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Evaluation of insurance products with guarantee in incomplete markets

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Abstract

Life insurance products are usually equipped with minimum guarantee and bonus provision options. The pricing of such claims is of vital importance for the insurance industry. Risk management, strategic asset allocation, and product design depend on the correct evaluation of the written options. Also regulators are interested in such issues since they have to be aware of the possible scenarios that the overall industry will face. Pricing techniques based on the Black & Scholes paradigm are often used, however, the hypotheses underneath this model are rarely met.

To overcome Black & Scholes limitations, we develop a stochastic programming model to determine the fair price of the minimum guarantee and bonus provision options. We show that such a model covers the most relevant sources of incompleteness accounted in the financial and insurance literature. We provide extensive empirical analyses to highlight the effect of incompleteness on the fair value of the option, and show how the whole framework can be used as a valuable normative tool for insurance companies and regulators.

Keywords: Life insurance; Policies with minimum guarantee; Option pricing; Incomplete markets

1. Introduction

In recent years, embedded options in life insurance contracts became subject to increasing concern for the academic world as well as for practitioners. The consequences of failing to properly price the embedded options in insurance contracts became evident after the case "Equitable Life vs Hyman", where the insurance company had to close its funds after suffering substantial losses due to a decision of the House of Lords interpreting negatively the discretion with which Equitable had structured the bonus to the policyholders. In order to avoid such occurrences, the new International Financial Reporting Standards for insurance contracts (IFRS 4) and Solvency II now require insurance companies to measure and price embedded derivatives in insurance contracts at a fair value.

In this paper we focus on the evaluation of life insurance products with embedded options originated by minimum guarantee returns and bonus provision. The option pricing approach has been widely used to determine the fair price of a large range of products marketed by life insurance companies and pension funds (see Babbel and Merril (1998), Boyle and Hardy (1997), Brennan and Schwartz (1979), Embrechts (2000), Vanderhoof and Altman (1998)).

The advance in this field has yielded numerous studies whose primary goal is to properly evaluate complex bonus mechanism, introducing surrender options (turning the option to an American-type), and refining the stochastic framework (see Bacinello (2003), Giraldi et al. (2003), Grosen and Jørgensen (2000, 2002), Miltersen and Persson (1999)).

All these authors develop their models within the framework outlined by the main assumptions of the option pricing theory, i.e., no-arbitrage, dynamic hedging, and market completeness. Of these three hypotheses, the least realistic one is that of market completeness, namely, it is possible to replicate the payoff of any claim in the market by means of a self-financing strategy.

There are manifold sources of market incompleteness. For example:

1. Jumps in the underlying stochastic process due to bubbles-economy crash, nature/weather-catastrophic large claim;
2. Heteroscedasticity of the processes for the underlying assets;
3. Market frictions: short sales, transaction costs, operational constraints;

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4. Non-tradeability of the underlying asset due to the absence or lack of liquidity of the reference market. This is especially true when the reference fund is an internal portfolio of the insurance company;
5. Discrete hedging, given that continuous rebalancing is unrealistic and expensive;
6. Mortality risk, that is, the risk associated with not knowing how many of the policyholders will survive.

With regard to the insurance field, only few authors concentrate their studies on the issue of market incompleteness. Møller (2001) determines risk-minimizing hedging strategies for equity-linked pure endowment contracts. In this case, the incompleteness arises from mortality risk, that is an additional risk factor, independent of the financial market risk. The financial market itself is assumed to be complete, and the guaranteed option written in the insurance contract is hedged as in the Black & Scholes model. Further extensions can be found in Møller (2002), where the author compares results obtained with super-replication (El Karoui and Quenez, 1995), mean–variance hedging (Duffie and Richardson, 1991), risk minimization (Föllmer and Sondermann, 1986), and indifference pricing. The latter approach is related to the indifference price of the contract under different filtrations, which are associated to different information sets (see Møller, 2003).

Moore and Young (2003) employ a utility method to determine the price of endowment contracts linked to risky index. In this case too, the source of incompleteness is the mortality risk. Under the principle of equivalent utility, the premium is that price which leaves the insurer indifferent between writing and not writing the endowment contract. They prove that, under the assumption of exponential utility, the indifference premium solves a nonlinear partial differential equation, where the nonlinear term reflects the additional mortality risk and the exponential risk preferences of the model.

Coleman et al. (2006) cope with the same problem and solve it in a more general setting by addressing market incompleteness in the many facets summarized above. They model the dynamics of the objective price measure by merging the traditional Black & Scholes price process with the Merton's jump diffusion process. They then hedge the insurance claim using the underlying asset and a set of standard options expiring before the maturity of the claim. The hedging strategy is determined by applying the minimum local hedging risk principle by Föllmer and Schweizer (1989). Through a Montecarlo simulation, they show that the risk-minimization hedging strategy delivers better performances with respect to the Black & Scholes delta hedging.

The main contribution of our paper is to extend the analysis developed in Briys and de Varenne (2001), and nailed down in Grosen and Jørgensen (2002), to encompass the sources of market incompleteness listed above. We assume that the equity holders of the insurance company have limited liability, and thus we properly model the issue of insolvency risk due to the bankruptcy event.

We use a stochastic programming model (King, 2002) to super-replicate the payoff generated by the bonus distribution scheme. As we will show, the model is general enough to deal with any complex final payoff generated by European path-dependent options. We account for the bankruptcy event by considering the liabilities of the company as a risky (defaultable) bond. Following Grosen and Jørgensen (2002), we introduce regulatory restrictions assuming that the solvency of the company is monitored at discrete points in time.

The paper is organized as follows: Section 2 defines the basic framework and the specifications of the insurance contract. Section 3 shows how stochastic programming models can handle option pricing in incomplete markets, and provides a framework to hedge the payoff generated by the insurance bonus scheme. Section 4 describes the experimental setting used to implement the model and discusses the results obtained. The final section contains our conclusions as well as some suggestions for future research.

