LEAST ENERGY SOLUTIONS WITH SIGN INFORMATION FOR PARAMETRIC DOUBLE PHASE PROBLEMS

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ABSTRACT. We consider a parametric double phase Dirichlet problem. In the reaction there is a superlinear perturbation term which satisfies a weak Nehari-type monotonicity condition. Using the Nehari manifold method, we show that for all parameters below a critical value, the problem has at least three nontrivial solutions all with sign information. The critical parameter value is precisely identified in terms of the spectrum of the lower exponent part of the differential operator.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. In this paper we study the following parametric double phase Dirichlet problem

$$\begin{cases} -\Delta_p^a u - \Delta_q u = \lambda|u|^{q-2}u + f(z,u) & \text{in } \Omega, \\ u\Big|_{\partial\Omega} = 0, \ 1 < q < p, \ \lambda \in \mathbb{R}. \end{cases} \tag{P_λ}$$

For $a \in L^\infty(\Omega)$ with $a(z) \geq 0$ for a.a. $z \in \Omega$, by Δ_p^a we denote the weighted p-Laplace differential operator defined by

$$\Delta_p^a u = \text{div } (a(z)|\nabla u|^{p-2}\nabla u).$$

In problem (P_λ) the differential operator is the sum of this weighted p-Laplacian with a q-Laplace differential operator, where $1 < q < p$. So, the differential operator of problem (P_λ), is not homogeneous and this makes the analysis of the problem more difficult. In the reaction (right hand side) of (P_λ), we have the combined effects of a parametric term $u \to \lambda|u|^{q-2}u$ and of a Carathéodory perturbation $f(z,x)$ (that is, for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z,x)$ is continuous). We assume that $f(z,\cdot)$ is $(p-1)$-superlinear as $x \to \pm\infty$. We point out that the exponent in the parametric term equals that of the unweighted part of the differential operator. This makes problem (P_λ) different from the well-known "concave-convex problem" where in the reaction we encounter the competing effects of sublinear and superlinear terms.

The differential operator of (P_λ) is related to the so-called "double phase functional" $\widehat{\rho}(\cdot)$ defined by

$$\widehat{\rho}(u) = \int_\Omega [a(z)|\nabla u|^p + |\nabla u|^q]dz.$$

The integrand of this functional is $\vartheta(z,y) = a(z)|y|^p + |y|^q$ for all $z \in \Omega$, all $y \in \mathbb{R}^N$. Since we do not assume that the weight $a(\cdot)$ is bounded away from zero (that is,

ess $\inf\limits_{\Omega} a = c > 0$), the integrand $\vartheta(z, \cdot)$ exhibits unbalanced growth, namely we have

$$|y|^q \le \vartheta(z, y) \le c_0[1 + |y|^p] \quad \text{for all } z \in \Omega, \text{ all } y \in \mathbb{R}^N, \text{ some } c_0 > 0.$$

Such functionals provide models describing strongly anisotropic materials. The modulating coefficient $a(\cdot)$ dictates the geometry of the composite made of two different materials. Marcellini [10] and Zhikov [24], [25] were the first to study such functionals in the context of problems of the calculus of variations and of nonlinear elasticity theory. Recently the interest for such functionals was revived by the work of Mingione and coworkers, who proved important local regularity results for the minimizers of such functionals. We mention the paper of Baroni-Colombo-Mingione [1] and the references therein. We also mention the recent work of Ragusa-Tachikawa [19], where the local regularity results are extended to anisotropic double phase functionals. However, we mention that a global regularity theory remains so far elusive and this is an additional difficulty in the study of problems like (P_λ).

Let $\widehat{\lambda}_1(q) > 0$ denote the principal eigenvalue of the operator $(-\Delta_q, W_0^{1,q}(\Omega))$. Using the Nehari manifold method along the lines of Szulkin-Weth [20] and Lin-Tang [8] (semilinear problems driven by the Dirichlet Laplacian), we show that for all $\lambda < \widehat{\lambda}_1(q)$ problem (P_λ) has at least three nontrivial solutions, all with sign information (positive, negative and nodal (sign-changing)) and with least energy (ground state solutions). Other existence and multiplicity results for different types of double phase equations, can be found in the papers of Colasuonno-Squassina [2], Gasiński-Papageorgiou [3], Gasiński-Winkert [4], [5], [6], Liu-Dai [9], Papageorgiou-Rădulescu-Repovš [11], [12], Papageorgiou-Repovš-Vetro [14], Papageorgiou-Vetro-Vetro [15], [16], Rădulescu [18], Zeng-Bai-Gasiński-Winkert [22], [23].

2. MATHEMATICAL BACKGROUND - HYPOTHESES

The analysis of problem (P_λ), uses Musielak-Orlicz spaces. A comprehensive treatment of such spaces can be found in the recent book of Harjulehto-Hästö [7].

Let $\vartheta : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ = [0, \infty)$) be the integrand defined by

$$\vartheta(z, x) = a(z)x^p + x^q.$$

Let $M(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable}\}$. As usual we identify two such functions which differ only on a Lebesgue-null set. Then the Musielak-Orlicz space $L^\vartheta(\Omega)$ is defined by

$$L^\vartheta(\Omega) = \{u \in M(\Omega) : \rho_\vartheta(u) < \infty\},$$

where $\rho_\vartheta(u) = \int_\Omega [a(z)|u|^p + |u|^q]dz$. We equip $L^\vartheta(\Omega)$ with the so-called "Luxemburg norm" defined by

$$\|u\|_\vartheta = \inf\left[\mu > 0 : \rho_\vartheta\left(\frac{u}{\mu}\right) \le 1\right].$$

With this norm $L^\vartheta(\Omega)$ becomes a separable Banach space which is uniformly convex (thus reflexive by the Milman-Pettis theorem, see Papageorgiou-Winkert [17], p. 225).

The corresponding Musielak-Orlicz-Sobolev space $W^{1,\vartheta}(\Omega)$ is defined by

$$W^{1,\vartheta}(\Omega) = \{u \in L^\vartheta(\Omega) : |\nabla u| \in L^\vartheta(\Omega)\},$$

with ∇u denoting the weak gradient of u. We equip this space with the following norm

$$\|u\|_{1,\vartheta} = \|u\|_\vartheta + \|\nabla u\|_\vartheta \quad \text{for all } u \in W^{1,\vartheta}(\Omega),$$

where $\|\nabla u\|_\vartheta = \| |\nabla u| \|_\vartheta$. Also, we set $W_0^{1,\vartheta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\vartheta}}$.

According to Theorem 6.2.8, p. 130, of Harjulehto-Hästö [7], the Poincaré inequality holds for $W_0^{1,\vartheta}(\Omega)$ and so

$$\|u\| = \|\nabla u\|_\vartheta \quad \text{for all } u \in W_0^{1,\vartheta}(\Omega),$$

is an equivalent norm for $W_0^{1,\vartheta}(\Omega)$. Equipped with these norms, the spaces $W^{1,\vartheta}(\Omega)$ and $W_0^{1,\vartheta}(\Omega)$ are separable Banach spaces which are uniformly convex (hence reflexive).

We impose the following conditions on the exponents p, q and the weight $a(\cdot)$.

H_0: $1 < q < p$, $\dfrac{p}{q} < 1 + \dfrac{1}{N}$ and $a \in L^\infty(\Omega)$, $a(z) \geq 0$ for a.a. $z \in \Omega$, $a \not\equiv 0$.

Remark 2.1. *The second inequality is common in double phase problems and guarantees that $W_0^{1,\vartheta}(\Omega) \hookrightarrow L^q(\Omega)$ compactly and densely.*

In general we have the following embeddings. Recall that $q^* = \begin{cases} \dfrac{Nq}{N-q} & \text{if } q < N \\ +\infty & \text{if } N \leq q \end{cases}$

(the critical Sobolev exponent corresponding to $q > 1$).

Proposition 2.1. *If hypotheses H_0 hold, then*
  (a) *$L^\vartheta(\Omega) \hookrightarrow L^r(\Omega)$, $W_0^{1,\vartheta}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ continuously and densely for all $1 \leq r \leq q^*$;*
  (b) *$W_0^{1,\vartheta}(\Omega) \hookrightarrow L^r(\Omega)$ continuously (resp. compactly) and densely for all $1 \leq r \leq q^*$ (resp. all $1 \leq r < q^*$);*
  (c) *$L^p(\Omega) \hookrightarrow L^\vartheta(\Omega)$ continuously and densely.*

There is a close relation between the norm $\|\cdot\|_\vartheta$ and the modular function $\rho_\vartheta(\cdot)$.

