POSITIVE SOLUTIONS FOR SINGULAR $(p, 2)$-EQUATIONS

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Abstract. We consider a nonlinear nonparametric Dirichlet problem driven by the sum of a p-Laplacian and of a Laplacian (a $(p, 2)$-equation) and a reaction which involves a singular term and a $(p-1)$-superlinear perturbation. Using variational tools and suitable truncation and comparison techniques, we show that the problem has two positive smooth solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear, nonparametric singular Dirichlet problem:

$$(1) \qquad -\Delta_p u(z) - \Delta u(z) = \mu(u(z)) + f(z, u(z)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0,\ 2 < p,\ u > 0,$$

where Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

When $p = 2$, we have the usual Laplacian denoted by Δ. So, in problem (1) we have a combination of two differential operators of different nature. Such situations arise in mathematical models of physical processes. We mention the works of Cherfils-Il′yasov [3] (reaction-diffusion systems) and Zhikov [22] (homogenization of composites made of two different materials with distinct hardening exponents, double phase problems). Equation (1), having as differential operator the sum of a p-Laplacian and a Laplacian, is called a "$(p, 2)$-equation". The differential operator is nonhomogeneous and this is a source of difficulties in the analysis of problem (1). In the reaction (right hand side of (1)), the function $\mu(\cdot)$ is singular at $x = 0$. In the literature we encounter the particular case $\mu(x) = x^{-\gamma}$ for $x > 0$. The perturbation $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.e. $z \in \Omega$, $x \to f(z, x)$ is continuous). We assume that for a.e. $z \in \Omega$, $f(z, \cdot)$ exhibits $(p-1)$-superlinear growth near $+\infty$. However, we do not employ the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). Using variational methods based on the critical point theory, together with suitable truncation techniques, we show the existence of a pair of positive smooth solutions.

Our work here is closely related to the recent paper of Papageorgiou-Rădulescu-Repovš [14], where the authors deal with a Dirichlet problem driven by the p-Laplacian and a reaction of the form $x^{-\gamma} + f(z, x)$. In that paper $f(z, \cdot)$ is a Carathéodory perturbation which is $(p-1)$-linear and possibly resonant at $+\infty$. In the literature most nonlinear singular problems studied are parametric with the parameter $\lambda > 0$ multiplying either the singular term or the perturbation f. The presence of the parameter, gives more flexibility in the equation and by varying the parameter, one can produce useful bounds

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and growth estimates and satisfy the geometry of various minimax theorems. Such parametric nonlinear singular problems were considered by Byun-Ko [2], Giacomoni-Schindler-Takáč [7], Papageorgiou-Rădulescu-Repovš [15], Papageorgiou-Smyrlis [17], Papageorgiou-Vetro-Vetro [18], Papageorgiou-Winkert [19], Saudi [21]. Finally we mention that double phase problems (but without singularities), can be found in Marano-Mosconi [12] and Papageorgiou-Rădulescu-Repovš [16].

2. Preliminaries-Hypotheses

In this analysis of problem (1) we will use the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\}$. By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. The Poincaré inequality (see, for example, Gasiński-Papageorgiou [4], p. 216), implies that

$$\|u\| = \|\nabla u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is ordered with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) \,:\, u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ \,:\, u(z) > 0 \text{ for all } z \in \Omega, \quad \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} < 0\right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. Also, we will use the Banach space $C^1(\overline{\Omega})$. This too is ordered with positive cone

$$\widehat{C}_+ = \{u \in C^1(\overline{\Omega}) \,:\, u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone too has a nonempty interior which is given by

$$D_+ = \left\{u \in \widehat{C}_+ \,:\, u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

In fact D_+ is also the interior of $\widehat{C}_+$ when the latter is equipped with the weaker $C(\overline{\Omega})$-norm topology.

The following simple fact about ordered Banach spaces will be used in the sequel (see Gasiński-Papageorgiou [5], Problem 4.180, p. 680).

