Musical pitch quantization as an eigenvalue problem

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May 28, 2020

Abstract

How can discrete pitches and chords emerge from the continuum of sound? Using a quantum cognition model of tonal music, we prove that the associated Schrödinger equation in Fourier space is invariant under continuous pitch transpositions. However, this symmetry is broken in the case of transpositions of chords, entailing a discrete cyclic group as transposition symmetry. Our research relates quantum mechanics with music and is consistent with music theory and seminal insights by Hermann von Helmholtz.

Keywords: scales, continuum, discrete, quantum cognition, transposition symmetry, circle of fifths, cyclic groups

1 Introduction

The dualism between the continuum and the discrete is ubiquitous in human culture. It involves mathematics, music, and thus mathematical music theory as well [25, 35]. According to Maier [35] the continuous versus the discrete aspects of melodic motion were highly disputed in the history of music and

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music psychology, as exemplified by the controversies between Riemann on the one hand and Stumpf, Lotze and Hornborstel on the other hand.

Discreteness and continuity also belong to advanced musical studies and to the daily practice of musicians. Belcanto singers are used to make a *portamento* [35], that is, a short slide from one note to another. Violinists can make continuous *glissandos* while pianists only discrete ones. Gregorian chant notation evolved from continuous and approximative signs (neumes) to a discrete and precise pitch notation. Indian singers use a discrete notation but they have practice in continuously connecting these points, and so on [14].

The same holds for musical instruments. For some of them, it can be more natural to produce discrete pitches than continuous ones, and vice versa. In the orchestra, strings or trombones can naturally produce continuous pitch frequencies. This is also the case for the theremin [55], where the selection of precise, discrete pitches requires a lot of training and expertise for the movement of hands and arms, even without any physical reference, such as positions on the neck of the violin, as moving along an invisible road [41]. Other instruments, such as keyboards or flutes, possess only a discrete spectrum of available pitch frequencies. Human voice can have the precision of discrete pitch as well as the fluency of continuous glides.

Questions of pitch continuity or discreteness regard performance practice as well as geometry: single notes can be seen as points or vertices, and continuous pitch connections as edges or segments of straight lines [41]. In musical performance, gestures [44, 37] also embody the continuity of motion through trajectories in space and time, while single notes can be seen as points that are touched during continuous movements [44].

The continuous-discreteness dichotomy is also important in the fields of biomusicology [22, 45] and ethnomusicology [11, 13, 14]. In the light of cultural studies, Burns [14, p. 217] therefore asked:

> Is this use of discrete scale steps universal? That is, are there musical cultures that use continuously variable pitches? The evidence from ethnomusicological studies indicates that the use of discrete pitch relationships is essentially universal. The only exceptions appear to be certain primitive musical styles [. . .] which are found in a few tribal cultures. Of course, pitch glides — glissandos, portamentos, trills, etc. are used as embellishment and ornamentation in most musical cultures. However, these embellishments are distinct from the basic scale structure of these musics.

For the emergence of discrete musical scales different approaches have been discussed in the literature. One of them, *salience* of musical categories, that has already been addressed by von Helmholtz [56, p. 252], was further leveraged in psychological theories of categorial perception [14, 45, 22]. Another explanation, also going back to von Helmholtz [56], is by means of the harmonic overtone series [56, 53, 24]. According to von Helmholtz [56, p. 253]:

> Recent theoreticians that have been born and bred in the system of harmonic music, have consequently supposed that they could explain the origin of the scales, by the assumption that all melodies arise from thinking of a harmony to them, and that the scale itself, considered as a melody of the key, arose from resolving the fundamental chords of the key into their separate tones. This view is certainly correct for modern scales; at least these have been modified to suit the requirements of the harmony. But scales existed long before there was any knowledge or experience of harmony at all. [. . .] It is clear that in the period of monophonic music, the scale could not have been constructed so as to suit the requirements of chordal connections unconsciously supplied. Yet a meaning may be assigned, in a somewhat altered form, to the views and hypotheses of musicians above mentioned, by supposing that the same physical and physiological relations of the tones, which become sensible when they are sounded together and determine the magnitude of the consonant intervals, might also have had an effect in the construction of the scale, although under somewhat different circumstances.

In this approach, a musical *tone* is considered as a periodic sound signal that can be decomposed into a discrete Fourier series of overtones [46]. These overtones obey certain frequency ratios, e.g., octave $2 : 1$, perfect fifth $3 : 2$, perfect fourth $4 : 3$ and so on. On the one hand, it is well known that two overtones forming an octave are perceptionally equivalent [14, 45, 22], which leads to the important concept of pitch classes, possessing a circular pitch symmetry that can be analyzed by means of group theory and its symmetries [3, 18]. On the other hand, perfect fourths and fifths appear perceptionally most similar [45, 22], giving rise to the structural organization of the circle of fifths [3].

However, explaining discrete scale emergence through the overtone series exhibits several peculiarities. One problem is that major scales are preferred against minor scales. Another one is that the argument applies to just intonation only. In order to account for equal temperament as well, several approaches have been suggested in the literature [57, 29, 21] that lead to the structure of a cyclic group $\mathbb{Z}_z$ (for $z \in \mathbb{N}$), describing transpositions jointly with inversions as basic musical symmetries [18]. Balzano [3] presented some evidence for the chromatic scale $z = 12$ of Western music, because two particular generators of the group $\mathbb{Z}_{12}$ generate the semitone steps of melodic dynamics on the one hand, and the fifth steps, important for harmonic composition, on the other hand. The resulting circle of fifths contains all diatonic scales as connected subsets.

Moreover, the decomposition $\mathbb{Z}_{12} = \mathbb{Z}_3 \times \mathbb{Z}_4$ explains the canonical construction of tertian harmonies. According to Balzano [3], these organizational principles could also be used for microtonal scales, such as $\mathbb{Z}_{20} = \mathbb{Z}_4 \times \mathbb{Z}_5$, or $\mathbb{Z}_{42} = \mathbb{Z}_6 \times \mathbb{Z}_7$.