Proposition 2.2. *If hypotheses H_0 hold, then*
  (a) *$\|u\|_\vartheta = \mu \Leftrightarrow \rho_\vartheta\left(\dfrac{u}{\mu}\right) = 1$;*
  (b) *$\|u\|_\vartheta < 1$ (resp. $= 1$, > 1) $\Leftrightarrow \rho_\vartheta(u) < 1$ (resp. $= 1$, > 1);*
  (c) *if $\|u\|_\vartheta < 1$, then $\|u\|_\vartheta^p \leq \rho_\vartheta(u) \leq \|u\|_\vartheta^q$,*
      if $\|u\|_\vartheta > 1$, then $\|u\|_\vartheta^q \leq \rho_\vartheta(u) \leq \|u\|_\vartheta^p$;
  (d) *$\|u_n\|_\vartheta \to 0$ (resp. $\to \infty$) $\Leftrightarrow \rho_\vartheta(u_n) \to 0$ (resp. $\to \infty$).*

Let $\langle\cdot,\cdot\rangle$ denote the duality brackets for the pair $(W_0^{1,\vartheta}(\Omega), W_0^{1,\vartheta}(\Omega)^*)$ and let $A: W_0^{1,\vartheta}(\Omega) \to W_0^{1,\vartheta}(\Omega)^*$ be the nonlinear operator defined by

$$\langle A(u), h \rangle = \int_\Omega (a(z)|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,\vartheta}(\Omega).$$

This operator has the following properties (see Liu-Dai [9]).

Proposition 2.3. *If hypotheses H_0 hold, then the operator $A: W_0^{1,\vartheta}(\Omega) \to W_0^{1,\vartheta}(\Omega)^*$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$ (that is, $A(\cdot)$ has the following property: $u_n \xrightarrow{w} u$ in $W_0^{1,\vartheta}(\Omega)$, $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$ imply that $u_n \to u$ in $W_0^{1,\vartheta}(\Omega)$).*

The hypotheses on the perturbation $f(z,x)$ are the following:

H_1: $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$, and

(i) $|f(z,x)| \leq a(z)[1+|x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^{\infty}(\Omega)$, $p < r < q^*$ (see hypotheses H_0);

(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim\limits_{x\to\pm\infty} \dfrac{F(z,x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$ and if $e(z,x) = f(z,x)x - pF(z,x)$, then

$$0 < \hat{c} \leq \liminf_{x\to\pm\infty} \frac{e(z,x)}{|x|^p} \text{ uniformly for a.a. } z \in \Omega;$$

(iii) $\lim\limits_{x\to 0} \dfrac{f(z,x)}{|x|^{q-2}x} = 0$ uniformly for a.a. $z \in \Omega$;

(iv) for a.a. $z \in \Omega$, the quotient function $x \to \dfrac{f(z,x)}{|x|^{p-1}}$ is increasing on $\mathring{\mathbb{R}}_+ = (0,\infty)$ and on $\mathring{\mathbb{R}}_- = (-\infty, 0)$.

Remark 2.2. *Hypothesis $H_1(ii)$ implies that for a.a. $z \in \Omega$, $f(z,\cdot)$ is $(p-1)$-superlinear as $x \to \pm\infty$. Hypothesis $H_1(iv)$ is weaker than the usual Nehari-type monotonicity condition which requires that the quotient function $x \to \dfrac{f(z,x)}{|x|^{p-1}}$ is strictly increasing on $\mathring{\mathbb{R}}_+$ and on $\mathring{\mathbb{R}}_-$ (see Gasiński-Winkert [4] and Liu-Dai [9]).*

For any function $u \in W_0^{1,\vartheta}(\Omega)$, we set

$$u^{\pm} = \max\{\pm u, 0\}.$$

We know that $u^{\pm} \in W_0^{1,\vartheta}(\Omega)$, $u = u^+ - u^-$ and $|u| = u^+ + u^-$.

We introduce the energy (Euler) functional $\varphi_\lambda : W_0^{1,\vartheta}(\Omega) \to \mathbb{R}$ for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\rho_a(\nabla u) + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{q}\|u\|_q^q - \int_\Omega F(z,u)dz$$

for all $u \in W_0^{1,\vartheta}(\Omega)$, with $\rho_a(\nabla u) = \int_\Omega a(z)|\nabla u|^p dz$. We know that $\varphi_\lambda \in C^1(W_0^{1,\vartheta}(\Omega))$.

Also, in order to produce constant sign solutions, we introduce the positive and negative truncations of $\varphi_\lambda(\cdot)$, namely the C^1-functionals defined by

$$\varphi_\lambda^{\pm}(u) = \frac{1}{p}\rho_a(\nabla u) + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{q}\|u^{\pm}\|_q^q - \int_\Omega F(z, \pm u^{\pm})dz$$

for all $u \in W_0^{1,\vartheta}(\Omega)$. We introduce the following Banach manifolds:

$$N = \{u \in W_0^{1,\vartheta}(\Omega) : \langle \varphi_\lambda'(u), u \rangle = 0,\ u \neq 0\},$$
$$N_+ = \{u \in W_0^{1,\vartheta}(\Omega) : \langle (\varphi_\lambda^+)'(u), u \rangle = 0,\ u \geq 0,\ u \neq 0\},$$
$$N_- = \{u \in W_0^{1,\vartheta}(\Omega) : \langle (\varphi_\lambda^-)'(u), u \rangle = 0,\ u \leq 0,\ u \neq 0\},$$
$$N_0 = \{u \in W_0^{1,\vartheta}(\Omega) : \langle \varphi_\lambda'(u), u^+ \rangle = \langle \varphi_\lambda'(u), u^- \rangle = 0,\ u^{\pm} \neq 0\}.$$

We see that N is the Nehari manifold for the energy functional $\varphi_\lambda(\cdot)$ and N_+, N_-, N_0 are submanifolds of N. Evidently every nontrivial solution of (P_λ) is in N. Similarly N_+ (resp. N_-) includes the positive (resp. negative) solutions of (P_λ), while N_0 contains the nodal (sign-changing) solutions of (P_λ).

In the next section we will prove a multiplicity theorem for (P_λ) under the strong Nehari-type monotonicity condition and then in Section 4 using an approximation argument, we will prove the multiplicity theorem under the relaxed monotonicity condition.

For this reason we introduce the following more restrictive set of hypotheses on the perturbation $f(z, x)$.

H_1': $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, hypotheses $H_1'(i), (ii), (iii)$ are the same as the corresponding hypotheses $H_1(i), (ii), (iii)$ and

(iv) for a.a. $z \in \Omega$, the quotient function $x \to \dfrac{f(z, x)}{|x|^{p-1}}$ is strictly increasing on $\mathring{\mathbb{R}}_+$ and on $\mathring{\mathbb{R}}_-$.

Finally recall that $\widehat{\lambda}_1(q)$ denotes the first eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$. We know that $\widehat{\lambda}_1(q) > 0$, it is isolated and simple and

$$\widehat{\lambda}_1(q) = \inf \left[\frac{\|\nabla u\|_q^q}{\|u\|_q^q} : u \in W_0^{1,q}(\Omega),\ u \neq 0 \right]. \tag{1}$$

The infimum on (1) is realized on the corresponding one dimensional eigenspace, the nontrivial elements of which have constant sign. If the boundary $\partial\Omega$ is a C^2-manifold, then the eigenfunctions of $\widehat{\lambda}_1(q) > 0$ belong in $C_0^1(\overline{\Omega})$.

3. Multiple Solutions - Strong Monotonicity

In this section we prove a multiplicity theorem for least energy solutions with sign information, using the strong Nehari-type monotonicity condition (see hypotheses H_1').

Actually, for the first results, we do not need this stronger monotonicity condition. In what follows $\beta(s) = \dfrac{1 - s^q}{q} - \dfrac{1 - s^p}{p}$ for all $s \geq 0$.

