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ALGEBRAS WITH SUPERAUTOMORPHISM: SIMPLE ALGEBRAS AND CODIMENSION GROWTH*

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ABSTRACT

Let A be an associative algebra endowed with a superautomorphism φ. In this paper we completely classify the finite-dimensional simple algebras with superautomorphism of order ≤ 2. Moreover, after generalizing the Wedderburn–Malcev Theorem in this setting, we prove that the sequence of φ-codimensions of A is polynomially bounded if and only if the variety generated by A does not contain the group algebra of $\mathbb{Z}_2$ and the algebra of 2×2 upper triangular matrices with suitable superautomorphisms.

1. Introduction

Let F be a fixed field of characteristic zero and let $A = A_0 \oplus A_1$ be an associative superalgebra over F, that is A is a direct sum of two vector subspaces such that $A_0A_0 + A_1A_1 \subseteq A_0$ and $A_0A_1 + A_1A_0 \subseteq A_1$.

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In this paper we assume that A is endowed with a superautomorphism $\varphi\colon A \to A$, that is a graded linear map such that for any homogeneous elements $a, b \in A_0 \cup A_1$

$$(ab)^\varphi = (-1)^{|a||b|} a^\varphi b^\varphi.$$

In recent years (see [5, 6]), superautomorphisms have been used in order to prove tight relations between some graded linear maps that can be defined on superalgebras, namely involutions, superinvolutions and pseudoinvolutions. These linear maps play a prominent role in the setting of Lie and Jordan superalgebras as can be seen for instance in [7, 12, 14].

One of the main results of this paper is the classification of the finite-dimensional simple superalgebras endowed with a superautomorphism of order ≤ 2. Such a classification also allows us to refine the Wedderburn–Malcev decomposition, obtained in Section 4.

In the final part of the paper we study this kind of algebras in the setting of the theory of polynomial identities. The great importance of superalgebras in such a field was revealed by Kemer (see [10]) who proved that every associative algebra satisfying a non-trivial polynomial identity over a field of characteristic zero satisfies the same polynomial identities as the Grassmann envelope of a finite-dimensional associative superalgebra.

Recall that a polynomial $f(x_1, \ldots, x_n) \in F\langle X\rangle$, the free algebra on the countable set $X = \{x_1, x_2, \ldots\}$ over F, is a polynomial identity for an F-algebra A if $f(a_1, \ldots, a_n) = 0$ for any choice of $a_i \in A$. The set of all identities satisfied by A forms a T-ideal of $F\langle X\rangle$ (an ideal invariant under all endomorphisms of the free algebra) and it is denoted by $\mathrm{Id}(A)$.

Knowing the polynomial identities satisfied by a given algebra is a very difficult problem in ring theory that was solved completely just in a few cases. So, in order to get some information about the polynomial identities satisfied by an algebra, in 1972, Regev introduced the so-called codimension sequence of an algebra A, denoted $c_n(A), n = 1, 2, \ldots$. Such a sequence gives an actual quantitative measure of the identities satisfied by an algebra. In [15], Regev proved the most important feature of $c_n(A)$: in case A is a PI-algebra, i.e., it satisfies a non-trivial polynomial identity, then $c_n(A)$ is exponentially bounded.

In [8, 9] Kemer showed that, given any PI-algebra A, $c_n(A)$, $n = 1, 2, \ldots$, cannot have intermediate growth, i.e., either is polynomially bounded or grows exponentially. Such a result is a consequence of the following theorem: $c_n(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if the Grassmann algebra G and the algebra of 2×2 upper triangular matrices UT_2 do not satisfy all the identities of A.

In the last part of the paper we obtain an analogous result for superalgebras with superautomorphism. More precisely, we shall prove that the corresponding φ-codimensions of A are polynomially bounded if and only if the variety generated by A does not contain the group algebra of $\mathbb{Z}_2$ and the algebra UT_2 with suitable superautomorphisms.

2. Preliminaries

Throughout this paper F will denote a field of characteristic zero and $A = A_0 \oplus A_1$ an associative superalgebra (also called ($\mathbb{Z}_2$-)graded algebra) over F. The elements of A_0 and A_1 are called homogeneous of degree zero (or even elements) and of degree one (or odd elements), respectively, and they satisfy the properties $A_0A_0 + A_1A_1 \subseteq A_0$ and $A_0A_1 + A_1A_0 \subseteq A_1$.