Proposition 1. *If X is an ordered Banach space with order cone K and $\operatorname{int} K \neq \emptyset$, then given $e \in \operatorname{int} K$, for every $u \in K$ we can find $\lambda_u > 0$ such that $\lambda_u e - u \in K$.*

By $\widehat{\lambda}_1$ we denote the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. We know that

(a) $\widehat{\lambda}_1 > 0$;
(b) $\widehat{\lambda}_1$ is simple (that is, if $\widehat{u}, \widehat{v} \in W_0^{1,p}(\Omega)$ are two eigenfunctions corresponding to $\widehat{\lambda}_1$, then $\widehat{u} = \xi\widehat{v}$ for some $\xi \in \mathbb{R} \setminus \{0\}$);
(c) $\widehat{\lambda}_1 = \inf \left[\dfrac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega),\, u \neq 0\right]$.

The infimum in (c) is realized on the corresponding one dimensional eigenspace, the elements of which have fixed sign. By $\widehat{u}_1$ we denote the positive, L^p-normalized (that is, $\|\widehat{u}_1\|_p = 1$) eigenfunction corresponding to $\widehat{\lambda}_1$. From the nonlinear regularity theory and the nonlinear maximum principle (see Gasiński-Papageorgiou [4], pp. 737-738), we have that $\widehat{u}_1 \in \operatorname{int} C_+$.

In the sequel we will use the following notation. If $x \in \mathbb{R}$, then we set $x^{\pm} = \max\{\pm x, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$ we define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We know that

$$u^{\pm} \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

By $A_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ we denote the nonlinear map defined by

$$\langle A_p(u), h\rangle = \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega).$$

If $p = 2$, then we write $A_2 = A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$. These maps are continuous and strictly monotone (hence maximal monotone too).

Given $q \in (1, +\infty)$, by q^* we denote the critical Sobolev exponent corresponding to q and defined by $q^* = \begin{cases} \frac{Nq}{N-q} & \text{if } q < N, \\ +\infty & \text{if } N \leq q. \end{cases}$

If $u, v : \Omega \to \mathbb{R}$ are measurable functions and $u \leq v$, then we define

$$[u, v] = \left\{y \in W_0^{1,p}(\Omega) : u(z) \leq y(z) \leq v(z) \quad \text{for a.e. } z \in \Omega\right\},$$
$$[u) = \left\{y \in W_0^{1,p}(\Omega) : u(z) \leq y(z) \quad \text{for a.e. } z \in \Omega\right\}.$$

Also by $\mathrm{int}_{C_0^1(\overline{\Omega})}[u, v]$ we will denote the interior in the $C_0^1(\overline{\Omega})$-norm topology of $[u, v] \cap C_0^1(\overline{\Omega})$.

For $h_1, h_2 \in L^\infty(\Omega)$, we write $h_1 \prec h_2$ if and only if for all $K \subseteq \Omega$ compact, there exists $\widehat{c}_K > 0$ such that $0 < \widehat{c}_K \leq h_2(z) - h_1(z)$ for a.e. $z \in K$. If $h_1, h_2 \in C(\Omega)$, then $h_1 \prec h_2$ if and only if $h_1(z) < h_2(z)$ for all $z \in \Omega$.

Given $\varphi \in C^1(W_0^{1,p}(\Omega))$, by K_φ we denote the critical set of φ, that is,

$$K_\varphi = \left\{u \in W_0^{1,p}(\Omega) : \varphi'(u) = 0\right\}.$$

Also we say that φ satisfies the C-condition if:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in $W^{-1,p'}(\Omega)$ as $n \to +\infty$, then $\{u_n\}_{n\geq 1}$ has a convergent subsequence".

Now we are ready to introduce our hypotheses on the data of (1).

H(μ): $\mu : (0, +\infty) \to (0, +\infty)$ is a locally Lipschitz function such that

(i) for some $\gamma \in (0, 1)$ we have

$$0 < c_0 \leq \liminf_{x \to 0^+} \mu(x)x^\gamma \leq \limsup_{x \to 0^+} \mu(x)x^\gamma \leq c_1;$$

(ii) $\mu(\cdot)$ is nonincreasing.

Remark 1. As we already mentioned in the introduction, in the literature we encounter the particular case $\mu(x) = x^{-\gamma}$ for all $x > 0$. However, our hypotheses H(μ) can also treat a singularity of the form $\mu(x) = cx^{-\gamma} + \sin x^{-\gamma}$ for all $x > 0$, with $c > 1$.