Proposition 3.1. *If hypotheses H_0, H_1 hold, then for all $u \in W_0^{1,\vartheta}(\Omega)$ and all $\tau, t \geq 0$, we have*

$$\varphi_\lambda(u) \geq \varphi_\lambda(\tau u^+ - t u^-) + \frac{1 - \tau^p}{p} \langle \varphi_\lambda'(u), u^+ \rangle - \frac{1 - t^p}{p} \langle \varphi_\lambda'(u), u^- \rangle$$
$$+ \beta(\tau) \left[\|\nabla u^+\|_q^q - \lambda \|u^+\|_q^q \right] + \beta(t) \left[\|\nabla u^-\|_q^q - \lambda \|u^-\|_q^q \right].$$

Proof. Let $u \in W_0^{1,\vartheta}(\Omega)$ and $\tau, t \geq 0$. We have

$$\varphi_\lambda(u) - \varphi_\lambda(\tau u^+ - t u^-)$$
$$= \varphi_\lambda(u^+) - \varphi_\lambda(\tau u^+) + \varphi_\lambda(-u^-) - \varphi_\lambda(t(-u^-))$$
$$= \frac{1 - \tau^p}{p} \rho_a(\nabla u^+) + \frac{1 - \tau^q}{q} \left[\|\nabla u^+\|_q^q - \lambda \|u^+\|_q^q \right] - \int_\Omega \left[F(z, u^+) - F(z, \tau u^+) \right] dz$$
$$+ \frac{1 - t^p}{p} \rho_a(\nabla u^-) + \frac{1 - t^q}{q} \left[\|\nabla u^-\|_q^q - \lambda \|u^-\|_q^q \right] - \int_\Omega \left[F(z, -u^-) - F(z, t(-u^-)) \right] dz. \tag{2}$$

For $\vartheta \geq 0$ and $x \neq 0$, we have

$$\frac{1 - \vartheta^p}{p} f(z, x) x + F(z, \vartheta x) - F(z, x)$$

$$= \int_\vartheta^1 f(z,x)xs^{p-1}ds - \int_\vartheta^1 \frac{d}{ds}F(z,sx)ds$$

$$= \int_\vartheta^1 f(z,x)xs^{p-1}ds - \int_\vartheta^1 f(z,sx)xds \quad \text{(using the chain rule)}$$

$$= \int_\vartheta^1 \left[\frac{f(z,x)}{|x|^{p-1}} - \frac{f(z,sx)}{(s|x|)^{p-1}}\right]x|x|^{p-1}s^{p-1}ds$$

$$\geq 0 \quad \text{(see hypothesis } H_1(iv)\text{)},$$

$$\Rightarrow \quad \frac{1-\vartheta^p}{p}f(z,x)x \geq F(z,x) - F(z,\vartheta x) \tag{3}$$

$$\text{for a.a. } z \in \Omega, \text{ all } x \neq 0 \text{ and all } \vartheta \geq 0.$$

Returning to (2) and using (3), we obtain

$$\varphi_\lambda(u) - \varphi_\lambda(\tau u^+ - tu^-)$$

$$\geq \frac{1-\tau^p}{p}\rho_a(\nabla u^+) + \frac{1-\tau^q}{q}\left[\|\nabla u^+\|_q^q - \lambda\|u^+\|_q^q\right] - \frac{1-\tau^p}{p}\int_\Omega f(z,u^+)u^+dz$$

$$+ \frac{1-t^p}{p}\rho_a(\nabla u^-) + \frac{1-t^q}{q}\left[\|\nabla u^-\|_q^q - \lambda\|u^-\|_q^q\right] - \frac{1-t^p}{p}\int_\Omega f(z,-u^-)(-u^-)dz$$

$$= \frac{1-\tau^p}{p}\langle\varphi_\lambda'(u), u^+\rangle + \beta(\tau)\left[\|\nabla u^+\|_q^q - \lambda\|u^+\|_q^q\right]$$

$$- \frac{1-t^p}{p}\langle\varphi_\lambda'(u), -u^-\rangle + \beta(t)\left[\|\nabla u^-\|_q^q - \lambda\|u^-\|_q^q\right].$$

$$\square$$

In a similar fashion, we show the same inequality for the functionals $\varphi_\lambda^\pm(\cdot)$.

Proposition 3.2. *If hypotheses H_0, H_1 hold, then for all $u \in W_0^{1,\vartheta}(\Omega)$ and all $\tau, t \geq 0$, we have*

$$\varphi_\lambda^\pm(u) \geq \varphi_\lambda^\pm(\tau u^+ - tu^-) + \frac{1-\tau^p}{p}\langle(\varphi_\lambda^\pm)'(u), u^+\rangle - \frac{1-t^p}{p}\langle(\varphi_\lambda^\pm)'(u), u^-\rangle$$

$$+ \beta(\tau)\left[\|\nabla u^+\|_q^q - \lambda\|u^+\|_q^q\right] + \beta(t)\left[\|\nabla u^-\|_q^q - \lambda\|u^-\|_q^q\right].$$

Note that $\beta(s) \geq \beta(1) = 0$ for all $s \geq 0$ and $\varphi_\lambda(u) = \varphi_\lambda(u^+ - u^-)$. Then using the above propositions and (1), we infer the following corollaries.

Corollary 3.1. *If hypotheses H_0, H_1 hold, $u \in N_0$ and $\lambda < \widehat{\lambda}_1(q)$, then $\varphi_\lambda(u) = \max_{\tau,t\geq 0} \varphi_\lambda(\tau u^+ - tu^-)$.*

Corollary 3.2. *If hypotheses H_0, H_1 hold, $u \in N$ and $\lambda < \widehat{\lambda}_1(q)$, then $\varphi_\lambda(u) = \max_{\tau\geq 0} \varphi_\lambda(\tau u)$.*

Corollary 3.3. *Suppose that hypotheses H_0, H_1 hold. We have*

(a) *if $u \in N_+$ and $\lambda < \widehat{\lambda}_1(q)$, then $\varphi_\lambda^+(u) = \max_{\tau\geq 0} \varphi_\lambda^+(\tau u)$;*

(b) *if $u \in N_-$ and $\lambda < \widehat{\lambda}_1(q)$, then $\varphi_\lambda^-(u) = \max_{t\geq 0} \varphi_\lambda^-(tu)$.*

The next two propositions establish the nonemptiness of the Nehari manifolds. Now we bring in the picture the stronger hypotheses H_1'.

Proposition 3.3. *If hypotheses* H_0, H_1' *hold,* $\lambda < \widehat{\lambda}_1(q)$ *and* $u \in W_0^{1,\vartheta}(\Omega)$, *then there exist unique* $\tau_u, t_u > 0$ *such that* $\tau_u u^+ - t_u u^- \in N_0$.

Proof. We consider the fibering function

$$\xi_+(t) = \varphi_\lambda(tu^+) \quad \text{for all } t > 0.$$

Using the chain rule, we see that

$$\xi_+'(t) = 0$$

$$\Leftrightarrow \quad \rho_a(\nabla(tu^+)) + \|\nabla(tu^+)\|_q^q = \lambda\|tu^+\|_q^q + \int_\Omega f(z, tu^+)(tu^+)dz$$

$$\Leftrightarrow \quad \rho_a(\nabla u^+) + \frac{1}{t^{p-q}}\left[\|\nabla u^+\|_q^q - \lambda\|u^+\|_q^q\right] = \int_\Omega \frac{f(z, tu^+)(tu^+)}{t^p}dz. \tag{4}$$

In (4) the left hand side is strictly decreasing in $t > 0$ (recall that $q < p$), while on account of hypothesis $H_1'(iv)$ the right hand side is strictly increasing in $t > 0$.

Note that because of hypotheses $H_1'(i), (iii)$, we see that given $\varepsilon > 0$, we can find $c_1 = c_1(\varepsilon) > 0$ such that

$$F(z, x) \le \frac{\varepsilon}{q}|x|^q + c_1|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{5}$$

Then we have

$$\xi_+(t) = \varphi_\lambda(tu^+)$$

$$\ge \frac{t^p}{p}\rho_a(\nabla u^+) + \frac{t^q}{q}\left[\|\nabla u^+\|_q^q - (\lambda + \varepsilon)\|u^+\|_q^q\right] - c_1 t^r\|u^+\|_r^r \quad \text{(see (5))}.$$

Choosing $\varepsilon \in (0, \widehat{\lambda}_1(q) - \lambda)$ (recall that $\lambda < \widehat{\lambda}_1(q)$), we have

$$\xi_+(t) \ge c_2 t^p - c_3 t^r \quad \text{for some } c_2, c_3 > 0, \text{ all } t \ge 0,$$

$$\Rightarrow \quad \xi_+(t) > 0 \quad \text{for all } t \in (0, 1) \text{ small (since } p < r).$$

On the other hand, hypotheses $H_1'(i), (ii)$ imply that given any $\eta > 0$, we can find $c_4 = c_4(\eta) > 0$ such that

$$F(z, x) \ge \frac{\eta}{p}|x|^p - c_4 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

We have

$$\xi_+(t) = \varphi_\lambda(tu^+)$$

$$\le \frac{t^p}{p}\rho_a(\nabla u^+) + \frac{t^q}{q}\left[\|\nabla u^+\|_q^q - \lambda\|u^+\|_q^q\right] - \frac{\eta t^p}{p}\|u^+\|_p^p + c_4|\Omega|_N$$

$$\text{(by } |\cdot|_N \text{ we denote the Lebesgue measure on } \mathbb{R}^N)$$

$$= \frac{t^p}{p}\left[\rho_a(\nabla u^+) - \eta\|u^+\|_p^p\right] + \frac{t^q}{q}\left[\|\nabla u^+\|_q^q - \lambda\|u^+\|_q^q\right] + c_4|\Omega|_N.$$

Since $\eta > 0$ is arbitrary, choosing $\eta > 0$ big and recalling that $\lambda < \widehat{\lambda}_1(q)$, we obtain

$$\xi_+(t) \le c_5 t^q - c_6 t^p + c_4|\Omega|_N \quad \text{for some } c_5, c_6 > 0, \text{ all } t \ge 0.$$

Since $q < p$, it follows that for $t > 0$ big we have

$$\xi_+(t) < 0.$$

Therefore we infer that there exists unique $\tau_u > 0$ such that

$$\xi_+(\tau_u) = \max_{\mathbb{R}_+} \xi_+.$$

Similarly working this time with the fibering function

$$\xi_-(t) = \varphi_\lambda(t(-u^-)) \quad \text{for all } t > 0,$$

we produce a unique $t_u > 0$ such that

$$\xi_-(t_u) = \max_{\mathbb{R}_+} \xi_-.$$

Then from Corollary 3.1, we conclude that

$$\tau_u u^+ - t_u u^- \in N_0.$$

$\square$

In a similar fashion we prove the analogous results for the functionals $\varphi_\lambda^\pm(\cdot)$.

