem* is usually associated with a class of strings of symbols, in which every string identifies a particular istance of the problem. Thus the 'tautology problem' (abbreviated with TAUT) can be associated with the class of strings of symbols that represent 'well-formed formulas' in a standard logical language, let us say the language of the *Principia Mathematica* of Russell and Whitehead. A solution to a particular istance of this problem consists, in this case, in an answer of the 'yes-or-no' type: 'yes' (usually encoded with the symbol '1') if the string in question is a tautology, 'no' (usually encoded with the symbol '0') if it is not. The string of symbols in (*), which we used above to encode a particular istance of the 'pigeonhole problem', represents a logically inconsistent formula, and since the negation of an inconsistent formula is a tautology, a correct algorithm solving TAUT has to produce the answer 'yes' whenever it receives as input the negation of (*). A problem like TAUT, whose solution consists in an affirmative or negative answer, is referred to as a *decision problem*. Not all interesting problems are decision problems. Indeed, many of the problems that are met in practice are described as *determination problems* ('what *is* the area of a circle of radius r?'), and among them a major role is played by *optimization problems*, problems that require finding an optimal solution within a set of possible solutions ('given a graph and a starting node s, find a minimum pathway from s to a given other node of the graph'). Nevertheless, these problems can be transformed into equivalent decision problems, so that it is possible to simplify the analysis by making reference only to decision problems and to the resources employed by the algorithms solving them.

The *running time* $T(n)$ of an algorithm measures the (maximum) number of steps that the algorithm has to perform as a function of the complexity of the input, that is to say, of the length of the string of symbols that encodes a particular instance of the problem. Likewise, the *running space* $S(n)$ can be defined

[21] For an excellent exposition, still valid after thirty years, see Garey and Johnson (1979).

Size of largest problem instance solvable in one hour

Running time	*With present computer*	*With computer 100 times faster*	*With computer 1000 times faster*
n	N1	100 N1	1000 N1
n^2	N2	10 N2	31.6 N2
n^3	N3	4.64 N3	10 N3
n^5	N4	2.5 N4	3.98 N4
2^n	N5	N5+6.64	N5+9.97
3^n	N6	N6+4.19	N6+6.29

FIG. 1: *Effect of improved technology on several polynomial and exponential time algorithms (from Garey and Johnson 1979).*

as the (maximum) number of units of memory that the algorithm has to use as a function of the complexity of the input. The problem is: how do these functions grow with an increase in the complexity n of the input? There is widespread agreement in defining as *feasible* (i.e. solvable in practice) problems that can be solved by algorithms working in polynomial time, i.e. ones for which a polynomial p exists such that, for any input of complexity n, the algorithm yields an answer in a number of steps $\leq p(n)$. (Solvability in polynomial time implies solvability in polynomial space, so that the time factor is privileged in the definition of the feasibility of a problem.) The underlying idea is that polynomial algorithms do not produce the kind of combinatorial explosion that we have observed in relation to truth-tables.

If, instead, the most efficient algorithm possible works in super-polynomial time, for example in *exponential* time (that is, described by a function like 2^n), the problem is considered *unfeasible.* Although certain polynomial time algorithms may well be highly inefficient and, for some inputs, even much more inefficient than exponential time algorithms that solve the same problem – for example when the running time is described by a polynomial of a very high degree, such as n^{100} – it must be observed that (a) their *asymptotic* behaviour is always enormously more efficient than that of exponential time algorithms; this also implies that, in practice, the actual time employed by these algorithms is highly sensitive to technological innovation, leading to the construction of faster and faster computers, while the latter is wholly irrelevant for exponential time algorithms (in this connection see Figure 1); (b) in actual fact the polynomial time algorithms that have emerged from attempts to solve natural problems arising in any research areas have a running time described by a polynomial of a low degree (usually not above three).

