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ARTICLE

Assessment of mathematical learning in a musical composition workshop applying tools from the onto-semiotic approach

Evaluación del aprendizaje de las matemáticas en un taller de composición musical aplicando las herramientas del enfoque Ontosemiótico

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Abstract

The mathematical music composition workshop, an endeavor that integrates mathematics and music majors, is a concrete example of a STEAM (science, technology, engineering, arts, and mathematics) project. In this article, the authors analyze how mathematics students and music composition students have worked together, have learned to interpret specialized languages from one another, and have presented their results to a public interested in the relationship that guards these two disciplines. The goal of improving the understanding of abstract mathematical concepts through the application to musical structures is analyzed using the Onto-semiotic Approach (OSA). This framework sheds light on some of the written and oral manifestations of the students who participated. The mathematical competence that the participants were expected to achieve through this interdisciplinary endeavor are privileged in this article over the music education goals (which were also present).

Keywords: Mathematics and music. STEAM projects. Onto-Semiotic Approach. Qualitative methods. Euclidean rhythms.

Resumen

El taller de composición musical matemática, un esfuerzo que integra a las carreras de matemáticas y música, es un ejemplo concreto de proyecto STEAM (ciencia, tecnología, ingeniería, artes y matemáticas). En este artículo, los autores analizan cómo los estudiantes de matemáticas y los de composición musical han trabajado juntos, han aprendido a interpretar los lenguajes especializados de unos y otros, y han presentado sus resultados a un público

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interesado en la relación que guardan estas dos disciplinas. El objetivo de mejorar la comprensión de los conceptos matemáticos abstractos mediante la aplicación a las estructuras musicales se analiza utilizando el Enfoque onto-semiótico (OSA). Este marco arroja luz sobre algunas de las manifestaciones escritas y orales de los alumnos que participaron. En este artículo se privilegian las competencias matemáticas que se esperaba que los participantes alcanzaran a través de este esfuerzo interdisciplinario por encima de los objetivos de educación musical (que también estaban presentes).

Palabras clave: Matemáticas y música. Proyectos STEAM. Enfoque Ontosemiótico. Métodos cualitativos. Ritmos euclidianos.

1 Introduction

The relation between mathematics and music has been acknowledged since ancient times. It has been said that mathematics is the formalization and abstraction of the patterns and symmetries that surround us, in both our internal and external worlds. Music also consists of symmetries and patterns, expressed through scores that, in turn, are interpreted by musicians who know the language and can transform the symbols that represent these operations in sound. The mathematician does not have a *performer* who can *interpret* algebra and symbology in general and, for this reason, even students who advance in the study of mathematics frequently do not learn how to find meaning and eventual applications of the abstractions they must master.

The symmetries and combinations implicit in musical structures have always intrigued both mathematicians and musicians. The Quadrivium was the model of higher learning in the Western world from antiquity through the Renaissance, and consisted in arithmetic, geometry, astronomy, and music; however, in modern times, the two disciplines have had very little academic interrelation. Recently, an area of research, pedagogy, and outreach, known as Mathematical Music Theory (MMT), has been witness to institutional collaboration among students and professors of both areas (Kovachi, 2014; Hall, 2014; Hughes, 2014; Mazzola; Mannone; Pang, 2016; Montiel; Gómez, 2014; Montiel, 2017, 2018).

Indeed, MMT studies relations and structures common to both subjects and much of the analysis carried out can be oriented towards teaching and learning mathematics (Bamberger; Disessa, 2003; Bensen, 2008; Johnson, 2008; Fiore, 2006, 2007, 2009; Bautista; Roth, 2012; Yust; Fiore, 2014). Once introduced to the content and philosophy of mathematical music theory, it appears logical that this area opens new horizons for research and pedagogy. It is worthwhile to mention that some of the relevant mathematical subjects are apt for the school level, although the present study is in the university context. Indeed, the pedagogy of MMT can be embraced from the primary and even preschool levels, all the way up to postgraduate students of mathematics and music (Bamberger; Disessa, 2003; Montiel; Gómez, 2018). Some

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of Bamberger's work, in which her Impromptu software (http://impromptu.moso.com.au/) is used to illustrate mathematical concepts through music with elementary school students, finds an echo in the utilization of the software Rubato Composer (https://www.rubato.org/) to reinforce advanced mathematical concepts (Montiel, 2011; Mazzola *et al.,* 2016). In consequence, the interdisciplinarity between mathematics and music extends to computer science and the competencies related to technology as well.

The present study is situated in a multidisciplinary context, where music is modeled by mathematics. In other words, it is a STEAM context, which is the acronym derived from STEM (Science, Technology, Engineering, and Mathematics) with the addition of the arts (the A). During the last approximately 15 years, there has been a growing, albeit slow, interest in the inclusion of the arts as a means of fomenting and including the force of creativity within scientific education. Creativity is essential to scientific activity, although it is not necessarily emphasized in its pedagogy (Watson, A.; Watson, G., 2013; Stewart; Mueller; Tippins, 2019; Sousa; Pilecki, 2013; Khine; Areepattamannil, 2019). On the other hand, creativity is part of innovation, whether scientific or artistic, and the familiarization with both languages can only broaden the possibilities of advancement. An interesting case is, for example, when researchers in genetics transform their data into musical notation to facilitate analysis such as decoding the sequence of genes in a chromosome (Root-Bernstein, R.; Root-Bernstein, M., 1997) or when sonification helps the understanding of viruses (Temple, 2020; Tirthankar; Vainio; Roning, 2021).

In general, it is not possible to conceive mathematical activity without a creative component. Hence, determining creative contexts should be an educational objective. The multidisciplinary STEAM projects foment creativity (Oner *et al.*, 2016; Chen; Lo, 2019) by generating a motivating context that better confidence in personal capacities (Conradty, Sotiriou; Bogner, 2020).

In this paper, we discuss a collaboration between the Mathematics and Statistics Department and the School of Music at a research university in the Southern United States, the Mathematical Music Composition Workshop, in which mathematics majors and music composition students have worked together, have learned to interpret specialized languages from one another, and have presented their results to a public interested in the relation that guards these two disciplines. We intend to show, using the Onto-semiotic approach (OSA) reference framework (Godino; Batanero; Font, 2007), that this experiment has produced results that reinforce STEAM projects as a form of mathematics pedagogy.