Proposition 3.4. *Suppose that hypotheses H_0, H_1' hold. We have*

(a) *if $u \in W_0^{1,\vartheta}(\Omega)$, $u^+ \not\equiv 0$ and $\lambda < \widehat{\lambda}_1(q)$, then there exists unique $\tau_u^+ > 0$ such that $\tau_u^+ u^+ \in N_+$;*

(b) *if $u \in W_0^{1,\vartheta}(\Omega)$, $u^- \not\equiv 0$ and $\lambda < \widehat{\lambda}_1(q)$, then there exists unique $t_u^- > 0$ such that $t_u^-(-u^-) \in N_-$.*

We set

$$\widehat{m}_\lambda^0 = \inf_{N_0} \varphi_\lambda, \quad \widehat{m}_\lambda^+ = \inf_{N_+} \varphi_\lambda^+, \quad \widehat{m}_\lambda^- = \inf_{N_-} \varphi_\lambda^-.$$

Also we introduce the following subsets of $W_0^{1,\vartheta}(\Omega)$:

$$W_n^{1,\vartheta}(\Omega) = \{u \in W_0^{1,\vartheta}(\Omega) : u^\pm \not\equiv 0\} \quad \text{(the nodal elements of } W_0^{1,\vartheta}(\Omega)\text{)},$$

$$W_+^{1,\vartheta}(\Omega) = \{u \in W_0^{1,\vartheta}(\Omega) : u^+ \not\equiv 0\},$$

$$W_-^{1,\vartheta}(\Omega) = \{u \in W_0^{1,\vartheta}(\Omega) : u^- \not\equiv 0\}.$$

Using these sets we can have minimax characterizations of $\widehat{m}_\lambda^0$ and $\widehat{m}_\lambda^\pm$.

Proposition 3.5. *If hypotheses H_0, H_1' hold and $\lambda < \widehat{\lambda}_1(q)$, then*

(a) $\widehat{m}_\lambda^0 = \displaystyle\inf_{u \in W_n^{1,\vartheta}(\Omega)} \max_{\tau,t \geq 0} \varphi_\lambda(\tau u^+ - tu^-)$;

(b) $\widehat{m}_\lambda^+ = \displaystyle\inf_{u \in W_+^{1,\vartheta}(\Omega)} \max_{\tau \geq 0} \varphi_\lambda^+(\tau u)$;

(c) $\widehat{m}_\lambda^- = \displaystyle\inf_{u \in W_-^{1,\vartheta}(\Omega)} \max_{t \geq 0} \varphi_\lambda^-(tu)$.

Proof. (a): Let $\mu_\lambda = \displaystyle\inf_{u \in W_n^{1,\vartheta}(\Omega)} \max_{\tau,t \geq 0} \varphi_\lambda(\tau u^+ - tu^-)$. Since $N_0 \subseteq W_n^{1,\vartheta}(\Omega)$ on account of Corollary 3.1 we have

$$\mu_\lambda \leq \inf_{u \in N_0} \max_{\tau,t \geq 0} \varphi_\lambda(\tau u^+ - tu^-) = \widehat{m}_\lambda^0. \tag{6}$$

On the other hand, if $u \in W_n^{1,\vartheta}(\Omega)$, then

$$\max_{\tau,t \geq 0} \varphi_\lambda(\tau u^+ - tu^-) \geq \varphi_\lambda(\tau_u u^+ - t_u u^-) \quad \text{(see Proposition 3.3)}$$

$$\geq \widehat{m}_\lambda^0 \quad \text{(since } \tau_u u^+ - t_u u^- \in N_0\text{)},$$

$$\Rightarrow \quad \mu_\lambda \geq \widehat{m}_\lambda^0. \tag{7}$$

From (6) and (7) we conclude that $\mu_\lambda = \widehat{m}_\lambda^0$.

(b) and (c): These parts are proved similarly using this time the functionals $\varphi_\lambda^+(\cdot)$, $\varphi_\lambda^-(\cdot)$, Corollary 3.3 and Proposition 3.4. $\hspace{2cm}\square$

The Nehari manifold N is much smaller that $W_0^{1,\vartheta}(\Omega)$ and so the functional $\varphi_\lambda(\cdot)$ restricted on N exhibits properties which fail to be true globally.

Proposition 3.6. *If hypotheses H_0, H_1' hold and $\lambda < \widehat{\lambda}_1(q)$, then $\varphi_\lambda\big|_N$ is coercive.*

Proof. We argue by contradiction. So, suppose that the assertion of the proposition is not true. Then we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq N$ such that

$$\begin{aligned} \varphi_\lambda(u_n) &\le c_7 \quad \text{for some } c_7 > 0, \text{ all } n \in \mathbb{N}, \\ \|u_n\| &\to \infty. \end{aligned} \tag{8}$$

Let $v_n = \dfrac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|v_n\| = 1$, $n \in \mathbb{N}$, and so we may assume that

$$v_n \xrightarrow{w} v \text{ in } W_0^{1,\vartheta}(\Omega) \text{ and } v_n \to v \text{ in } L^r(\Omega) \tag{9}$$

(recall that $W_0^{1,\vartheta}(\Omega) \hookrightarrow L^r(\Omega)$ compactly, see Proposition 2.1).

From (8) we have

$$\rho_a(\nabla v_n) + \frac{p}{q\|u_n\|^{p-q}}\left[\|\nabla v_n\|_q^q - \lambda\|v_n\|_q^q\right] - \int_\Omega \frac{pF(z,u_n)}{\|u_n\|^p}dz \le \frac{pc_7}{\|u_n\|^p} \quad \text{for all } n \in \mathbb{N}. \tag{10}$$

Since $u_n \in N$, we have

$$-\rho_a(\nabla v_n) - \frac{1}{\|u_n\|^{p-q}}\left[\|\nabla v_n\|_q^q - \lambda\|v_n\|_q^q\right] + \int_\Omega \frac{f(z,u_n)u_n}{\|u_n\|^p}dz = 0 \quad \text{for all } n \in \mathbb{N}. \tag{11}$$

Adding (10) and (11) and recalling that $\lambda < \widehat{\lambda}_1(q)$ (see (1)), $q < p$, we obtain

$$\int_\Omega \frac{f(z,u_n)u_n - pF(z,u_n)}{\|u_n\|^p}dz \le \varepsilon_n \quad \text{with } \varepsilon_n \downarrow 0,$$

$$\Rightarrow \quad \int_\Omega \frac{f(z,u_n)u_n - pF(z,u_n)}{u_n^p}v_n dz \le \varepsilon. \tag{12}$$

We claim that $v \ne 0$. To see this, suppose that $v = 0$. Then for $\ell > 0$, on account of Corollary 3.2, we have

$$\begin{aligned} c_7 &\ge \varphi_\lambda(u_n) \\ &\ge \varphi_\lambda\left(\frac{\ell}{\|u_n\|}u_n\right) \quad (\text{recall } u_n \in \mathbb{N}) \\ &= \frac{\ell^p}{p}\rho_a(\nabla v_n) + \frac{\ell^q}{q}\left[\|\nabla v_n\|_q^q - \lambda\|u_n\|_q^q\right] - \int_\Omega F(z,\ell v_n)dz \\ &\ge \frac{\ell^p}{p} - \int_\Omega F(z,\ell v_n)dz \end{aligned}$$

(recall that $\lambda < \widehat{\lambda}_1(q)$, $\|v_n\| = 1$ and see Proposition 2.2).

Passing to the limit as $n \to \infty$, using (9) and recalling that we have assumed that $v = 0$, we obtain

$$\ell^p \le pc_7.$$

But $\ell > 0$ is arbitrary. So, we let $\ell \to \infty$ and have contradiction. This proves that $v \neq 0$.