The class **P** is the class of all *tractable* decision problems, i.e. those that can be solved through an algorithm working in polynomial time. The bad news is that many interesting problems met in mathematical and technological research, for which a decision procedure is known, do *not* belong to **P**. Among these there stands out the problem of establishing whether a certain proposition is a theorem of elementary Euclidean geometry. Although, as we have already mentioned, this problem was 'solved' by Tarski in 1951, twenty-three years later Fischer and Rabin proved that it is in fact an intractable problem.[22] Other decidable problems whose intractability has been shown are:

- The arithmetic of the addition of natural numbers (Fischer and Rabin 1974; decidability shown by Pressburger in 1929).
- The arithmetic of the multiplication of natural numbers (Fischer and Rabin 1974).
- The theory of linear orders (which is obtained by the first-order axioms that express transitivity, totality and anti-symmetry; non-elementary lower bound demonstrated by A. R. Meyer in 1975; decidability shown by Rabin in 1969).[23]

What can one say about TAUT, our problem of recognizing tautologies? It is licit to require that a 'logically perfect language' should be a language *L* in which:

1. Inside *L* the problem of tautology should be solvable in polynomial time: if it has to be possible to recognize a tautology 'immediately', simply by examining the propositional sign expressing it, that is to say, if one must be in a condition of 'seeing' from the sign itself that a certain expression is a tautology, then a minimum requirement is that there exists a polynomial time algorithm that allows one to make such recognition;
2. The translation from ordinary logical language into *L should* be feasible, that is to say, it should be possible to express in *L* what can be expressed in the ordinary logical language without producing any combinatorial explosion that would make the logically perfect language *L* 'perfectly' useless for the purposes of deductive practice.

Well, Wittgenstein's 'logically perfect language', in which the propositional signs coincide with the truth-tables, satisfies the first requirement – given any

[22] See Rabin (1977).
[23] For all these results, see Rabin (1977).

formal language in which it is possible to represent the truth-tables, it is possible to recognize in polynomial time whether a string of symbols in this language encodes the truth-table of a tautology[24] – but does not satisfy the second, since the length of the translation increases, as we have seen, exponentially in relation to the length of the proposition translated. It is to be observed that this does not only happen for artificial examples – for 'logical monsters' constructed *ad hoc* in order to obtain the desired result – but is a general phenomenon that manifests itself as the number of atomic propositions involved increases and therefore, as we have seen, also concerns tautologies expressing entirely natural logical principles, of great utility in both mathematical and ordinary reasoning. Thus the 'adequate' notation proposed by Wittgenstein in the *Tractatus* is not a logically perfect language. But does another formal language exist that satisfies both our requirements? We can call this question 'the problem of the logically perfect language'.

6. Cook's theorem and the inevitable imperfection of logical languages

In 1971 Stephen Cook proved a result from which it immediately follows that the problem in question is very probably unsolvable (Cook 1971). To outline the meaning of this result we have first to introduce a new concept: that of *non-deterministic algorithm.* Let us consider the complementary problem of TAUT, i.e. the *satisfiability* problem, which we can abbreviate as SAT: given any string of symbols expressing a proposition (a 'well-formed formula' again) in a standard Boolean language, the problem is to determine whether an assignment of truth values (true or false) to the atomic propositions that occur in it exists – or, as Leibniz would have put it, whether there is 'a possible world' – that renders this proposition true. In such a case the formula in question is said to be *satisfiable.* Since a tautology is a formula that is true for all assignments of this type (true in 'all possible worlds'), and given that a formula is true if and only if its negation is false, it follows that a formula is satisfiable if and only if its negation is not a tautology, so that SAT and TAUT are complementary problems.

Let us now imagine an 'algorithm' attempting to determine whether a given formula is satisfiable using 'guesswork', that is to say by assigning at random a truth value to each atomic proposition and verifying whether the assignment thus obtained makes the given proposition true. An algorithm of the

[24] For this purpose it is sufficient to ensure that the truth values are assigned correctly on each line and that the final column only contains "1".

kind is 'non-deterministic' in that there are no instructions that exactly specify how the truth values that are assigned to the elementary propositions are to be chosen. Hence, the computation is no longer expressed as a sequence of steps, but as a *tree* of possible choices. Well, the running time of such a non-deterministic algorithm is the (maximum) number $T(n)$ of steps that must be performed for any input of length n, in the case in which the choices made are always the 'luckiest' possible, that is to say, those that lead to recognizing the yes-instances of the problem (in our case to an assignment, if any, that makes the input formula true). In the case of SAT, supposing that the luckiest choices of truth values are always made for the elementary propositions, it can be verified in linear time that the assignment thus obtained is correct, and therefore SAT can be 'solved' in polynomial time by a non-deterministic algorithm that simply consists in guessing an assignment and then verify that it is correct.