An even more crucial problem with the overtone model is the silent assumption that a tone must be considered as a periodic sound signal of infinite duration in order to justify expansion into a discrete Fourier series. This is a rough idealization [7] since tones appear as whole, half, quarter or even shorter notes in musical scores. Therefore, the mathematical prerequisite for Fourier expansion is not fulfilled and instead of Fourier series, one would have to compute Fourier integrals of an acoustic signal with finite duration. This, however, leads immediately to a continuous overtone spectrum that is not able to explain pitch quantization anymore. Moreover, its frequency bandwidth is inversely related to the signal duration by means of the *uncertainty relation* for signal processing [23]. Thus, the shorter a note, the more continuous frequencies must be superimposed to yield the expected sound signal.

Therefore, the problem of discrete pitch emergence can be connected with *quantization* in physics. In fact, the contraposition between continuity and discreteness is a relevant part of quantum mechanics, and it is at the base of the dualism between particles and waves, between the corpuscular and undulatory nature of light and matter [54]. While the wave formalism is well-established in the domain of sound studies, discreteness and quantization is still a fertile research field there.

In fact, after the pioneering work by Gabor [23] in the domain of sound quantization, to the best of our knowledge, most relevant applications of the quantum paradigm to sound studies and music theory are relatively recent, and concentrated in the latest years. This represents an increased attention towards this topic, from music theory to sound processing and other technological applications. And yet, the wave formalism developed for sound seems to show significative analogies with quantum wave mechanics [12]. The idea of sound quantization, in terms of mechanical oscillation quantization, led to the conceptualization of the *phonon*. According to Perkovitz [48], "a phonon is a definite discrete unit or quantum of vibrational mechanical energy, just as a photon is a quantum of electromagnetic or light energy." Regarding the application of the quantum paradigm to sound, recent applications in the domain of quantum acoustics and technology include the storing quantum states within macroscopic devices and performing quantum operations on qubit-phonon systems [16]. Very recently, scientists have been able to "listen to" phonons via "quantum microphones" [2].

The paradigms of quantum mechanics have further been informing music and shaping research in music and cognition [19], human voice as a probe to investigate the world of sound with the *quantum vocal theory of sound* [52], the evaluation of musical memory [38], the quantitative and experimental investigation of tonal models [6], and developments of studies in mathematical theory of musical gestures first described by Mazzola and Andreatta [44].

In this article, instead of considering signals in the time domain, we regard tones as cognitive entities or psychological Gestalts [26, 59], namely as wave functions over the continuation of the circle of fifths. This requires harmonic analysis over the circle group and hence proper Fourier series with discrete spectrum. From that assumption, we prove the emergence of a cyclic group $\mathbb{Z}_z$ (Theorem 3.4). Cyclic groups such as $\mathbb{Z}_z$ are heavily used in mathematical music theory [1, 3, 18, 47]; in our paper, we actually join current research in the domain of mathematical music theory, acoustics, and quantum mechanics, by proving the emergence of discrete pitches and cyclic groups from the continuity of sound. Thus, our approach offers a new solution to the old problem, addressed by Maier [35].

Specifically, we start from the quantum cognition approach [15, 49] to tonal attraction [30, 6]. We investigate musical transformations such as transpositions, and we formulate the tonal attraction phenomenon as an eigenvalue problem, according to a fundamental method in quantum mechanics [54]. Thus, from very general considerations and evaluations about energy and symmetry breaking, we reach fundamental topics in mathematical music theory. This would also make more concrete some hints of application of quantum mechanics and in general physics results and methods to music [43].

The structure of the paper is the following. In Section 2 we list some required preliminaries about the quantum cognition approach to tonal attraction, presenting it later as an eigenvalue problem. In Section 3 we deliver the harmonic analysis, also discussing tonal and chordal transposition symmetry. After a short Discussion in Section 4 including possible future developments of this research, we conclude the paper in Section 5.

2 Musical Quantum Models

Quantum cognition has evolved as a powerful mathematical instrument to treat several puzzles and paradoxes of bounded rationality in cognitive psychology and decision theory [15, 49, 9]. It became increasingly popular in mathematical music theory in recent times [8, 19, 20, 5, 6, 51, 40] because they perfectly suit to the group-theoretical methods applied to investigate musical symmetries [3, 18, 47, 17].

The probable most important musical symmetry is transposition invariance, generalizing the basic concept of octave equivalence. According to McDermott and Hauser [45, p. 33]:

> In music, the relationships between pitches are generally more important than the absolute values of the pitches that are used. A melody will be recognized effortlessly even if it is transposed up or down by a fixed amount, a manipulation that alters the absolute pitch but preserves the relative pitch distances.

Transposition invariance in general, and octave equivalence in particular have been investigated in numerous experimental studies in humans and other species (see McDermott and Hauser [45], Fitch [22] for overviews and Wagner et al. [58] for a recent study). Also the quantum cognition approach for tonal attraction essentially builds upon transpositional symmetry [8, 10, 5, 6].

The starting point of the theoretical approach of Blutner [8], Blutner and beim Graben [10], beim Graben and Blutner [5], and beim Graben and Blutner [6] are experimental findings on static and dynamic tonal attraction [31, 30, 60, 61]. These are music-psychological priming experiments where subjects are asked to rate how well probe tone pitches fit into an earlier presented priming context. These contexts which are given by chords, cadences, ascending or descending scales, establish a tonal scale. Depending on their instruction, subjects indicate whether a probe either statically fits or dynamically resolves the prime. With their static tonal attraction experiments, Krumhansl and Kessler [30] laid the ground for Lerdahl's hierarchical model of the diatonic scales [32, 33]. As probe tones, the tonic received the highest rating, followed by the tones completing the tonic triad and then by the diatonic scale tones. The non-scale tones were ranked at the bottom.