H(f): $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.e. $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z)[1 + x^{r-1}]$ for a.e. $z \in \Omega$, all $x \geq 0$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;

(ii) set $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x\to+\infty} \dfrac{F(z,x)}{x^p} = +\infty$ uniformly for a.e. $z \in \Omega$;

(iii) there exists $\tau \in \left((r-p)\max\left\{\dfrac{N}{p}, 1\right\}, p^*\right)$ such that

$$0 < c_2 \leq \liminf_{x\to+\infty} \frac{f(z,x)x - pF(z,x)}{x^\tau} \text{ uniformly for a.e. } z \in \Omega;$$

(iv) there exists $\delta_0 > 0$ such that $0 < m_s \leq \inf[f(z,x) : s \leq x \leq \delta_0]$ for a.e. $z \in \Omega$;

(v) there exists $\eta_0 > \delta_0$ such that $\eta_0^{-\gamma} + f(z,\eta_0) \leq 0$ for a.e. $z \in \Omega$;

(vi) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.e. $z \in \Omega$ the function $x \to f(z,x) + \widehat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 2. Since we look for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality we may assume that

$$f(z,x) = 0 \quad \text{for a.e. } z \in \Omega, \text{ all } x \leq 0. \tag{2}$$

Hypotheses H(f)(ii),(iii) imply that

$$\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty \quad \text{uniformly for a.e. } z \in \Omega.$$

So, the perturbation $f(z,\cdot)$ is $(p-1)$-superlinear. Usually elliptic problems with a superlinear reaction are studied using the AR-condition which in our setting has an unilateral form due to (2). So, in this form the AR-condition says that there exist $q > p$ and $M > 0$ such that

$$0 < qF(z,x) \leq f(z,x)x \quad \text{for a.e. } z \in \Omega, \text{ all } x \geq M, \tag{3}$$

$$0 < \operatorname*{ess\,inf}_\Omega F(\cdot, M). \tag{4}$$

Integrating (3) and using (4), we obtain the weaker condition

$$c_3 x^q \leq F(z,x) \quad \text{for a.e. } z \in \Omega, \text{ all } x \geq M, \text{ some } c_3 > 0,$$
$$\Rightarrow \quad c_4 x^{q-1} \leq f(z,x) \quad \text{for a.e. } z \in \Omega, \text{ all } x \geq M, \text{ some } c_4 > 0 \text{ (see (3))}.$$

So, the AR-condition implies that for a.e. $z \in \Omega$ the perturbation $f(z,\cdot)$ has at least $(q-1)$-polynomial growth. In our case hypothesis H(f)(iii) which replaces the AR-condition, is less restrictive and permits superlinear nonlinearities with "slower" growth near $+\infty$. For example, consider the following function (for the sake of simplicity we drop the z-dependence):

$$f(x) = \begin{cases} x^{q-1} - x^{\tau-1} & \text{if } 0 \leq x \leq 1, \\ x^{p-1}\ln x & \text{if } 1 < x, \end{cases}$$

with $1 < p < q < \tau < +\infty$ (see (2)). This function satisfies hypotheses H(f), but fails to satisfy the AR-condition (see (3)).

3. Positive Solution

Let $\vartheta \in (0,+\infty)$. We start by considering the following auxiliary Dirichlet problem:

$$-\Delta_p u(z) - \Delta u(z) = \vartheta \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{Au_ϑ}$$

Proposition 2. *For all $\vartheta > 0$, problem (Au_ϑ) has a unique solution $\widetilde{u}_\vartheta \in \operatorname{int} C_+$, the function $\vartheta \to \widetilde{u}_\vartheta$ is strictly increasing from $(0, +\infty)$ into C_+ (that is, $\vartheta_1 < \vartheta_2 \Rightarrow \widetilde{u}_{\vartheta_2} - \widetilde{u}_{\vartheta_1} \in \operatorname{int} C_+$) and $\widetilde{u}_\vartheta \to 0$ in $C_0^1(\overline{\Omega})$ as $\vartheta \to 0^+$.*