2. Insurance products with guarantee

We assume that an insurance company issues contracts that promise to pay some benefits, at the end of a specified maturity time T, contingent to the value of a reference fund I_T. More general payout schedules can be introduced without changing the main body of the model and its tractability.

We denote by I_0 the value of the fund at the inception of the contract, and we let $L_0 \equiv \alpha\, I_0$ be the premium paid by the policyholders to enter the contract; the initial investment by the equityholders is then given by $E_0 \equiv (1-\alpha)\, I_0$.

Note that, unlike Briys and de Varenne (2001) and Grosen and Jørgensen (2002), the reference fund could be any index used to determine the contractual obligations of the company. Broadly speaking, what matters for the company are the liabilities generated by the final payoff, and its main concern is the hedging of such a claim.

As stated above, a major source of incompleteness is the non-tradeability of the underlying asset or liquidity restrictions on it. For this reason, the hedging portfolio will, in general, consist of liquid assets (stock, bonds, options or futures) other than the underlying asset. In case of illiquidity or non-tradeability of the underlying, this hypothesis is more realistic than assuming that the hedging is performed by trading the underlying and the risk-free.

The insurance contract is equipped with a minimum guarantee provision. In particular, at maturity, the policyholder will receive an amount of money, L_T^G, obtained by compounding the initial premium, L_0, at the rate r_G,

$$L_T^G = L_0\, \mathrm{e}^{r_G\, T}. \tag{1}$$

Besides the final maturity guarantee, a bonus provision entitles policyholders to receive a share of the upside potential over the guarantee payment. The payoff of the bonus option is given by

$$\delta \left[\alpha I_T - L_T^G\right]^+, \tag{2}$$

where δ is the participation coefficient and $[\cdot]^+$ indicates the positive part of its content.

Since we are assuming that the equityholders have limited liability, in the event that at maturity the level of the fund is lower than the guaranteed payment, namely $I_T < L_T^G$, the equityholders will declare bankruptcy and the policyholders will walk away with the remaining assets. As we will show in Section 3, since the company invests the premium received in the hedging portfolio, and given that the latter is a super-replicating portfolio, in case of defaults, its value will be worth at least I_T.

In summary, the payoff of the policyholder is

$$\Phi(I_T) = \begin{cases} I_T & I_T \leq L_T^G \\ L_T^G & L_T^G \leq I_T \leq L_T^G/\alpha \\ L_T^G + \delta\left[\alpha\, I_T - L_T^G\right] & I_T > L_T^G/\alpha. \end{cases} \tag{3}$$

In a more compact form we have that

$$\Phi(I_T) = \underbrace{\delta\left[\alpha\, I_T - L_T^G\right]^+}_{\text{Bonus option}} + \underbrace{L_T^G - \left[L_T^G - I_T\right]^+}_{\text{Defaultable bond payoff}}. \tag{4}$$

Thus, purchasing an insurance policy with minimum guarantee and bonus provision is equivalent to taking a long position on the bonus option, to benefit of the potential upsides over the final maturity guarantee, and a long position on a defaultable bond.

We do not discuss here whether policyholders have enough information to properly price the default option. We align with the premises advanced in Briys and de Varenne (2001) and Grosen and Jørgensen (2002), and claim that our methodology is able to encompass their propositions and others proposed in literature.

We also extend our analysis to comprise regulatory restrictions. These are made explicit by imposing a barrier that forces the option to expire if I_t touches the barrier,

$$I_t \leq \lambda L_0 e^{r_G t} \equiv B_t \quad t \in [0, T[. \tag{5}$$

As explained in Grosen and Jørgensen (2002), this is equivalent to the monitoring, on the regulators' side, of the insurance assets value. Only in case that I_t is above the barrier, the option will be allowed to expire at maturity.

The parameter λ controls the sensitivity of the regulators to the risk of defaults. If $\lambda > 1$, the regulatory authorities prevent defaults imposing a *buffer* between the market value of company's assets and the nominal obligations to policyholders. If $\lambda < 1$, the regulators allow temporary and limited deficits. Clearly, in case of defaults, the recovered assets will not be sufficient to cover the policyholders' initial investments plus the minimum guarantee interests matured up to the date of defaults.

An alternative crediting scheme is to distribute the bonus at multiple periods until maturity. We denote by $t = 1, 2, \ldots, T$, discrete points in time from today ($t = 0$) until maturity T, and by R_t the rate of return of I_t during the period $t-1$ to t. The liability at each point in time is given by the following dynamic equation (see Consiglio et al. (2001))

$$L_t^G = L_{t-1}^G e^{[\delta R_t - r_G]^+ + r_G}, \tag{6}$$

where $L_0 \equiv \alpha A_0$. Broadly speaking, at each anniversary date, the rate of return of the policy is the maximum between the risky rate, δR_t, and the minimum guarantee rate, r_G. The initial capital, $A_0 = L_0 + E_0$, is invested in the risky fund to back the final liability, L_T^G. The dynamics of the asset fund is given by

$$A_t = A_{t-1} e^{R_t}. \tag{7}$$

The final payoff of the policyholder is

$$\Phi(A_T) = \begin{cases} L_T^G & A_T \geq L_T^G \\ A_T & A_T < L_T^G \end{cases} \tag{8}$$

or more compactly,

$$\Phi(A_T) = A_T - \left[A_T - L_T^G\right]^+. \tag{9}$$

In this case the policyholder is short of a call option on the potential upside over the liability. This is justified by the fact that the surplus, $A_T - L_T^G$, is the reward for the equityholders and it must be passed to them. Note that, given the crediting scheme (6), the bonus for the policyholders is already accounted in L_T^G, provided that the asset value is sufficient to back it.

Alternative schemes that use a combination of distribution plans can be implemented as well. See for details Consiglio et al. (2006), Grosen and Jørgensen (2000).

3. Super-replication via stochastic programming

We describe here a stochastic programming model to price the bonus and default option discussed in the previous section. Details about the model and its use for option pricing can be found in King (2002), King et al. (2005).