In their original publication, Krumhansl and Kessler [30] reported their results graphically with interval size at the x- and attraction value at the y-axis. The resulting plots show large fluctuations according to the arrangement of tones along the chroma circle. However, when plotting the results after rearrangement along the circle of fifths, the Krumhansl and Kessler data appear much smoother because the diatonic and the non-scale tones are separated in two disjoint connected sets. beim Graben and Blutner [5] and beim Graben and Blutner [6] have realized that the static attraction profile can be described — in first order — as cosine similarity [50] between tones along the circle of fifths. Therefore, the circle of fifths appears to be more suitable for cognitive representations than the chroma circle of physical pitch heights. This conforms with ideas about the height-width duality and Handschin's tone character [17], where tonal representation space is regarded as a two-dimensional continuum spanned by "tone height" (corresponding to the octave dimension) and "tone character" (tone width, corresponding to the fifth dimension), the latter representing the music-psychological qualia [17].

2.1 Preliminaries

In musical quantum cognition tones are considered as *Gestalts* [26, 59], i.e., as wave functions (probability amplitudes) over the continuation of the circle of fifths. Although this continuation can be music-theoretically motivated through the constructions of Clampitt and Noll [17], here we simply postulate the existence of this one-dimensional continuous manifold as cognitive representation space, assuming equal temperament.

Definition 2.1. Let $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ be the unit circle as *configuration space.* S^1 is parameterized by a real variable, the arc angle $x \in [0, 2\pi[$ measured in radians. The circle of fifths is then a sampling of S^1 obtained from $x_k = k\pi/6$ for $k \in \mathbb{Z}_{12}$. A *tone* is a wave function $\psi : S^1 \to \mathbb{C}$, such that $\int_{S^1} \psi^*(x)\psi(x)\,\mathrm{d}x = 1$. Therefore, a tone is a state in the complex Hilbert space $L^2(S^1)$.[1]

This definition leads immediately to the notion of an attraction kernel.

Definition 2.2. Let $\psi \in L^2(S^1)$ be a tone over the unit circle S^1. The squared modulus $p(x) = |\psi(x)|^2 = \psi^*(x)\psi(x)$ is the *attraction rate* — or *anchoring strength* according to [33] — of the probe x by the *context tone* ψ. The probability density function $p(x)$ is called *attraction kernel.*

Next, we define the important concepts of musical transposition and the tonic.

Definition 2.3. Let $\psi \in L^2(S^1)$ be a tone over the unit circle S^1. Moreover, let $a \in S^1$ be an arbitrary context. The *transposition* of ψ by a is obtained by applying a transposition operator T_a to ψ through

$$\psi_a(x) = T_a\psi(x) = \psi(x - a)\,. \tag{1}$$

Definition 2.4. The transposition by $a = 0$ gives the distinguished *tonic* ψ_0,

$$\psi_0(x) = T_0\psi(x) = \psi(x - 0) = \psi(x)\,. \tag{2}$$

Because any transposition is parameterized by some $a \in S^1$, we state our first result.

Proposition 2.5. *The transpositions T_a with $a \in S^1$ are represented through the continuous circle group $U(1)$.*

Proof. Let T_a, T_b be two transpositions with $a, b \in S^1$. Then $T_a \circ T_b = T_{a+b}$, where $a + b$ is addition modulo 2π in S^1. Then, $\rho(T_a) = \mathrm{e}^{\mathrm{i}\hat{a}}$ is a representation

[1] This definition can be justified by an experimental finding of Krumhansl and Shepard [31] using quartertones as probes in a tonal attraction experiment. The attraction rates at quartertones interpolate those of neighboring semitones.

of T_a in $U(1)$ where $\hat{a} \in \mathbb{R}$ is the realization of $a \in S^1$ such that

$$\rho(T_a) \cdot \rho(T_b) = \mathrm{e}^{\mathrm{i}\hat{a}}\mathrm{e}^{\mathrm{i}\hat{b}} = \mathrm{e}^{\mathrm{i}(\hat{a}+\hat{b})} = \rho(T_{a+b}) \,.$$

□

Often one wants to assess the anchoring strength of a probe tone exerted by a chord context instead of a single tone. For this aim, Woolhouse [60] and Blutner [8] suggested to sum or to average the attraction profiles of individual pairs of tones over all possible pairings. We therefore formulate the *Woolhouse Conjecture* 2.6 as follows.

Conjecture 2.6. *Let $C = \{a_i \in S^1 | i = 1, \ldots, n\}$ be a chord context of n context tones and let $\rho(a_i) \in \mathbb{R}$ be their respective weights. Then, the attraction rate of probe $x \in S^1$ is obtained through discrete convolution [1]*

$$p_C(x) = \sum_{i=1}^{n} \rho(a_i) p(x - a_i) \tag{3}$$

with kernel function $p(x) = |\psi(x)|^2$.

In the following, we assume equal weights $1/n$ of all chord members and get for a simple dyad $C = \{a, b\}$,

$$p_{ab}(x) = |\psi_{ab}(x)|^2 = \frac{1}{2}\left(|\psi_a(x)|^2 + |\psi_b(x)|^2\right) , \tag{4}$$

where ψ_{ab} is a tentative wave function for the dyad C that will be explored subsequently.

2.2 The eigenvalue problem of tonal attraction

beim Graben and Blutner [5] have shown that tonal attraction can be formulated as an eigenvalue problem

$$H\psi(x) = E\psi(x) \tag{5}$$

with real eigenvalue E for a hermitian differential operator H, the *Hamiltonian*, acting on Hilbert space $L^2(S^1)$. Equation (5) is essentially a stationary Schrödinger equation for the wave function ψ of a single quantum particle [54].