Proof. The operator $A_p + A : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ (recall $p > 2$) is continuous, strictly monotone (hence maximal monotone too) and coercive. So, it is surjective (see Gasiński-Papageorgiou [4], p. 319). Hence we can find $\widetilde{u}_\vartheta \in W_0^{1,p}(\Omega)$, $\widetilde{u}_\vartheta \neq 0$ such that

$$\langle A_p(\widetilde{u}_\vartheta), h\rangle + \langle A(\widetilde{u}_\vartheta), h\rangle = \vartheta \int_\Omega h dz \quad \text{for all } h \in W_0^{1,p}(\Omega).$$

Choosing $h = -\widetilde{u}_\vartheta^- \in W_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned} & \|\nabla \widetilde{u}_\vartheta^-\|_p^p + \|\nabla \widetilde{u}_\vartheta^-\|_2^2 \leq 0, \\ \Rightarrow \quad & \widetilde{u}_\vartheta \geq 0,\ \widetilde{u}_\vartheta \neq 0. \end{aligned}$$

We have

$$-\Delta_p \widetilde{u}_\vartheta(z) - \Delta \widetilde{u}_\vartheta(z) = \vartheta \quad \text{for a.e. } z \in \Omega, \quad \widetilde{u}_\vartheta\big|_{\partial\Omega} = 0.$$

Theorem 7.1, p. 286, of Ladyzhenskaya-Ural′tseva [9], implies that $\widetilde{u}_\vartheta \in L^\infty(\Omega)$. So, we can apply Theorem 1 of Lieberman [11] and have $\widetilde{u}_\vartheta \in C_+ \setminus \{0\}$.

Moreover, the nonlinear maximum principle of Pucci-Serrin [20] (pp. 111, 120) guarantees that $\widetilde{u}_\vartheta \in \operatorname{int} C_+$.

This solution is unique on account of the strict monotonicity of the map $u \to A_p(u) + A(u)$.

Now suppose that $\vartheta_1 < \vartheta_2$ and let $\widetilde{u}_{\vartheta_1}, \widetilde{u}_{\vartheta_2} \in \operatorname{int} C_+$ be the corresponding unique solutions of problems (Au_{ϑ_1}) and (Au_{ϑ_2}). We have

$$\begin{aligned} & -\Delta_p \widetilde{u}_{\vartheta_1} - \Delta \widetilde{u}_{\vartheta_1} = \vartheta_1 < \vartheta_2 = -\Delta_p \widetilde{u}_{\vartheta_2} - \Delta \widetilde{u}_{\vartheta_2} \quad \text{a.e. in } \Omega, \\ \Rightarrow \quad & \widetilde{u}_{\vartheta_2} - \widetilde{u}_{\vartheta_1} \in \operatorname{int} C_+ \quad \text{(see Gasiński-Papageorgiou [6], Proposition 3.2)}, \\ \Rightarrow \quad & \vartheta \to \widetilde{u}_\vartheta \text{ is strictly increasing.} \end{aligned}$$

Finally let $\vartheta_n \to 0^+$ and let $\widetilde{u}_n = \widetilde{u}_{\vartheta_n} \in \operatorname{int} C_+$ be the corresponding unique solution of (Au_{ϑ_n}), $n \in \mathbb{N}$. As before, from Ladyzhenskaya-Ural′tseva [9] (p. 286), we know that there exists $c_4 > 0$ such that

$$\|\widetilde{u}_n\|_\infty \leq c_4 \quad \text{for all } n \in \mathbb{N}.$$

Hence Theorem 1 of Lieberman [11] implies that we can find $\alpha \in (0,1)$ and $c_5 > 0$ such that

$$\widetilde{u}_n \in C_0^{1,\alpha}(\overline{\Omega}) = C^{1,\alpha}(\overline{\Omega}) \cap C_0^1(\overline{\Omega}),\ \|\widetilde{u}_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c_5 \quad \text{for all } n \in \mathbb{N}.$$

Exploiting the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and since for $\vartheta = 0$, $\widetilde{u}_0 \equiv 0$ is the unique solution of the auxiliary problem, we have that $\widetilde{u}_n \to 0$ in $C_0^1(\overline{\Omega})$. □