A European contingent claim (ECC) is a security associated to the stochastic process $S = (S_t)_{t=0}^T$, that gives to its owner a stochastic cashflow $F = (F_t)_{t=0}^T$. This definition is quite general to include put and call options with barriers, lookback and Asian payoff, futures and any derivatives whose cashflow does not depend on decisions taken before the final maturity. Note that such definition covers claims written on multiple underlyings or non-tradeable instruments, and claims whose payouts depend on risk factors independent of financial markets. With regard to insurance products, a cashflow $F = (F_t)_{t=0}^T$, which is nonzero for some $t < T$, embraces features such as mortality risk or lapse. Both events generate payouts during the life of the contract that are not related to the movements of the underlying, or at least, not directly. In detail, lapses are due to the surrender of the policy because of better returns offered by comparable investments. It is possible to study a statistical model that relates the lapse rate to some reference index, and adapt the cashflow to the stochastic process generated by the chosen index (see Consiglio and Zenios (2001)). In this case, the relation with the underlying is mediated through the estimated lapse function. Note that the decision to surrender the policy is exogenous to the pricing model, and therefore it is not necessary to model the claim as an American option (see Bacinello (2003)).

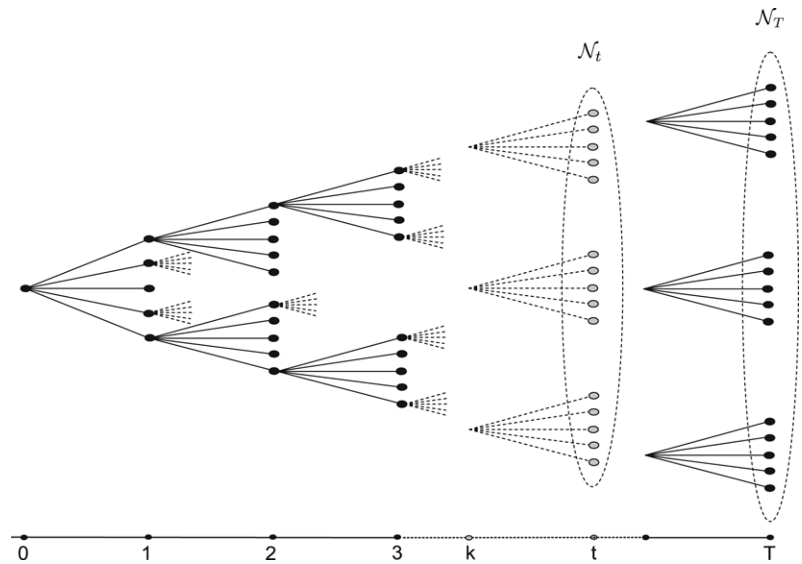

Fig. 1. A graphic representation of a non-recombinant tree.

As far as mortality risk is concerned, this is a phenomenon associated to the change of the mortality intensity over time. As a consequence, the payouts generated by the attached policy claim will not depend on a deterministic function any longer. Following Dahl and Møller (2005), one could model a stochastic process capturing the random dynamics of the mortality intensity, and then adapt the cashflow $F = (F_t)_{t=0}^T$ to the stochastic process describing the new mortality law.

We base our computational machinery on a finite-dimensional market model. Following King's notation, the market consists of J securities with prices $S_t = \left(S_t^1, \ldots, S_t^J\right)$; security prices and other payments are discrete random variables supported on a finite probability space $(\Omega, \mathcal{F}, P)$ whose atoms ω are sequences of real-valued vectors (asset values) over the discrete time periods $t = 0, 1, \ldots, T$. The path histories of the security prices up to time t correspond one-to-one with nodes $n \in \mathcal{N}_t$. The set $\mathcal{N}_0$ consists of the root node $n = 0$, and the leaf nodes $n \in \mathcal{N}_T$ correspond one-to-one with the probability atoms $\omega \in \Omega$. In the tree, every node $n \in \mathcal{N}_t$, $t = 1, \ldots, T$, has a unique ancestor node $a(n) \in \mathcal{N}_{t-1}$, and every node $n \in \mathcal{N}_t$, $t = 0, \ldots, T-1$, has a non-empty set of child nodes $\mathcal{C}(n) \subset \mathcal{N}_t$. The collection of all the nodes is denoted by $\mathcal{N} \equiv \bigcup_{t=0}^T \mathcal{N}_t$.

The probabilistic structure implies that the market evolves as a discrete, non-recombinant tree (hence, suitable for incomplete markets). In this context, a European-style contingent claim is represented by a stochastic cashflow F with payouts $\{F_n\}_{n\in\mathcal{N}}$ that depends on the underlying securities S. In the presence of risk factors other than the traded securities, the process S_t is augmented by K real-valued variables $\xi_t = \left(\xi_t^1, \ldots, \xi_t^K\right)$ whose path histories match the nodes $n \in \mathcal{N}_t$, for each $t = 0, 1, 2, \ldots, T$.

This is the case of mortality risk or surrender option when adapted to the stochastic structure of the tree. For example, if we denote by $\{q_n\}_{n\in\mathcal{N}}$ the process of the probability of death, and by $\{l_n\}_{n\in\mathcal{N}}$ the process of the probability of lapse, the liability process, $\{L_n\}_{n\in\mathcal{N}}$, is given by

$$L_n = (1 - \Lambda_n) L_{a(n)} \mathrm{e}^{r_G}, \tag{10}$$

where, at the initial node, $L_n = L_0$, and $\Lambda_n = q_n + l_n$ is the probability of "abandoning" the policy (we assume that the death and lapse event are independent).

In Fig. 1, we show a graphic representation of a tree with five arcs springing from each node.

When the markets are incomplete, an interval between the buyer and writer's prices describes the possible range of arbitrage-free evaluations. Since we assume that the policyholders are price-takers, we focus our attention on the writer's price, which will be greater than the buyer's price (see King (2002) for a formal proof of such statements).

We can define the writer's price of the contingent claim as the smallest amount of current cash, V, needed to start a trading strategy to back the payout process $\{F_n\}_{n\in\mathcal{N}}$ with no risk.