Now assume that the superalgebra A is endowed with a superautomorphism, that is a graded linear map $\varphi\colon A \longrightarrow A$ (i.e. a map preserving the grading) such that

$$(ab)^\varphi = (-1)^{|a||b|} a^\varphi b^\varphi,$$

for any homogeneous elements $a, b \in A$. Here $|c|$ denotes the homogeneous degree of $c \in A_0 \cup A_1$.

Notice that A_0 is just an algebra with an automorphism.

In what follows we shall consider only superautomorphisms φ of order ≤ 2, i.e., $(a^\varphi)^\varphi = a$ for all $a \in A$ and we shall say that A is a superalgebra with a superautomorphism (we shall omit the order of φ) or a φ-superalgebra.

Since $\operatorname{char} F = 0$, we can write $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, where for $i = 0, 1$, $A_i^+ = \{a \in A_i \mid a^\varphi = a\}$ and $A_i^- = \{a \in A_i \mid a^\varphi = -a\}$ denote the sets of symmetric and skew elements of A_i, respectively.

One can define in a natural way a superautomorphism on the free associative superalgebra $F\langle Y \cup Z\rangle = \mathcal{F}_0 \oplus \mathcal{F}_1$ on the countable set $Y \cup Z$ over F, where we regard the variables of Y as even and those of Z as odd. Here $\mathcal{F}_0$ is the subspace of $F\langle Y \cup Z\rangle$ spanned by all monomials in the variables of $Y \cup Z$ having an even number of variables of Z and $\mathcal{F}_1$ is the subspace spanned by all monomials having an odd number of variables of Z.

We shall write $F\langle Y\cup Z,\varphi\rangle$ for the free superalgebra with superautomorphism on the countable set $Y\cup Z$ over F. It is useful to regard $F\langle Y\cup Z,\varphi\rangle$ as generated by even and odd symmetric variables and by even and odd skew variables: if for $i=1,2,\dots$, we let $y_i^+=y_i+y_i^\varphi$, $y_i^-=y_i-y_i^\varphi$, $z_i^+=z_i+z_i^\varphi$ and $z_i^-=z_i-z_i^\varphi$, then

$$F\langle Y\cup Z,\varphi\rangle = F\langle y_1^+,y_1^-,z_1^+,z_1^-,y_2^+,y_2^-,z_2^+,z_2^-,\dots\rangle.$$

A polynomial $f(y_1^+,\dots,y_m^+,y_1^-,\dots,y_n^-,z_1^+,\dots,z_r^+,z_1^-,\dots,z_s^-)\in F\langle Y\cup Z,\varphi\rangle$ is a φ-polynomial identity of A (or simply a φ-identity), and we write $f\equiv 0$, if, for all $u_1^+,\dots,u_m^+\in A_0^+$, $u_1^-,\dots,u_n^-\in A_0^-$, $v_1^+,\dots,v_r^+\in A_1^+$ and $v_1^-,\dots,v_s^-\in A_1^-$, we have that

$$f(u_1^+,\dots,u_m^+,u_1^-,\dots,u_n^-,v_1^+,\dots,v_r^+,v_1^-,\dots,v_s^-)=0.$$

We denote by $\mathrm{Id}^\varphi(A)=\{f\in F\langle Y\cup Z,\varphi\rangle \mid f\equiv 0 \text{ on } A\}$ the T_2^φ-ideal of φ-identities of A, i.e., $\mathrm{Id}^\varphi(A)$ is an ideal of $F\langle Y\cup Z,\varphi\rangle$ invariant under all graded endomorphisms of $F\langle Y\cup Z\rangle$ commuting with the superautomorphism φ.

As in the super case, it is easily seen that in characteristic zero, every φ-identity is equivalent to a system of multilinear φ-identities. Hence if we denote by

$$P_n^\varphi=\mathrm{span}_F\{w_{\sigma(1)}\cdots w_{\sigma(n)} \mid \sigma\in S_n,\, w_i\in\{y_i^+,y_i^-,z_i^+,z_i^-\},\, i=1,\dots,n\}$$

the space of multilinear polynomials of degree n in the variables y_1^+, y_1^-, z_1^+, z_1^-, …, y_n^+, y_n^-, z_n^+, z_n^- (i.e., y_i^+ or y_i^- or z_i^+ or z_i^- appears in each monomial at degree 1), the study of $\mathrm{Id}^\varphi(A)$ is equivalent to the study of $P_n^\varphi\cap\mathrm{Id}^\varphi(A)$, for all $n\geq 1$. The non-negative integer

$$c_n^\varphi(A)=\dim_F\frac{P_n^\varphi}{P_n^\varphi\cap\mathrm{Id}^\varphi(A)},\quad n\geq 1,$$

is called the n-th φ-codimension of A.

