Iterated Conditionals, Trivalent Logics, and Conditional Random Quantities

Lydia Castronovo*[0000−0001−5521−646] and Giuseppe Sanfilippo* **[0000−0002−0657−3833]

Department of Mathematics and Computer Science, University of Palermo, Palermo, Italy
{lydia.castronovo,giuseppe.sanfilippo}@unipa.it

Abstract. We consider some notions of iterated conditionals by checking the validity of some desirable basic logical and probabilistic properties, which are valid for simple conditionals. We consider de Finetti's notion of conditional as a three-valued object and as a conditional random quantity in the betting framework. We recall the notions of conjunction and disjunction among conditionals in selected trivalent logics. Then, we analyze the two notions of iterated conditional introduced by Calabrese and de Finetti, respectively. We show that the compound probability theorem and other basic properties are not preserved by these objects, by also computing some probability propagation rules. Then, for each trivalent logic we introduce an iterated conditional as a suitable random quantity which satisfies the compound prevision theorem and some of the desirable properties. Finally, we remark that all the basic properties are satisfied by the iterated conditional mainly developed in recent papers by Gilio and Sanfilippo in the setting of conditional random quantities.

Keywords: Coherence · Conditional events · Conditional random quantities· Conditional previsions· Conjoined and disjoined conditionals· Iterated conditionals· Compound probability theorem· Lower and upper bounds· Import-Export principle

1 Introduction

The study of conditionals is a relevant research topic in many fields, like philosophy of science, psychology of uncertain reasoning, probability theory, conditional logics, knowledge representation (see, e.g., [1,2,11,12,13,15,16,18,31,32,35,36,37]). Usually, conjunctions and disjunctions among conditionals have been introduced in tri-valued logics (see, e.g., [1,4,8,9,31]) In particular, de Finetti in 1935 ([17]) proposed a three-valued logic (which coincides with Kleen-Lukasiewicz-Heyting logic [9]) for conditional events by also introducing suitable notions of conjunction and disjunction. Calabrese in

* Both authors contributed equally to the article and are listed alphabetically

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[5] introduced an algebra of conditionals by using the notions of quasi conjunction and quasi disjunction also studied in ([1]). Both, de Finetti and Calabrese, introduced a notion of iterated conditional as a suitable conditional event with the requirement, among other properties, that the Import-Export principle be satisfied, which means that the iterated conditional $(B|K)|A$ coincides with the conditional event $B|AK$. The validity of such a principle, jointly with the requirement of preserving the classical probabilistic properties, leads to the well-known Lewis' triviality results ([34]). Moreover, by defining conjunctions and disjunctions as conditional events (in a trivalent-logic) it follows that some classical probabilistic properties are lost; for instance, the lower and upper probability bounds for the conjunction are no more the Fréchet-Hoeffding bounds ([39]). A different approach to compound conditionals has been given in [35,32]. A related approach has been developed in the setting of coherence in recent papers by Gilio and Sanfilippo (see, e.g., [24,25,26,27,28]), where compound conditionals are defined as suitable conditional random quantities with a finite number of possible values in the interval $[0,1]$. The advantage of this approach is that all the basic logical and probabilistic properties are preserved, for instance De Morgan's Laws and Fréchet-Hoeffding bounds; for a synthesis see [28]. The iterated conditional, named here $|_{gs}$, is defined by means of the structure $\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\overline{\bigcirc}$ (where $\mathbb{P}$ is the symbol of prevision) and hence satisfies the compound prevision theorem. Moreover, Lewis' triviality results are avoided because the Import-Export Principle is not satisfied (see [24,40,41]).

The purpose of this paper is to investigate some of the basic properties valid for events and conditional events with a view to different notions of iterated conditionals. Indeed, things get more problematic when we replace events with conditional events and we move to the properties of iterated conditionals. After recalling some trivalent logics (Kleene-Lukasiewicz-Heyting-de Finetti, Lukasiewicz, Bochvar-Kleene, and Sobocinski) we study basic properties for the notions of iterated conditional introduced by Calabrese and by de Finetti. We also compute some sets of coherent assessments on families of conditional events involving the previous two iterated conditionals. Among other things, we observe that the compound probability theorem is not preserved by these objects. Then, by exploiting the structure $\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\overline{\bigcirc}$, for each trivalent logic we introduce a suitable notion of iterated conditional which satisfies the compound prevision theorem and some (but not all) of the selected basic properties. Finally, we remark that, among the selected iterated conditionals, $|_{gs}$ is the only one which satisfies all the basic properties.

The paper is organized as follows. In Section 2 we first recall some preliminary notions and results on coherence, conditional events and conditional random quantities; then, we recall the logical operations in some trivalent logics and in the context of conditional random quantities. In Section 3 we check the validity of some logical and probabilistic properties satisfied by events and conditional events for the iterated conditional defined by Calabrese (Section 3.1) and by de Finetti (Section 3.2). We also compute the lower and upper probability bounds for both the iterated conditionals. In Section 4, we recall some results on $|_{gs}$

and for each trivalent logic, we introduce and briefly study the iterated conditional defined as a suitable conditional random quantity. We also consider a generalized version of Bayes formula and give some results on the set of coherent assessments. Finally, we give some conclusion remarks.

2 Preliminary notions and results

In this section we recall some basic notions and results which concern coherence and logical operations among conditional events.
Events, conditional events, conditional random quantities and coherence. An event A is a two-valued logical entity which is either *true* (T), or *false* (F). We use the same symbol to refer to an event and its indicator. We denote by Ω the sure event and by $\varnothing$ the impossible one. We denote by $A \wedge B$ (resp., $A \vee B$), or simply by AB, the conjunction (resp., disjunction) of A and B. By $\bar{A}$ we denote the negation of A. We simply write $A \subseteq B$ to denote that A logically implies B, i.e., $A\bar{B} = \varnothing$. Given two events A and H, with $H \neq \varnothing$, the conditional event $A|H$ is a three-valued logical entity which is *true*, or *false*, or *void* (V), according to whether AH is true, or $\bar{A}H$ is true, or $\bar{H}$ is true, respectively. We recall that, given any conditional event $A|H$, the negation $\overline{A|H}$ is defined as $\overline{A|H} = \bar{A}|H$. The notion of logical inclusion among events has been generalized to conditional events by Goodman and Nguyen in [30]. Given two conditional events $A|H$ and $B|K$, we say that $A|H$ logically implies $B|K$, denoted by $A|H \subseteq B|K$, if and only if AH logically implies BK and $\bar{B}K$ logically implies $\bar{A}H$, that is

$$A|H \subseteq B|K \iff AH \subseteq BK \text{ and } \bar{B}K \subseteq \bar{A}H. \tag{1}$$