The class of problems analogous to SAT, i.e. the ones that can be solved in polynomial time by a similar ideal 'algorithm' always making the best choices, is named **NP** (in which the letter 'N' stands for 'non-deterministic' and the letter 'P' for 'polynomial'). The class **NP** can also be characterized, as emerges from our discussion, as the class of problems for which it is possible to verify in polynomial time whether a putative proof that the answer is "yes" is indeed correct. For example, a given assignment to the atomic formulas that makes a complex formula true can be seen as a proof that the formula in question is satisfiable (that is, it belongs to SAT), and it is easy to verify that the proof, once found, is correct. It has emerged that the answer to many important problems is very difficult to find through ordinary deterministic algorithms, although the yes-instances admit of proofs that can be easily verified once they have been found. These problems are therefore in **NP**, but have eluded so far every attempt to show that they are also in **P**. One of these recalcitrant problems is precisely SAT.

It is legitimate to wonder whether luck really is so essential in solving problems in **NP**. Given a problem in **NP** is it perhaps always possible to find a deterministic algorithm that solves it in polynomial time? If the answer were affirmative then **P** would be equal to **NP**; in the opposite case the two classes would be different. Given its enormous scientific and technological impact, this is one of the seven 'problems of the millennium' for whose solution the Clay Mathematics Institute has put up a prize of seven million dollars (one million for each problem).[25] The point is that, among the problems in **NP**, there are some that are *really* hard: not only have they always eluded the efforts of

[25] See the site of the Clay Institute: http://www.claymath.org/millennium.

researchers to find deterministic algorithms solving them in polynomial time, but it is also possible to show that if one of these were solved feasibly, then *all* problems in **NP** would be. An example is the famous 'travelling salesman problem', abbreviated as TSP: given a network of towns connected by roads and a numerical bound B, the problem is finding a tour, if any, that visits all the towns exactly once and has lenght no more than B. It can be shown that, for any problem Π in **NP**, a feasible procedure exists (i.e. one with polynomial running time) for translating any instance I of Π into an instance I' of TSP in such a way that I' is a yes-instance of TSP if and only if I is a yes-instance of Π. Therefore, if TSP were in **P**, that is to say, if it were a tractable problem, then every problem in **NP** would also be tractable and therefore **P** would be equal to **NP**. Problems of this type are called **NP**-*complete*. Another example of an **NP**-complete problem is the 'problem of the partial sums': given a finite set of integers, the problem is to determine whether it includes a subset such that the sum of its elements is zero. It can quickly be verified whether or not a given subset yields a solution to the problem, but no method is known for finding a solution that is significantly more efficient than checking, one by one, all the subsets (which are exponential in number). The list of the **NP**-complete problems increases very fast[26] and covers hundreds of interesting issues, belonging to a variety of research fields, which have always eluded any attempt to find an efficient algorithmic solution. Recently the problem of *sudoku*, well known to puzzle-solvers, has joined the list.[27]

Since all problems in **NP** are polynomially reducible to each of the **NP**-complete problems, the latter are all equivalent to one another (that is to say, polynomially reducible to one another). Therefore, either they are all tractable and $\mathbf{P} = \mathbf{NP}$, or they are all intractable and $\mathbf{P} \neq \mathbf{NP}$. The dominant conjecture among researchers is that $\mathbf{P} \neq \mathbf{NP}$ and, accordingly, that all **NP**-complete problems are intractable. This conjecture is used in numerous application areas (also ones that are critical from the security point of view, such as cryptography) *as if* it were demonstrated. From this point of view, its epistemological status is no different than that of a well corroborated hypothesis in a theory belonging to the empirical sciences: we behave as if it were true, though aware that one day or another it could prove false.

[26] For an ample overview check the latest edition of Garey and Johnson (1979).

[27] In its most general version the *sudoku* problem requires inserting numbers between 1 and n^2 in a matrix of $n^2 \times n^2$ elements subdivided into n^2 sub-matrixes of dimension n^2, so that no number appears more than once in every line, column and sub-matrix. For a proof of its **NP**-completeness see Yato (2003).

Cook's theorem, which opened up the way to the theory of **NP**-completeness, consists precisely in the statement that in a standard Boolean language:

*SAT is **NP**-complete.*

It is therefore highly plausible (though unproven) that in a standard Boolean language like that of Frege and Russell, SAT is an intractable problem. Since a proposition is a tautology if and only if it is not satisfiable, any solution to SAT is also a solution to TAUT, the tautology problem; therefore, in a standard Boolean language, we must expect TAUT to be intractable too.