Let $n\geq 1$ and write $n=n_1+\cdots+n_4$ as a sum of four non-negative integers. We denote by $P_{n_1,\dots,n_4}\subseteq P_n^\varphi$ the vector space of the multilinear polynomials in which the first n_1 variables are even symmetric, the next n_2 variables are odd symmetric, the next n_3 variables are even skew and the last n_4 variables are odd skew. Now if we set

$$c_{n_1,\dots,n_4}(A)=\dim_F\frac{P_{n_1,\dots,n_4}}{P_{n_1,\dots,n_4}\cap\mathrm{Id}^\varphi(A)}$$

it is immediate to see that

$$c_n^{\varphi}(A) = \sum_{n_1+\cdots+n_4=n} \binom{n}{n_1,\ldots,n_4} c_{n_1,\ldots,n_4}(A) \tag{1}$$

where $\binom{n}{n_1,\ldots,n_4} = \frac{n!}{n_1!\cdots n_4!}$ stands for the multinomial coefficient. Hence the growth of $c_n^{\varphi}(A)$ is related to the growth of multinomial coefficients and of $c_{n_1,\ldots,n_4}(A)$, for any $n = n_1 + \cdots + n_4$. Since for any $n = n_1 + \cdots + n_4$,

$$c_{n_1,n_2,n_3,n_4}(A) \leq c_n(A),$$

where $c_n(A)$ is the n-th ordinary codimension of A (see [1, Remark 2.1]), we get that $c_n(A) \leq c_n^{\varphi}(A) \leq 4^n c_n(A)$, $n = 1, 2, \ldots$; hence the following corollary holds.

Corollary 1: *If A is a PI-superalgebra with superautomorphism φ, then $c_n^{\varphi}(A)$, $n = 1, 2, \ldots$, is exponentially bounded.*

3. Superalgebras with superautomorphism generating varieties of almost polynomial growth

In this section we shall construct finite-dimensional superalgebras with superautomorphism generating varieties of almost polynomial growth.

We recall that given a variety of algebras with superautomorphism $\mathcal{V}$, the growth of $\mathcal{V}$ is defined as the growth of the sequence of φ-codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}^{\varphi}(A)$. Then we say that $\mathcal{V}$ has polynomial growth if $c_n^{\varphi}(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^{\varphi}(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

The proof of the following remark is immediate.

Remark 2: Let $A = A_0 \oplus A_1$ be a superalgebra.

(1) If $A_1^2 = 0$ then the superautomorphisms on A coincide with the graded automorphisms on A, i.e. automorphisms preserving the grading. In particular, if $A_1 = 0$ then the superautomorphisms on A coincide with the automorphisms on A.
(2) If A is commutative then the superautomorphisms on A of order ≤ 2 coincide with the superinvolutions on A.

In [1, 2] the authors classified the varieties of superalgebras with superinvolution of almost polynomial growth, by giving a complete list of algebras generating them. By the previous remark, any commutative algebra appearing in such a list will generate a variety of superalgebras with superautomorphism of almost polynomial growth. This is the case of the two-dimensional algebra $F \oplus F$ with trivial grading and exchange superinvolution $*$ given by $(a, b)^* = (b, a)$. In the language of superalgebras with superautomorphism we get following.

Theorem 3 ([1, 3]): *The superalgebra $F \oplus F$ endowed with trivial grading and exchange superautomorphism φ given by $(a, b)^\varphi = (b, a)$ generates a variety of almost polynomial growth.*

Given polynomials $f_1, \ldots, f_n \in F\langle Y \cup Z, \varphi\rangle$ let us denote by $\langle f_1, \ldots, f_n\rangle_{T_2^\varphi}$ the T_2^φ-ideal generated by $f_1, \ldots, f_n$. Hence we have that

$$\mathrm{Id}^\varphi(F \oplus F) = \langle [x_1, x_2], z^+, z^-\rangle_{T_2^\varphi},$$

for any variable $x \in Y \cup Z$.