In the betting framework, to assess $P(A|H) = x$ amounts to say that, for every real number s, you are willing to pay an amount $s\,x$ and to receive s, or 0, or $s\,x$, according to whether AH is true, or $\bar{A}H$ is true, or $\bar{H}$ is true (bet called off), respectively. Hence, for the random gain $G = sH(A - x)$, the possible values are $s(1 - x)$, or $-s\,x$, or 0, according to whether AH is true, or $\bar{A}H$ is true, or $\bar{H}$ is true, respectively. We denote by X a *random quantity*, that is an uncertain real quantity, which has a well determined but unknown value. We assume that X has a finite set of possible values. Given any event $H \neq \varnothing$, agreeing to the betting metaphor, if you assess that the prevision of "X *conditional on* H" (or short: "X *given* H"), $\mathbb{P}(X|H)$, is equal to μ, this means that for any given real number s you are willing to pay an amount $s\mu$ and to receive sX, or $s\mu$, according to whether H is true, or false (bet called off), respectively. The random gain is $G = s(XH + \mu\bar{H}) - s\mu = sH(X - \mu)$. In particular, when X is (the indicator of) an event A, then $\mathbb{P}(X|H) = P(A|H)$. Given a conditional event $A|H$ with $P(A|H) = x$, the indicator of $A|H$, denoted by the same symbol, is

$$A|H = AH + x\bar{H} = AH + x(1 - H) = \begin{cases} 1, \text{ if } AH \text{ is true,} \\ 0, \text{ if } \bar{A}H \text{ is true,} \\ x, \text{ if } \bar{H} \text{ is true.} \end{cases} \tag{2}$$

Notice that it holds that $\mathbb{P}(AH + x\bar{H}) = xP(H) + xP(\bar{H}) = x$. The third value of the random quantity $A|H$ (subjectively) depends on the assessed probability $P(A|H) = x$. When $H \subseteq A$ (i.e., $AH = H$), it holds that $P(A|H) = 1$; then, for the indicator $A|H$ it holds that $A|H = AH + x\bar{H} = H + \bar{H} = 1$, (when $H \subseteq A$). Likewise, if $AH = \emptyset$, it holds that $P(A|H) = 0$; then $A|H = 0 + 0\bar{H} = 0$, (when $AH = \emptyset$). For the indicator of the negation of $A|H$ it holds that $\bar{A}|H = 1 - A|H$. Given two conditional events $A|H$ and $B|K$, for every coherent assessment (x, y) on $\{A|H, B|K\}$, it holds that ([28, formula (15)])

$$AH + x\bar{H} \leqslant BK + y\bar{K} \iff A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } K \subseteq B,$$

that is, between the numerical values of $A|H$ and $B|K$, under coherence it holds that

$$A|H \leqslant B|K \iff A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } K \subseteq B. \tag{3}$$

By following the approach given in [10,24,33], once a coherent assessment $\mu = \mathbb{P}(X|H)$ is specified, the conditional random quantity $X|H$ (is not looked at as the restriction to H, but) is defined as X, or μ, according to whether H is true, or $\bar{H}$ is true; that is,

$$X|H = XH + \mu\bar{H}. \tag{4}$$

As shown in (4), given any random quantity X and any event $H \neq \emptyset$, in the framework of subjective probability, in order to define $X|H$ we just need to specify the value μ of the conditional prevision $\mathbb{P}(X|H)$. Indeed, once the value μ is specified, the object $X|H$ is (subjectively) determined. We observe that (4) is consistent because

$$\mathbb{P}(XH+\mu\bar{H}) = \mathbb{P}(XH)+\mu P(\bar{H}) = \mathbb{P}(X|H)P(H)+\mu P(\bar{H}) = \mu P(H)+\mu P(\bar{H}) = \mu.$$

By (4), the random gain associated with a bet on $X|H$ can be represented as $G = s(X|H - \mu)$, that is G is the difference between what you receive, $sX|H$, and what you pay, $s\mu$. In what follows, for any given conditional random quantity $X|H$, we assume that, when H is true, the set of possible values of X is finite. In this case we say that $X|H$ is a finite conditional random quantity. Denoting by $\mathcal{X}_H = \{x_1, \ldots, x_r\}$ the set of possible values of X restricted to H and by setting $A_j = (X = x_j)$, $j = 1, \ldots, r$, it holds that $\bigvee_{j=1}^{r} A_j = H$ and $X|H = XH + \mu\bar{H} = x_1A_1 + \cdots + x_rA_r + \mu\bar{H}$. Given a prevision function $\mathbb{P}$ defined on an arbitrary family $\mathcal{K}$ of finite conditional random quantities, consider a finite subfamily $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\} \subseteq \mathcal{K}$ and the vector $\mathcal{M} = (\mu_1, \ldots, \mu_n)$, where $\mu_i = \mathbb{P}(X_i|H_i)$ is the assessed prevision for the conditional random quantity $X_i|H_i$, $i \in \{1, \ldots, n\}$. With the pair $(\mathcal{F}, \mathcal{M})$ we associate the random gain $G = \sum_{i=1}^{n} s_iH_i(X_i - \mu_i) = \sum_{i=1}^{n} s_i(X_i|H_i - \mu_i)$. We denote by $\mathcal{G}_{\mathcal{H}_n}$ the set of values of G restricted to $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$. Then, the notion of coherence is defined as below.

Definition 1. The function $\mathbb{P}$ defined on $\mathcal{K}$ is coherent if and only if, $\forall n \geqslant 1$, $\forall\, s_1, \ldots, s_n$, $\forall\, \mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\} \subseteq \mathcal{K}$, it holds that: *min* $\mathcal{G}_{\mathcal{H}_n} \leqslant 0 \leqslant$ *max* $\mathcal{G}_{\mathcal{H}_n}$.

In other words, $\mathbb{P}$ on $\mathcal{K}$ is incoherent if and only if there exists a finite combination of n bets such that, after discarding the case where all the bets are called off, the values of the random gain are all positive or all negative. In the particular case where $\mathcal{K}$ is a family of conditional events, then Definition 1 becomes the well known definition of coherence for a probability function, denoted as P.

Given a family $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$, for each $i \in \{1, \ldots, n\}$ we denote by $\{x_{i1}, \ldots, x_{ir_i}\}$ the set of possible values of X_i when H_i is true; then, we set $A_{ij} = (X_i = x_{ij})$, $i = 1, \ldots, n$, $j = 1, \ldots, r_i$. We set $C_0 = \bar{H}_1 \cdots \bar{H}_n$ (it may be $C_0 = \emptyset$) and we denote by $C_1, \ldots, C_m$ the constituents contained in $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$. Hence $\bigwedge_{i=1}^{n}(A_{i1} \vee \cdots \vee A_{ir_i} \vee \bar{H}_i) = \bigvee_{h=0}^{m} C_h$. With each C_h, $h \in \{1, \ldots, m\}$, we associate a vector $Q_h = (q_{h1}, \ldots, q_{hn})$, where $q_{hi} = x_{ij}$ if $C_h \subseteq A_{ij}$, $j = 1, \ldots, r_i$, while $q_{hi} = \mu_i$ if $C_h \subseteq \bar{H}_i$; with C_0 we associate $Q_0 = \mathcal{M} = (\mu_1, \ldots, \mu_n)$. Denoting by $\mathcal{I}$ the convex hull of $Q_1, \ldots, Q_m$, the condition $\mathcal{M} \in \mathcal{I}$ amounts to the existence of a vector $(\lambda_1, \ldots, \lambda_m)$ such that: $\sum_{h=1}^{m} \lambda_h Q_h = \mathcal{M}\,,\ \sum_{h=1}^{m} \lambda_h = 1\,,\ \lambda_h \geqslant 0\,,\ \forall\, h$; in other words, $\mathcal{M} \in \mathcal{I}$ is equivalent to the solvability of the system (Σ), associated with $(\mathcal{F}, \mathcal{M})$,

$$(\Sigma) \quad \textstyle\sum_{h=1}^{m} \lambda_h q_{hi} = \mu_i\,,\ i \in \{1, \ldots, n\}\,, \sum_{h=1}^{m} \lambda_h = 1,\ \ \lambda_h \geqslant 0\,,\ \ h \in \{1, \ldots, m\}\,. \quad (5)$$