The amount of cash V represents the initial cash infusion needed to start the self-financing process that will deliver, for each final node $n \in \mathcal{N}_T$, a set of portfolios whose values super-replicates the final payout. Note that, since it is requested that portfolios are self-financing, the super-replication property is guaranteed by imposing that the portfolio value at the final nodes is greater or equal to zero.

Let us denote by X_n^j, Y_n^j, and Z_n^j, respectively, the amount of share purchased, sold, and held at each node $n \in \mathcal{N}$, and for each security $j = 1, 2, \ldots, J$. By definition, X_n^j and Y_n^j are always greater than or equal to zero, while, if we allow for short positions, $Z_n^j \in \mathbb{R}$.

By assuming that no holdings are available at the initial node $n = 0$, the inventory constraints, for each $j = 1, 2, \ldots, J$, are given by

$$Z_0^j = X_0^j - Y_0^j. \tag{11}$$

To balance out, the value of the amount purchased plus any payout must be equal to the value of the amount sold, that is,

$$\sum_{j=1}^{J} S_0^j(1-\kappa)Y_0^j = V + \sum_{j=1}^{J} S_0^j(1+\kappa)X_0^j + F_0, \tag{12}$$

where κ is the proportional transaction cost for buying and selling. Observe that, at the initial node, a nonzero value of Y_n^j is possible only if we allow for short positions. This is also due to the presence of the transaction costs which avoid that buying and selling of the same security occurs at each node.

At each node $n \in \mathcal{N}_k$, $k = 1, 2, \ldots, T$, and for each security $j = 1, 2, \ldots, J$, the inventory constraints will depend on the holdings at the ancestor node $a(n)$, plus any proceeds from buying or selling securities. So we have that

$$Z_n^j = Z_{a(n)}^j + X_n^j - Y_n^j. \tag{13}$$

The balance constraints follow consequently,

$$\sum_{j=1}^{J} S_n^j(1-\kappa)Y_n^j = \sum_{j=1}^{J} S_n^j(1+\kappa)X_n^j + F_n. \tag{14}$$

To sum up, the stochastic programming model for the insurance evaluation problem can be written as follows

$$\underset{X_n^j, Y_n^j, Z_n^j}{\text{Minimize}}\ V \tag{15}$$

$$\text{s.t.} \quad Z_0^j = X_0^j - Y_0^j, \quad \text{for all } j = 1, 2, \ldots, J \tag{16}$$

$$\sum_{j=1}^{J} S_0^j(1-\kappa)Y_0^j = V + \sum_{j=1}^{J} S_0^j(1+\kappa)X_0^j + F_0, \tag{17}$$

$$Z_n^j = Z_{a(n)}^j + X_n^j - Y_n^j, \quad \text{for all } n \in \mathcal{N}_k, \; k = 1, 2, \ldots, T, \; j = 1, 2, \ldots, J \tag{18}$$

$$\sum_{j=1}^{J} S_n^j(1-\kappa)Y_n^j = \sum_{j=1}^{J} S_n^j(1+\kappa)X_n^j + F_n, \quad \text{for all } n \in \mathcal{N}_k, k = 1, 2, \ldots, T \tag{19}$$

$$\sum_{j=1}^{J} S_n^j Z_n^j \geq 0, \quad \text{for all } n \in \mathcal{N}_T, \tag{20}$$

$$X_n^j, Y_n^j \geq 0, \quad \text{for all } n \in \mathcal{N}, \; j = 1, 2, \ldots, J \tag{21}$$

$$Z_n^j \in \mathbb{R} \quad \text{for all } n \in \mathcal{N}, \; j = 1, 2, \ldots, J. \tag{22}$$

We highlight here some important points:

1. Problem (15)–(22) is a linear programming model where the objective function minimizes the value of the hedging portfolio, or rather the cost of the option.
2. The payout process $\{F_n\}_{n\in\mathcal{N}}$ is a parameter of the problems (15)–(22). This implies that any complicated structure for F_n does not change the complexity of the model. For example, the option with payoff given by (4) corresponds to the process $F_n = 0$, for all $n \in \mathcal{N}_k$, $k = 0, 1, 2, \ldots, T-1$, and $F_n = \delta\left[\alpha I_n - L_T^G\right]^+ + L_T^G - \left[L_T^G - I_n\right]^+$, for all $n \in \mathcal{N}_T$, where I_n is the reference fund process adapted to the stochastic structure of the tree.
3. The payouts due to the change in the mortality intensity and to the surrender option follow from Eq. (10). In particular, the expected outflow due to such events is obtained by the probability of abandoning the policy, Λ_n, times the liability matured at the ancestor node, therefore, $F_n = \Lambda_n L_{a(n)} e^{rG}$, for all $n \in \mathcal{N}_k$, $k = 1, 2, \ldots, T-1$.
4. Constraints (17) and (19) correspond to the self-financing equations described in King (2002). They are easily obtained by substituting the holding variables, Z_n^j, defined by Eqs. (16) and (18), in King's model equations.
5. Constraints (20) ensure that at each final node the total position of the hedging portfolio is not short. In other words, if short positions are allowed, the portfolio process must end up with enough long positions such that a positive portfolio value is delivered. Note that transaction costs at the final nodes $n \in \mathcal{N}_T$ are accounted in the balance constraints (19).

4. Implementation notes and results

We run our experiments assuming a policy horizon $T = 10$ years. The time interval between two periods is set to 1.67 years, for a total of 6 time periods. To run our experiments in a reasonable time, we assume that our market is made up of 4 assets plus one risk-free ($J = 4 + 1$). Note that the computational time depends on the number of assets, the discretization adopted and the operational constraints. In the worst case – the problem with transaction costs, that basically tripled the number of variables – the computational time amounts to approximately half an hour (CPU Pentium 4, 2.4 GHz).

To encompass the more general case, that is when the underlying asset is not tradeable, the reference fund, I, is not included among the J assets, therefore, the hedging portfolio is formed by the four risky assets plus the risk-free.