Now consider the algebra $UT_2 = UT_2(F)$ of 2×2 upper-triangular matrices over the field F :

$$UT_2 = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a, b, c \in F \right\}.$$

We can see UT_2 as a superalgebra with the only two non-isomorphic $\mathbb{Z}_2$-gradings:

Trivial grading: $UT_2 = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\};$

Natural grading: $UT_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \right\}.$

The superalgebra UT_2 with trivial grading will be denoted by UT_2 whereas we shall use the symbol UT_2^{gr} to indicate the superalgebra UT_2 with natural grading.

Moreover, let us consider the following two automorphisms on UT_2 :

Trivial automorphism id: $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{\mathrm{id}} = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix};$

Natural automorphism φ: $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{\varphi} = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}.$

Notice that both automorphisms are graded automorphisms on UT_2 and UT_2^{gr}. Hence, since in both gradings $(UT_2)_1^2 = 0$, by Remark 2, we get four superalgebras with superautomorphism:

- UT_2 is the algebra UT_2 with trivial grading and trivial superautomorphism;
- UT_2^{sup} is the algebra UT_2 with trivial grading and natural superautomorphism;
- UT_2^{gr} is the algebra UT_2 with natural grading and trivial superautomorphism;
- $UT_2^{\mathrm{gr,sup}}$ is the algebra UT_2 with natural grading and natural superautomorphism.

Notice that the above algebras are not T_2^{φ}-equivalent, i.e., their T_2^{φ}-ideals are different, as it is easily seen:

$$\begin{aligned}
\mathrm{Id}^{\varphi}(UT_2) &= \langle [y_1^+, y_2^+][y_3^+, y_4^+], y^-, z^+, z^- \rangle_{T_2^{\varphi}}, \\
\mathrm{Id}^{\varphi}(UT_2^{\mathrm{sup}}) &= \langle [y_1^+, y_2^+], y_1^- y_2^-, z^+, z^- \rangle_{T_2^{\varphi}}, \\
\mathrm{Id}^{\varphi}(UT_2^{\mathrm{gr}}) &= \langle [y_1^+, y_2^+], y^-, z_1^+ z_2^+, z^- \rangle_{T_2^{\varphi}}, \\
\mathrm{Id}^{\varphi}(UT_2^{\mathrm{gr,sup}}) &= \langle [y_1^+, y_2^+], y^-, z^+, z_1^- z_2^- \rangle_{T_2^{\varphi}}.
\end{aligned}$$

The first three algebras, seen as ordinary algebras or superalgebras or algebras with an automorphism of order 2, were proved to generate varieties of almost polynomial growth (see [8, 9] and [16]). With the same approach it can be proved that also the last one generates a variety of almost polynomial growth. We summarize these results in the following.

THEOREM 4: *The algebras UT_2, UT_2^{sup}, UT_2^{gr} and $UT_2^{\mathrm{gr,sup}}$ generate varieties of superalgebras with superautomorphism of almost polynomial growth.*

4. A Wedderburn–Malcev decomposition

In this section we prove a Wedderburn–Malcev theorem for finite-dimensional superalgebras with superautomorphism. Recall that if A is a superalgebra, a subset (subalgebra, ideal) $S \subseteq A$ is a graded subset (subalgebra, ideal) of A if $S = (S \cap A_0) \oplus (S \cap A_1)$. The following remark holds.

Remark 5: Let A be a superalgebra with superautomorphism φ and let $B \subseteq A$ be a subalgebra.

(1) If $B = B_0 \oplus B_1$ is a graded subalgebra of A then $B^\varphi = B_0^\varphi \oplus B_1^\varphi$ is a graded subalgebra of A.
(2) If $I = I_0 \oplus I_1 \subseteq B$ is a graded ideal (subset) of B then $I^\varphi = I_0^\varphi \oplus I_1^\varphi$ is a graded ideal (subset) of B^φ.
(3) If I is a minimal graded ideal of B then I^φ is a minimal graded ideal of B^φ.

From now on $A = A_0 \oplus A_1$ denotes a finite-dimensional superalgebra with superautomorphism φ of order ≤ 2 and $J(A)$ its Jacobson radical.