Let $\widehat{\Omega} = \{z \in \Omega : v(z) \neq 0\}$. We know that $|\widehat{\Omega}|_N > 0$. Then from (12), passing to the limit as $n \to \infty$ and using Fatou's lemma and hypothesis $H_1'(ii)$, we obtain

$$0 < \widehat{c} \int_\Omega |v|^p dz \leq 0,$$

a contradiction. This proves that $\varphi_\lambda\big|_N$ is coercive. □

With minor modifications in the above proof we can prove the same result for $\varphi_\lambda^+\big|_{N_+}$, $\varphi_\lambda^-\big|_{N_-}$ ($\lambda < \widehat{\lambda}_1(q)$).

Proposition 3.7. *If hypotheses H_0, H_1' hold and $\lambda < \widehat{\lambda}_1(q)$, then $\varphi_\lambda^\pm\big|_{N_\pm}$ are both coercive.*

Proof. We do the proof for $\varphi_\lambda^+\big|_{N_+}$, the proof for $\varphi_\lambda^-\big|_{N_-}$ being similar. Again we proceed indirectly. So, suppose that $\varphi_\lambda^+\big|_{N_+}$ is not coercive. Then we can find $\{u_n\}_{n \in \mathbb{N}} \subseteq N_+$ such that

$$\varphi_\lambda^+(u_n) \leq c_8 \quad \text{for some } c_8 > 0, \text{ all } n \in \mathbb{N},$$
$$\|u_n\| \to \infty. \tag{13}$$

Freom (13) we have

$$\rho_a(\nabla u_n^+) + \frac{p}{q}\left[\|\nabla u_n^+\|_q^q - \lambda\|u_n^+\|_q^q\right] - \int_\Omega pF(z, u_n^+)dz \leq pc_8 \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

Moreover, since $u_n \in N_+$, we have

$$\rho_a(\nabla u_n^+) + \left[\|\nabla u_n^+\|_q^q - \lambda\|u_n^+\|_q^q\right] = \int_\Omega f(z, u_n^+)u_n^+ dz \quad \text{for all } n \in \mathbb{N}. \tag{15}$$

From (14), (15) and since $q < p$, we infer that

$$\int_\Omega e(z, u_n^+)dz = \int_\Omega [f(z, u_n^+)u_n^+ - pF(z, u_n^+)]dz \leq pc_8. \tag{16}$$

Suppose that $\|u_n^+\| \to \infty$ and set $v_n = \dfrac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then we may assume that

$$v_n \xrightarrow{w} v \text{ in } W_0^{1,\vartheta}(\Omega) \text{ and } v_n \to v \text{ in } L^r(\Omega) \text{ (see Proposition 2.1).}$$

From (16) we have

$$\int_\Omega \frac{f(z, u_n^+)u_n^+ - pF(z, u_n^+)}{\|u_n^+\|^p}dz \leq \varepsilon_n' \quad \text{with } \varepsilon_n' \downarrow 0. \tag{17}$$

As in the proof of Proposition 3.6 and using the fact that for all $y \in W_0^{1,\vartheta}(\Omega)$, $\varphi_\lambda^+(y^+) \leq \varphi_\lambda^+(y)$, we show that $v \neq 0$ and from that we derive a contradiction as in the proof of Proposition 3.6. Therefore $\{u_n^+\}_{n \in \mathbb{N}} \subseteq W_0^{1,\vartheta}(\Omega)$ is bounded. This fact and (13) imply that $\{u_n^-\}_{n \in \mathbb{N}} \subseteq W_0^{1,\vartheta}(\Omega)$ is bounded (see Proposition 2.2 and hypothesis $H_1'(i)$). Therefore $\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,\vartheta}(\Omega)$ is bounded and this contradicts (13). Hence $\varphi_\lambda^+\big|_{N_+}$ is coercive. Similarly we show that $\varphi_\lambda^-\big|_{N_-}$ is coercive. □

Using Propositions 3.6 and 3.7 we will show that $\widehat{m}_\lambda^0$ is realized on N_0, while $\widehat{m}_\lambda^\pm$ are realized on $N_\pm$.

Proposition 3.8. *If hypotheses H_0, H_1' hold and $\lambda < \widehat{\lambda}_1(q)$, then*

(a) *there exists $y_0 \in N_0$ such that $\varphi_\lambda(y_0) = \inf\limits_{N_0} \varphi_\lambda = \widehat{m}_\lambda^0 > 0$;*

(b) *there exists $u_0 \in N_+$ such that $\varphi_\lambda^+(u_0) = \inf\limits_{N_+} \varphi_\lambda = \widehat{m}_\lambda^+ > 0$;*

(c) *there exists $v_0 \in N_-$ such that $\varphi_\lambda^-(v_0) = \inf\limits_{N_-} \varphi_\lambda = \widehat{m}_\lambda^- > 0$.*

Proof. (a): Let $\{y_n\}_{n \in \mathbb{N}} \subseteq N_0$ be a minimizing sequence for $\inf\limits_{N_0} \varphi_\lambda$, that is,

$$\varphi_\lambda(y_n) \downarrow \widehat{m}_\lambda^0 \quad \text{as } n \to \infty.$$

From Proposition 3.6, we know that $\{y_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,\vartheta}(\Omega)$ is bounded. So, we may assume that

$$y_n \xrightarrow{w} y_0 \text{ in } W_0^{1,\vartheta}(\Omega) \text{ and } y_n \to y_0 \text{ in } L^r(\Omega), \tag{18}$$

$$\Rightarrow \quad y_n^\pm \xrightarrow{w} y_0^\pm \text{ in } W_0^{1,\vartheta}(\Omega) \text{ and } y_n^\pm \to y_0^\pm \text{ in } L^r(\Omega). \tag{19}$$

Recall that $y_n \in N_0$ for all $n \in \mathbb{N}$. So, we have

$$\langle \varphi_\lambda'(y_n), y_n^+ \rangle = 0,$$

$$\Rightarrow \quad \rho_a(\nabla y_n^+) + \left[\|\nabla y_n^+\|_q^q - \lambda\|y_n^+\|_q^q\right] = \int_\Omega f(z, y_n^+)y_n^+ \, dz,$$

$$\Rightarrow \quad \rho_a(\nabla y_n^+) + c_9\|\nabla y_n^+\|_q^q \leq \int_\Omega f(z, y_n^+)y_n^+ \, dz, \tag{20}$$

$$\text{for some } c_9 > 0, \text{ all } n \in \mathbb{N} \text{ (recall that } \lambda < \widehat{\lambda}_1(q)\text{)}.$$

On account of hypotheses $H_1'(i), (iii)$, given $\varepsilon > 0$, we can find $c_{10} = c_{10}(\varepsilon) > 0$ such that

$$f(z, x)x \leq \varepsilon|x|^q + c_{10}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{21}$$

We use (21) in (20) and choosing $\varepsilon > 0$ small, we obtain

$$\rho_a(\nabla y_n^+) + c_{11}\|\nabla y_n^+\|_q^q \leq c_{10}\|y_n^+\|_r^r \quad \text{for some } c_{11} > 0, \text{ all } n \in \mathbb{N},$$

$$\Rightarrow \quad \rho_a(\nabla y_n^+) \leq c_{12}\|y_n^+\|_r^r \quad \text{for some } c_{12} > 0, \text{ all } n \in \mathbb{N}.$$

Using Proposition 2.2 and the fact that $W_0^{1,\vartheta}(\Omega) \hookrightarrow L^r(\Omega)$ (see Proposition 2.1), we obtain

$$\min\left\{\|y_n^+\|_r^p, \|y_n^+\|_r^q\right\} \leq c_{13}\|y_n^+\|_r^r \quad \text{for some } c_{13} > 0, \text{ all } n \in \mathbb{N},$$

$$\Rightarrow \quad c_{14} \leq \|y_n^+\|_r \quad \text{for some } c_{14} > 0, \text{ all } n \in \mathbb{N} \text{ (recall } q < p < r\text{)}.$$

Passing to the limit as $n \to \infty$ and using (19), we obtain

$$c_{14} \leq \|y_0^+\|_r,$$

$$\Rightarrow \quad y_0^+ \neq 0 \quad \text{and in a similar way we show that } y_0^- \neq 0. \tag{22}$$

Since $y_n \in N_0$, $n \in \mathbb{N}$, we have

$$\langle \varphi_\lambda'(y_n), y_n^+ \rangle = 0 \quad \text{for all } n \in \mathbb{N},$$

$$\Rightarrow \quad \rho_a(\nabla y_n^+) + \|\nabla y_n^+\|_q^q = \lambda\|y_n^+\|_q^q + \int_\Omega f(z, y_n^+)y_n^+ \, dz \quad \text{for all } n \in \mathbb{N}.$$