Given the assessment $\mathcal{M} = (\mu_1, \ldots, \mu_n)$ on $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$, let S be the set of solutions $\Lambda = (\lambda_1, \ldots, \lambda_m)$ of system (Σ). We point out that the solvability of system (Σ) is a necessary (but not sufficient) condition for coherence of $\mathcal{M}$ on $\mathcal{F}$. When (Σ) is solvable, that is $S \neq \emptyset$, we define:

$$\begin{array}{l} \Phi_i(\Lambda) = \Phi_j(\lambda_1, \ldots, \lambda_m) = \sum_{r:C_r \subseteq H_i} \lambda_r\,;\ \Lambda \in S\,, M_i = \max_{\Lambda \in S}, \Phi_i(\Lambda),\ i \in \{1, \ldots, n\}\,; \\ I_0 = \{i\,:\, M_i = 0\}\,, \mathcal{F}_0 = \{X_i|H_i\,, i \in I_0\},\ \ \mathcal{M}_0 = (\mu_i,\, i \in I_0). \end{array} \quad (6)$$

For what concerns the probabilistic meaning of I_0, it holds that $i \in I_0$ if and only if the (unique) coherent extension of $\mathcal{M}$ to $H_i|\mathcal{H}_n$ is zero. Then, the following theorem can be proved ([3, Theorem 3]):

Theorem 1. A conditional prevision assessment $\mathcal{M} = (\mu_1, \ldots, \mu_n)$ on the family $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$ is coherent if and only if the following conditions are satisfied: (i) the system (Σ) defined in (5) is solvable; (ii) if $I_0 \neq \emptyset$, then $\mathcal{M}_0$ is coherent.

Of course, the previous results can be used in the case of conditional events. In particular, given a probability assessment $\mathcal{P} = (p_1, \ldots, p_n)$ on a family of n conditional events $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$, we can determine the constituents $C_0, C_1, \ldots, C_m$, where $C_0 = \bar{H}_1 \cdots \bar{H}_n$, and the associated points $Q_0, Q_1, \ldots, Q_m$, where $Q_0 = \mathcal{P}$. We observe that $Q_h = (q_{h1}, \ldots, q_{hn})$, with $q_{hi} \in \{1, 0, p_i\}$, $i = 1, \ldots, n$, $h = 1, \ldots, m$.

Trivalent Logics, Logical Operations of Conditionals and Conditional Random Quantities. We recall some notions of conjunction among conditional events in some trivalent logics: Kleene-Lukasiewicz-Heyting conjunction ($\wedge_K$), or de Finetti conjunction ([14]); Lukasiewicz conjunction ($\wedge_L$); Bochvar internal conjunction, or Kleene weak conjunction ($\wedge_B$); Sobocinski conjunction, or quasi conjunction ($\wedge_S$). In all these definitions the result of the conjunction is still a conditional event with set of truth values {true, false, void} (see, e.g., [8,9]). We also recall the notions of conjunction among conditional events, $\wedge_{gs}$, introduced as a suitable conditional random quantity in a betting-scheme context([24,25], see also [32,35]). We list below in an explicit way the five conjunctions and the associated disjunctions obtained by De Morgan's law ([22]):

1. $(A|H) \wedge_K (B|K) = AHBK|(HK \vee \bar{A}H \vee \bar{B}K)$,
   $(A|H) \vee_K (B|K) = (AH \vee BK)|(\bar{A}H\bar{B}K \vee AH \vee BK)$;
2. $(A|H) \wedge_L (B|K) = AHBK|(HK \vee \bar{A}\,\bar{B} \vee \bar{A}\,\bar{K} \vee \bar{B}\,\bar{H} \vee \bar{H}\,\bar{K})$,
   $(A|H) \vee_L (B|K) = (AH \vee BK)|(\bar{A}H\bar{B}K \vee AH \vee BK \vee \bar{H}\,\bar{K})$;
3. $(A|H) \wedge_B (B|K) = AHBK|HK$,
   $(A|H) \vee_B (B|K) = (A \vee B)|HK$;
4. $(A|H) \wedge_S (B|K) = ((AH \vee \bar{H}) \wedge (BK \vee \bar{K}))|(H \vee K)$,
   $(A|H) \vee_S (B|K) = (AH \vee BK)|(H \vee K)$;
5. $(A|H) \wedge_{gs} (B|K) = (AHBK + P(A|H)\bar{H}BK + P(B|K)AH\bar{K})|(H \vee K)$,
   $(A|H) \vee_{gs} (B|K) = (AH \vee BK + P(A|H)\bar{H}\bar{B}K + P(B|K)\bar{A}H\bar{K})|(H \vee K)$.

The operations above are all commutative and associative. By setting $P(A|H) = x$, $P(B|K) = y$, $P[(A|H) \wedge_i (B|K)] = z_i$, $i \in \{K, L, B, S\}$, and $\mathbb{P}[(A|H) \wedge_{gs} (B|K)] = z_{gs}$, based on (2) and on (4) the conjunctions $(A|H) \wedge_i (B|K)$, $i \in \{K, L, B, S, gs\}$ can be also looked at as random quantities with set of possible value illustrated in Table 1. A similar interpretation can also be given for the associated disjunctions. Notice that, differently from conditional events which

	$A\|H$	$B\|K$	$\wedge_K$	$\wedge_L$	$\wedge_B$	$\wedge_S$	$\wedge_{gs}$
$AHBK$	1	1	1	1	1	1	1
$AH\bar{B}K$	1	0	0	0	0	0	0
$AH\bar{K}$	1	y	z_K	z_L	z_B	1	y
$\bar{A}HBK$	0	1	0	0	0	0	0
$\bar{A}H\bar{B}K$	0	0	0	0	0	0	0
$\bar{A}H\,\bar{K}$	0	y	0	0	z_B	0	0
$\bar{H}BK$	x	1	z_K	z_L	z_B	1	x
$\bar{H}\,\bar{B}K$	x	0	0	0	z_B	0	0
$\bar{H}\,\bar{K}$	x	y	z_K	0	z_B	z_S	z_{gs}