Lemma 6: *If $B \subseteq A$ is a semisimple graded subalgebra of A, then also B^φ is a semisimple graded subalgebra of A.*

Proof. By Remark 5, B^φ is a graded subalgebra of A and we are left to prove that B^φ is semisimple, i.e., $J(B^\varphi) = 0$, where $J(B^\varphi)$ denotes the Jacobson radical of B^φ. It is well known that $J(B^\varphi)$ is a graded nilpotent ideal of B^φ. We claim that $J(B^\varphi)^\varphi$ is a nilpotent ideal of B. Let m be the smallest positive integer such that $J(B^\varphi)^m = 0$ and let $a_1, \dots, a_m \in J(B^\varphi)^\varphi$. Since $J(B^\varphi)$ is a graded ideal of B^φ we get that, for all i, $a_i = (b_i + c_i)^\varphi$, where b_i and c_i are homogeneous elements of $J(B^\varphi)$ of degree zero and one, respectively. Then

$$a_1 \cdots a_m = \sum \alpha_j (d_1^j \cdots d_m^j)^\varphi$$

where either $d_i^j = b_i$ or $d_i^j = c_i$ and $\alpha_j = \pm 1$. But $J(B^\varphi)^m = 0$ and, so, we get that $a_1 \cdots a_m = 0$ and $J(B^\varphi)^\varphi$ is nilpotent. Hence $J(B^\varphi)^\varphi \subseteq J(B)$. But since B is semisimple we get that $J(B^\varphi)^\varphi = J(B) = 0$ and, so, $J(B^\varphi) = 0$. ∎

By the Wedderburn–Malcev theorem for superalgebras [4], we can write

$$A = B + J,$$

where B is a semisimple graded subalgebra of A and $J = J(A) = J_0 \oplus J_1$ is a graded ideal. Moreover

$$B = B_1 \oplus \cdots \oplus B_k,$$

where $B_1, \dots, B_k$ are simple graded algebras. The following result was proved in [1, Lemma 4.2].

LEMMA 7: *If B and B' are semisimple graded subalgebras of A such that $A = B + J = B' + J$, with $J^2 = 0$, then there exists $x_0 \in J_0$ such that*

$$B' = (1 + x_0)B(1 - x_0).$$

An ideal (subalgebra) I of A is a φ-ideal (-subalgebra) of A if it is a graded ideal (subalgebra) and $I^\varphi = I$.

Definition 8: Let A be a superalgebra with superautomorphism φ such that $A^2 \neq 0$. We say that A is

- simple, as an ordinary algebra, if it has no non-trivial ideals.
- simple, as a superalgebra, or super simple if it has no non-trivial graded ideals.
- simple, as a superalgebra with superautomorphism, or φ-simple if it has no non-trivial φ-ideals.

Now we are in a position to prove the Wedderburn–Malcev Theorem for superalgebras with superautomorphism.

THEOREM 9: *Let A be a finite-dimensional superalgebra with superautomorphism over a field F of characteristic zero. Then there exists a semisimple φ-subalgebra B such that*

$$A = B + J(A)$$

and $J(A)$ is a φ-ideal of A. Moreover

$$B = B_1 \oplus \cdots \oplus B_k$$

where $B_1, \ldots, B_k$ are φ-simple algebras.

Proof. By the Wedderburn–Malcev theorem for superalgebras we can write

$$A = B + J$$

where B is a semisimple graded subalgebra of A and $J = J(A)$, its Jacobson radical, is a graded ideal of A. Since J is nilpotent, as in the proof of Lemma 6, we have that J^φ is a nilpotent ideal of A. But being J the maximal nilpotent ideal of A, we get $J^\varphi \subseteq J$ and, so, $J = J^\varphi$. Hence J is a φ-ideal of A.

If $J = 0$ or $B = B^\varphi$ then B is a semisimple superalgebra with superautomorphism and we are done. So assume that $J \neq 0$ and $B \neq B^\varphi$.