Using hypothesis H(μ)(ii), we can find $\vartheta_0 > 0$ such that

$$0 < \frac{\vartheta}{\mu(\widetilde{u}_\vartheta)} \leq 1 \text{ and } 0 \leq \widetilde{u}_\vartheta \leq \delta_0 \text{ on } \overline{\Omega} \quad \text{for all } \vartheta \in (0, \vartheta_0], \tag{5}$$

We fix $\vartheta \in (0, \vartheta_0]$ and introduce the following two Carathéodory functions

$$g_\vartheta(z,x) = \begin{cases} \mu(\widetilde{u}_\vartheta(z)) + f(z, \widetilde{u}_\vartheta(z)) & \text{if } x \leq \widetilde{u}_\vartheta(z), \\ \mu(x) + f(z,x) & \text{if } \widetilde{u}_\vartheta(z) < x, \end{cases} \tag{6}$$

$$\widehat{g}_\vartheta(z,x)=\begin{cases} g_\vartheta(z,x) & \text{if } x\le\eta_0,\\ g_\vartheta(z,\eta_0) & \text{if } \eta_0<x.\end{cases} \tag{7}$$

We set $G_\vartheta(z,x)=\int_0^x g_\vartheta(z,s)ds$ and $\widehat{G}_\vartheta(z,x)=\int_0^x \widehat{g}_\vartheta(z,s)ds$.

Note that on account of hypotheses H(μ), we can find $c_6>0$ and $\delta>0$ such that

$$\mu(x)\le c_6x^{-\gamma} \text{ for all } 0<x\le\delta \text{ and } \mu(y)\le\mu(\delta) \text{ for all } y\ge\delta. \tag{8}$$

Let $s>N$ and $\varepsilon>0$. We have $(\widetilde{u}_\vartheta+\varepsilon)^s\in D_+$ and $\widehat{u}_1\in C_+\subseteq\widehat{C}_+$. So, according to Proposition 1, we can find $c_7>0$ such that $\widehat{u}_1\le c_7(\widetilde{u}_\vartheta+\varepsilon)^s$.

Then we have

$$\begin{aligned} & \widehat{u}_1^{1/s}\le c_7^{1/s}(\widetilde{u}_\vartheta+\varepsilon),\\ \Rightarrow\ & (\widetilde{u}_\vartheta+\varepsilon)^{-\gamma}\le c_8\widehat{u}_1^{-\gamma/s} \quad \text{for some } c_8>0,\\ \Rightarrow\ & \widetilde{u}_\vartheta^{-\gamma}\le c_8\widehat{u}_1^{-\gamma/s} \text{ on } \Omega \text{ (just let } \varepsilon\to0^+).\end{aligned}$$

From the Lemma in Lazer-McKenna [10], we have that

$$\begin{aligned} & \widehat{u}_1^{-\gamma/s}\in L^s(\Omega),\\ \Rightarrow\ & \widetilde{u}_\vartheta^{-\gamma}\in L^s(\Omega),\\ \Rightarrow\ & \mu(\widetilde{u}_\vartheta)\in L^s(\Omega) \quad \text{(see (8))}. \end{aligned} \tag{9}$$

So, we can consider the following two functionals $\varphi_\vartheta,\widehat{\varphi}_\vartheta:W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\begin{aligned} \varphi_\vartheta(u)&=\frac{1}{p}\|\nabla u\|_p^p+\frac12\|\nabla u\|_2^2-\int_\Omega G_\vartheta(z,u)dz,\\ \widehat{\varphi}_\vartheta(u)&=\frac{1}{p}\|\nabla u\|_p^p+\frac12\|\nabla u\|_2^2-\int_\Omega \widehat{G}_\vartheta(z,u)dz \quad \text{for all } u\in W_0^{1,p}(\Omega).\end{aligned}$$

Proposition 3 of Papageorgiou-Smyrlis [17] implies that $\varphi_\vartheta,\widehat{\varphi}_\vartheta\in C^1(W_0^{1,p}(\Omega))$.