The article will be developed according to the following scheme. In section II,

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Theoretical framework, a presentation of the Ontosemiotic Approach is made, with emphasis on the dimensions that are relevant to this study. Section III, Mathematical content and musical context, specifies the mathematical subjects included in the present study and how they guide the creation of musical compositions. In section IV, Experimentation, the population, and our *sample* of that population are described. Additionally, a description of the *teaching and learning* processes is explained concerning design and techniques, and a description of the context and genesis of the Mathematical Music Composition Workshop is given.

The experimental design can be framed in terms of a case study, where *internal validity* is based on the coherence of the information collected through different instruments (*methodological triangulation*). In addition, validity is controlled by the contrast between the experimental groups in this research and similar groups of students with the same characteristics in previous years (*cohort groups*) or students simultaneously involved in a similar instructional process in the institution. The method of analysis is qualitative. In section V, Results and Discussion, the data will be presented and analyzed and, finally, in section VI, Conclusions, conclusions are drawn as to the effectivity of the workshop regarding mathematics teaching and learning, through the optic of the OSA.

2 Theoretical Framework and Research Methods

The collaboration between mathematics and music students is complex. Each group of students has a reasonable degree of familiarity and competence within their discipline, including the use of a particular language, while lacking the same control in the "other" area. This way, the professor must ensure certain interactions that allow the transference of information coherently between both types of students. There is a need for explicit interventions by the professor, which regulate the process and situate the knowledge in an institutional context, for it to be shared by the entire group. However, if the professor carries out all sessions in a lecture format, there is no interaction among the students. These interactions are necessary both for the adaptation of the students' knowledge to the multidisciplinary context and for the management of the learning process by the teacher. Thus, moments with a constructivist approach are also needed, where students experiment, inquire, and debate without explicit control by the teacher. In other words, a balance is required between inquiry and transmission, between creativity and autonomy of the students, always within the available time frame of the project.

The educational model is situated somewhere between the design based on *Inquiry-*

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Based Education, which proposes a learning environment based on exploration, with minimal or null guidance from the professor (Artigue; Blomhøj, 2013), and an *objectivist* educational model based on the transmission of knowledge by the professor directly, in a lecture format (Mayer, 2004; Boghossian, 2006).

Godino, Burgos and Wilhelmi (2020) propose a dialogic-collaborative model that looks for the appropriation of institutional practices by the student as the mathematical object O (the content which is being taught) passes through a series of problem-situations. The professor, as an institutional expert, should update these practices permanently. The student learns while participating with the professor and with other students in the joint work that is carried out. However, the question is how such a process of teaching and learning can be analyzed to see if it has been developed "adequately" in a specific context. A possible answer to this generic question demands the analysis of the different dimensions that are involved in the educational processes: the type of mathematics necessary in a particular context and situation (epistemic and ecological dimensions), the way of teaching and the methods employed (instructional and methodological dimensions) and the cognitive development of the students together with the creation of conditions that motivate and interest them (cognitive and affective dimensions). The notion of didactical suitability (Godino; Wilhelmi; Bencomo, 2005; Godino; Bencomo; Font; Wilhelmi, 2006) supposes the evaluation of the study process in terms of these 6 dimensions.

Therefore, the epistemic suitability allows the evaluation of the quality of the mathematical design, that is, of the objects, processes, and meanings that are put into play in the study process itself, and the pragmatic function that they carry out during that study process. The cognitive suitability looks to evaluate the study process in terms of the cognitive capacities of the students and should allow the use of previous knowledge to accumulate until the new concepts emerge. Additionally, apart from the students' cognitive stage, their personal, affective, and motivational interests should be considered in a way that *engages* them in the proposed mathematical activity (affective suitability). The instructional suitability refers to the capacity of the instructor to discern what is pertinent in the instructional process, that is, the type of delivery, the way of interaction with the students, the way the interactions among the students themselves can be integrated, the results obtained, etc. Additionally, the evaluation of the methods utilized and the distribution of different activities in time is of special interest (methodological suitability). All this is related, of course, to the instructional project at hand, which will determine the necessary conditions (possibilities and restrictions) to carry it out (ecological suitability).

The analysis of suitability refers to effective teaching and learning and should be

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designed previously to allow the evaluation of the hypothetical or preplanned trajectories, where "not only do we take into account the objectives, instructional tasks and hypothesis about the learning process, but also the roles of the teachers and learners and the instructional resources used" (Godino; Rivas; Burgos; Wilhelmi, 2019, p. 150). The trajectory must include epistemic, instructional, and cognitive configurations.

The epistemic configuration includes the mathematical objects and processes needed for carrying out tasks and solving problems. The instructional configuration determines the role of the professor and the students in the study process. Finally, the cognitive configuration describes the personal practices that are necessary for learning to occur, as well as the affective (emotional) aspects that can condition this learning. Beltrán-Pellicer and Godino (2020) identify a set of components and criteria of affective suitability of a study process in mathematics that they relate to: the type of language used, the emotions provoked and expressed, the attitudes developed and confronted, the beliefs (founded or not) that condition the mathematical activity, the values attributed to mathematics for daily or professional life, and the relationship of the affective domain with the rest of the didactic dimensions. The coordinated study of these configurations implies the analysis of the processes that relate them to each other. Sanchez, Font and Breda (2022) identify key aspects of mathematical creativity with the versatility of notions and processes, the ability to find connections among objects and contexts, and the exploration and validation of properties, all of this through an open perspective both in problem-solving and in proposing new problems. Identifying these aspects is key in the evaluation of the significance of certain educational experiences.

The analysis of the trajectory effectively implemented, contrasting it with the hypothetical planned trajectory, reveals significant didactical facts (SDF), i.e., didactic actions or practices that play a role or admit an interpretation in terms of the intended instructional meaning (Godino; Rivas; Arteaga; Lasa; Wilhelmi, 2014). These SDFs relate to the epistemic, instructional, and cognitive dimensions, as well as to observed semiotic conflicts (Godino; Font; Wilhelmi; Lurduy, 2011) and they have a contextual interpretation, given that they refer to the contrast between the planned and implemented trajectories.