Suppose first that $J^2 = 0$. By Lemma 6, B^φ is a semisimple graded subalgebra of A. Hence, by Lemma 7, we have that

$$B^\varphi = (1 + x_0)B(1 - x_0),$$

for some $x_0 \in J_0$. For any homogeneous element $b \in B$ we have that

$$b^\varphi = (1 + x_0)\bar{b}(1 - x_0)$$

for some homogeneous element $\bar{b} \in B$ with the same homogeneous degree as b^φ and b. Hence

$$\begin{aligned} b = (b^\varphi)^\varphi &= ((1 + x_0)\bar{b}(1 - x_0))^\varphi \\ &= (1 + x_0^\varphi)\bar{b}^\varphi(1 - x_0^\varphi) \\ &= (1 + x_0^\varphi)(1 + x_0)\tilde{b}(1 - x_0)(1 - x_0^\varphi) \\ &= (1 + x_0^\varphi + x_0)\tilde{b}(1 - x_0^\varphi - x_0) \end{aligned}$$

for some $\tilde{b} \in B_0 \cup B_1$ with the same homogeneous degree as b. Since $J \cap B = 0$ we obtain that

$$b = \tilde{b} \quad \text{and} \quad (x_0 + x_0^\varphi)\tilde{b} = \tilde{b}(x_0 + x_0^\varphi).$$

It follows that, for any $b \in B_0 \cup B_1$,

$$\begin{aligned} b^\varphi &= (1 + x_0)\bar{b}(1 - x_0) \\ &= \Big(1 + \frac{x_0 + x_0^\varphi}{2} + \frac{x_0 - x_0^\varphi}{2}\Big)\bar{b}\Big(1 - \frac{x_0 + x_0^\varphi}{2} - \frac{x_0 - x_0^\varphi}{2}\Big) \\ &= \Big(1 + \frac{x_0 - x_0^\varphi}{2}\Big)\bar{b}\Big(1 - \frac{x_0 - x_0^\varphi}{2}\Big) \\ &= (1 + x_0^-)\bar{b}(1 - x_0^-), \end{aligned}$$

where $x_0^- = \frac{x_0 - x_0^\varphi}{2} \in J_0^-$.

Consider the subalgebra $C = (1 + \frac{x_0^-}{2})B(1 - \frac{x_0^-}{2})$ of A. Clearly C is a graded subalgebra of A and by the above C is a φ-subalgebra. Also, since C is isomorphic to B, it is a semisimple φ-subalgebra of A.

Suppose now that $J^2 \neq 0$ and choose $m \geq 2$ such that $J^m \neq 0$ and $J^{m+1} = 0$. It is easy to see that J^m is a φ-ideal of A and, so, A/J^m is a superalgebra with induced superautomorphism. Its Jacobson radical $J(A/J^m) = J(A)/J^m$ is such that $J(A/J^m)^m = 0$. Hence, by induction on m, we have that there exists a semisimple φ-subalgebra B'/J^m such that

$$A/J^m = B'/J^m + J/J^m.$$

From $J(B'/J^m) = 0$ it follows that $J(B') = J^m$ and so, we can write

$$B' = C + J^m,$$

where C is a semisimple graded subalgebra of B'. Since $(J^m)^2 \subseteq J^{2m} = 0$, by the first part of the proof we can assume $C^\varphi = C$, i.e., C is a semisimple φ-subalgebra of A and we are done since

$$A = B' + J = C + J^m + J = C + J.$$

By the Wedderburn–Malcev theorem for superalgebras, we can write

$$C = D_1 \oplus \cdots \oplus D_h,$$

where $D_1, \ldots, D_h$ are all the minimal graded ideals of C. Hence, by Remark 5, for every i, D_i^φ is also a minimal graded ideal of C and, so, $D_i^\varphi = D_j$, for some $j \in \{1, \ldots, h\}$.

We now rename $D_1, \ldots, D_h$ and we write

$$C = C_1 \oplus \cdots \oplus C_k$$

where either $C_i = D_j$ with $D_j = D_j^\varphi$ or $C_i = D_j \oplus D_j^\varphi$, with $D_j \neq D_j^\varphi$. Thus $C_1, \ldots, C_k$ are minimal φ-ideals of C, i.e. φ-simple algebras. ∎

5. Classifying simple superalgebras with superautomorphism

This section is devoted to the classification of simple superalgebras with superautomorphism. Recall that we consider only superautomorphisms of order ≤ 2.

The following lemma goes in this direction.

Lemma 10: *Let A be a finite-dimensional simple superalgebra with superautomorphism φ. Then A is either*

- *simple as a superalgebra or*
- *$A = B \oplus B^\varphi$, for some simple superalgebra B.*

Proof. If A is super simple we have nothing to prove.