Proposition 3. *If hypotheses $H(\mu)$, $H(f)$ hold, then*

(a) $K_{\varphi_\vartheta}\subseteq[\widetilde{u}_\vartheta)\cap\operatorname{int}C_+$;
(b) $K_{\widehat{\varphi}_\vartheta}\subseteq[\widetilde{u}_\vartheta,\eta_0]\cap\operatorname{int}C_+$.

Proof. (a) Let $u\in K_{\varphi_\vartheta}$. Then

$$\langle A_p(u),h\rangle+\langle A(u),h\rangle=\int_\Omega g_\vartheta(z,u)hdz \quad \text{for all } h\in W_0^{1,p}(\Omega). \tag{10}$$

We choose $h=(\widetilde{u}_\vartheta-u)^+\in W_0^{1,p}(\Omega)$. Then we have

$$\begin{aligned} & \langle A_p(u),(\widetilde{u}_\vartheta-u)^+\rangle+\langle A(u),(\widetilde{u}_\vartheta-u)^+\rangle\\ &=\int_\Omega[\mu(\widetilde{u}_\vartheta)+f(z,\widetilde{u}_\vartheta)](\widetilde{u}_\vartheta-u)^+dz \quad \text{(see (6))}\\ &\ge\int_\Omega\mu(\widetilde{u}_\vartheta)(\widetilde{u}_\vartheta-u)^+dz \quad \text{(see (5) and hypothesis H}(f)\text{(iv))}\\ &\ge\int_\Omega\vartheta(\widetilde{u}_\vartheta-u)^+dz \quad \text{(see (5))}\\ &=\langle A_p(\widetilde{u}_\vartheta),(\widetilde{u}_\vartheta-u)^+\rangle+\langle A(\widetilde{u}_\vartheta),(\widetilde{u}_\vartheta-u)^+\rangle \quad \text{(see Proposition 2)},\end{aligned}$$

$$\Rightarrow\ \widetilde{u}_\vartheta\le u. \tag{11}$$

From (6), (10), (11), we infer that

$$-\Delta_p u(z) - \Delta u(z) = \mu(u(z)) + f(z,u(z)) \quad \text{for a.e. } z \in \Omega. \tag{12}$$

As before, from (12) and Ladyzhenskaya-Ural′tseva [9] (p. 286), we have $u \in L^\infty(\Omega)$. Let $k(z) = \mu(u(z)) + f(z,u(z))$. Hypothesis H$(f)$(i) and (9) imply that $k \in L^s(\Omega)$. Let $y \in H_0^1(\Omega)$ be the unique solution of the linear Dirichlet problem

$$-\Delta y(z) = k(z) \quad \text{in } \Omega, \quad y\big|_{\partial\Omega} = 0.$$

Invoking Theorem 9.15, p. 241, of Gilbarg-Trudinger [8], we have that $y \in W^{2,s}(\Omega)$. From the Sobolev embedding theorem we know that

$$W^{2,s}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega}) \text{ with } \alpha = 1 - \frac{N}{s} > 0.$$

Therefore, if we set $\eta(z) = \nabla y(z)$, then $\eta \in C^\alpha(\overline{\Omega}, \mathbb{R}^N)$ and from (12) we have

$$-\text{div}\left(|\nabla u(z)|^{p-2}\nabla u(z) - \nabla u(z) - \eta(z)\right) = 0 \quad \text{for a.e. } z \in \Omega.$$

But then we can apply Theorem 1 of Lieberman [11] and conclude that $u \in \text{int}\, C_+$ (see (11)). So, we have proved that $K_{\varphi_\vartheta} \subseteq [\widetilde{u}_\vartheta) \cap \text{int}\, C_+$.

(b) Let $u \in K_{\widehat{\varphi}_\vartheta}$. We have

$$\langle A_p(u), h\rangle + \langle A(u), h\rangle = \int_\Omega \widehat{g}_\vartheta(z,u) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{13}$$

From part (a) we already have that $\widetilde{u}_\vartheta \le u$ (see (7)).