In the situation at hand, where mathematics and music students interact, the language employed and the intelligibility in the communication are crucial. For this reason, the SDF refers to relations established between the two disciplines. In the Ontological and Semiotic Approach (OSA) (Godino; Batanero; Font, 2007) a semiotic function refers to any relation between an object (content) and its representation (expression). According to the OSA, the relations between expression and content can be presented as representational (an object is

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substituted for another), instrumental (an object is employed as an instrument for another object), or structural (two or more objects compose a system out of which new objects emerge). The determination of semiotic functions and their contextual significance provide SDF, due to their value in the shared construction of knowledge.

The qualitative research method is used in the experiment. Hence, the experimentation is centered around the internal validity (Bostic *et al*., 2021), that is, around the degree of consistency between the plans and the observations relative to the impact on the teaching (cause) and learning (effect) processes. The case study is longitudinal and cross-sectional controlled. On the one hand, the longitudinal comparison is carried out by contrasting the participants with students of previous generations. As the music composition professor had taught the seminar for approximately two years, there were about 20 students to compare with. As to mathematics, one of the authors compared students in her *Bridge to High Mathematics* and *Modern Algebra* courses, also during two years before the study, about 35 students. On the other hand, the transversal comparison is done by contrasting the participants with those of their generation who did not participate in the STEAM project. This implied 15 mathematics students and 12 music composition students. In other words, qualitative contrasts were carried out around the learning and assimilation of mathematical concepts by students who participated in the mathematical music composition workshop (experimental group), taking as a reference the level of the control groups.

The method of triangulation was implemented, as several instruments allowed the control of internal validity. The oral and written manifestations of the participants in their meetings with the professors, their dress rehearsal and actual lecture-concert, the final compositions, and talks are all registered and include the verbalization of the mathematical subjects and the explanation of how they were incorporated into the resulting musical works.

The qualitative research method, with emphasis on the search for internal comparisons, is used for all the teams and all the subjects involved. Therefore, general data will come from all the teams, while the one selected group should be considered an educational and action-research case study (Bassey, 1999; Gerring, 2004), given that it is not the objective of this study to develop the educational theory implied in the processes of teaching and learning mathematics, nor do we pretend to evaluate with the motive of prescribing actions and methods to be implemented. The objective is the comprehension of the action itself (what has occurred?) so that this understanding can contribute to the development of the broadening of competencies for mathematics teachers and professors. Hence, as an educational case study, our contribution consists of information, results, and a discussion of impact. Additionally, as an action-research

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case study, this work contains useful ideas when analyzing other types of activities as well as information that can guide practices in similar contexts.

In summary, the research pentagon (Bikner-Ahsbahs, 2019) consists of the following components:

- *Research object*. On the one hand, the mathematical objects (greatest common divisor, Euclid's algorithm, etc.) and, on the other hand, the musical objects (e.g., Euclidian rhythms).
- *Research aim*. The detection of semiotic functions between mathematical and musical objects. In other words, the description of the expression-content relation amongst these objects and the negotiation of meanings that the subjects determine in the teaching and learning process.
- *Research question*. In a multidisciplinary context, to what extent does the didactic transposition of a notion contribute to the learning of mathematics, both for mathematics and non-mathematics majors? That is, do those with a mathematical background construct some type of *novel* meaning of mathematical notions by necessity when interacting with a *non-mathematician*?
- *Research method*. Qualitative methods using triangulation as understood in this context.
- *Research situation*. The STEAM philosophy, where the semiotic interactions among subjects with heterogenous and interdisciplinary backgrounds are conceptualized, emphasizing the significant didactical facts, and assessing suitability in the epistemic, ecological, cognitive, affective, instructional, and methodological dimensions.

3 Mathematical content and musical context

By no means it is necessary to be versed in mathematical music theory to understand the intention and reach of the present work. However, we will present a very brief idea of some of the concepts that cross over between mathematics and music, so that the reader can gain even more insight into the possibilities of this combination.

In the Mathematical Music Composition Workshop and, specifically, its second version that took place during the fall of 2019 (the last pre-pandemic COVID-19 face-to-face workshop), the concepts from mathematical music theory that were finally implemented included: the concepts of mathematical music theory which were notions and procedures such as: a) the identification and construction of musical patterns using mathematical algorithms

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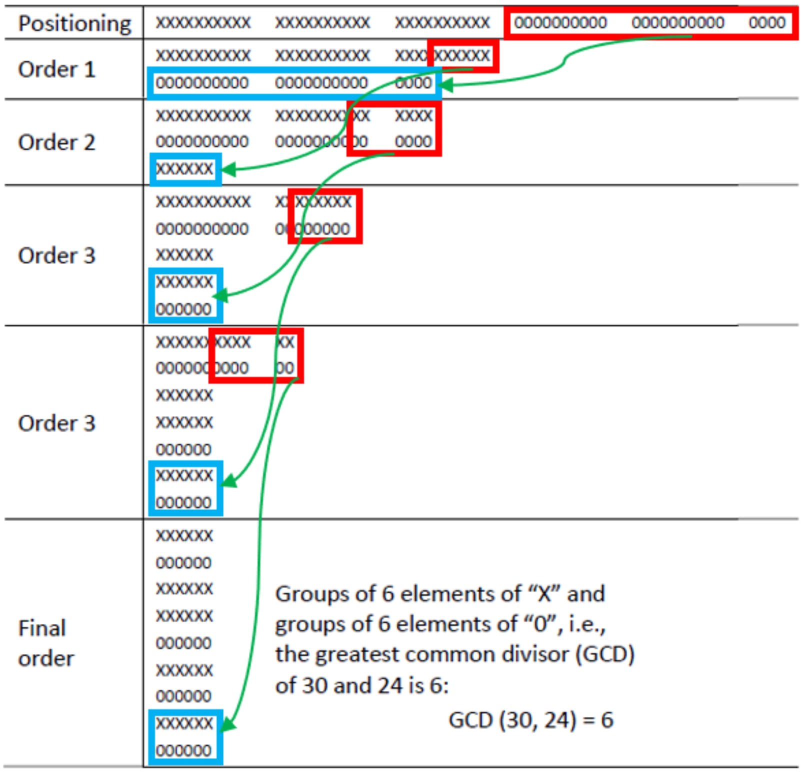
(e.g., Euclidean division as the foundation of Euclidean Rhythms); b) musical set theory; c) probability and Markov chains; d) contextual transformations; e) Nzakarian canons and algebraic combinatorics on words; f) Neo-Riemannian transformations.