We solve the optimization models using the algebraic modelling language GAMS by Brooke et al. (1992). The most complex task concerned the solution of the nonlinear goal programming model to generate trees. Under the GAMS platform, we use the CONOPT solver and, in case it failed to converge to a solution, the SNOPT solver. We employ CPLEX to solve the linear programming model to determine the price of the option. The average problem had 130,000 rows, 290,000 columns and 660,000 nonzero entries.

4.1. The tree generation model

We generate the tree of the underlying price process S by matching the first M moments of its unknown distribution. Our approach is based on the model by Høyland and Wallace (2001), where the user provides a set of moments, $\mathcal{M}$, of the underlying distribution (mean, variance, skewness, covariance, or quantiles), and then, prices and probabilities are jointly determined by solving either a nonlinear system of equations, or a nonlinear optimization problem. The method also allows for intertemporal dependencies, such as mean reverting or volatility clumping effect.

Observe that, we do not sample from a distribution to generate our tree, but we build a discrete probability distribution

Table 1
Statistical properties of the securities used in the experiments. We built trees by matching means, variances and covariances shown in the table

	Mean	Variance–covariance matrix					
		RiskFree	Asset-1	Asset-2	Asset-3	Asset-4	Reference fund
RiskFree	0.02	0.0001	–	–	–	–	–
Asset-1	0.03	0.5	0.04	–	–	–	–
Asset-2	0.035	0.15	0.31	0.0484	–	–	–
Asset-3	0.04	0.2	0.21	0.021	0.0625	–	–
Asset-4	0.045	0.25	0.12	0.19	0.31	0.0225	–
Reference fund	0.04	0.14	0.2	0.1	0.24	0.212	0.01

with given moments, and feed it to the stochastic programming model. For this reason, the number of arcs from each node is a matter related to the solvability of the matching problem. On the contrary, when sampling from a distribution, the number of scenarios and the tree structure contribute to the stability of the solution, and a sensitivity analysis is necessary. In such cases, a scenario reduction procedure can help to shrink the size of the tree, while maintaining a reasonable accuracy (see Dupačová et al. (2003)).

For a review on alternative scenario generation methods, also see Dupačová et al. (2000) and references therein.

We cast the tree generation model as a nonconvex weighted least-squares minimization problem,

$$\underset{S,p}{\text{Minimize}} \sum_{j=1}^{J} \left\{ \sum_{i \in \mathcal{M}} \alpha_i \left[\sum_{m \in \mathcal{C}(n)} p_m \left(\ln \frac{S_m^j}{S_n^j} \right)^i - \mu_i^j \right]^2 \right\} + \sum_{j}^{J} \sum_{\substack{h \\ h>j}}^{J} \left\{ \gamma_{jh} \left[\sum_{m \in \mathcal{C}(n)} p_m \left(\ln \frac{S_m^j}{S_n^j} - \mu_1^j \right) \times \left(\ln \frac{S_m^h}{S_n^h} - \mu_1^h \right) - \rho \right]^2 \right\}. \quad (23)$$

$$\text{s.t.} \quad \sum_{m \in \mathcal{C}(n)} p_m = 1 \quad (24)$$

$$p_m \geq 0 \quad m \in \mathcal{C}(n). \quad (25)$$

The goal programming model (23)–(25) is much more flexible than solving a system of nonlinear equations matching the moments of the unknown distribution. As underlined in Høyland and Wallace (2001), the solvability of the nonlinear system of equations increases with the number of arcs. However, to be tractable, the set of arcs that springs out from each node has to be of limited size. In fact, the size of the tree grows exponentially with the number of branches springing out from each node. In particular, if $|\mathcal{C}(n)| = \nu$, the total number of nodes is $\sum_t \nu^t + 1$.

Note, however, that infeasibility could also arise from an inconsistency in the specification of the moments (see Høyland and Wallace (2001) for a discussion on this topic).

In this paper we use the model (23)–(25) to match the parameters of the price process. We restrict the matching problem to means, variances and covariances, and we let them be constant over time. Our decision is motivated by the need to keep the experimental settings as simple as possible to highlight the features of our model as an evaluation tool for insurance policies.

In Table 1 we display the statistical properties for each of the J assets and the reference fund.

4.2. Non-arbitrage constraints

To be consistent with financial asset pricing theory, the tree has to exclude arbitrage opportunities. This is a very important feature, since in the presence of arbitrage scenarios the model (15)–(22) will end up with an unbounded solution.

Arbitrage opportunities can be avoided either by adding non-arbitrage constraints or by increasing the number of arcs. The latter solution is often impracticable due to the exponential growth of the tree size.

Following Klaassen (2002), it is possible to prove that arbitrage opportunities of the *first type* are prevented if and only if,

$$\pi_n S_n^j - \sum_{m \in \mathcal{C}(n)} \pi_m S_m^j = \sum_{m \in \mathcal{C}(n)} p_m S_m^j \quad \text{for all } j = 1, \dots J \quad (26)$$

$$\pi_m \geq 0 \quad m \in \mathcal{C}(n) \quad (27)$$

$$\pi_0 \in \mathbb{R}. \quad (28)$$

Likewise, arbitrage opportunities of the *second type* are excluded if and only if,

$$\sum_{m \in \mathcal{C}(n)} \nu_m S_m^j = S_n^j \quad \text{for all } j = 1, \dots J \quad (29)$$

$$\nu_m \geq 0 \quad m \in \mathcal{C}(n). \quad (30)$$

The set of Eqs. (26)–(28) and (29) and (30) can be added as constraints to the goal programming model (23)–(25) to preclude arbitrage opportunities of both types in the tree that is generated. Note that Eqs. (26) and (29) are nonlinear constraints and must be handled with care. In our experiments we added only first type arbitrage constraints, and those were sufficient to guarantee trees without arbitrage opportunities.