Note that the modular function $\rho_a(\cdot)$ is continuous, convex, hence sequentially weakly lower semicontinuous. Therefore if we pass to the limit as $n \to \infty$ and use (19), we obtain

$$\rho_a(\nabla y_0^+) + \|\nabla y_0^+\|_q^q \leq \lambda \|y_0^+\|_q^q + \int_\Omega f(z, y_0^+) y_0^+ dz,$$

$$\Rightarrow \quad \langle \varphi_\lambda'(y_0), y_0^+ \rangle \leq 0. \tag{23}$$

In a similar fashion, we show that

$$\varphi_\lambda'(y_0), -y_0^- \rangle \leq 0. \tag{24}$$

Since $y_n \in N_0 \subseteq N$, we have

$$\widehat{m}_\lambda^0 = \lim_{n \to \infty} \left[\varphi_\lambda(y_n) - \frac{1}{p} \langle \varphi_\lambda'(y_n), y_n \rangle \right]$$

$$= \lim_{n \to \infty} \left[\left(\frac{1}{q} - \frac{1}{p} \right) \left(\|\nabla y_n\|_q^q - \lambda \|y_n\|_q^q \right) + \int_\Omega \left(\frac{1}{p} f(z, y_n) y_n - F(z, y_n) \right) dz \right]$$

$$\geq \left(\frac{1}{q} - \frac{1}{p} \right) \left(\|\nabla y_0\|_q^q - \lambda \|y_0\|_q^q \right) + \int_\Omega \left(\frac{1}{p} f(z, y_0) y_0 - F(z, y_0) \right) dz \quad \text{(see (18))}$$

$$= \varphi_\lambda(y_0) - \frac{1}{p} \langle \varphi_\lambda'(y_0), y_0 \rangle$$

$$\geq \varphi_\lambda(\tau_0 y_0^+ - t_0 y_0^-) + \frac{1 - \tau_0^p}{p} \langle \varphi_\lambda'(y_0), y_0^+ \rangle - \frac{1 - t_0^p}{p} \langle \varphi_\lambda'(y_0), y_0^- \rangle - \frac{1}{p} \langle \varphi_\lambda'(y_0), y_0 \rangle$$

with $\tau_0 = \tau_{y_0}$, $t_0 = t_{y_0}$ (see Propositions 3.1, 3.3 and recall that $\beta \geq 0$, $\lambda < \widehat{\lambda}_1(q)$)

$$\geq \widehat{m}_\lambda^0 - \frac{\tau_0^p}{p} \langle \varphi_\lambda'(y_0), y_0^+ \rangle + \frac{t_0^p}{p} \langle \varphi_\lambda'(y_0), y_0^- \rangle \quad \text{(see Proposition 3.3)}$$

$$\geq \widehat{m}_\lambda^0 \quad \text{(see (23), (24))}.$$

It follows that

$$\langle \varphi_\lambda'(y_0), y_0^+ \rangle = \langle \varphi_\lambda'(y_0), y_0^- \rangle = 0,$$

$$\Rightarrow \quad y_0 \in N_0 \quad \text{(see (22))}. \tag{25}$$

From the sequential weak lower semicontinuity of $\varphi_\lambda(\cdot)$, we have

$$\varphi_\lambda(y_0) \leq \lim_{n \to \infty} \varphi_\lambda(y_n) = \widehat{m}_\lambda^0,$$

$$\Rightarrow \quad \varphi_\lambda(y_0) = \widehat{m}_\lambda^0 \quad \text{(see (25))}.$$

Moreover, we have

$$\widehat{m}_\lambda^0 = \varphi_\lambda(y_0) = \frac{1}{p} \rho_a(\nabla y_0) + \frac{1}{q} \left[\|\nabla y_0\|_q^q - \lambda \|y_0\|_q^q \right] - \int_\Omega F(z, y_0) dz$$

$$> \frac{1}{p} \left[\rho_a(\nabla y_0) + \|\nabla y_0\|_q^q - \lambda \|y_0\|_q^q - \int_\Omega f(z, y_0) y_0 dz \right]$$

$$= 0.$$

Here first we have used (3) with $\vartheta = 0$ and with the remark that if $H_1'(iv)$ holds, then the inequality is strict (see the proof of Proposition 3.1).

Then we used that $q < p$, $\lambda < \widehat{\lambda}_1(q)$ and $y_0 \in N_0 \subseteq N$ (see (25)). So, finally we conclude that $\widehat{m}_\lambda^0 > 0$.

(b) and (c): These parts are proved in a similar fashion. $\square$

Remark 3.1. *The above proof also shows that there exists $c_{15} > 0$ such that*

$$0 < c_{15} \leq \|u\| \quad \text{for all } u \in N. \tag{26}$$

Indeed, if $u \in N$, $\|u\| \leq 1$, from (21) and Proposition 2.1, we have

$$\varepsilon c_{16} \|u\|^q + c_{17} \|u\|^r$$

$$\geq \int_\Omega f(z, u) u \, dz \quad \text{for some } c_{16}, c_{17} > 0$$

$$= \rho_a(\nabla u) + \left[\|\nabla u\|_q^q - \lambda \|u\|_q^q \right] \quad \text{(since $u \in N$)}$$

$$\geq \rho_a(\nabla u) + c_{18} \|\nabla u\|_q^q \quad \text{for some } c_{18} > 0 \text{ (recall that $\lambda < \widehat{\lambda}_1(q)$ and see (1))}$$

$$\geq c_{19} \|u\|^q \quad \text{for some } c_{19} > 0 \text{ (see Proposition 2.2),}$$

$$\Rightarrow \quad c_{20} \|u\|^q \leq \|u\|^r \quad \text{for some } c_{20} > 0 \text{ (choosing $\varepsilon > 0$ small).}$$

So, (26) holds, since $q < p < r$.

Next following the arguments of Willem [21] (p. 74) and of Szulkin-Weth [20] (p. 612), we show that N_0 is a natural constraint for the functional $\varphi_\lambda(\cdot)$ (see Papageorgiou-Rădulescu-Repovš [13], Definition 5.5.11, p. 425). This way we can show that $y_0 \in N_0$ from Proposition 3.8 is a nodal solution of (P_λ) where $\lambda < \widehat{\lambda}_1(q)$.

Proposition 3.9. *If hypotheses H_0, H_1' hold, $\lambda < \widehat{\lambda}_1(q)$ and $y_0 \in N_0$ is as in Proposition 3.8 (a), then $y_0 \in K_{\varphi_\lambda} = \{y \in W_0^{1,\vartheta}(\Omega) : \varphi_\lambda'(y) = 0\}$ and so $y_0 \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$ is a nodal solution of (P_λ).*

Proof. Since $y_0 \in N_0$ (see (25)), we have

$$\langle \varphi_\lambda'(y_0^+), y_0^+ \rangle = 0 = \langle \varphi_\lambda'(-y_0^-), -y_0^- \rangle.$$

For $\tau, t \in \mathring{\mathbb{R}}_+ \setminus \{1\}$ (recall $\mathring{\mathbb{R}}_+ = (0, \infty)$), we have

$$\varphi_\lambda(\tau y_0^+ - t y_0^-) = \varphi_\lambda(\tau y_0^+) + \varphi_\lambda(t(-y_0^-))$$

$$\leq \varphi_\lambda(y_0^+) + \varphi_\lambda(-y_0^-) \quad \text{(see Corollary 3.3)}$$

$$= \varphi_\lambda(y_0) = \widehat{m}_\lambda^0 \quad \text{(see Proposition 3.8).} \tag{27}$$

Arguing by contradiction, suppose that $\varphi_\lambda'(y_0) \neq 0$. Consider the parallelogram $P = \left(\frac{1}{2}, \frac{3}{2} \right)^2$ and the function $\gamma(\tau, t) = \tau y_0^+ - t y_0^-$, $\tau, t \geq 0$. From (27), we have

$$\mu = \max[\varphi_\lambda(\gamma(\tau, t)) : (\tau, t) \in \partial P] < \widehat{m}_\lambda^0. \tag{28}$$

We apply Lemma 2.3, p. 38, of Willem [21] (the quantitative deformation lemma), with $\varepsilon = \min\left\{ \frac{\widehat{m}_\lambda^0 - \mu}{4}, \frac{\eta \delta}{8} \right\}$ and $\mathcal{S} = \overline{B}_\delta(y_0) = \{y \in W_0^{1,\vartheta}(\Omega) : \|y - y_0\| \leq \delta\}$, for some $\delta > 0$ and $\eta > 0$, and obtain a deformation $h_0(s, u)$ such that

$$h_0(1, u) = u \quad \text{for all } u \in \varphi_\lambda^{-1}([\widehat{m}_\lambda^0 - 2\varepsilon, \widehat{m}_\lambda^0 + 2\varepsilon]),$$

$$h_0(1, \varphi_\lambda^{\widehat{m}_\lambda^0 + \varepsilon} \cap \overline{B}_\delta(y_0)) \subseteq \varphi_\lambda^{\widehat{m}_\lambda^0 - \varepsilon}$$

where for every $c \in \mathbb{R}$, $\varphi_\lambda^c = \{u \in W_0^{1,\vartheta}(\Omega) : \varphi_\lambda(u) \leq c\}$, $\varphi_\lambda(h_0(1,u)) \leq \varphi_\lambda(u)$ for all $u \in W_0^{1,\vartheta}(\Omega)$.