The probable intractability of the tautology problem in a standard Boolean language immediately implies that our two requirements for a logically perfect language cannot *simultaneously* be satisfied. If in a given language *L* tautologies can immediately be recognized through mere *inspection* of the symbols (that is to say in linear or, at worst, polynomial time), it is then highly implausible, via Cook's theorem and the related conjecture that $\mathbf{P} \neq \mathbf{NP}$, that there is a feasible translation from the ordinary logical language to *L*. For, its existence would also imply the existence of an efficient deterministic algorithm to solve the tautology problem (in the standard language), which instead, according to the currently accepted conjecture, does not exist. It therefore seems that the dream of a perfect language, in which the answer to a question is contained in its very 'clear and distinct' formulation, is unattainable in practice even in the case of apparently very simple problems for which it has long been known that it can be attained in principle. Hence there is no such thing as *instant rationality* (even for the most basic problems).

6. Logic in a network of imperfect languages

The conclusion that we reached in the previous section forces us to revisit the old idea of the presumed 'tautological' character of propositional logic. How is it possible for a logical truth 'to say nothing' (T. 6.11) if recognizing the fact that it 'says nothing' is such a hard problem that, with all probability, it does not admit any practical algorithmic solution? We should conclude that what a proposition 'says' cannot be fully understood or communicated, except in the simplest cases (in which only few atomic propositions occur), even with the help of the fastest computers that can be built compatibly with the laws of nature. But such a conclusion certainly appears paradoxical and constitutes a particularly strong version of what the philosopher Jaakko Hintikka has called 'the scandal of deduction':

> C.D. Broad has called the unsolved problems concerning induction a scandal of philosophy. It seems to me that in addition to this scandal of induction there is an equally disquieting scandal of deduction. Its urgency can be brought home to each of us by any clever freshman who asks, upon being told that deductive reasoning is 'tautological' or 'analytical' and that logical truths have no 'empirical content' and cannot be used to make 'factual assertions': in what other sense, then, does deductive reasoning give us new information? Is it not perfectly obvious there is some such sense, for what point would there otherwise be to logic and mathematics? (Hintikka 1973, p. 222)

Hintikka's thesis, in brief, it is that the idea of the 'tautological' or 'analytical' character of deductive reasoning clashes both with intuition and with the discovery, by Church and Turing in the 1930s, that quantificational logic is undecidable.

Hintikka proposes to resolve the scandal through a revision of the traditional notion of semantic information (from which it would result that logical reasoning does not produce any new information), on whose basis it is possible to argue that the truths of quantificational logic are *not* tautological.[28] Nevertheless, this conclusion cannot be extended to propositional logic, given that for the latter a mechanical decision procedure exists. Thus, according to Hintikka, "the term 'tautology' does characterize very aptly the truth and inferences of propositional logic. One reason for its one-time appeal to philosophers was undoubtedly its success in this limited area" (Hintikka 1973, p. 154). It is precisely this conclusion of Hintikka's that is challenged, according to our reconstruction, by the consequences of Cook's result discussed in the previous sections.[29]

The thesis that deductive logic (in general) is 'tautological' has been taken up more recently by Carlo Cellucci (1998, 2000) in the context of his criticism of the 'closed world conception', that is to say the traditional point of view according to which mathematical theories are closed systems that cannot exchange information with the environment and are based on the axiomatic method. According to this orthodox view, mathematics is correctly represent-

[28] For a detailed criticism of Hintikka's proposal, which however does not address the themes discussed in this paper, see Sequoiah-Grayson (2008).

[29] A thorough discussion of the consequences of Cook's theorem on the very idea that propositional logic is tautological, inspired by some ideas in D'Agostino and Mondadori (2000), can be found in D'Agostino and Floridi (2009). On the related theme of 'logical omniscience' see also D'Agostino (2010).

ed by formal systems in which, according to a famous analogy by Gottlob Frege, theorems are contained in axioms "like plants in seeds" (Frege 1884, § 8), and its truths are established in a rigorously deductive way. But, since deduction cannot increase information (given that logic is 'tautological'), and indeed generally reduces it (the conclusions are usually weaker than the premises), the closed world conception fails to explain the utility and creativeness of mathematical reasoning. According to Cellucci, this *reductio ad absurdum* of the closed world conception shows, (a) that mathematics cannot be adequately described by formal systems, and (b) that the methods of mathematics cannot be contained in the narrow confines of traditional deductive logic, but must be sought in alternative logics, able to represent reasoning processes that increase the information content. To the 'closed world conception' Cellucci opposes an 'open world conception', according to which mathematical knowledge is more similar to a 'distributed environment' of open systems, each of which affords a partial representation of the corresponding domain and is able to exchange information with the others.