4.3. Fair contracts

The price of the insurance contract described in Section 2 depends on many parameters. The most important ones are: the minimum guarantee rate (r_G), the participation coefficient (δ), the leverage (α), the barrier buffer parameter (λ). Not

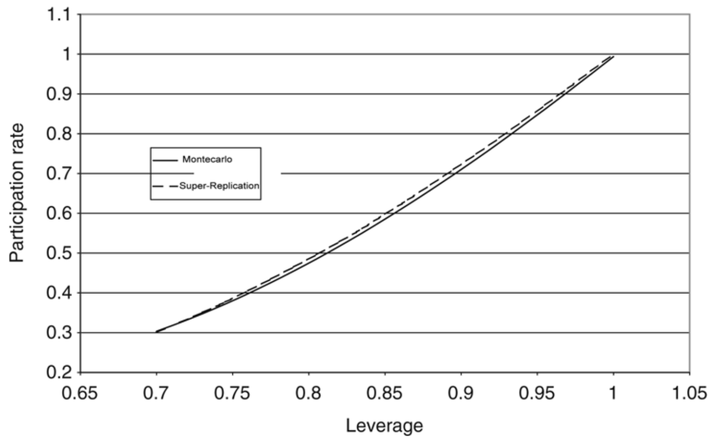

Fig. 2. The relationship between leverage and participation rate. The two curves are determined assuming that the market is complete. The minimum guarantee rate is fixed at 2% per year, the time horizon is 10 years, and the volatility of the reference fund is 10% per year.

all the combinations of these parameters determine a fair value of the insurance contract. In particular, let us denote by $V(0, I_0, r_G, \delta, \alpha, \lambda)$ the value at time 0 of the insurance contract. The latter is said to be *fair* if the initial policyholder's contribution, L_0, is equal to the initial market value of the purchased claim, that is,

$$L_0 \equiv \alpha I_0 = V(0, I_0, r_G, \delta, \alpha, \lambda). \tag{31}$$

We carry out our analysis by fixing all the parameters except one, and by determining the value of the contract such that Eq. (31) is satisfied. This is accomplished through a numerical search procedure. If, for example, Eq. (31) has to be analyzed as a function of the minimum guarantee rate, we reduce the matching problem to finding zeros of the function $\Upsilon(r_G) = \alpha I_0 - V(r_G)$. Using a secant rule, r_G at iteration $m+1$ is given by the following formula,

$$r_{G,m+1} = r_{G,m} - \Upsilon(r_{G,m}) \left[\frac{r_{G,m} - r_{G,m-1}}{\Upsilon(r_{G,m}) - \Upsilon(r_{G,m-1})} \right]. \tag{32}$$

Note that every step of the search procedure implies the solution of a large linear programming model, as $V(\cdot)$ is the price of the insurance claim.

4.4. The effect on regulatory requirements

We start our analyses by looking at the effect of market incompleteness on the requirements imposed by the regulators. As pointed out by Briys and de Varenne (2001), regulators usually affect insurance design by imposing a minimum capital requirement, by controlling the portfolio composition to prevent investments in high-risk assets, and by introducing a ceiling on the minimum guarantee rate. It is a common standpoint that the risk of bankruptcy decreases as the level of the capital to asset ratio ($E_0/A_0 = 1 - \alpha$) increases. In such a case, the policyholders carry less leverage risk ($L_0/A_0 = \alpha$), and require a lower participation rate (δ).

We check our results against a benchmark that is the pricing model in complete markets. By assuming market completeness, we determine the fair price of the insurance contract having payoff specified by (4), and written on the underlying I. We price the bonus and default options using a standard Montecarlo approach, by simulating 100,000 trajectories from the risk-neutral distribution of the reference fund, with volatility equal to 10% per year. We then solve Eq. (31) as function of δ, for fixed values of α ranging from 0.7 to 1, while keeping fixed $r_G = 2\%$ per year.

We expect that when the market is complete the super-replication model delivers the same results. Hence, we build a tree with the same number of branches as the number of securities ($\nu = J = 5$), and determine those values of δ such that the contract is fair. As shown in Fig. 2, the two curves almost overlap and we can attribute the difference to numerical approximations. As observed by Briys and de Varenne (2001),[1] the participation rate decreases as the leverage value is reduced, due to the lower risk carried by the policyholder.

Observe that, in this case, we are not assuming that the underlying is included in the set of assets. Therefore, the hedging portfolio will consist of long and short positions of the J assets spanning the market. In practice, if we have enough liquid assets (stock, futures, options), we can super-replicate a claim written on a non-tradeable reference funds. In this way, we overcome one of the most controversial aspect of the Black & Scholes model – non-tradeability of the reference fund – when used for insurance pricing.

In Fig. 3, we display the hedging portfolios for different levels of the leverage. For simplicity, we assume that at time $t = 0$ the value of the reference fund and the prices of the assets are equal to 100. We note that the risk-free component decreases as the leverage increases. This occurs because the final payoff is an increasing function of the leverage, and therefore, the higher the leverage the greater the payments due at maturity. We also observe a stronger growth of the `Asset-4` with respect to the rest of the assets. This is a consequence of the fact that the expected value and the volatility of `Asset-4` are, among all assets, the closest to those of the underlying asset. Recall that these are fair contracts, and that to a higher value of the leverage

[1] Note that in their experiments Briys and de Varenne assume that the time horizon is one year. This is the reason why they obtain fair contracts with a minimum guarantee between 8% and 11%.