Table 1. Numerical values (of the indicator) of the conjunctions $\wedge_i$, $i \in \{K, L, B, S, gs\}$. The triplet (x, y, z_i) denotes a coherent assessment on $\{A|H, B|K, (A|H) \wedge_i (B|K)\}$.

are three-valued objects, the conjunction $(A|H) \wedge_{gs} (B|K)$ (and the associated disjunction) is no longer a three-valued object, but a five-valued object with values in $[0, 1]$. In betting terms, the prevision $z_{gs} = \mathbb{P}[(A|H) \wedge_{gs} (B|K)]$ represents the amount you agree to pay, with the proviso that you will receive the random quantity $AHBK + x\bar{H}BK + yAH\bar{K}$, if $H \vee K$ is true, z_{gs} if $\bar{H}\,\bar{K}$ is true. In other words by paying z_{gs} you receive: 1, if both conditional events are true; 0, if at least one of the conditional event is false; the probability of the conditional event that is void if one conditional event is void and the other one is true; the amount z_{gs} you paid if both conditional events are void. The notion of conjunction $\wedge_{gs}$ (and disjunction $\vee_{gs}$) among conditional events has been generalized to the case of n conditional events in [25]. For some applications see, .e.g., [21,40,41]. Developments of this approach to general compound conditionals has been given in [20]. Differently from the other notions of conjunctions, $\wedge_{gs}$ preserves the classical logical and probabilistic properties valid for unconditional events (see, e.g.,[28]). In particular, the Fréchet-Hoeffding bounds, i.e., the lower and upper bounds $z' = \max\{x+y-1,0\}$, $z'' = \min\{x,y\}$, obtained under logical independence in the unconditional case for the coherent extensions $z = P(AB)$ of $P(A) = x$ and $P(B) = y$, when A and B are replaced by $A|H$ and $B|K$, are only satisfied by z_{gs} (see Table 2).

	$\wedge_K$	$\wedge_L$	$\wedge_B$	$\wedge_S$	$\wedge_{gs}$
z'	0	0	0	$\max\{x+y-1,0\}$	$\max\{x+y-1,0\}$
z''	$\min\{x,y\}$	$\min\{x,y\}$	1	$\begin{cases} \frac{x+y-2xy}{1-xy}, & \text{if } (x,y) \neq (1,1) \\ 1, & \text{if } (x,y) = (1,1) \end{cases}$	$\min\{x,y\}$

Table 2. Lower and upper bounds z', z'' for the selected conjunctions $\wedge_K, \wedge_L, \wedge_B, \wedge_S, \wedge_{gs}$, for the given assessment $x = P(A|H)$ and $y = P(B|K)$ ([39]).

3 Some basic properties and iterated conditionals

We recall some basic logical and probabilistic properties satisfied by events and conditional events. Notice that, from (2), $B|A = AB + P(B|A)\bar{A}$.
(P1) $B|A = AB|A$;
(P2) $AB \subseteq B|A$, and $P(AB) \leqslant P(B|A)$;
(P3) $P(AB) = P(B|A)P(A)$ (*compound probability theorem*);
(P4) given two logical independent events A, B, with $P(A) = x$ and $P(B) = y$, the extension $\mu = P(B|A)$ is coherent if and only if $\mu \in [\mu', \mu'']$, where (see, e.g. [41, Theorem 6])

$$\mu' = \begin{cases} \frac{\max\{x+y-1,0\}}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} , \quad \mu'' = \begin{cases} \frac{\min\{x,y\}}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases} \tag{7}$$

We will check the validity of the properties above when replacing events A, B by conditional events $A|H, B|K$ for the notion of iterated conditional introduced in the trivalent logics by Calabrese and by de Finetti, respectively.

3.1 The Iterated conditional of Calabrese

We analyze the iterated conditional, here denoted by $(B|K)|_C(A|H)$, introduced by Calabrese in [5] (see also [6,7]).

Definition 2. *Given any pair of conditional events $A|H$ and $B|K$, the iterated conditional $(B|K)|_C(A|H)$ is defined as*

$$(B|K)|_C(A|H) = B|(K \wedge (\bar{H} \vee A)). \tag{8}$$

We observe that in (8) the conditioning event is the conjunction of the conditioning event K of the consequent $B|K$ and the material conditional $\bar{H} \vee A$ associated with the antecedent $A|H$. By applying Definition 2 with $H = \Omega$, it holds that $(B|K)|_C A = ABK|AK = B|AK$, which shows that the Import-Export principle (i.e., $(B|K)|A = B|AK$, see, e.g., [35]) is satisfied by the iterated conditional $|_C$.

By recalling that the notions of conjunction and disjunction of conditionals used by Calabrese ([5]) coincide with $\wedge_S$ and $\vee_S$, respectively, we observe that

$$[(A|H) \wedge_S (B|K)]|_C(A|H) = [(AHBK \vee AH\bar{K} \vee \bar{H}BK)|(H \vee K)]|_C(A|H) = \\ = (AHBK \vee AH\bar{K} \vee \bar{H}BK)|(AK \vee \bar{H}K \vee AH\bar{K}). \tag{9}$$

From (8) and (9) it follows that $(B|K)|_C(A|H) \neq ((A|H) \wedge_S (B|K))|_C(A|H)$. Indeed, as illustrated by Table 3, when the constituent $AH\bar{K}$ is true, it holds that $(B|K)|_C(A|H)$ is void, while $[(A|H) \wedge_S (B|K)]|_C(A|H)$ is true. Then, property (P1) is not satisfied by the pair $(\wedge_S, |_C)$. From Table 3 we also obtain that

C_h	$(A\|H) \wedge_S (B\|K)$	$(B\|K)\|_C(A\|H)$	$[(A\|H) \wedge_S (B\|K)]\|_C(A\|H)$
$AHBK \vee \bar{H}BK$	True	True	True
$AH\bar{B}K \vee \bar{H}\bar{B}K$	False	False	False
$AH\bar{K}$	True	Void	True
$\bar{A}H$	False	Void	Void
$\bar{H}\,\bar{K}$	Void	Void	Void

Table 3. Truth values of $(A|H) \wedge_S (B|K)$, $(B|K)|_C(A|H)$, and $[(A|H) \wedge_S (B|K)]|_C(A|H)$.

property (P2) is not satisfied by $(\wedge_S, |_C)$. Indeed, when $AH\bar{K}$ is true, it holds that $(A|H) \wedge_S (B|K)$ is true, while $(B|K)|_C(A|H)$ is void and hence $((A|H) \wedge_S (B|K)) \nsubseteq (B|K)|_C(A|H)$.

Now let us focus our attention on the following results regarding the coherence of a probability assessment on $\{A|H, (B|K)|_C(A|H), (A|H) \wedge_S (B|K)\}$ (Theorem 2) and on $\{A|H, B|K, (B|K)|_C(A|H)\}$ (Theorem 3).