In what follows, we will focus only on the teaching and learning process of Euclidian rhythms, to concentrate the interest and reach of this multidisciplinary design. As was seen in the Introduction, Euclidean rhythms are generated by using Euclid's algorithm to calculate the greatest common divisor (GCD) of two numbers. For *small* numbers, this algorithm has a graphical representation (Figure 1).



Red rectangle: the elements identified in each step are relocated in the next step
Blue rectangle: the relocated elements from the previous step
Green arrow: the flow of the elements between the steps.

Figure 1 – Calculating the GCD (30, 24) through a graphical algorithm
Source: Own elaboration.

The musical aspect of the calculation carried out in Figure 1 is arrived to at the last ordering of the sequence of sounds *1* and silence *0*, from left to right according to columns, that is:

101101010 101101010 101101010 101101010 101101010 101101010

This distribution contains cycles of 9 pulses, formed by 5 notes and 4 rests, given that all the columns are equal (101101010).

In the example that will be used as a case study, the rhythmic aspect of music is

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privileged. As mentioned, one way of modeling rhythm is as a distribution of strong beats (notes) on a timeline of pulses, that is, a combination of strong beats and silences (rests). For example, if our timeline has 12 pulses and 3 strong beats (which means 9 rests to complete the 12 pulses), we can distribute the strong beats in an even manner and we will hear a strong beat every 4 pulses. This is the Euclidean rhythm E(3,12). As there is a cyclic repetition, the rhythm E(3,12) can be interpreted by the arithmetic notion of modulus. In this case, the strong beats fall on 0, 4, and 8, only to begin again on 0, conceived as 12, 24, 36, etc. modulus 12 (Figure 2).

1	0	0	0	1	0	0	0	1	0	0	0

Figure 2 – Completely even distribution for E(3,12)
Source: Own elaboration.

However, if the number of strong beats is 7 and we want to distribute them as evenly as possible in 12 pulses, the Euclidean algorithm can be used to make the calculation (Toussaint, 2005, 2013) (Figure 3). Thus, the concept of Euclidean rhythms is formalized and the theory behind this process is related to the ubiquitous notion of maximal evenness (Gómez; 2018; Clough; Douthett, 1991).

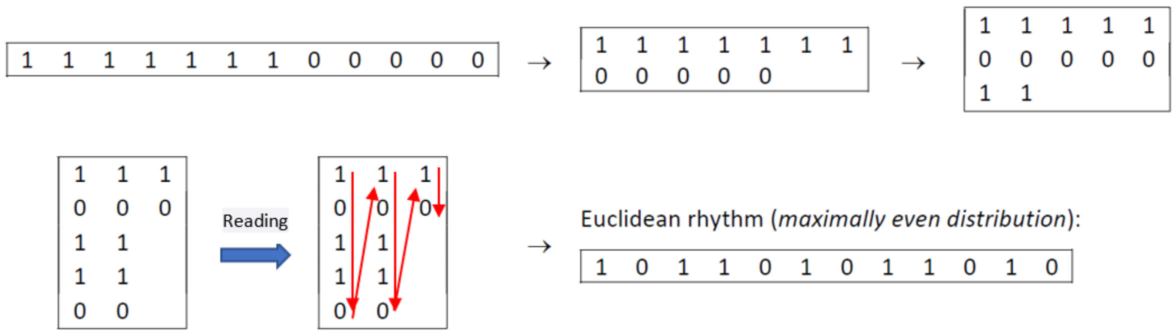

Figure 3 – Maximally even distribution for E(7,12)
Source: Own elaboration.

Indeed, Toussaint (2005) showed that the Euclidean algorithm, used to calculate the greatest common divisor of two given integers, can be employed to generate a family of rhythms that he called Euclidean rhythms. The concept of maximal evenness implies the search for the most *regular* distribution possible and arose in the pitch domain to explain the distribution of black and white notes on the piano keyboard. In the case of rhythm, in Euclidean rhythms, the patterns of strong beats are distributed as evenly as possible (for example, 7 strong beats in a timeline of 12 beats, as in Figure 3). Toussaint realized that the same Euclidean algorithm could explain the Björklund algorithm in the context of nuclear physics accelerators. Additionally, it has been seen that Bresenham's line algorithm (https://n9.cl/5h9k) and other techniques, such

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as Christoffel Words (Hollos, S.; Hollos, J.R., 2014) can be used to generate Euclidean rhythms. These connections, plus the algorithm itself and its application, defined the mathematical competencies for this subject.

4 Experimentation

During the spring and fall semesters, two interdisciplinary projects titled the *Mathematical Music Composition Workshop* were carried out at a large research university in the Southern United States. Students from the Department of Mathematics and Statistics participated together with music composition students from the Composition Seminar at the School of Music, from the same university. The idea was that the music composition students would employ some of the composition techniques based on mathematics that they had studied in their seminar to compose a piece, while the mathematics students would work with them to clarify the mathematical theory behind the algorithmic procedure.

During both semesters, but especially during the second project, the result was very illustrative of a) the relevance of the dialogic-collaborative method in the instructional process; b) the usefulness of identifying significative didactical facts (SDF) as well as semiotic functions of diverse nature among the two disciplines (mathematics and music) to evaluate the study process and, in consequence, determine its didactic suitability.

On both occasions, the result materialized in a *lecture-concert* in which the mathematics students (and some of the music composition students) gave presentations about the mathematical techniques employed in the pieces, and the composition students explained how and where these techniques were incorporated. Immediately after the explanations, the pieces were played by music students.

The lecture-concert has been the culmination of the process up until now, as well as the way to do outreach and create interest in the experiment. All the participating music students were registered in the Composition Seminar, a mandatory course for composition majors and an optative course for performance or music education majors. The professor is very versed in mathematical techniques and has advanced studies in physics. During the two semesters in which the mathematical music composition workshop was carried out, the participating music students offered themselves as volunteers and their mastery of the pertinent mathematical concepts could be contrasted with that of the remaining members of the composition seminar who did not participate.