From the above properties of the deformation, we infer that

$$\max[\varphi_\lambda(h_0(1,\gamma(\tau,t))) : (\tau,t) \in P] < \widehat{m}_\lambda^0. \tag{29}$$

Let $k(\tau,t) = h_0(1,\gamma(\tau,t))$ and set

$$\vartheta_0(\tau,t) = \left(\langle \varphi_\lambda'(\tau y_0), y_0^+ \rangle, \langle \varphi_\lambda'(t y_0), -y_0^- \rangle \right),$$

$$\vartheta_1(\tau,t) = \left(\frac{1}{\tau}\langle \varphi_\lambda'(k(\tau,t)), k(\tau,t)^+ \rangle, \frac{1}{t}\langle \varphi_\lambda'(k(\tau,t)), -k(\tau,t)^- \rangle \right) \quad \text{for all } (\tau,t) \in P.$$

By $\widehat{d}_B(\cdot,\cdot,\cdot)$ we denote the Brouwer degree. From the proof of Proposition 3.3 and the homotopy invariance property of the degree, we have

$$\widehat{d}_B(\vartheta_0, P, 0) = 1. \tag{30}$$

Note that $\gamma|_{\partial P} = k|_{\partial P}$ (see (28), (29) and recall the choice of ε). Then using the properties of the Brouwer degree (see [13], p. 178), we have

$$\widehat{d}_B(\vartheta_0, P, 0) = \widehat{d}_B(\vartheta_1, P, 0),$$

$$\Rightarrow \quad \widehat{d}_B(\vartheta_1, P, 0) = 1 \quad \text{(see (30))},$$

$$\Rightarrow \quad h_0(t, \gamma(P)) \cap N_0 \neq \emptyset.$$

But this contradicts (29). Therefore $y_0 \in K_{\varphi_\lambda}$ and so we have that $y_0 \in W_0^{1,\varphi}(\Omega)$ is a nodal solution of problem (P_λ). Moreover, from Gasiński-Winkert [4] (Theorem 3.1), we have that $y_0 \in L^\infty(\Omega)$. $\qquad\square$

Next using the functionals φ_λ^+ and φ_λ^-, we will produce two nontrivial, bounded, constant sign solutions of (P_λ) (a positive solution and a negative solution). The proof follows the arguments used in the proof of Proposition 3.9.

Proposition 3.10. *If hypotheses H_0, H_1' hold, $\lambda < \widehat{\lambda}_1(q)$ and $u_0 \in N_+$, $v_0 \in N_-$ are as in Proposition 3.5 (b), (c) respectively, then $u_0 \in K_{\varphi_\lambda^+} = \{u \in W_0^{1,\vartheta}(\Omega) : (\varphi_\lambda^+)'(u) = 0\}$, $v_0 \in K_{\varphi_\lambda^-} = \{v \in W_0^{1,\vartheta}(\Omega) : (\varphi_\lambda^-)'(v) = 0\}$ and so $u_0 \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$ is a positive solution of (P_λ) and $v_0 \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$ is a negative solution of (P_λ).*

Proof. We do the proof for $u_0 \in N_+$, the proof for $v_0 \in N_-$ being similar. As we already mentioned, we follow the reasoning in the proof of Proposition 3.9. So, we proceed indirectly and assume that $(\varphi_\lambda^+)'(u_0) \neq 0$. Then we can find $\delta > 0$ and $\eta > 0$ such that

$$\|u - u_0\| \leq 3\delta \Rightarrow \|(\varphi_\lambda^+)'(u)\| \geq \eta > 0.$$

Let $D = \left(\frac{1}{2}, \frac{3}{2}\right)$ and consider the function $\gamma_+(\tau) = \tau u_0^+$, $\tau \geq 0$. We know that $\varphi_\lambda^+(\gamma_+(\tau)) = \widehat{m}_\lambda^+$ if and only if $\tau = 1$ and $\varphi_\lambda^+(\gamma_+(\tau)) < \widehat{m}_\lambda^+$ for all $\tau \in \mathbb{R}_+ \setminus \{1\}$. Therefore

$$\mu = \max_{\partial D} \varphi_\lambda^+(\gamma_+(\tau)) < \widehat{m}_\lambda^+.$$

As before, we use the quantitative deformation lemma of Willem [21] (p. 38), with $\varepsilon = \min\left\{\frac{\widehat{m}_\lambda^+ - \mu}{4}, \frac{\eta\delta}{8}\right\}$ and $\mathcal{S} = \overline{B}_\delta(u_0)$. We obtain a transformation $h_+(t,u)$ such

that

$$h_+(1, u) = u \quad \text{for all } u \in (\varphi_\lambda^+)^{-1}([\widehat{m}_\lambda^+ - 2\varepsilon, \widehat{m}_\lambda^+ + 2\varepsilon]),$$

$$h_+(1, (\varphi_\lambda^+)^{\widehat{m}_\lambda^+ + \varepsilon} \cap \overline{B}_\delta(u_0)) \subseteq (\varphi_\lambda^+)^{\widehat{m}_\lambda^+ - \varepsilon},$$

$$\varphi_\lambda^+(h_+(t, u)) \leq \varphi_\lambda^+(h_+(s, u)) \quad \text{for all } 0 \leq s \leq t \leq 1, \text{ all } u \in W_0^{1,\vartheta}(\Omega).$$

It follows that

$$\max_{\tau \in D} \varphi_\lambda(h_+(1, \gamma_+(\tau))) < \widehat{m}_\lambda^+. \tag{31}$$

We introduce the following functions

$$k(\tau) = h_+(1, \gamma_+(\tau)),$$

$$\vartheta_0(\tau) = \langle (\varphi_\lambda^+)'(\tau u_0^+), u_0^+ \rangle,$$

$$\vartheta_1(\tau) = \frac{1}{\tau} \langle (\varphi_\lambda^+)'(k(\tau)), k(\tau)^+ \rangle \quad \text{for all } \tau \in D.$$

We know that

$$\widehat{d}_B(\vartheta_0, D, 0) = 1,$$

$$\widehat{d}_B(\vartheta_1, D, 0) = \widehat{d}_B(\vartheta_0, D, 0) = 1 \quad (\text{since } \gamma_+|_{\partial D} = k|_{\partial D}),$$

$$\Rightarrow \quad h_+(t, \gamma_+(D)) \cap N_+ \neq \emptyset,$$

wich contradicts (31). Therefore $u_0 \in K_{\varphi_\lambda^+}$. Similarly we show that $v_0 \in K_{\varphi_\lambda^-}$.

We have

$$\langle (\varphi_\lambda^+)'(u_0), h \rangle = 0 \quad \text{for all } h \in W_0^{1,\varphi}(\Omega).$$

We choose $h = -u_0^- \in W_0^{1,\varphi}(\Omega)$ and obtain

$$\rho_a(\nabla u_0^-) + \left[\|\nabla u_0^-\|_q^q - \lambda \|u_0^-\|_q^q \right] = 0,$$

$$\Rightarrow \quad c_{21} \rho_a(\nabla u_0^-) \leq 0 \quad \text{for some } c_{21} > 0,$$

$$\Rightarrow \quad u_0 \geq 0, \ u_0 \neq 0 \quad (\text{see Proposition 2.2}).$$

So, u_0 is a nontrivial positive solution of (P_λ) and $u_0 \in W_0^{1,\varphi}(\Omega) \cap L^\infty(\Omega)$ (see [4]). Similarly for v_0 using this time the functional $\varphi_\lambda^-(\cdot)$. □

4. Multiple Solutions - Relaxed Monotonicity

In this section, we relax the strong Nehari-type monotonicity condition $H_1'(iv)$ and use hypothesis $H_1(iv)$. Via an approximation argument, we show that we still have three nontrivial bounded solutions of (P_λ) $(\lambda < \widehat{\lambda}_1(q))$, all with sign information (positive, negative and nodal).