Theorem 2. *Let A, B, H, K be any logically independent events. A probability assessment $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, (B|K)|_C(A|H), (A|H) \wedge_S (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where $z' = xy$ and $z'' = max(x, y)$.*

Proof. Due to lack of space we illustrate the proof only for the lower bound $z' = xy$. The constituents C_h's and the points Q_h's associated with the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F}$ are $C_1 = AHBK, C_2 = \bar{A}H, C_3 = \bar{H}BK, C_4 = AH\bar{B}K, C_5 = \bar{H}\bar{B}K, C_6 = AH\bar{K}, C_0 = \bar{H}\,\bar{K}$ and $Q_1 = (1, 1, 1)$, $Q_2 = (0, y, 0)$, $Q_3 = (x, 1, 1), Q_4 = (1, 0, 0), Q_5 = (x, 0, 0)$, $Q_6 = (1, y, 1)$, $\mathcal{P} = Q_0 = (x, y, z)$. The system (Σ) in (5) associated with the pair $(\mathcal{F}, \mathcal{P})$ becomes

$$\begin{cases} \lambda_1 + x\lambda_3 + \lambda_4 + x\lambda_5 + \lambda_6 = x, \ \lambda_1 + y\lambda_2 + \lambda_3 + y\lambda_6 = y, \\ \lambda_1 + \lambda_3 + \lambda_6 = z, \lambda_1 + \cdots + \lambda_6 = 1, \ \lambda_i \geqslant 0 \ \forall i = 1, \ldots, 6. \end{cases} \tag{10}$$

We observe that, for every $(x,y) \in [0,1]^2$, it holds that $\mathcal{P} = (x,y,xy) = xyQ_1 + (1-x)Q_2 + x(1-y)Q_4$. Then, $\mathcal{P} \in \mathcal{I}$, where $\mathcal{I}$ is the convex hull of $Q_1, \cdots Q_6$, with a solution of (10) given by $\Lambda = (xy, 1-x, 0, x(1-y), 0, 0)$. For the functions ϕ_j, as defined in (6), it holds that $\phi_1(\Lambda) = \sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_6 = xy + (1-x) + x(1-y) = 1 > 0$, $\phi_2(\Lambda) = \sum_{h:C_h \subseteq (A \vee \overline{H}) \vee K} \lambda_h = x$, $\phi_3(\Lambda) = \sum_{h:C_h \subseteq H \vee K} \lambda_h = 1 > 0$. We distinguish two cases: (i) $x > 0$, (ii) $x = 0$. In the case (i) we get $\phi_1 = \phi_3 = 1 > 0$ and $\phi_2 > 0$; then $\mathcal{I}_0 = \emptyset$. By Theorem 1, the assessment (x,y,xy) is coherent $\forall (x,y) \in [0,1]^2$. In the case (ii) we get $\mathcal{I}_0 \subseteq \{2\}$, with the sub-assessment $\mathcal{P}_0 = y$ on $\mathcal{F}_0 = \{(B|K)|_C(A|H)\}$ coherent for every $y \in [0,1]$. Then, by Theorem 1, the assessment (x,y,xy) on $\mathcal{F}$ is coherent for every $(x,y) \in [0,1]^2$. In order to prove that $z' = xy$ is the lower bound for $z = P((A|H) \wedge_S (B|K))$, we verify that (x,y,z), with $(x,y) \in [0,1]^2$ and $z < z' = xy$, is not coherent because $(x,y,z) \notin \mathcal{I}$. We observe that the points Q_1, Q_2, Q_4 belong to the plane $\pi : yX + Y - Z = y$. We set $f(X,Y,Z) = yX + Y - Z$ and we obtain $f(Q_1) = f(Q_2) = f(Q_4) = y$, $f(Q_3) = f(Q_5) = xy \leqslant y$, $f(Q_6) = f(1,y,1) = y+y-1 = 2y-1 \leqslant y$. Then, by considering $\mathcal{P} = (x,y,z)$, with $z < xy$, it holds that $f(\mathcal{P}) = f(x,y,z) = xy+y-z > y \geqslant f(Q_h)$, $h = 1, \ldots, 6$, and hence $\mathcal{P} = (x,y,z) \notin \mathcal{I}$. Indeed, if it were $\mathcal{P} \in \mathcal{I}$, that is $\mathcal{P}$ linear convex combination of $Q_1, \ldots, Q_6$, it would follow that $f(\mathcal{P}) = f(\sum_{h=1}^{6} \lambda_h Q_h) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leqslant y$. Thus, the lower bound for $z = P((A|H) \wedge_S (B|K))$ is $z' = xy$. □

From Theorem 2 any probability assessment (x,y,z) on $\mathcal{F} = \{A|H, (B|K)|_C(A|H), (A|H) \wedge_S (B|K)\}$, with $(x,y) \in [0,1]^2$ and $xy \leqslant z \leqslant max(x,y)$ is coherent. Thus, as $z = xy$ is not the unique coherent extension of the conjunction $(A|H) \wedge_S (B|K)$, in general the quantity $P[(A|H) \wedge_S (B|K)]$ do not coincide with the product $P[(B|K)|_C(A|H)]P(A|H)$. For example, it could be that $P[(B|K)|_C(A|H)]P(A|H) = 0 < P[(A|H) \wedge_S (B|K)] = 1$, because the assessment $(1,0,1)$ on $\mathcal{F}$ is coherent (while it is not coherent on $\{A, B|A, AB\}$). Then, property (P3) is not satisfied by the pair $(\wedge_S, |_C)$.

Theorem 3. *Let A, B, H, K be any logically independent events. The probability assessments $\mathcal{P} = (x,y,z)$ on the family of conditional events $\mathcal{F} = \{A|H, B|K, (B|K)|_C(A|H)\}$ is coherent for every $(x,y,z) \in [0,1]^3$.*

Proof. The proof is omitted due to lack of space.

We observe that the probability propagation rule valid for unconditional events (property (P4)) is no longer valid for Calabrese's iterated conditional. Indeed, from Theorem 3, any probability assessment (x,y,z) on $\mathcal{F} = \{A|H, B|K, (B|K)|_C(A|H)\}$, with $(x,y,z) \in [0,1]^3$ is coherent. For instance, the assessment $(1,1,0)$ on $\mathcal{F}$ is coherent, while it is not coherent on $\{A, B, B|A\}$.

3.2 The Iterated conditional of de Finetti.

We now analyze the iterated conditional introduced by de Finetti in [14].

Definition 3. *Given any pair of conditional events $A|H$ and $B|K$, de Finetti iterated conditional, denoted by $(B|K)|_{df}(A|H)$, is defined as*

$$(B|K)|_{df}(A|H) = B|(AHK). \tag{11}$$

By applying Definition 3 with $H = \Omega$, it holds that $(B|K)|_{df}A = B|AK$, which shows that the Import-Export principle [35] is satisfied by $|_{df}$. We recall that the notion of conjunction and disjunction of conditionals introduced by de Finetti in [14] coincide with $\wedge_K$ and $\vee_K$ recalled in Section 2. From (11) it holds that

$$\begin{array}{l}[(A|H) \wedge_K (B|K)]|_{df}(A|H) = [AHBK|(HK \vee \bar{A}H \vee \bar{B}K)]|_{df}(A|H) = \\ AHBK|(AHK \vee AH\bar{B}K) = AHBK|AHK = (B|K)|_{df}(A|H).\end{array} \tag{12}$$

Then, property (P1) is satisfied by the pair $(\wedge_K, |_{df})$ (see also Table 4). From Table 4 we also observe that relation (P2) is satisfied by $(\wedge_K, |_{df})$. Indeed, according to (1), if $(A|H) \wedge_K (B|K)$ is true, then $(B|K)|_{df}(A|H)$ is true; if $(B|K)|_{df}(A|H)$ is false, then $(A|H) \wedge_K (B|K)$ is false. We consider now the following results regarding the coherence of a probability assessment on $\{A|H, (B|K)|_{df}(A|H), (A|H) \wedge_K (B|K)\}$ (Theorem 4) and on $\{A|H, B|K, (B|K)|_{df}(A|H)\}$ (Theorem 5).