Next in (13) we choose $h = (u-\eta_0)^+ \in W_0^{1,p}(\Omega)$ (recall $u \in W_0^{1,p}(\Omega)$). We have

$$\begin{aligned}
&\langle A_p(u), (u-\eta_0)^+\rangle + \langle A(u), (u-\eta_0)^+\rangle \\
&= \int_\Omega [\mu(\eta_0) + f(z,\eta_0)]\,(u-\eta_0)^+ dz \quad \text{(see (7) and (6))} \\
&\le 0 = \langle A_p(\eta_0), (u-\eta_0)^+\rangle + \langle A(\eta_0), (u-\eta_0)^+\rangle \quad \text{(see hypothesis H}(f)\text{(v))}, \\
\Rightarrow \quad & u \le \eta_0.
\end{aligned}$$

So, we have proved that $u \in [\widetilde{u}_\vartheta, \eta_0]$.

As before (see part (a)), we can prove that $u \in \text{int}\, C_+$, so we conclude that $K_{\widehat{\varphi}_\vartheta} \subseteq [\widetilde{u}_\vartheta, \eta_0] \cap \text{int}\, C_+$.

□

Now we are ready for our multiplicity theorem producing two positive smooth solutions.

Theorem 1. *If hypotheses $H(\mu)$, $H(f)$ hold, then problem* (1) *has two positive solutions u_0, $\widehat{u} \in \text{int}\, C_+$ with $\widehat{u} \neq u_0$.*

Proof. From (7) it is clear that $\widehat{\varphi}_\vartheta$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}
&\widehat{\varphi}_\vartheta(u_0) = \inf\left[\widehat{\varphi}_\vartheta(u) : u \in W_0^{1,p}(\Omega)\right], \\
\Rightarrow \quad & u_0 \in K_{\widehat{\varphi}_\vartheta} \subseteq [\widetilde{u}_\vartheta, \eta_0] \cap \text{int}\, C_+ \quad \text{(see Proposition 3)}.
\end{aligned} \tag{14}$$

Let $\rho = \eta_0$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis H(f)(vi). Then for $\widetilde{\xi}_\rho > \widehat{\xi}_\rho$ we have

$$-\Delta_p u_0 - \Delta u_0 + \widetilde{\xi}_\rho u_0^{p-1} - \mu(u_0)$$

$$= f(z,u_0) + \widetilde{\xi}_\rho u_0^{p-1}$$

$$\geq f(z,\widetilde{u}_\vartheta) + \widetilde{\xi}_\rho \widetilde{u}_\vartheta^{p-1} \quad \text{for a.e. } z \in \Omega \text{ (see hypothesis H}(f)\text{(vi) and (14)).} \tag{15}$$

Using (5) we have

$$-\Delta_p \widetilde{u}_\vartheta - \Delta \widetilde{u}_\vartheta = \vartheta \leq \mu(\widetilde{u}_\vartheta) \leq \mu(\widetilde{u}_\vartheta) + f(z,\widetilde{u}_\vartheta) \quad \text{for a.e. } z \in \Omega$$
$$\text{(see hypothesis H}(f)\text{(iv) and recall that } 0 < \widetilde{u}_\vartheta \leq \delta_0).$$

We return to (15) and use the above inequality. We obtain

$$\begin{aligned} & -\Delta_p u_0 - \Delta u_0 + \widetilde{\xi}_\rho u_0^{p-1} - \mu(u_0) \\ & = f(z,u_0) + \widetilde{\xi}_\rho u_0^{p-1} \\ & \geq f(z,\widetilde{u}_\vartheta) + \widetilde{\xi}_\rho \widetilde{u}_\vartheta^{p-1} \\ & \geq -\Delta_p \widetilde{u}_\vartheta - \Delta \widetilde{u}_\vartheta + \widetilde{\xi}_\rho \widetilde{u}_\vartheta^{p-1} - \mu(\widetilde{u}_\vartheta). \end{aligned} \tag{16}$$