The composition seminar usually has a population of 15 students per semester. During

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the first semester of the experiment, 8 students volunteered to participate in the mathematical music composition workshop, although only 5 of these students completed their pieces with the necessary mathematical content; 4 of these 5 students also participated in the lecture-concert. During the second semester, 8 students also participated, although 2 of them repeated their participation in the workshop and the concert with new material. Five teams participated in the lecture-concert during the second experiment.

All the Mathematics students who participated in the workshops were recruited from the Mathematics and Statistics Club, an undergraduate student organization that foments conferences and outreach, and whose members are usually -but not exclusively- mathematics majors. Consequently, they did not all have the same level of studies, although the 8 mathematics students that participated (3 during the first semester and 7 in the second semester, with 2 students who repeated their participation in the workshop) 6 were in their last 3 semesters of their undergraduate studies in mathematics. Of the eight participating mathematics volunteers, 5 were enrolled in one or more classes taught by one of the authors during the period of the experiment. This allowed for additional data, as the evaluations could be added to the observations made in the context of the workshop. Once the students were accepted in the workshop, their participation was recognized by the Department of Mathematics and Statistics in a similar way to the undergraduate research program at the same institution. Occasionally, there should be a specific “call” for participants in which professors from relevant courses announce the opportunity and advantages of participating in the Workshop. Unfortunately, due to the Covid interruption and some changes at the School of Music, these plans have not yet gone into effect, but there is consensus at both the Department of Mathematics and Statistics and the School of Music, that they represent desirable interdisciplinary knowledge.

The team members were assigned by the mathematics and music professors, given that the students from mathematics and those from music did not know each other previously. However, before forming the teams, the composition seminar students had to choose the techniques they wished to use in their pieces. Hence the mathematics students could choose the subjects that most interested them, and the teams were formed in function of such preferences.

There was a calendar for the different stages of the experiment. Once the music students chose the mathematical techniques they wanted to use in their pieces, they met with one of the professors to design the work plan. As the music students began to produce musical material, the mathematics students developed the analogies and metaphors they would use to explain these ideas, the mathematical techniques, and the resulting structures. Indeed, the mathematics students needed to carry out a genuine didactical transposition (Chevallard, 1989) of the

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concepts, often sophisticated and new for them, to explain them to the music students and prepare their outreach exposition for the lecture-concert.

5 Results and Discussion

The activities of all the teams can be classified as a mixture of guided learning by the instructor and a didactic model with dialogic-collaborative characteristics. The subjects were clearly defined in the composition seminar and, once the teams chose their areas, the students of music and mathematics met with the composition professor to design their work plan. However, the teams carried out most of their sessions without the presence of an instructor, although the composition students showed their progress to the music professor and the mathematics students came to the mathematics professor to ask questions referent to the mathematical content and to the way of synthesizing it for the lecture-concert presentation.

A general feature that could be verified during the interviews and during the sessions that the mathematics professor held with the mathematics students was how these students realized that they use a very specialized language. This specialization, of course, occurs in all disciplines. However, while the music students were very conscious of the fact that they work with super-specialized symbols and language, the mathematics students were prone to suppose, in the beginning, that the musicians would understand the notation and mathematical ideas, given that *everyone takes algebra* (in Secondary Education), in the words of one of the mathematics students.

This attitude towards the complexity of mathematical language and mathematics itself is not surprising, given that it is common for mathematics teachers to suppose that students will understand sophisticated concepts by formal definition. In other words, it is an example of the illusion of transparency, that is, the phenomenon according to which the professor believes that students will share the meaning of a mathematical object, derived from its definition, with him (Brousseau, 1997). In consequence, the preparation of the talks at the lecture-concert, with a general public as audience, demanded from them an exercise in didactic transposition of very technical mathematical notions.

In the following paragraphs, we will present an analysis of the significant didactical facts (SDF) related to the epistemic and cognitive dimensions. This analysis situates the acquisition of the SDFs in a dialogic-collaborative model (Godino; Burgos; Wilhelmi, 2020), which takes elements from the constructivist and inquiry-based approaches, as was mentioned in section II. Some of the SDFs were identified in the process of knowledge transmission from

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professors to students, while others arose during the group work, either in the presence of one or both instructors or in the students' autonomous sessions, and they are identified as student constructions.

5.1 Some examples of SDF in the epistemic-ecological and cognitive-affective dimensions

The SDF identified with the epistemic and ecological dimension is differentiated when they appear as transmission professor-student (objectivist model, OM), when the content emerges during peer interaction (constructivist model, CM), or when the content, processes or meanings emerge in a mixed context, and learning by transmission and autonomous activity interact in the didactic trajectory (DM). The following are some SDFs related to the epistemic-ecological dimension.

- (OM) All the students, from mathematics and music, received an introductory presentation about the mathematics that form (and explain) certain musical expressions. For the music students, this fact implied new "rules" to work with as, contrary to a romantic vision of music composition, these students study the rules behind the different formal styles throughout the history of Western and other musical traditions, even if some of them then incline to the *Avant Gard*.

For the mathematics students, especially those with little or no musical background, it was this presentation that allowed them to understand how they could play a fundamental role in the composition of a musical piece by using their mathematics knowledge.

Before the Lecture-Concert, during the *dress rehearsals,* the instructors encouraged a conversation about the process and what they had learned, both in terms of mathematics and music.

- (CM) The music students, together with the mathematics students, had to use mathematical techniques in the creation of their compositions without sacrificing aesthetic value. In addition, they were supposed to augment the aesthetic value of their pieces through the incorporation of new musical ideas resulting from mathematical knowledge and in accord with music composition theory. The mathematics students, more than the music students, expressed their surprise through comments such as

> *I never thought that we would really make beautiful music and we did* (Comments from mathematics students, 2019).

- (DM) The professors introduced the notions of Euclidean rhythms, together with subjects such as modular arithmetic, the Euclidean algorithm, the Björkland algorithm, the

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combinatorial concept of necklace, as well as the identification of maximal evenness through the sum of chords in a circle representation (geometric language!), using the formula 2sin($\frac{c\pi}{d}$), where c is the number of strong beats and d is the total number of pulses. To reinforce ideas, the students created timelines and necklaces with different combinations of relative primes and listened to their maximally even distributions (and non-maximally even distributions) with handclapping and percussion instruments. This concrete *back and forth* from the mathematical geometric language to the musical rhythmic language is of cognitive importance (the extension of mathematical knowledge -Euclidean algorithm- as well as to an extra-mathematical context – musical rhythms). The affective aspect could be identified by students' gestures and comments, as well as discussions. The transformation of mathematics to rhythm, and the gestural effect it had, is described in the affective domain in a general manner when "attention is paid to non-verbal language to foster immediacy" (Beltran-Pellicer; Godino, 2020, p. 17).