C_h	$(A\|H) \wedge_K (B\|K)$	$(B\|K)\|_{df}(A\|H)$	$[(A\|H) \wedge_K (B\|K)]\|_{df}(A\|H)$
$AHBK$	True	True	True
$AH\bar{B}K$	False	False	False
$AH\bar{K} \vee \bar{H}BK \vee \bar{H}\,\bar{K}$	Void	Void	Void
$\bar{A}H \vee \bar{H}\bar{B}K$	False	Void	Void

Table 4. Truth table of $(A|H) \wedge_K (B|K)$, $(B|K)|_{df}(A|H)$, and $[(A|H) \wedge_K (B|K)]|_{df}(A|H)$.

Theorem 4. *Let A, B, H, K be any logically independent events. A probability assessment $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, (B|K)|_{df}(A|H), (A|H) \wedge_K (B|K)\}$ is coherent if and only if $(x, y) \in [0, 1]^2$ and $z \in [z', z'']$, where $z' = 0$ and $z'' = xy$.*

Proof. The proof is omitted due to lack of space.

From Theorem 4 any probability assessment (x, y, z) on $\mathcal{F} = \{A|H, (B|K)|_{df}(A|H), (A|H) \wedge_K (B|K)\}$, with $(x, y) \in [0, 1]^2$ and $0 \leqslant z \leqslant xy$, is coherent. Thus, as $z = xy$ is not the unique coherent extension of the conjunction $(A|H) \wedge_K (B|K)$, the quantity $P[(A|H) \wedge_K (B|K)]$ could not coincide with the product $P[(B|K)|_{df}(A|H)]P(A|H)$. For example, if we choose the probability assessment $\mathcal{P} = (1, 1, 0)$, it is coherent on $\mathcal{F}$ but not on $\{A, B|A, AB\}$ because $P[(A|H) \wedge_K (B|K)] = 0 < P[(B|K)|_{df}(A|H)]P(A|H) = 1$.
Then, property (P3) is not satisfied by the pair $(\wedge_K, |_{df})$.

Theorem 5. *Let A, B, H, K be any logically independent events. The probability assessments $\mathcal{P} = (x, y, z)$ on the family of conditional events $\mathcal{F} = \{A|H, B|K, (B|K)|_{df}(A|H)\}$ is coherent for every $(x, y, z) \in [0, 1]^3$.*

Proof. The proof is omitted due to lack of space.

We observe that the probability propagation rule valid for unconditional events (P4) is no longer valid for de Finetti's iterated conditional. Indeed, from Theorem 5, any probability assessment (x, y, z) on $\mathcal{F} = \{A|H, B|K, (B|K)|_{df}(A|H)\}$, with $(x, y, z) \in [0, 1]^3$ is coherent. For instance, the assessment $(1, 1, 0)$ is coherent on $\mathcal{F}$ but it is not coherent on $\{A, B, B|A\}$.

4 Iterated conditionals and compound prevision theorem

In [23] (see also [40]), by using the structure

$$\Box|\bigcirc = \Box \wedge \bigcirc + \mathbb{P}(\Box|\bigcirc)\overline{\bigcirc}, \tag{13}$$

which reduces to formula (2) when $\Box = A, \bigcirc = H$, given two conditional events $A|H$, $B|K$, with $AH \neq \emptyset$, the iterated conditional $(B|K)|_{gs}(A|H)$ has been defined as the following conditional random quantity

$$(B|K)|_{gs}(A|H) = (A|H) \wedge_{gs} (B|K) + \mu_{gs}(\bar{A}|H). \tag{14}$$

We now examine the different definitions of iterated conditional (see Table 5), beyond $|_{gs}$, we can obtain by using the structure (13) and each conjunction: $\wedge_K, \wedge_L, \wedge_B, \wedge_S$.

Definition 4. *Given two conditional events $A|H$, $B|K$, with $AH \neq \emptyset$, for each $i \in \{K, L, B, S\}$, we define the iterated conditional $(B|K)|_i(A|H)$ as*

$$(B|K)|_i(A|H) = (A|H) \wedge_i (B|K) + \mu_i(\bar{A}|H), \tag{15}$$

where $\mu_i = \mathbb{P}[(B|K)|_i(A|H)]$.

Remark 1. We remind that, in agreement with [1,32] and differently from [35], for the iterated conditional $(B|K)|_{gs}(A|H)$ the Import-Export principle is not valid. As a consequence, as shown in [24] (see also [40,41]), Lewis' triviality results ([34]) are avoided by $|_{gs}$. It can be easily proved that $(B|K)|_iA \neq B|AK$, $i \in \{K, L, B, S\}$. Then, the Import-Export principle is not satisfied by any of the iterated conditional $|_K, |_L, |_B, |_S, |_{gs}$.

For each pair $(\wedge_i, |_i)$, $i \in \{K, L, B, S, gs\}$, we show the validity of properties (P1)–(P3) introduced in Section 3, where the events A, B are replaced by the conditional events $A|H$, $B|K$, respectively. Then, we discuss the validity of generalized versions of Bayes's Theorem and the validity of property (P4).
(P1). We recall that the pair $(\wedge_{gs}, |_{gs})$ satisfies property (P1) because $((A|H) \wedge_{gs} (B|K))|_{gs}(A|H) = (B|K)|_{gs}(A|H)$ ([29, Theorem 5]). Moreover, each pair $(\wedge_i, |_i)$, $i \in \{K, L, B, S\}$, also satisfies property (P1) as shown by the following result.