The image of the growth of mathematical knowledge emerging from this analysis is undoubtedly more realistic and interesting than the traditional one. Nevertheless, one of the premises on which it is founded, that is to say the presumed 'tautological' character of deductive reasoning, appears to be difficult to justify in the light of our discussion. If it is true that mathematics is a network of open systems, this is largely independent of the thesis that deductive reasoning is tautological, which can hardly be defended even in the elementary domain of propositional logic. Nevertheless, the consequences of the theory of **NP**-completeness that we have discussed here lend further support to Cellucci's open world conception. In this connection, Cook's theorem can be exploited to maintain that the 'closed world conception' is untenable even within (traditional) deductive logic! It is not possible – or more exactly it is highly unlikely – that there exists a *single* formal system for classical propositional logic that can be used *in practice* to solve *all* problems in this domain. We have seen that one of these formal systems, the method of truth-tables, fails miserably in the practical attempt to recognize the tautological character of a fundamental combinational principle, the so-called 'pigeonhole principle'. We chose this principle, by way of example, because it is an 'intuitively obvious fact' widely used in mathematical practice, not an artificial problem concocted for the sole purpose of challenging a particular formal system. Furthermore, the class of tautologies that express this principle is difficult even to *prove* for most complete formal systems used in automated deduction (in the technical sense that the shortest proof has exponential length). We can therefore maintain that all these formal systems are *practically incomplete* and that their incompleteness emerges in relation to classes of logical truths whose mathematical meaning is unquestionable, quite independently of the intractability proof.

On the other hand, it is not difficult to construct formal systems for propositional logic in which the pigeonhole problem *can* 'easily' be solved (in polynomial time).[30] Cook's theorem implies, however, that all these formal systems must also, with all probability, be 'practically incomplete' too, in that they will not succeed in recognizing in polynomial time other (infinite) classes of tautologies. Therefore the only hope, even in this elementary domain, is to build up a 'distributed environment' of logical systems that are able to exchange information with one another, each of which yields a partial but feasible representation of a fragment of propositional logic. It follows from Cook's theorem that this 'network' of systems cannot be finite,[31] so that the search for a solution to our logical problems will necessarily be, in accordance with the open world conception, a potentially infinite process. The myth of the logically perfect language, of a characteristic language for logic that can save us from the toil of deduction, has to give way to a pluralistic vision in which, instead of a *single* 'perfect' language, there is a (potentially infinite) *variety* of logical languages, each of which is inevitably 'imperfect' and can only attain perfection in relation to a partial domain, to a tiny fragment of the universe of logical relations. It is only from such a continual interaction of imperfect languages, from the full heuristic unfolding of the 'perfection/imperfection' opposition, that there can gradually (and never completely) emerge a practical solution to the tautology problem, as well as to other general problems that for centuries have challenged the ability of logicians and mathematicians.

REFERENCES

BLACK, Max (1964): *Companion to Wittgenstein's Tractatus*, Cambridge: Cambridge University Press.

CELLUCCI, Carlo (1998): *Le ragioni della logica*, Bari: Laterza.

——— (2000): "The Growth of Mathematical Knowledge: An Open World View", in E. Grosholz and H. Berger (eds.), *The Growth of Mathematical Knowledge*, Dordrecht: Kluwer, pp. 153-176.

[30] For an example, see Dunham and Wang (1976).

[31] Otherwise the combination of these systems would afford a polynomial decision procedure for the tautology problem.

CHAITIN, Gregory (2009): *The Search for the Perfect Language*, lecture delivered to the Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada, on September 21, 2009; published in Portuguese in *Dicta & Contradicta*, 4, 2009, http://www.cs.auckland.ac.nz/~chaitin/pi.html.

CONANT, James (2000): "Elucidation and Nonsense in Frege and Early Wittgenstein" in A. Crary and R. Read (eds.), *The New Wittgenstein*, London: Routledge, pp. 174-217.