One simple model was obtained by the choice

$$H = T = -\frac{\partial^2}{\partial x^2} \,, \tag{6}$$

corresponding to the operator of *kinetic energy* T of a particle freely moving across the circle of fifths S^1. Equation (5) is solved by the wave function

$$\psi(x) = \frac{1}{\sqrt{\pi}} \cos \frac{x}{2} \tag{7}$$

for eigenvalue $E = 1/4$, leading to a cosine similarity [50] attraction kernel

$$p(x) = \frac{1}{\pi} \cos^2 \frac{x}{2} \tag{8}$$

that reflects the perceptional similarity of fifths and fourths along the circle of fifths according to Krumhansl and Kessler [30].

A more appropriate model with Hamiltonian

$$H = T + M + U \tag{9}$$

and differential operators

$$M = m(x) \frac{\partial}{\partial x} \tag{10}$$

and

$$U = E + g(x) \tag{11}$$

for some real functions m, g defined upon S^1 leads to the *quantum deformation model* [5, 6, 10]

$$-\psi''(x) + m(x)\psi'(x) + g(x)\psi(x) = 0\,. \tag{12}$$

When

$$m(x) = \frac{\gamma''(x)}{\gamma'(x)}\,, \tag{13}$$

and

$$g(x) = -\gamma'(x)^2 \tag{14}$$

for some deformation function γ, this differential equation is solved by

$$\psi(x) = A \cos \gamma(x) \tag{15}$$

where A is an appropriate normalization constant, and hence by a deformed cosine similarity kernel

$$p(x) = A^2 \cos^2 \gamma(x)\,. \tag{16}$$

Choosing several polynomial deformations γ, beim Graben and Blutner [6] were able to fit both static [30] and dynamic [60] tonal attraction data. Moreover, Blutner and beim Graben [10] have shown how different deformations can be ubiquitously unified as local gauge symmetries.

3 Harmonic Analysis

Since the above functions m, g, ψ are defined over the configuration space of the unit circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, namely the continuation of the circle of fifths $\mathbb{Z}_{12}$, they obey its circular symmetry and must hence be 2π-periodic functions over $\mathbb{C}$.[2] The musical interpretation of this periodicity is octave equivalence or rather its harmonic analogue at the circle of fifths, that is, enharmonic equivalence.[3] Thus we may apply the framework of harmonic analysis [1], and developing them into their Fourier series (provided their existence):

$$m(x) = \sum_k M_k \mathrm{e}^{\mathrm{i}kx} \tag{17}$$

$$g(x) = \sum_k G_k \mathrm{e}^{\mathrm{i}kx} \tag{18}$$

$$\psi(x) = \sum_k P_k \mathrm{e}^{\mathrm{i}kx} \tag{19}$$

where the indices extend over all integers $-\infty < k < \infty$. Their Fourier coefficients are given as

$$M_k = \frac{1}{2\pi} \int_0^{2\pi} m(x) \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x \tag{20}$$

$$G_k = \frac{1}{2\pi} \int_0^{2\pi} g(x) \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x \tag{21}$$

$$P_k = \frac{1}{2\pi} \int_0^{2\pi} \psi(x) \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x \, . \tag{22}$$

Proposition 3.1. *In Fourier space the Schrödinger Equation (12) is*

$$k^2 P_k + \sum_l (\mathrm{i} l M_{k-l} + G_{k-l}) P_l = 0 \tag{23}$$

for all $k \in \mathbb{Z}$.

Proof. Inserting Equations (17 – 19) into the Schrödinger Eq. (12) yields Eq. (23). □

[2] This has an interesting physical interpretation: Periodic potentials in the Hamiltonian describe the Bravais lattices of crystals in solid state physics. The Schrödinger equation in configuration space is solved by Bloch waves [27, 28] for canonically conjugated lattice sites and wave vectors. In particular, Kramers [28] proved the emergence of the Lie group $\mathrm{SL}(2, \mathbb{C})$ as an essential symmetry. This group is also relevant for the canonical transformation of intervals and their respective multiplicities in mathematical musicology [17, 47].

[3] With a small caveat: enharmonic equivalence leads to exactly the same sounds but with different names for single notes or for intervals, while octave equivalence leads to different sounds, having an n-octave distance between them, but considered within the same pitch class and hence with the same note names.

3.1 Tonal transposition symmetry

First, we prove that musical transposition symmetry holds also in Fourier space. To this end, we recognize that a (continuous) transposition $a \in S^1$ turns out to be equivalent up to multiplication by a constant phase factor.

Proposition 3.2. *Let the periodic functions m, g, ψ be given as their Fourier series (17 – 19). Then the transposed functions are obtained as*

$$m(x-a) = \sum_k \tilde{M}_k e^{ikx} \tag{24}$$

$$g(x-a) = \sum_k \tilde{G}_k e^{ikx} \tag{25}$$

$$\psi(x-a) = \sum_k \tilde{P}_k e^{ikx} \tag{26}$$

with Fourier coefficents

$$\tilde{M}_k = M_k e^{-ika} \tag{27}$$

$$\tilde{G}_k = G_k e^{-ika} \tag{28}$$

$$\tilde{P}_k = P_k e^{-ika} \,. \tag{29}$$

for $a \in \mathbb{R}$.