As we already mentioned, our approach is based on an approximation of the superlinear perturbation $f(z, \cdot)$. So, for every $\varepsilon > 0$, we consider the function

$$f_\varepsilon(z, x) = f(z, x) + \varepsilon r |x|^{r-2} x.$$

This is a Carathéodory function which satisfies the strong Nehari-type monotonicity condition $H_1'(iv)$. We set $F_\varepsilon(z, x) = \int_0^x f_\varepsilon(z, s) ds$ and for every $\lambda > 0$, we introduce the C^1-functional $\varphi_\lambda^\varepsilon : W_0^{1,\vartheta}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda^\varepsilon(u) = \frac{1}{p} \rho_a(\nabla u) + \frac{1}{q} \|\nabla u\|_q^q - \frac{\lambda}{q} \|u\|_q^q - \int_\Omega F_\varepsilon(z, u) dz \quad \text{for all } u \in W_0^{1,\vartheta}(\Omega).$$

Evidently, we have

$$\varphi_\lambda^\varepsilon(u) = \varphi_\lambda(u) - \varepsilon\|u\|_r^r \quad \text{for all } u \in W_0^{1,\vartheta}(\Omega). \tag{32}$$

As before, we also consider the positive and negative truncations of $\varphi_\lambda^\varepsilon(\cdot)$, denoted by $(\varphi_\lambda^\varepsilon)^{\pm}(\cdot)$. For these functionals, we consider the corresponding Nehari-type manifolds denoted by N^ε, N_+^ε, N_-^ε, N_0^ε.

Proposition 4.1. *If hypotheses H_0, H_1 hold, $\varepsilon \in (0,1]$ and $\lambda < \widehat{\lambda}_1(q)$, then $\varphi_\lambda^\varepsilon(u) \geq \delta_0 > 0$ for all $u \in N^\varepsilon$.*

Proof. On account of hypotheses $H_1(i), (iii)$, given any $\vartheta > 0$, we can find $c_{22} = c_{22}(\vartheta) > 0$ such that

$$F(z,x) \leq \frac{\vartheta}{q}|x|^q + c_{22}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ (see also (21))}. \tag{33}$$

From Corollary 3.2, we know that for all $u \in N^\varepsilon$, we have

$$\varphi_\lambda^\varepsilon(u) = \max_{\tau \geq 0} \varphi_\lambda^\varepsilon(\tau u)$$

$$\geq \max_{\tau \geq 0} \left[\frac{\tau^p}{p}\rho_a(\nabla u) + \frac{\tau^q}{q}\left(\|\nabla u\|_q^q - (\lambda + \vartheta)\|u\|_q^q\right) - c_{23}\tau^r\|u\|^r \right]$$

for some $c_{23} > 0$ (see (33)).

Choosing $\vartheta > 0$ small (recall that $\lambda < \widehat{\lambda}_1(q)$) and using the fact that $q < p < r$, we obtain that

$$\varphi_\lambda^\varepsilon(u) \geq \max_{0 \leq \tau \leq 1} \left[\frac{\tau^p}{p}\rho_a(\nabla u) - c_{23}\tau^r\|u\|^r \right]$$

$$\geq \delta_0 > 0 \quad \text{(see Proposition 2.2)}.$$

$\square$

Now we are ready to produce nodal and constant sign solutions for problem (P_λ) $(\lambda < \widehat{\lambda}_1(q))$ under the relaxed Nehari-type monotonicity condition.

Theorem 4.1. *If hypotheses H_0, H_1 hold and $\lambda < \widehat{\lambda}_1(q)$, then problem (P_λ) has at least three nontrivial solutions $u_0 \in N_+ \cap L^\infty(\Omega)$, $v_0 \in N_- \cap L^\infty(\Omega)$, $y_0 \in N_0 \cap L^\infty(\Omega)$ (nodal).*

Proof. First we produce the nodal solution y_0.

Let $\varepsilon_n \downarrow 0$. From Proposition 3.9 we know that there exists $y_n \in N^{\varepsilon_n} \cap L^\infty(\Omega)$ such that

$$\widehat{m}_\lambda^{\varepsilon_n} = \varphi_\lambda^{\varepsilon_n}(y_n) > 0 \text{ and } (\varphi_\lambda^{\varepsilon_n})'(y_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{34}$$

Let $u \in N_0$. For every $n \in \mathbb{N}$, we can find unique $\tau_n, t_n > 0$ such that

$$\tau_n u^+ - t_n u^- \in N_0^{\varepsilon_n} \quad \text{(see Proposition 3.3)}.$$

Then we have

$$\varphi_\lambda(u) \geq \varphi_\lambda(\tau_n u^+ - t_n u^-) \quad \text{(see Corollary 3.1)}$$
$$\geq \varphi_\lambda^{\varepsilon_n}(\tau_n u^+ - t_n u^-) \quad \text{(see (32))}$$
$$\geq \widehat{m}_\lambda^{\varepsilon_n}.$$

Since $u \in N_0$ is arbitrary, it follows that

$$\widehat{m}_\lambda^0 \geq \widehat{m}_\lambda^{\varepsilon_n} \geq \delta_0 > 0 \quad \text{(see Proposition 4.1)}.$$

So, we may assume that

$$\widehat{m}_\lambda^{\varepsilon_n} \to \widehat{m}_\lambda^* > 0 \quad \text{as } n \to \infty, \ \widehat{m}_\lambda^0 \geq \widehat{m}_\lambda^*. \tag{35}$$

A contradiction argument as in the proof of Proposition 3.6 shows that $\{y_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,\vartheta}(\Omega)$ is bounded. So, we may assume that

$$y_n \xrightarrow{w} y_0 \text{ in } W_0^{1,\vartheta}(\Omega) \text{ and } y_n \to y_0 \text{ in } L^r(\Omega). \tag{36}$$

From (34) we have

$$\langle A(y_n), h \rangle = \lambda \int_\Omega |y_n|^{q-2} y_n h dz + \int_\Omega f(z, y_n) h dz \quad \text{for all } h \in W_0^{1,\vartheta}(\Omega). \tag{37}$$

In (37) we choose $h = y_n - y_0 \in W_0^{1,\vartheta}(\Omega)$, pass to the limit as $n \to \infty$ and use (36). We obtain

$$\lim_{n \to \infty} \langle A(y_n), y_n - y_0 \rangle = 0,$$
$$\Rightarrow \quad y_n \to y_0 \text{ in } W_0^{1,\vartheta}(\Omega) \text{ (see Proposition 2.3)},$$
$$\Rightarrow \quad y_n^\pm \to y_0^\pm \text{ in } W_0^{1,\vartheta}(\Omega). \tag{38}$$

Then we have

$$\varphi_\lambda(y_0^\pm) = \lim_{n \to \infty} \varphi_\lambda^{\varepsilon_n}(y_n^\pm) \geq \delta_0 > 0 \quad \text{(see Proposition 4.1 and (32))},$$
$$\Rightarrow \quad y_0^\pm \neq 0.$$

Also since $y_n \in N_0^{\varepsilon_n}$, $n \in \mathbb{N}$, we have

$$\langle \varphi_\lambda^{\varepsilon_n}(y_n), y_n^+ \rangle = 0 = \langle \varphi_\lambda^{\varepsilon_n}(y_n), y_n^- \rangle \quad \text{for all } n \in \mathbb{N},$$
$$\Rightarrow \quad \langle \varphi_\lambda(y_0), y_0^+ \rangle = 0 = \langle \varphi_\lambda(y_0), y_0^- \rangle \quad \text{(see (38))}.$$

Therefore we have

$$y_0 \in N_0 \text{ and } \widehat{m}_\lambda^* = \varphi_\lambda(y_0) \geq \widehat{m}_\lambda^0,$$
$$\Rightarrow \quad y_0 \in N_0 \text{ and } \widehat{m}_\lambda^* = \widehat{m}_\lambda^0 = \varphi_\lambda(y_0), \ \varphi_\lambda'(y_0) = 0 \text{ (see (34) and (35))},$$
$$\Rightarrow \quad y_0 \text{ is a nodal solution of } (P_\lambda) \text{ and } y_0 \in L^\infty(\Omega).$$

Similarly working with $\{(\varphi_\lambda^+)^{\varepsilon_n}, \varphi_\lambda^+\}_{n \in \mathbb{N}}$ we obtain a positive solution $u_0 \in N_+ \cap L^\infty(\Omega), \varphi_\lambda^+(u_0) = \widehat{m}_\lambda^+$, while working with $\{(\varphi_\lambda^-)^{\varepsilon_n}, \varphi_\lambda^-\}_{n \in \mathbb{N}}$ we obtain a negative solution $v_0 \in N_- \cap L^\infty(\Omega), \varphi_\lambda^-(v_0) = \widehat{m}_\lambda^-$. $\qquad\square$

DECLARATIONS

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