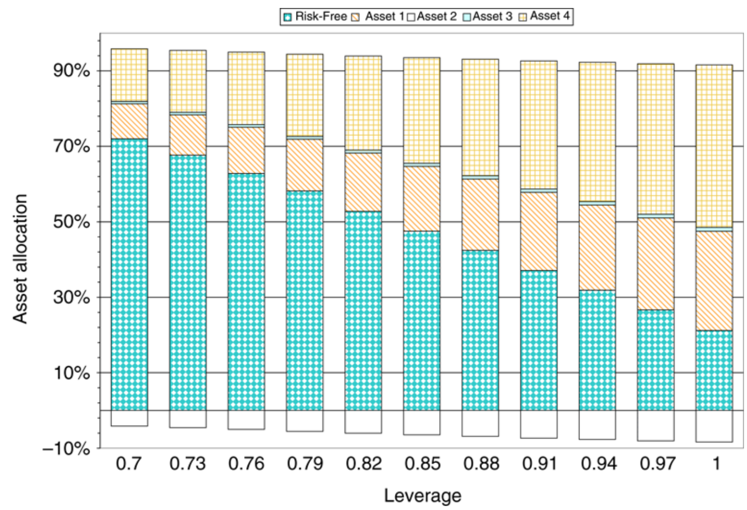

Fig. 3. Hedging portfolios for different levels of the leverage. Higher values of the leverage imply higher values of the participation rate. In these cases, the bonus provision will prevail on the minimum guarantee, and the hedging portfolios will be more shifted towards those assets with statistical properties more similar to the underlying asset.

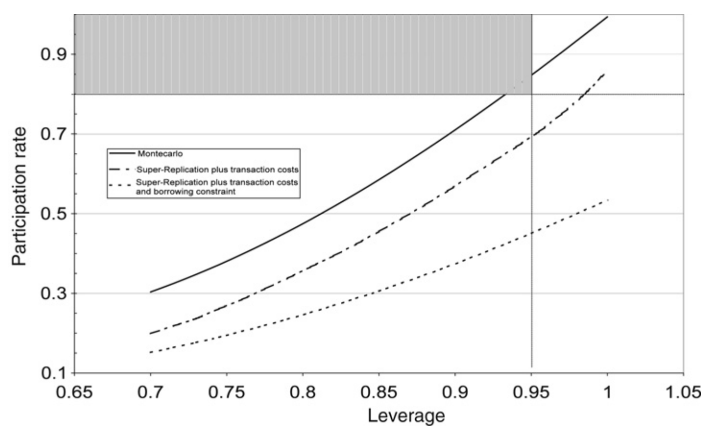

Fig. 4. The relationship between leverage and participation rate when transaction costs and borrowing constraints are introduced. The shaded area represents the feasibility region where both leverage and participation rate are effective.

corresponds a higher value of the participation rate (see Fig. 3). Hence, the payoff of the insurance policy is more affected by the floating components (bonus provision), driven by I, than the fixed component (the minimum guarantee).

Assuming the complete market model as benchmark, we can now study how the relation leverage–participation rate is affected by one or more sources of incompleteness. In Fig. 4, we show such relation for the benchmark and the super-replication model with transaction costs and borrowing constraint. To this purpose, proportional transaction costs are set to 0.3% and the spread between lending and borrowing rate is set to 2%. The model with borrowing and lending rate is obtained by slightly modifying model (15)–(22). We basically split the variable concerning the risk-free, and then we constrain these two variables – one carrying the amount to borrow, and the other carrying the amount to lend – to take non-negative values.

As it can be observed, the two curves are pushed downwards with respect to the benchmark. This is caused by the increment of the cost of the claim due to the transaction costs and to the higher rate for borrowing. Given the augmented cost of the claim, it will be that $\alpha I_0 - V^*(\cdot) < 0$, where $V^*(\cdot)$ is the option value in case of complete market. In other words, the cost of the claim is greater than the premium received (αI_0), and to restore the equivalence, either the leverage has to be increased or the participation rate has to be lowered.

Through this analysis, we can gauge the effect on regulatory restrictions. If, for example, the maximum leverage allowed is 95% and the minimum participation rate is 80% (these are typical values in Europe), the shaded area in Fig. 4 delimits the effectiveness of the insurance company for the class of policies with a minimum guarantee $r_G = 2\%$, and a 10% volatility of the reference fund. It is evident that the unrealistic hypothesis of absence of transaction costs and borrowing at the risk-free rate will deem as effective a wrong level of the participation rate and of the leverage.

We believe that this is a valuable tool for regulators to ascertain the fairness of the policies sold in the market, and for insurance managers to optimally design their products. However, as highlighted by Briys and de Varenne, it might be also the case that the levels imposed by the regulators are inconsistent.

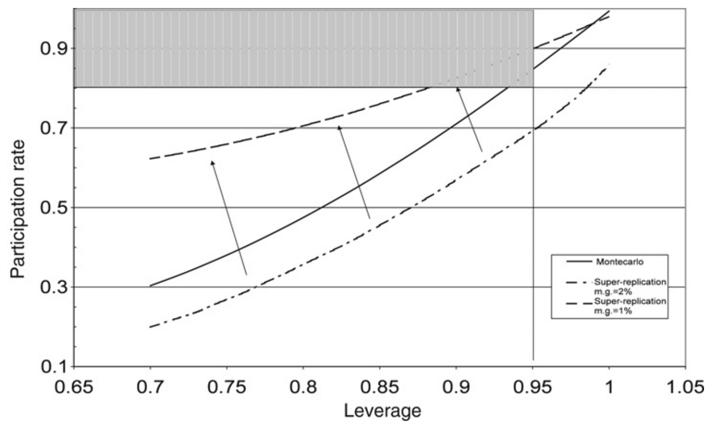

Fig. 5. The effect of market incompleteness on regulatory constraints. The insurance contract evaluated taking into account the transaction costs lies well below the feasibility area. A possible solution would be to lower the minimum guarantee – from 2% to 1% – so that the curve moves upwards.