Proof. Since the transposition operator T_a is linear it can be applied to the Fourier series (17 – 19),

$$m(x-a) = \sum_k M_k e^{ik(x-a)} = \sum_k M_k e^{ikx} e^{-ika} \tag{30}$$

$$g(x-a) = \sum_k G_k e^{ik(x-a)} = \sum_k G_k e^{ikx} e^{-ika} \tag{31}$$

$$\psi(x-a) = \sum_k P_k e^{ik(x-a)} = \sum_k P_k e^{ikx} e^{-ika} \,. \tag{32}$$

□

Theorem 3.3. *The Schrödinger equation in Fourier space* (23) *is invariant under continuous transpositions T_a for $a \in S^1$ (corresponding to $\rho(T_a) = e^{i\hat{a}} \in$ U(1) with $\hat{a} \in \mathbb{R}$).*

Proof. Replacing all Fourier coefficients M_k, G_k, P_k in (23) by the transposed ones in (27 — 29), where we deliberately equate $\hat{a} = a \in \mathbb{R}$ in the following for the sake of simplicity, yields

$$\begin{aligned}
k^2\tilde{P}_k + \sum_l (\mathrm{i}l\tilde{M}_{k-l} + \tilde{G}_{k-l})\tilde{P}_l &= 0 \\
k^2 P_k \mathrm{e}^{-\mathrm{i}ka} + \sum_l (\mathrm{i}lM_{k-l}\mathrm{e}^{-\mathrm{i}(k-l)a} + G_{k-l}\mathrm{e}^{-\mathrm{i}(k-l)a})P_l\mathrm{e}^{-\mathrm{i}la} &= 0 \\
k^2 P_k \mathrm{e}^{-\mathrm{i}ka} + \sum_l (\mathrm{i}lM_{k-l} + G_{k-l})P_l\mathrm{e}^{-\mathrm{i}ka} &= 0 \\
k^2 P_k + \sum_l (\mathrm{i}lM_{k-l} + G_{k-l})P_l &= 0\,,
\end{aligned}$$

i.e., continuous transposition invariance. □

3.2 Chordal transposition symmetry

Next, we consider a simple dyad $C = \{a, b\}$ and assume that the Woolhouse Conjecture 2.6 is valid for the anchoring strength of chord contexts. Then, Eq. (4) holds:

$$p_{ab}(x) = |\psi_{ab}(x)|^2 = \frac{1}{2}\left(|\psi_a(x)|^2 + |\psi_b(x)|^2\right).$$

Theorem 3.4. *The Schrödinger equation in Fourier space (23) is only invariant under discrete transpositions T_a for $a \in \mathbb{Z}_z$ ($z \in \mathbb{N}$) for chord contexts. Hence continuous transposition symmetry is broken.*

Proof. Under the assumption that the wave function ψ_{ab} has the Fourier series

$$\psi_{ab}(x) = \sum_k Q_k \mathrm{e}^{\mathrm{i}kx} \tag{33}$$

we obtain on the one hand

$$|\psi_{ab}(x)|^2 = \sum_{kl} Q_k Q_l^* \mathrm{e}^{\mathrm{i}(k-l)x}$$

for the convolution. On the other hand, we have

$$\begin{aligned}
|\psi_a(x)|^2 &= \sum_{kl} P_k P_l^* \mathrm{e}^{\mathrm{i}(k-l)x}\mathrm{e}^{-\mathrm{i}(k-l)a} \\
|\psi_b(x)|^2 &= \sum_{kl} P_k P_l^* \mathrm{e}^{\mathrm{i}(k-l)x}\mathrm{e}^{-\mathrm{i}(k-l)b}
\end{aligned}$$

due to Eq. (32). Their mixture yields then the identity

$$Q_k Q_l^* = \frac{1}{2}\left(\mathrm{e}^{-\mathrm{i}(k-l)a} + \mathrm{e}^{-\mathrm{i}(k-l)b}\right) P_k P_l^*\,. \tag{34}$$

Next, we need the complex conjugated Schrödinger equation

$$k^2 P_k^* + \sum_l (-\mathrm{i} l M_{k-l}^* + G_{k-l}^*) P_l^* = 0 \tag{35}$$

such that the product of (23) and (35) yields

$$k^2 m^2 P_k P_m^* + k^2 \sum_n (-\mathrm{i} n M_{m-n}^* + G_{m-n}^*) P_k P_n^* +$$
$$+ m^2 \sum_l (\mathrm{i} l M_{k-l} + G_{k-l}) P_l P_m^* + \sum_{ln} (\mathrm{i} l M_{k-l} + G_{k-l})(-\mathrm{i} n M_{m-n}^* + G_{m-n}^*) P_l P_n^* = 0. \tag{36}$$

Now we assume that the chord wave function ψ_{ab} obeys a structurally similar product equation

$$k^2 m^2 Q_k Q_m^* + k^2 \sum_n (-\mathrm{i} n \hat{M}_{m-n}^* + \hat{G}_{m-n}^*) Q_k Q_n^* +$$
$$+ m^2 \sum_l (\mathrm{i} l \hat{M}_{k-l} + \hat{G}_{k-l}) Q_l Q_m^* + \sum_{ln} (\mathrm{i} l \hat{M}_{k-l} + \hat{G}_{k-l})(-\mathrm{i} n \hat{M}_{m-n}^* + \hat{G}_{m-n}^*) Q_l Q_n^* = 0 \tag{37}$$

in Fourier space, where $\hat{M}_k, \hat{G}_k$ are the Fourier coefficients of the yet unknown energy operators resulting from the interaction of two context tones a and b.

Inserting (34) into (37) gives

$$k^2 m^2 \left(\mathrm{e}^{-\mathrm{i}(k-m)a} + \mathrm{e}^{-\mathrm{i}(k-m)b} \right) P_k P_m^* +$$
$$+ k^2 \sum_n (-\mathrm{i} n \hat{M}_{m-n}^* + \hat{G}_{m-n}^*) \left(\mathrm{e}^{-\mathrm{i}(k-n)a} + \mathrm{e}^{-\mathrm{i}(k-n)b} \right) P_k P_n^* +$$
$$+ m^2 \sum_l (\mathrm{i} l \hat{M}_{k-l} + \hat{G}_{k-l}) \left(\mathrm{e}^{-\mathrm{i}(l-m)a} + \mathrm{e}^{-\mathrm{i}(l-m)b} \right) P_l P_m^* +$$
$$\sum_{ln} (\mathrm{i} l \hat{M}_{k-l} + \hat{G}_{k-l})(-\mathrm{i} n \hat{M}_{m-n}^* + \hat{G}_{m-n}^*) \left(\mathrm{e}^{-\mathrm{i}(l-n)a} + \mathrm{e}^{-\mathrm{i}(l-n)b} \right) P_l P_n^* = 0 \,. \tag{38}$$

In case of transposition invariance, Eq. (38) must reduce to (36) in analogy to the proof of Theorem 3.3.