COOK, Stephen (1971): "The Complexity of Theorem-Proving Procedures", in *Conference Record of Third Annual ACM Symposium on Theory of Computing*, New York: ACM, pp. 151-158.

COUTURAT, Luis (1903a): *Histoire de la langue universelle*, with Léopold Leau, Paris: Hachette.

——— (1903b): *Opuscules et fragments inédits de Leibniz*, Paris: Alcan.

D'AGOSTINO, Marcello (2010): "Depth-Bounded Boolean Logic and the Problem of Logical Omniscience", in H. Hosni and F. Montagna (eds.), *Probability, Uncertainty and Rationality*, Berlin: Springer (SNS Series).

D'AGOSTINO, Marcello and FLORIDI, Luciano (2009): "The Enduring Scandal of Deduction. Is Propositional Logic Really Uninformative?", *Synthese*, 167 (2), pp. 271-315.

D'AGOSTINO, Marcello and MONDADORI, Marco (1999): "La logica è davvero analitica?", in *Studi in Onore di Franco Crispini*, Bollettino Filosofico dell'Università della Calabria, 15.

DUNHAM, Bradford and WANG, Hao (1976): "Toward Feasible Solutions of the Tautology Problem", *Annals of Mathematical Logic*, 10, pp. 117-154.

DIAMOND, Cora (1991): *The Realistic Spirit*, Cambridge (Mass): The MIT Press.

ECO, Umberto (1993): *La ricerca della lingua perfetta*, Roma-Bari: Laterza (English translation: *The Search for the Perfect Language*, Oxford: Blackwell, 1997).

FEFERMAN, Solomon (2006): "The Impact of the Incompleteness Theorems on Mathematics", *Notices of the AMS*, 53 (4), pp. 434-439.

FRASCOLLA, Pasquale (2006): *Understanding Wittgenstein's Tractatus*, London: Routledge.

FREGE, Gottlob (1879): *Begriffsschrift, eine der arithmetischen Nachgebildete Formelsprache des reinen Denkens* (English translation, by S. Bauer-Mengelberg: *Begriffsschrift, a Formula Language Modelled upon that of Arithmetic, for Pure Thought*, in Van Heijenoort 1967a, pp. 1-82).

——— (1880-81) "Booles rechnende Logik und die Begriffsschrift" (English translation, by P. Long and R. M. White: "Boole's Logical Calculus and the Concept-script", in Frege 1991).

——— (1884): *Die Grundlagen der Arithmetik* (English translation, by J. L. Austin: *The Foundations of Arithmetic*, Oxford: Blackwell, 1980).

——— (1897): "Über die Begriffsschrift des Herrn Peano und meine eigene", in *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Mathematisch-Physische Klasse*, 48, pp. 361-378 (English translation: "On Mr. Peano's Conceptual Notation and My Own", in McGuinness 1984, pp. 234-248).

——— (1915): "Meine Grundlegenden logischen Einsichten" (English translation: "My Basic Logical Insights", in Frege 1991, pp. 251-252).

——— (1991): *Posthoumus Writings*, edited by H. Hermes, F. Kambartel and F. Kaulbach, New York: Wiley & Sons.

GAREY, Michael and JOHNSON, David S. (1979): *Computers and Intractability. A Guide to the Theory of NP-Completeness*, New York: W.H. Freeman.

GENSINI, Stefano (1991): *Il naturale ed il simbolico*, Roma: Bulzoni.

HINTIKKA, Jaakko (1973): *Logic, Language Games and Information. Kantian Themes in the Philosophy of Logic*, Oxford: Clarendon Press.

——— (1997): "Lingua Universalis vs. Calculus Ratiocinator. An Ultimate Presupposition of Twentieth-Century Philosophy", in *Selected Papers*, Dordrecht: Kluwer, vol. II, pp. IX-X.

KNEALE, William and KNEALE, Martina (1962): *The Development of Logic*, Oxford: Oxford University Press.

KORTE, Tapio (2010): "Frege's Begriffsschrift as a *lingua characteristica*", *Synthese*, 174, pp. 283-294.

KREISEL, George and KRIVINE, Jean Louis (1971): *Elements of Mathematical Logic*, Amsterdam: North Holland.

LEIBNIZ, Gottfried Wilhelm (1679a): "Elementa Characteristicae universalis", in Couturat (1903b, pp. 42-49).