Let $a(y) = |y|^{p-2}y + y$ for all $y \in \mathbb{R}^N$. Then $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ (recall that $p > 2$) and $\operatorname{div} a(\nabla u) = \Delta_p u + \Delta u$ for all $u \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} & \nabla a(y) = |y|^{p-2}\left[I + (p-2)\frac{y \otimes y}{|y|^2}\right] + I \quad \text{for all } y \in \mathbb{R}^N, \\ \Rightarrow \quad & (\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq |\xi|^2 \quad \text{for all } y, \xi \in \mathbb{R}^N, \\ \Rightarrow \quad & \nabla a(\nabla u_0(\cdot)) \text{ is positive definite on } \Omega. \end{aligned}$$

Also, we have

$$-\Delta_p u_0 - \Delta u_0 = \mu(u_0) + f(z,u_0) \quad \text{for a.e. } z \in \Omega,$$
$$-\Delta_p \eta_0 - \Delta \eta_0 = 0 \geq \mu(\eta_0) + f(z,\eta_0) \quad \text{for a.e. } z \in \Omega \text{ (see hypothesis H}(f)\text{(v)).}$$

Recall that $\mu(\cdot)$ is locally Lipschitz. On account of hypothesis H(f)(vi), $f(z,\cdot)$ is lower locally Lipschitz. So, we can apply the tangency principle of Pucci-Serrin [20] (see Theorem 2.5.2, p. 35) and infer that

$$u_0(z) < \eta_0 \quad \text{for all } z \in \overline{\Omega}. \tag{17}$$

In a similar way (see (16)) we get that

$$\widetilde{u}_\vartheta(z) < u_0(z) \quad \text{for all } z \in \Omega. \tag{18}$$

From (18), hypothesis H(f)(vi) and since $\widetilde{\xi}_\rho > \widehat{\xi}_\rho$, we see that

$$f(\cdot, u_0(\cdot)) + \widetilde{\xi}_\rho u_0(\cdot)^{p-1} \succ f(\cdot, \widetilde{u}_\vartheta(\cdot)) + \widetilde{\xi}_\rho \widetilde{u}_\vartheta(\cdot)^{p-1}. \tag{19}$$

From (16), (19) and Proposition 4 of Papageorgiou-Smyrlis [17] (singular strong comparison principle), it follows that

$$u_0 - \widetilde{u}_\vartheta \in \operatorname{int} C_+. \tag{20}$$

From (17) and (20) we have $u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\widetilde{u}_\vartheta, \eta_0]$.

Note that

$$\begin{aligned} & \widehat{\varphi}_\vartheta\Big|_{[\widetilde{u}_\vartheta,\eta_0]} = \varphi_\vartheta\Big|_{[\widetilde{u}_\vartheta,\eta_0]} \quad \text{(see (6), (7))}, \\ \Rightarrow \quad & u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi_\vartheta, \\ \Rightarrow \quad & u_0 \text{ is a local } W_0^{1,p}(\overline{\Omega})\text{-minimizer of } \varphi_\vartheta \text{ (see [7]).} \end{aligned} \tag{21}$$

From Proposition 3 it is clear that we may assume that K_{φ_ϑ} is finite (otherwise we have already infinitely many positive smooth solutions (see Proposition 3 and (6)) and so we are done). Then on account of (21), we can find $\rho \in (0,1)$ small such that

$$\varphi_\vartheta(u_0) < \inf\left[\varphi_\vartheta(u) : \|u - u_0\| = \rho\right] = m \tag{22}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 29).

Hypothesis H(f)(ii) implies that

$$\varphi_\vartheta(t\widehat{u}_1) \to -\infty \quad \text{as } t \to +\infty. \tag{23}$$

Finally hypothesis H(f)(iii) and Proposition 9 of Papageorgiou-Rădulescu [13] imply that

$$\varphi_\vartheta(\cdot) \text{ satisfies the C-condition.} \tag{24}$$

Then (22), (23), (24) permit the use of the mountain pass theorem (see Gasiński-Papageorgiou [4], p. 648). So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\varphi_\vartheta} \subseteq [\widetilde{u}_\vartheta) \cap \operatorname{int} C_+ \text{ (see Proposition 3)}, \; m \leq \varphi_\vartheta(\widehat{u}). \tag{25}$$

From (6), (22) and (25) we conclude that $\widehat{u}$ is a second positive smooth solution of (1) and $\widehat{u} \neq u_0$. □

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