When $\mathrm{e}^{-\mathrm{i}(k-m)a} + \mathrm{e}^{-\mathrm{i}(k-m)b} \neq 0$, we can divide by this term and obtain

$$k^2 m^2 P_k P_m^* + k^2 \sum_n (-\mathrm{i}n\hat{M}_{m-n}^* + \hat{G}_{m-n}^*) \frac{\mathrm{e}^{-\mathrm{i}(k-n)a} + \mathrm{e}^{-\mathrm{i}(k-n)b}}{\mathrm{e}^{-\mathrm{i}(k-m)a} + \mathrm{e}^{-\mathrm{i}(k-m)b}} P_k P_n^* +$$
$$+ m^2 \sum_l (\mathrm{i}l\hat{M}_{k-l} + \hat{G}_{k-l}) \frac{\mathrm{e}^{-\mathrm{i}(l-m)a} + \mathrm{e}^{-\mathrm{i}(l-m)b}}{\mathrm{e}^{-\mathrm{i}(k-m)a} + \mathrm{e}^{-\mathrm{i}(k-m)b}} P_l P_m^* +$$
$$\sum_{ln} (\mathrm{i}l\hat{M}_{k-l} + \hat{G}_{k-l})(-\mathrm{i}n\hat{M}_{m-n}^* + \hat{G}_{m-n}^*) \frac{\mathrm{e}^{-\mathrm{i}(l-n)a} + \mathrm{e}^{-\mathrm{i}(l-n)b}}{\mathrm{e}^{-\mathrm{i}(k-m)a} + \mathrm{e}^{-\mathrm{i}(k-m)b}} P_l P_n^* = 0 \,. \tag{39}$$

As all the fractions above are similar, we discuss their general form

$$F_{ab}(p,q) = \frac{\mathrm{e}^{-\mathrm{i}pa} + \mathrm{e}^{-\mathrm{i}pb}}{\mathrm{e}^{-\mathrm{i}qa} + \mathrm{e}^{-\mathrm{i}qb}} \tag{40}$$

for independent $p, q \in \mathbb{Z}$. First, we use Euler's formula for rewriting

$$\mathrm{e}^{-\mathrm{i}pa} + \mathrm{e}^{-\mathrm{i}pb} = 2\cos\frac{p(a-b)}{2} \exp\left[-\mathrm{i}\frac{p(a+b)}{2}\right]$$

and thus

$$F_{ab}(p,q) = \frac{\cos\frac{p(a-b)}{2}}{\cos\frac{q(a-b)}{2}} \exp\left[-\mathrm{i}\frac{(p-q)(a+b)}{2}\right] \,.$$

To evaluate the first term, we substitute $p - q = u$ and insert $p = u + q$ into the denominator

$$\cos\frac{p(a-b)}{2} = \cos\frac{u(a-b)}{2}\cos\frac{q(a-b)}{2} - \sin\frac{u(a-b)}{2}\sin\frac{q(a-b)}{2}$$

by virtue of the trigonometric addition theorems. Inserting this into F_{ab} again, yields

$$F_{ab}(p,q) = \cos\frac{(p-q)(a-b)}{2} \exp\left[-\mathrm{i}\frac{(p-q)(a+b)}{2}\right] -$$
$$- \sin\frac{(p-q)(a-b)}{2} \tan\frac{q(a-b)}{2} \exp\left[-\mathrm{i}\frac{(p-q)(a+b)}{2}\right] \,, \tag{41}$$

after reverting the substitution.

This function depends only on the interval $p - q$ if

$$\tan\frac{q(a-b)}{2} = 0 \tag{42}$$

for all $q \in \mathbb{Z}$. Moreover, the poles q_∞ of

$$\tan \frac{q_\infty(a-b)}{2} = \pm\infty \tag{43}$$

must be excluded in order to permit the division in Eq. (39).

Now, we have to discuss these *quantization conditions* [54]. Consider (42), which holds for all $q \in \mathbb{Z}$ if there is another $p \in \mathbb{Z}$ with[4]

$$\frac{q(a-b)}{2} = p\pi \; . \tag{44}$$

Because of the periodicity of (42), we choose p as a multiple of q, i.e. $p = jq$ with some $j \in \mathbb{Z}$.

Equation (44) is solvable in $\mathbb{Z}$ only when the interval $a - b$ is a rational multiple of 2π. We therefore assume the existence of fixed integers $r, s \in \mathbb{Z}$ and $z \in \mathbb{N}$, such that $a = 2\pi r/z, b = 2\pi s/z$ and obtain

$$a - b = 2\pi \frac{r-s}{z} \; . \tag{45}$$

Inserting (45) into (44) yields

$$r - s = jz \; , \tag{46}$$

which means that r, s are congruent modulo z. Thereby $r, s \in \mathbb{Z}_z$ with the cyclic group $\mathbb{Z}_z$.

For Western music we have particularly $z = 12$, and hence $a, b \in \frac{\pi}{6}\mathbb{Z}_{12}$. Thus, the originally assumed continuous transposition symmetry U(1) breaks down into the cyclic group of the circle of fifths, leading to the emergence of the chromatic scale from musical transposition invariance. Note that the same argument also applies to contemporary approaches for microtonality [14] which give rise to other cyclic groups $\mathbb{Z}_z$, with $z = 20, 30$, or $z = 42$ [3].