——— (1679b): "Elementa Calculi", in Couturat (1903b, pp. 49-57) (English translation in Leibniz 1969, pp. 235-239).

——— (1679c): *Sine titulo*, in Leibniz (1890, pp. 184-189) (English translation: "On the general characteristic", in Leibniz 1969, pp. 221-228).

——— (1684): *Sine titulo* ("De Scientia universali seu calcolo philosophico"), in Leibniz (1890, pp. 198-203) (English translation: "On the General Characteristic", in Leibniz 1969, pp. 221-228).

——— (1696): "Letter to Gabriel Wagner", in Leibniz (1969, p. 475).

——— (1969): *Philosophical Papers and Letters*, edited by L. L. Loemker, London: Kluwer.

——— (1890): *Philosophische Schriften*, edited by C. I. Gerhardt, vol. VII, Berlin.

——— (2000): *Logical Papers*, Oxford: Oxford University Press.

MARCONI, Diego (2005): "Frascolla on Logic in the *Tractatus*", in *Dialectica*, 57, pp. 97-107.

MANGIONE, Corrado and BOZZI, Silvio (1993): *Storia della Logica. Da Boole ai nostri giorni*, Milano: Garzanti.

MCGUINNES, Brian (ed.) (1984): *Collected Papers on Mathematics, Logic, and Philosophy*, Oxford: Basil Blackwell.

MILKOV, Nikolay (2006): "A New Interpretation of Leibniz's Concept of *characteristica universalis*", in H. Poser (ed.), *Einheit in der Vielheit*. *Proceedings of the 8th International Leibniz-Congress*, Hanover, 2006, pp. 606-614.

MONDADORI, Marco (1986): "Il sogno di Leibniz"; reprinted in M. D'Agostino, G. Giorello and S. Veca (eds.), *Logica e politica. Per Marco Mondadori*, Milano: Fodazione Arnoldo e Alberto Mondadori, 2001, pp. 15-25.

MUGNAI, Massimo (1976): *Astrazione e realtà*, Milano: Feltrinelli.

PEANO, Giuseppe (1908): *Formulario Mathematico* [1895-1908], Torino: Bocca.

——— (1925): *Giochi di aritmetica e problemi interessanti*, Torino: Paravia.

PIANA, Giovanni (1973): *Interpretazione del "Tractatus" di Wittgenstein*, Milano: Il Saggiatore.

RABIN, Michael O. (1977): "Decidable theories", in J. Barwise (ed.), *Handbook of Mathematical Logic*, Amsterdam: North-Holland.

ROSSI, Paolo (2000): *Logic and the Art of Memory. The Quest for a Universal Language*, London: Athlone Contemporary European Thinkers (English translation of *Clavis universalis*, 2nd edition, Bologna: il Mulino, 1983).

RUSSELL, Bertrand (1901): "Mathematicians and the Metaphysicians", in *Mysticism and Logic and Other Essays*, London: G. Allen & Unwin, 1917.

——— (1918): *The Collected Papers of Bertrand Russell. Vol. VIII: The Philosophy of Logical Atomism and Other Essays (1914-1919)*, London: Routledge, 1986, pp. 157-246.

STENLUND, Soren (2002): "On the Linguistic Turn in Philosophy", in M. Gustafsson and L. Hertzberg (eds.), *The Practice of Language*, Dodrecht: Kluwer, pp. 11-51.

TARSKI, Alfred (1951): *A Decision Method for Elementary Algebra and Geometry*, Berkeley: University of California Press.

VAN HEIJENOORT, Jean (ed.) (1967a): *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, Cambridge (Mass.): Harvard University Press.

——— (1967b): "Logic as Calculus and Logic as Language", *Synthese*, 17, pp. 324-330.

WITTGENSTEIN, Ludwig (1921): *Logisch-philosophisce Abhandlung/Tractatus Logico-philosophicus*, edited by B. F. McGuinness and J. Schulte, Frankfurt:

Surkamp, 2001 (English translation by D. F. Pears and B. F. McGuinnes, London: Routledge, 1974).

YATO, Takayuki (2003): *Complexity and Completeness of Finding Another Solution and its Application to Puzzles*, Master Thesis, Graduate School of Science, University of Tokyo, http://www-imai.is.s.u-tokyo.ac.jp/~yato/data2/MasterThesis.pdf