Theorem 6. *Given two conditional events $A|H$, $B|K$, with $AH \neq \emptyset$, it holds that*

$$((A|H) \wedge_i (B|K))|_i(A|H) = (B|K)|_i(A|H), \quad i \in \{K, L, B, S\}. \tag{16}$$

Proof. Let be given $i \in \{K, L, B, S\}$. We set $\mu_i = \mathbb{P}[(B|K)|_i(A|H)]$ and $\nu_i = \mathbb{P}[((A|H) \wedge (B|K))|_i(A|H)]$. By Definition 4, as $(A|H) \wedge_i (A|H) \wedge_i (B|K) = (A|H) \wedge_i (B|K)$, it holds that

$$((A|H) \wedge_i (B|K))|_i(A|H) = (A|H) \wedge_i (B|K) + \nu_i(\bar{A}|H). \tag{17}$$

From (15) and (17), in order to prove (16) it is enough to verify that $\nu_i = \mu_i$. We observe that $((A|H) \wedge_i (B|K))|_i(A|H) - (B|K)|_i(A|H) = (\nu_i - \mu_i)(\bar{A}|H)$, where $\nu_i - \mu_i = \mathbb{P}[((A|H) \wedge_i (B|K))|_i(A|H) - (B|K)|_i(A|H)]$. By setting $P(A|H) = x$, it holds that

$$(\nu_i - \mu_i)(\bar{A}|H) = \begin{cases} 0, \text{ if } A|H = 1, \\ \nu_i - \mu_i, \text{ if } A|H = 0, \\ (\nu_i - \mu_i)(1 - x), \text{ if } A|H = x, \ 0 < x < 1. \end{cases}$$

Notice that, in the betting scheme, $\nu_i - \mu_i$ is the amount to be paid in order to receive the random amount $(\nu_i - \mu_i)(\bar{A}|H)$. Then, by coherence, $\nu_i - \mu_i$ must be a linear convex combination of the possible values of $(\nu_i - \mu_i)(\bar{A}|H)$, by discarding the cases where the bet called off, that is the cases where you receive back the paid amount $\nu_i - \mu_i$, whatever $\nu_i - \mu_i$ be. In other words, coherence requires that $\nu_i - \mu_i$ must belong to the convex hull of the set $\{0, (\nu_i - \mu_i)(1 - x)\}$, that is $\nu_i - \mu_i = \alpha \cdot 0 + (1 - \alpha)(\nu_i - \mu_i)(1 - x)$, for some $\alpha \in [0, 1]$. Then, as $0 < x < 1$, we observe that the previous equality holds if and only if $\nu_i - \mu_i = 0$, that is $\nu_i = \mu_i$. Therefore, equality (16) holds. □

(P2). Coherence requires that $\mu_i \geqslant 0$, $i \in \{K, L, B, S, gs\}$. Then, from (15) it holds that $(A|H) \wedge_i (B|K) \leqslant (B|K)|_i(A|H)$, $i \in \{K, L, B, S, gs\}$ and hence $P[(A|H) \wedge_i (B|K)] \leqslant \mathbb{P}[(B|K)|_i(A|H)]$, $i \in \{K, L, B, S, gs\}$. Therefore, each pair $(\wedge_i, |_i)$, $i \in \{K, L, B, S, gs\}$ satisfies the numerical counterpart of (P2), where, based on (3), the symbol $\subseteq$ is replaced by $\leqslant$.

(P3). We recall that the pair $(\wedge_{gs}, |_{gs})$ satisfies (P3) because, by exploiting the

	$(B\|K)\|_K(A\|H)$	$(B\|K)\|_L(A\|H)$	$(B\|K)\|_B(A\|H)$	$(B\|K)\|_S(A\|H)$	$(B\|K)\|_{gs}(A\|H)$
$AHBK$	1	1	1	1	1
$AH\bar{B}K$	0	0	0	0	0
$AH\bar{K}$	$x\mu_K$	$x\mu_L$	$x\mu_B$	1	y
$\bar{A}HK$	μ_K	μ_L	μ_B	μ_S	μ_{gs}
$\bar{A}H\bar{K}$	μ_K	μ_L	$\mu_B(1+x)$	μ_S	μ_{gs}
$\bar{H}BK$	μ_K	μ_L	μ_B	$1+\mu_S(1-x)$	$x+\mu_{gs}(1-x)$
$\bar{H}\bar{B}K$	$\mu_K(1-x)$	$\mu_L(1-x)$	μ_B	$\mu_S(1-x)$	$\mu_{gs}(1-x)$
$\bar{H}\bar{K}$	μ_K	$\mu_L(1-x)$	μ_B	μ_S	μ_{gs}

Table 5. Numerical values of $(B|K)|_i(A|H)$, $i \in \{K, L, B, S, gs\}$. We denotes $x = P(A|H)$, $y = P(B|K)$, and $\mu_i = \mathbb{P}[(B|K)|_i(A|H)]$, $i \in \{K, L, B, S, gs\}$.

structure (13), it holds that $\mathbb{P}[(A|H) \wedge_{gs} (B|K)] = \mathbb{P}[(B|K)|_{gs}(A|H)]P(A|H)$ ([23]). Concerning the pairs $(\wedge_i, |_i)$, $i \in \{K, L, B, S\}$ we show below that (P3) is also valid. Indeed, for the linearity of prevision, from (15), we obtain that

$$\begin{aligned} \mu_i = \mathbb{P}[(B|K)|_i(A|H)] &= \mathbb{P}[(B|K) \wedge_i (A|H)] + \mu_i \mathbb{P}(\bar{A}|H) = \\ &= \mathbb{P}[(B|K) \wedge (A|H)] + \mu_i\, P(\bar{A}|H) = z_i + \mu_i(1-x)\,, \end{aligned} \tag{18}$$

where $z_i = \mathbb{P}((A|H) \wedge_i (B|K))$, $i \in \{K, L, B, S\}$. As $\mu_i = z_i + \mu_i(1-x)$, it follows that $z_i = \mu_i x$, for $i \in \{K, L, B, S\}$. In other words, coherence requires that

$$\mathbb{P}[(A|H) \wedge_i (B|K)] = \mathbb{P}[(B|K)|_i(A|H)]P(A|H), \;\; i \in \{K, L, B, S\}, \tag{19}$$

which states that the compound prevision theorem (property (P3)) is valid for each pair $(\wedge_i, |_i)$, $i \in \{K, L, B, S\}$.

Remark 2. By exploiting the compound prevision theorem, we analyze generalized versions of Bayes' Theorem for the iterated conditionals $|_K, |_L, |_B, |_S, |_{gs}$. As $\mathbb{P}[(B|K) \wedge_i (A|H)] = \mathbb{P}[(B|K)|_i(A|H)]P(A|H) = \mathbb{P}[(A|H)|_i(B|K)]P(B|K)$, $i \in \{K, L, B, S, gs\}$, when $P(A|H) > 0$ it holds that

$$\mathbb{P}[(B|K)|_i(A|H)] = \frac{\mathbb{P}[(A|H)|_i(B|K)]P(B|K)}{P(A|H)},\; i \in \{K, L, B, S, gs\}, \tag{20}$$

which generalizes the Bayes's formula $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$. We now analyze the validity of the generalization of Bayes's Theorem given in the following version: $P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B)+P(A|\bar{B})P(\bar{B})}$. We recall that, given two events A and B, it holds that $A = AB \vee A\bar{B}$, and hence $P(A) = P(AB) + P(A\bar{B}) = P(A|B)P(B) + P(A|\bar{B})P(B)$. However, when A, B are replaced by the conditional events $A|H, B|K$, respectively, we obtain that (see [22])

- $[(A|H) \wedge_K (B|K)] \vee_K [(A|H) \wedge_K (\bar{B}|K)] = AHK|(AHK \vee \bar{A}H) \neq A|H$;
- $[(A|H) \wedge_L (B|K)] \vee_L [(A|H) \wedge_L (\bar{B}|K)] = AHK|(H \vee \bar{K}) \neq A|H$;
- $[(A|H) \wedge_B (B|K)] \vee_B [(A|H) \wedge_B (\bar{B}|K)] = A|(HK) \neq A|H$;
- $[(A|H) \wedge_S (B|K)] \vee_S [(A|H) \wedge_S (\bar{B}|K)] = (A \vee \bar{H})|(H \vee K) \neq A|H$;
- $[(A|H) \wedge_{gs} (B|K)] \vee_{gs} [(A|H) \wedge_{gs} (\bar{B}|K)] = A|H$.