- (OM) The mathematics professor institutionalized the notion of isomorphism as a mathematical model for the combinations of bichords through modular arithmetic. Equivalence classes, functions between sets of equivalence classes, and the addition of other restrictive properties to transform sets in groups, are all competence that mathematics majors should acquire formally. By having to explain these concepts in an informal way both to the music composition students and to the general public, there was an indication (through comments to the professor) of enhancement of understanding. One student said:

> *I received A's in Bridge to Higher Mathematics and Modern Algebra I, and I am only now understanding what this all means* (Mathematics student's comment, 2019).

The following are some SDFs related to the cognitive-affective dimension.

- The music students knew that their pieces would be played in the lecture-concert and that a heterogeneous public would attend, some having sophisticated musical criteria (professors and students from the School of Music). These students put a lot of effort into the incorporation of mathematical material in their pieces, looking for coherence. With this motivation, they were open to new mathematical knowledge that, in another context, they might have rejected. This could be verified through the comparisons made with the control groups, both longitudinal and transversal. In terms of the affective domain, Beltrán-Pellicer and Godino's criteria of the component *Interrelation with other domains/facets* were met, that is: "(1) The affective component is planned in the teaching-learning process. (2) Positive emotions are related to mathematical attitudes and to the successful resolution of tasks, fostering the emotional reflection of the student in this regard" (Beltrán-Pellicer; Godino, 2020, p. 17).

- Because of the above stated, there was also a special motivation for the mathematics

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students, as they had to make a presentation of the mathematical subjects to the general public. This resulted in reinforcement of what they had learned and could be measured and compared, in five of the eight cases (those that were students of the mathematics professor) with their performance on exams and other assignments. The ability to explain and show understanding of the mathematics involved served to promote self-esteem, avoiding the phobias and rejection that often occur when confronting mathematical notions. Additionally, the participation in the creation and communication of the music and the mathematical ideas transferred the responsibility to the students, fostering a mathematical attitude.

- The work in mixed groups of mathematics and music students fomented an interchange of knowledge and language in the context of a common goal: the lecture-concert. A mutual appreciation amongst both types of students was also created, synthesized perfectly by the following comment

> *I only learned recently how busy the schedule of a music major actually is* (Student's comment, 2019).

Here one could see that certain beliefs and stereotypes were contrasted with real situations in which both mathematics and music students could take a glimpse into the daily routine of each other's area.

5.2 Semiotic function

The identification of the SDF allows a better comprehension of the interaction between the workshop participants. We will refer to the subject of the case study, as an example of the Mathematical Music Composition Workshop as team 1. Team 1 is formed by two mathematics students and one music student. The subjects this group chose are Euclidean rhythms and contextual transformations; however, due to space constraints, we will only analyze the Euclidean rhythms and only use one of the mathematics students' presentations. We will refer to the mathematics student as S1 and the music student as S2.

> *S1: Toussaint used the Euclidean algorithm to distribute the number of onsets and offsets as evenly as possible… Using the Bjorklund algorithm we will be able to generate a binary sequence of 1's and 0's. The 1's will represent onsets and the 0's will represent rests. Remember the goal is to distribute the number of 1's and 0's evenly throughout the sequence… An example is using the Bjorklund algorithm for 5 and 12… We can actually express this sequence as 100101001010 (he put the audience to clap the onsets and offsets). Let us try 4 and 10. Four is the onset and six remaining 0's which are the rests* (Mathematics student's comment, 2019).

In Figure 4 the new example given by S1 is shown. It is the Euclidian rhythm E(4,10) that can be obtained easily through the graphical algorithm (Figure 1).

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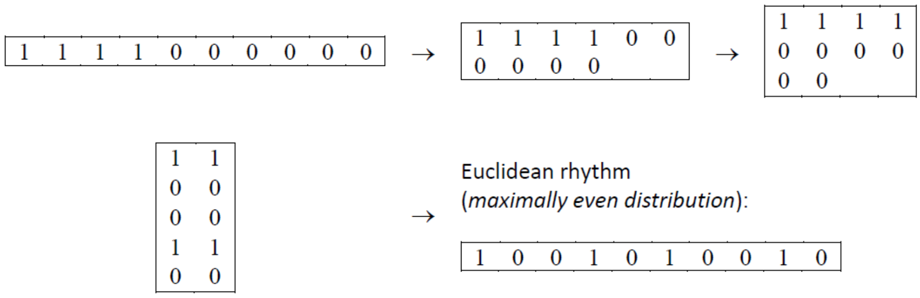

Figure 4 - Maximally even distribution for E(4,10)
Source: Own elaboration.

The most relevant aspect here is the fact that, from the beginning, the semiotic functions have a marked structural character as the student apparently passes seamlessly between mathematical and musical expressions and content. From the analysis made with all the groups, the tentative conclusion is that this happens more frequently when the musical content is rhythm rather than pitch. As a result, a new object emerges (rhythm) which dotes a numerical sequence obtained by a mathematical algorithm (structural semiotic function *rhythm-algorithm*) with musical meaning. However, this new structure can be implemented to obtain new sequences and, consequently, new rhythms.

Thus, S1 employs a first example in which he states that

> *[...] using the Bjorklund algorithm for 5 and 12…we can actually express this sequence as 100101001010* (Mathematics student's comment, 2019).

which is a clear instrumental semiotic function. However, to a public not versed in the underlying mathematics it seems very unclear. S1 seems to realize this when presenting his second example and employs a representational semiotic function:

> *Let us try 4 and 6. Four is the onset and the six remaining 0's which are the rests* (Mathematics student's comment, 2019).

In this case, his expression is numerical, and the content is musical, which is usually more common in the music students than the mathematics students. He also illustrates what the two numbers represent (the smaller number is the strong beats (onsets) and the larger number is the total amount of pulses counting strong beats and rests).