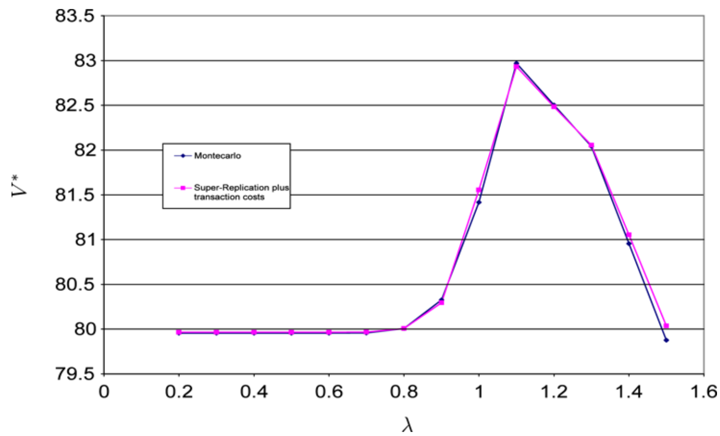

Fig. 6. The effect of market incompleteness and regulators' intervention policy on contract value. The parameter λ measures the reactivity of the regulators to downfalls of the insurance assets. The two contracts share the same parameters except δ. No significant changes can be observed when the policy is negotiated taking into account the transaction costs.

In this example, a possible solution would be to lower the minimum guarantee to push the curve upwards (See Fig. 5), so as to restore a level of leverage and participation rate compatible with the given regulation.

4.5. The effect on default barrier

As shown in Section 3, our model is flexible enough to include the pricing of path-dependent options. In Section 2 we pointed out that regulators usually monitor insurance assets during the life of the policy, and, prior to the maturity of the policy, they might take their measures against possible defaults. This is equivalent to introducing a barrier whose level is controlled by the buffer parameter λ.

In Fig. 6 we show the relation between the buffer control parameter, λ, and the value of the contract. The policy is negotiated with $\alpha = 80\%$, $r_G = 2\%$ and $\lambda = 0.8$. Recall that we analyse only fair contracts, therefore, the two contracts must differ from one parameter. In this case we choose δ as free parameter, and we obtained a $\delta^C = 0.46$ for the complete market, and a $\delta^I = 0.397$ for the incomplete case.

We then change λ to show the effect of a possible revision of the regulators' intervention policy. We display the curve derived assuming a complete market, and the curve relative to the case of an incomplete market due to transaction costs. We observe that the two curves follow the same trend found in Jørgensen (2001).

The two curves almost overlap and this indicates that the policy of the regulators has the same impact either when the market is complete, or when frictions, like transaction costs, are present. Note however, that the two contracts differ from the parameter δ and the impact of incompleteness on this parameter is significant, as already shown in Section 4.4.

4.6. The effect on policy design parameters

The analysis carried out so far can be extended to include other combinations of the design parameters. Figs. 7 and 8 are useful aids to design insurance contracts according to the company needs, ensuring, at the same time, that the value of the embedded options are properly evaluated. In Fig. 7 we show the relationship between the volatility and the participation level. The minimum guarantee and the leverage are respectively set to 2% and 0.95. An alternative view is the relationship between the leverage and the minimum guarantee, while keeping fixed the participation rate ($\delta = 0.6$) and the volatility ($\sigma = 10\%$).

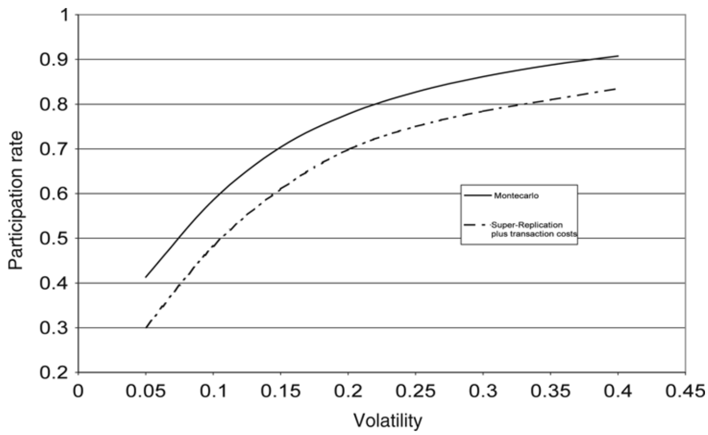

Fig. 7. The relationship between volatility and participation rate. Due to the long call position of the policyholders, the higher the volatility, the higher the participation level allowed. In incomplete markets, a lower level of the participation level must be delivered to maintain the fairness of contract.

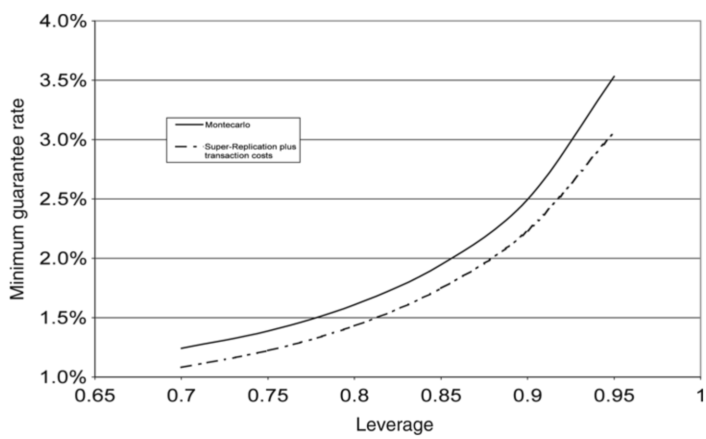

Fig. 8. The relationship between leverage and minimum guarantee rate. A higher minimum guarantee rate compensates for the higher financial risk due to the greater leverage levels. However, in incomplete markets the true curve is the one that lies below, and therefore, the true compensation, in terms of minimum guarantee rate, is relatively lower.

The same pattern as before applies here. Transaction costs increase the cost of the option, and, to restore the fairness of the contract, the policy designer has to cut down either the participation level or the minimum guarantee rate.

5. Conclusions

We developed and tested a super-replication model for evaluating insurance products with guarantee. We showed that the stochastic programming model obtained is flexible enough to encompass the most relevant sources of incompleteness encountered in the financial and insurance literature. The model improved upon the classical Black & Scholes model in providing a valuable tool for designing insurance policies and ascertaining their fair value under realistic hypotheses.

An interesting extension of the model would be to the introduction of endogenous lapse decisions. As is known, a surrender option of the policy can be modelled as an American option. This implies that the stochastic programming model has to take into account, at each rebalancing time, the so-called continuation value.

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