Then, for each $i \in \{K, L, B, S\}$, $P(A|H)$ cannot be decomposed as $\mathbb{P}((A|H) \wedge_i (B|K)) + \mathbb{P}((A|H) \wedge_i (\bar{B}|K)) = \mathbb{P}((A|H)|_i(B|K))P(B|K) + \mathbb{P}((A|H)|_i(\bar{B}|K))P(\bar{B}|K)$, while $P(A|H) = \mathbb{P}((A|H) \wedge_{gs} (B|K)) + \mathbb{P}((A|H) \wedge_{gs} (\bar{B}|K)) = \mathbb{P}((A|H)|_{gs}(B|K))P(B|K) + \mathbb{P}((A|H)|_{gs}(\bar{B}|K))P(\bar{B}|K)$. Hence, for each $i \in \{K, L, B, S\}$, $\mathbb{P}[(B|K)|_i(A|H)]$ does not in general coincide with $\mathbb{P}((A|H)|_i(B|K))P(B|K)/(\mathbb{P}((A|H)|_i(B|K))P(B|K) + \mathbb{P}((A|H)|_i(\bar{B}|K))P(\bar{B}|K))$, while

$$\mathbb{P}[(B|K)|_{gs}(A|H)] = \frac{\mathbb{P}((A|H)|_{gs}(B|K))P(B|K)}{\mathbb{P}((A|H)|_{gs}(B|K))P(B|K) + \mathbb{P}((A|H)|_{gs}(\bar{B}|K))P(\bar{B}|K)}. \tag{21}$$

Therefore, the generalization of the second version of Bayes' formula only holds for $|_{gs}$ and does not hold for $|_K, |_L, |_B$, and $|_S$.

(P4). Concerning the pair $(\wedge_K, |_K)$, we have the following result

Theorem 7. *Let A, B, H, K be any logically independent events. The set Π of all the coherent assessment (x, y, z, μ) on the family $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_K (B|K), (B|K)|_K(A|H)\}$ is $\Pi = \Pi' \cup \Pi''$, where $\Pi' = \{(x, y, z, \mu) : x \in (0, 1], y \in [0, 1], z \in [z', z''], \mu = \frac{z}{x}\}$ with $z' = 0$, $z'' = min\{x, y\}$, and $\Pi'' = \{(0, y, 0, \mu) : (y, \mu) \in [0, 1]^2\}$.*

Proof. The proof is omitted due to lack of space.

Based on Theorem 7, as the assessment $(1, 1, 0, 0)$ on $\{A|H, B|K, (A|H) \wedge_K (B|K), (B|K)|_K(A|H)\}$ is coherent, it follows that the sub-assessment $(1, 1, 0)$ on $\{A|H, B|K, (B|K)|_K(A|H)\}$ is coherent too. However, the assessment $(1, 1, 0)$ on $\{A, B, B|A\}$ is not coherent because by (7) it holds that $0 = \mu < \mu' = \frac{\max\{1+1-1,0\}}{1} = 1$. Then, formula (P4) is not satisfied by $|_K$.
Concerning the pair $(\wedge_L, |_L)$ it can be easily shown that statement of Theorem 7 also holds when $\wedge_K, |_K$ are replaced by $\wedge_L, |_L$, respectively. Then, also $|_L$ does not satisfy property (P4).
We now focus on the pair $(\wedge_B, |_B)$. We recall that the assessment $(x, y, 1)$ on $\{A|H, B|K, (A|H) \wedge_B (B|K)\}$ is coherent for every $(x, y) \in [0, 1]^2$ (see Table 2). Then, when $0 < x < 1$, the extension $\mu = \mathbb{P}[(B|K)|_B(A|H)] = \frac{\mathbb{P}[(A|H) \wedge_B (B|K)]}{P(A|H)} = \frac{1}{x} > 1$ is coherent and hence property (P4) is not satisfied by $|_B$, because by (7) it holds that $\mu > 1 \geqslant \mu''$.
Likewise, we observe that the assessment $(x, 1, 1)$ on $\{A|H, B|K, (A|H) \wedge_S (B|K)\}$ is coherent for every $x \in [0, 1]$ (see Table 2). Then, when $0 < x < 1$, as $\mathbb{P}[(B|K)|_S(A|H)] = \frac{1}{x}$, the extension $\mu = \frac{1}{x} > 1$ on $(B|K)|_S(A|H)$ is coherent. That is, it is coherent to assess $\mathbb{P}[(B|K)|_S(A|H)] > 1$ and hence property (P4) is not satisfied by $|_S$.
Finally, differently from the other iterated conditionals, we recall that $|_{gs}$ satisfies (P4) ([41, Theorem 4]). Indeed, given a coherent assessment (x, y) on $\{A|H, B|K\}$, under logical independence, for the iterated conditional $(B|K)|_{gs}(A|H)$ the extension $\mu = \mathbb{P}((B|K)|_{gs}(A|H))$ is coherent if and only if $\mu \in [\mu', \mu'']$, where $\mu' = \begin{cases} \frac{\max\{x+y-1,0\}}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$, $\mu'' = \begin{cases} \frac{\min\{x,y\}}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$

5 Conclusions

We recalled some trivalent logics (Kleene-Lukasiewicz-Heyting-de Finetti, Lukasiewicz, Bochvar-Kleene, and Sobocinski) and the notion of compound conditional as conditional random quantity. We considered some basic logical and probabilistic properties, valid for events and conditional events, by checking their validity for selected notions of iterated conditional. In particular, we studied the iterated conditionals introduced in trivalent logics by Calabrese and by de Finetti, by also focusing on the numerical representation of the truth-values. For both the iterated conditionals we computed the lower and upper bounds and we showed that some basic properties are not satisfied. Then, for each trivalent logic, we introduced the iterated conditional $(|_K, |_L, |_B, |_S)$ defined by exploiting the same structure used in order to define $|_{gs}$, that is as a suitable random quantity which satisfies the compound prevision theorem. We observed that all the basic properties are satisfied only by the iterated conditional $|_{gs}$. Future work will concern the deepening of other logical and probabilistic properties of the iterated conditionals $|_K, |_L, |_B$, and $|_S$ in the framework of nonmonotonic reasoning, Boolean algebras of conditionals ([18,19]), and in other non-classical logics, like connexive logic ([38]).

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