Once again, after the mathematical presentation, S2 explains how he used the mathematical notions in his composition.

> *S2: My piece can be expressed in two words: minimalist and programmatic. With these two concepts I tested Euclidean rhythms. For example, (4,9) is very common in the Balkan region, (5,12) is used by Bernstein in West Side Story…* (Music student's comment, 2019).

S2, by associating his piece with Euclidean rhythms and mentioning other composers and their works, *translates* S1's representational semiotic function to his own particular context

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and shows the professor how he has understood the content. This way he fulfills an ecological norm (Godino; Font; Wilhelmi; De Castro, 2009) concerning the evaluation: the program (the established institutional guideline) must be obeyed, and the professor must evaluate while the student should demonstrate that he has mastered the content. However, this demonstration does not deter S2 from his main objective, which is that of implementing the mathematical content in his composition. He soon leaves the rhythmic domain and translates the ideas to the pitch domain.

> *S2: What I did was look at the pitch numbers. I tried to play around with seventh chords, but with that you get a 4D cube [figure 6a], and it is really hard to conceptualize… my highest grade in high school math was an 80, so I just look at it as a 3D tonnetz, like this [figure 5b]* (Music student's comment, 2019).

There is some semiotic conflict and misconception here, but it can easily be detected through a linguistic analysis. As can be seen in Figure 6b, the tonnetz is 2D. However, it is in fact a flat torus and the torus itself is a surface in 3 dimensions, the same way as the real 3D tonnetz represents the 4D object S2 referred to. At the same time, further work with this team gave indications to the mathematics students so that this aspect could be clarified among peers. In the end, this conflict did not affect the progress of the study process centered on Euclidean rhythms.

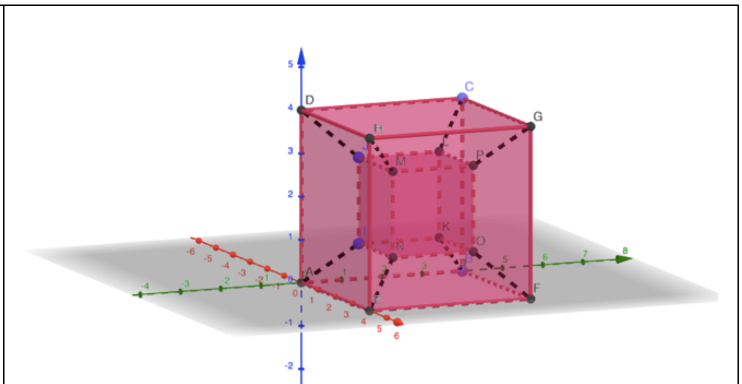
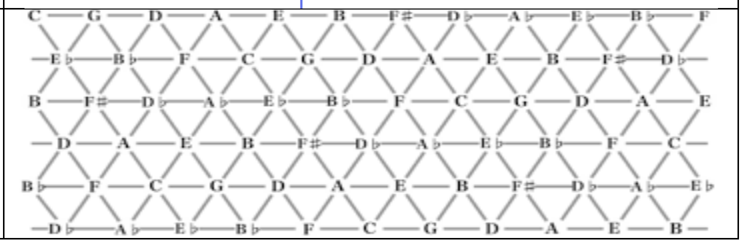
(a) 4D cube	
(b) 2D Tonnetz	

Figure 5 - 4D cube by GeoGebra software and the 2D Tonnetz
Source: Own elaboration.

Under the usual conditions, and without access to the mathematical music composition workshop, it is improbable that music composition students would come into contact with geometric and topological concepts such as dimension, even when they use some of the techniques that arise from math-music research in their compositions. The contact with

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mathematics majors and the motivation to prepare a presentation about their pieces in the lecture-concert augments the affective suitability and positively influences mathematics learning.

5.3 Didactic suitability

Finally, the previous analysis allows us to evaluate the didactical suitability according to the dimensions mentioned in section II. We will briefly sketch this evaluation in the following list, using a scale of high, medium, or low, and the indicators of the didactic suitability introduced by Godino (2013), to show how we situate the past workshops and what we think is necessary to work on to improve them:

Epistemic suitability: Medium. The evaluation is in terms of the different mathematical objects involved: situations, language, rules, arguments, and relations. An articulated and representative sample of situations is presented to allow contextualization and to shed light on the practices and applications. Translations from verbal to formal language are established to understand the discourse based on intelligible arguments for both groups of students (mathematics and music). Indeed, the lecture-concert is a situation in which the students need to exhibit and argue their decisions and achievements. However, even though relations among certain musical compositions and the mathematics employed can be seen, the relations between mathematical objects, important for more general and applicable rules (for example, in other STEM areas) were not touched on.

Ecological suitability: High. The evaluation of this suitability refers to the adaption of the curriculum, the development of professional and cultural facets, and the ability to innovate and establish interdisciplinary connections. Indeed, the mathematics composition workshop is a way of introducing and reinforcing both the mathematics and music students' core curriculum, as well as extra material that is not often covered in standard courses but is useful in their professional training. Students' learning is enhanced with this educational innovation in an interdisciplinary context, which foments the development of communication and argumentation competencies for both specialized and non-specialized publics, demanding cultural adaptation.

Cognitive suitability. High. The evaluation is carried out in terms of previous knowledge, individual adaptation, and the learning indeed achieved. The workshop participants' profile guaranteed that the students possessed the necessary previous knowledge. The organization of the study process through a dialogic-collaborative model permitted individual adaptation, reinforced by the organization in mixed teams (mathematics and music students),

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fomenting a positive outcome. The students could present the lecture-concert, where their concrete results could be exhibited both in the talks and through the actual product, that is, the musical pieces that were performed. These final products were a measure of the learning achieved, given that the students showed their conceptual, communication, and argumentation competencies.

Affective suitability: High. The evaluation is carried out according to the components: emotions, interests, and attitudes. The students were very motivated. In the end, they participated voluntarily, without any sign of emotions in terms of phobias, fear, or rejection of mathematics; even the composition seminar students were those who had chosen to use the mathematical techniques in their end-of-semester pieces, emphasizing the aesthetic qualities and precision involved. They all were committed to the project and to the lecture-concert and were interested in explaining the concepts in the best possible way and transmitting this knowledge to their peers, professors, and the public. The students had an open and unbiased attitude, both in the study process and on the lecture-concert: the discourse itself was valued, not the person or the profile (mathematician or musician) who carried it out.