In order to finalize the invariance proof, we introduce a function

$$H_{ab}(z) = \cos \frac{z(a-b)}{2} \exp \left[-\mathrm{i} \frac{z(a+b)}{2} \right] \tag{47}$$

reproducing $F_{ab}(z)$ [Eq. (41)] in the Schrödinger product equation (39) under

[4] Condition (43) leads accordingly to

$$\frac{q(a-b)}{2} = \frac{\pi}{2} + p\pi$$

that is solved under the same considerations below.

the quantization condition (42). Then

$$
\begin{aligned}
k^2 m^2 P_k P_m^* + k^2 \sum_n (-\mathrm{i}n\hat{M}_{m-n}^* + \hat{G}_{m-n}^*) H_{ab}(k-n-k+m) P_k P_n^* + \\
+ m^2 \sum_l (\mathrm{i}l\hat{M}_{k-l} + \hat{G}_{k-l}) H_{ab}(l-m-k+m) P_l P_m^* + \\
\sum_{ln} (\mathrm{i}l\hat{M}_{k-l} + \hat{G}_{k-l})(-\mathrm{i}n\hat{M}_{m-n}^* + \hat{G}_{m-n}^*) H_{ab}(l-n-k+m) P_l P_n^* = 0 \,. \quad (48)
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
k^2 m^2 P_k P_m^* + k^2 \sum_n (-\mathrm{i}n\hat{M}_{m-n}^* + \hat{G}_{m-n}^*) H_{ab}(m-n) P_k P_n^* + \\
+ m^2 \sum_l (\mathrm{i}l\hat{M}_{k-l} + \hat{G}_{k-l}) H_{ab}(l-k) P_l P_m^* + \\
\sum_{ln} (\mathrm{i}l\hat{M}_{k-l} + \hat{G}_{k-l})(-\mathrm{i}n\hat{M}_{m-n}^* + \hat{G}_{m-n}^*) H_{ab}(l-n-k+m) P_l P_n^* = 0 \,, \quad (49)
\end{aligned}
$$

and after complex conjugation

$$
\begin{aligned}
k^2 m^2 P_k P_m^* + k^2 \sum_n (-\mathrm{i}n\hat{M}_{m-n}^* + \hat{G}_{m-n}^*) H_{ab}(m-n) P_k P_n^* + \\
+ m^2 \sum_l (\mathrm{i}l\hat{M}_{k-l} + \hat{G}_{k-l}) H_{ab}^*(k-l) P_l P_m^* + \\
\sum_{ln} (\mathrm{i}l\hat{M}_{k-l} + \hat{G}_{k-l})(-\mathrm{i}n\hat{M}_{m-n}^* + \hat{G}_{m-n}^*) H_{ab}(l-n-k+m) P_l P_n^* = 0 \,, \quad (50)
\end{aligned}
$$

Equations (36) and (50) lead to the following invariance constraints

$$
\begin{aligned}
\hat{M}_{m-n}^* H_{ab}(m-n) &= M_{m-n}^* && (51)\\
\hat{G}_{m-n}^* H_{ab}(m-n) &= G_{m-n}^* && (52)\\
\hat{M}_{k-l} H_{ab}^*(k-l) &= M_{k-l} && (53)\\
\hat{G}_{k-l} H_{ab}^*(k-l) &= G_{k-l} && (54)\\
\hat{M}_{k-l}\hat{M}_{m-n}^* H_{ab}(l-n-k+m) &= M_{k-l} M_{m-n}^* && (55)\\
\hat{M}_{k-l}\hat{G}_{m-n}^* H_{ab}(l-n-k+m) &= M_{k-l} G_{m-n}^* && (56)\\
\hat{G}_{k-l}\hat{M}_{m-n}^* H_{ab}(l-n-k+m) &= G_{k-l} M_{m-n}^* && (57)\\
\hat{G}_{k-l}\hat{G}_{m-n}^* H_{ab}(l-n-k+m) &= G_{k-l} G_{m-n}^* \,. && (58)
\end{aligned}
$$

These constraints that are highly redundant are identically fulfilled when

$$
\begin{aligned}
\hat{M}_k H_{ab}^*(k) &= M_k && (59)\\
\hat{G}_k H_{ab}^*(k) &= G_k && (60)
\end{aligned}
$$

for the transposed Fourier coefficients $\hat{M}_k$ and $\hat{G}_k$ of the dyad $C = \{a, b\}$. Thus, the transposed product equation (38) reduces to (36) in the case when the interval $a - b$ is a rational multiple of 2π. Hence discrete transposition invariance holds. □

4 Discussion

In this study, we have used harmonic analysis of tonal Gestalt patterns over the continuation of the circle of fifths to the unit circle. Formulating tonal attraction as an eigenvalue problem for a musical wave function [8, 10, 5, 6], we transformed the resulting Schrödinger equation into Fourier space, as all involved functions and differential operators must exactly be periodic over their cyclic configuration space. We investigated musical transposition symmetry through linear translations of wave functions along the unit circle.

Our study entailed two main results. First, considering monophonic music consisting only of simple tones, melodies can be continuously transposed (Theorem 3.3). Moreover, we have proven in Theorem 3.4 that the continuous transposition symmetry is broken for polyphonic music consisting of chords. This actually proves the emergence of discrete pitches (organized within musical chords) from the continuum of pitch frequencies. Such a result appears as a connecting anchor within different topics in the framework of mathematical music theory.