Instructional suitability: High. The evaluation was conducted according to the type of interaction (instructor-student or among peers) and the autonomy of the students. The teaching and learning processes were facilitated by the students' motivation throughout the entire project, as well as through the negotiation of meanings, both in the plenary discussions with the professors and in the peer groups. Indeed, the reduced group size allowed a very high level of commitment. The students worked autonomously and were very implicated in achieving an intelligible discourse during the lecture-concert.

Methodological suitability: Medium. This dimension is analyzed in terms of both material and temporary aspects, as well as the institutional conditions (number of students, schedule, classroom conditions, etc.). There was some difficulty in coordinating meetings due to the very different schedules and types of majors of the two disciplines involved. Additionally, the demand of finding interpreters for the pieces, and the coordination of the necessary technology needed in presenting the mathematical talks with the realization of the musical performances, are examples of unique issues related to resources and planning that must be addressed. Some of these difficulties were "compensated" through the small number of students, allowing a collaborative dynamic and a use of time that was adaptable to the necessities of the teams.

Figure 6 summarizes the didactic suitability of the study process.

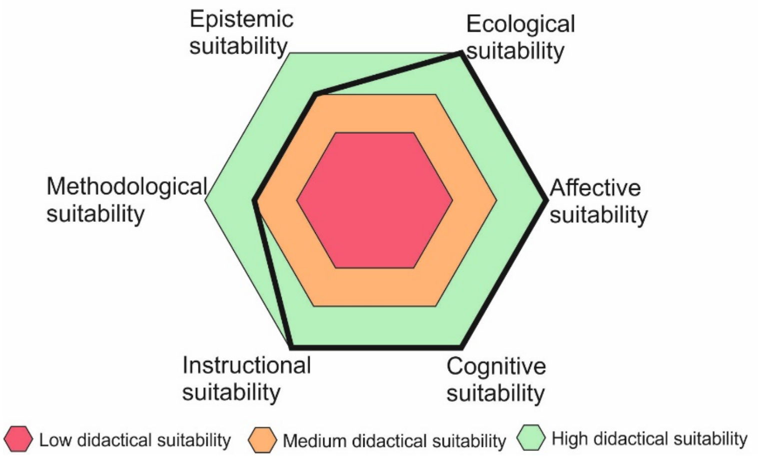

Figure 6 - Didactical suitability
Source: Own elaboration.

6 Conclusions

Through the analysis of semiotic functions (Godino; Batanero; Font, 2007), the authors have tried to convey how interdisciplinary collaboration using mathematical music theory can play an essential role in STEAM education. In this interdisciplinary context, the organization of the educational aspects is carried out by projects that generate conditions that permit each discipline to contribute with the necessary notions, processes, and meanings. The didactic trajectory that is implemented through a dialogic-collaborative model (Godino; Burgos; Wilhelmi, 2020) has allowed a balance between class lectures by the professors and the autonomous activity of the students. The professors' presentations fulfill a regulatory function (in the socialization or institutionalization of the content) or act as catalysts (stimulating the process, given the time limitations). The students' autonomous activity facilitates the emergence of meanings both for the mathematical notions and the development of the creative process.

The control of the epistemic and ecological suitability (Breda; Pino-Fan; Font, 2017) is key in this case, where differentiated student profiles are part of the educational project. The Mathematical Music Composition Workshop paired mathematics and music students, allowing them to share their expertise in the learning process while defining specific mathematical competencies to be mastered.

Through measured goals, it is safe to state that the outcome has been successful in terms of the desired objectives. For the mathematics students who participated in the Mathematical Music Composition workshop, the experience actually expanded their mathematical horizons; apart from the exotic application, they saw some mathematical notions that are not necessarily part of the standard curriculum. As an added benefit, they also gained sensitivity toward musical

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creation and an appreciation of music as a very precise discipline. The music students in both experiments were highly motivated by the eventual premiere of their works and were capable of learning and assimilating the mathematical concepts, as was evidenced not only by their discourse but also by the way the students used the concepts and techniques in the development of their concert pieces.

The success in the consecution of mathematical knowledge is due, to a large degree, to the communication that was achieved between the two student groups. This communication exercise demands deep knowledge of the mathematical content to transmit it with less formalisms, looking for metaphors and analogies that can build bridges between different ways of attacking the same problem. It is for this reason that the identification of semiotic functions determines didactically significant facts which are key to understanding the expression-content duality. The lens of semiotic functions fomented the analysis of the participants' written and oral expressions as follows. The representative semiotic function captured the students' transition between the two disciplines and how the use of mathematical and musical language can enhance mathematical thinking. The instrumental and structural semiotic functions show how mathematical objects and structures (sequences, points in the Cartesian plane, sets of equivalence classes) and musical structures (rhythm, contours) can give rise to new ways of understanding both disciplines.

The decision to include only the subject of Euclidean rhythms in the case study was due to the fact that this subject can be incorporated into the mathematics curriculum even when the instructor does not have musical knowledge. As can be witnessed in Wilhelmi and Montiel (2018), it is possible to create a study unit around maximal evenness and the algorithms, including the Euclidean algorithm, plus the graphical and algebraic representations, by an instructor who does not have a musical background. What is more, using easily accessible free programs on the internet, the class can generate music through these activities.

It Is obvious that there cannot be an indiscriminate reproduction of the complete model of the mathematics-music composition workshop, given that special conditions are needed for students from both disciplines to truly benefit. However, if there is the availability of researchers and instructors in both mathematics departments and schools of music or, if at a school level, there is a collaboration between mathematics and music teachers, the possibility of valuable results is high. There are also ways, as mentioned above, to incorporate some of these techniques into the mathematics classroom without needing musical expertise. To the extent that the conditions lend themselves and that these STEAM projects are implemented, it will be necessary to carry out studies that will offer guidance on how to guarantee that the

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desired competencies are achieved. In this vein, the analysis of ecological suitability in potential study processes should contribute to framing the feasibility of the educational projects, given that this suitability links the teaching objectives and the content with the logical institutional restrictions.

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