In fact, developments in mathematical music theory involve several areas such as tuning, chords, musical structures, and they exploit different mathematical and computational tools, ranging from signal processing to most abstract algebra. One of the contemporary challenges in this field is rejoining all these studies within a unified vision [43, 39]. Our investigation presented in this paper connects some thoughts by von Helmholtz [56] regarding the sound with the abstraction and formalization of music theory. Interestingly, the same paradigm of Fourier analysis is present at the level of sound within acoustics, at the level of music theory of chords and rhythms with complex exponentials [1], and in our research, that aims to connect these two worlds. In fact, the emergence of discrete pitches (and chords) from the continuum can be seen as a first, but essential step towards all Western (but not only) music theory.

Starting point of our argumentation was a one-dimensional, continuous and periodic manifold, the unit circle S^1. A coordinate $x \in S^1$ refers to a probe tone at the circle of fifths which can be identified with the cognitively relevant width dimension [17]. In quantum mechanics, positions are eigenvalues of a corresponding operator $\hat{x}$. Canonically conjugated to the position operator is a momentum operator $\hat{p}$, such that both obey the Heisenberg commutation relation. Thus, position and momentum form Fourier pairs such that our method applies. For a free quantum system, the joint spectrum of position and momentum is then $S^1 \times \mathbb{R}$. In order to obtain a musical interpretation it is tempting to regard the x-dimension still as tone width but the p-dimension as tone height in the sense of Clampitt and Noll [17]. As we have proven here, in a first step momentum becomes quantized by octave equivalence or enharmonic equivalence, respectively, for monophonic music, leading to $S^1 \times \mathbb{Z}$. In a second step, continuous transposition symmetry is broken for polyphonic music such that position and momentum are then restricted to the discrete space $\mathbb{Z}_z \times \mathbb{Z}$.

In future research, canonical transformations of position and momentum (or of tone width and tone height) could be examined. These would describe how different harmonic basis systems are mapped onto each other, e.g. perfect fifths and fourths onto major and minor seconds. We hope that our approach could therefore lead to new insights about the psychological representations of tonal music and the emergence of pentatonic, diatonic and chromatic scales [47, 17].

Our theory essentially rests upon equal temperament of the underlying tonal space. Therefore one might ask how it could apply to other tunings, such as just intonation or Pythagorean temperament as well. In a recent contribution, Baroin and Calvet [4] have demonstrated how different tuning systems could be related to each other by deformations of tonal spaces, such as the circle of fifths or a *tonnetz*. Yet, deformation transformations are at the core of our quantum models. beim Graben and Blutner [5] and beim Graben and Blutner [6] introduced deformations of pure cosine similarity to describe the tonal attraction data of Krumhansl and Kessler [30] and [60], respectively. Moreover, Blutner and beim Graben [10] have shown how different deformation models could be transformed by means of gauge symmetries. Future research in mathematical music theory may show whether and how those gauge transformations could be related to canonical transformations of tonal space [47, 17].

Future developments can further involve more quantitative evaluations, cognitive experiments, as well as an overall formalization of these topics. Although musical transposition invariance has been extensively studied in psychological experiments [45, 22, 58], our study poses new interesting questions for experimental investigation. Our first result, demonstrated in Theorem 3.3, states continuous transposition symmetry for monophonic music. Thus we suggest experiments where the acoustic pitch frequencies of monophonic melodies could be electronically manipulated to obtain continuous transpositions. Yet our second result, proven in Theorem 3.4, requires discrete transposition symmetry for polyphonic and hence harmonic music. Controlling monophony against polyphony could yield new experimental insights about musical transposition symmetry in humans and other species.

Moreover, we could, for example, define suitable functors[5] connecting 'nat-

[5] A *functor* is a morphism between category. A category is constituted by objects and morphisms between them, satisfying associative and identity properties [34].

ural' sounds, with emerging 'musical' sounds, with abstract objects of music theory, and their organization within (Western) musical scores. The inverse construction, that can be obtained by reversing all the arrows, can lead from written scores to more and more 'isolated' theoretical objects in music (scales, chords, notes), to finally reach the physical reality of sounds. Such a construction can be easily related with Mazzola's performance theory [42] and with mathematical theory of musical gestures [44]. The relationship between the overall movement 'physical reality of sound to written scores' and the vice versa, 'written scores to sounds,' can be seen as a categorical adjunction [34]. In this framework, our research might play a decisive role to strengthen the connection between passages, and to stress the relevance of mathematical paradigms such as the Fourier formalism, convolutions, gauge transformations, and symmetry breaking [10, 5, 6]. Some of the concepts, such as transposition, are familiar to musicians, mathematicians, and math-musicians; others, such as symmetry breaking, are more deeply connected with physics. In particle physics, e.g., spontaneous symmetry breaking is connected with the emergence of matter according to the Higgs mechanisms [36, pp. 253 – 265]. By contrast, we demonstrated the emergence of musical harmony here, thereby offering a new solution to the old problem, discussed by Maier [35].

5 Conclusion

In this paper, we investigated the emergence of discrete pitches and chords from the continuum of sound. This is strictly related with the dualism between discreteness and continuity in physics. Thus, the use of quantum formalism with quantization rules appeared as a natural and helpful research tool. Applying harmonic analysis to the quantum deformation model, we derived quantization conditions for musical wave functions in infinite-dimensional Fourier space. Combining musical transposition symmetry with the discrete convolution model for chordal contexts, we derived the emergence of the chromatic 12-tone cyclic group through symmetry breaking. Other possible symmetries are cyclic groups as considered in microtonality approaches. This research opens up new interdisciplinary and collaborative scenarios, where scientists and musicians can work closely, highlighting the relevance of paradigms from theoretical physics, and thus from nature, for the definition of music fundamentals. We can wonder if, once more, the roots of the best human creations might be hidden within the rules and the beauty of nature.

Acknowledgments

We thank Reinhard Blutner and Thomas Noll for helpful comments. We are greatfully indebted to the helpful comments of two anonymous referees, and acknowledge the conspicuous service of the editorial board.

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