Assume then that A is φ-simple but not super simple. Consider B a proper non-zero graded ideal of A. It is not difficult to see that both $B+B^\varphi$ and $B\cap B^\varphi$ are graded ideals of A stable under the action of the superautomorphism φ. Since A is φ-simple we get that $A = B + B^\varphi$ and that $B \cap B^\varphi = \{0\}$. Hence $A = B \oplus B^\varphi$.

We are left to show that B is super simple. To this end, let I be a proper non-zero graded ideal of B. Then $I \oplus I^{\varphi}$ is a proper graded ideal of A stable under φ, a contradiction. ∎

In order to get a complete classification of the φ-simple superalgebras we need the classification of the simple superalgebras.

Theorem 11: *Let A be a finite-dimensional simple superalgebra over an algebraically closed field F of characteristic zero. Then A is isomorphic to one of the following:*

- *$Q(n) = M_n(F \oplus cF) = Q(n)_0 \oplus Q(n)_1$, where $Q(n)_0 = M_n(F)$ and $Q(n)_1 = cM_n(F)$ with $c^2 = 1$;*
- *$M_{k,h}(F)$, the algebra of $n \times n$ matrices, $n = k + h$, $k \geq h \geq 0$, with the following $\mathbb{Z}_2$-grading:*

$$M_{k,h}(F) = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} cc0 & Y \\ Z & 0 \end{pmatrix} \right\},$$

where X, Y, Z, T are $k \times k,\ k \times h,\ h \times k,\ h \times h$ matrices, respectively.

It is well-known that there is a one-to-one correspondence between $\mathbb{Z}_2$-gradings and automorphisms of order ≤ 2. If $A = A_0 \oplus A_1$ is a superalgebra then A can be endowed with an automorphism ψ as follows: $\psi\colon A \to A$ such that $\varphi(a_0 + a_1) = a_0 - a_1$, for all $a_0 \in A_0, a_1 \in A_1$. Conversely, let A be endowed with an automorphism ψ of order ≤ 2 and let

$$A_0^{\psi} = \{a \in A \mid \psi(a) = a\} \quad \text{and} \quad A_1^{\psi} = \{a \in A \mid \psi(a) = -a\}.$$

Then $A = A_0 \oplus A_1$ is a superalgebra with grading $A_0 = A_0^{\psi}$ and $A_1 = A_1^{\psi}$. Hence, in the language of algebras with an automorphism the previous theorem can be rewritten as follows.

Theorem 12: *Let A be a simple algebra over an algebraically closed field F of characteristic zero endowed with an automorphism of order ≤ 2. Then A is isomorphic to one of the following:*

- *$M_n(F) \oplus M_n(F)$ with the exchange automorphism $(a, b)^{\psi} = (b, a)$;*
- *$M_{k,h}(F) = \{\left(\begin{smallmatrix} X & Y \\ Z & T \end{smallmatrix}\right)\}$, where $k \geq h \geq 0$ and*

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{\psi} = \begin{pmatrix} X & -Y \\ -Z & T \end{pmatrix}.$$

Definition 13: Given a superalgebra B, we define $\bar{B}$ to be the superalgebra with the same graded vector space structure as B but with product $\circ$ given on homogeneous elements a, b by the formula

$$a \circ b := (-1)^{|a||b|} ab.$$

The following remark can be easily proved.

Remark 14: The algebras $\bar{B}$ and B are isomorphic as superalgebras.

Now consider the superalgebra $B \oplus \bar{B}$ with grading induced by the grading on B and define on it the following superautomorphism:

$$ex \colon B \oplus \bar{B} \to B \oplus \bar{B}$$

with $(a, b)^{ex} = (b, a)$.

Given two superalgebras with superautomorphism (A, φ) and (C, ψ) we say that they are isomorphic (as superalgebras with superautomorphism) if there exists a graded isomorphism of algebras $\tau \colon A \longrightarrow C$ such that $\tau(a^{\varphi}) = \tau(a)^{\psi}$, for any $a \in A$.

Let $A = B \oplus B^{\varphi}$ be a finite-dimensional simple superalgebra with superautomorphism φ. An easy computation shows that the map

$$\psi \colon (B \oplus B^{\varphi}, \varphi) \longrightarrow (B \oplus \bar{B}, ex)$$
$$a + b^{\varphi} \longmapsto (a, b)$$

is an isomorphism of superalgebras with superautomorphism.

Hence, as a consequence of this result and Lemma 10, by taking into account the classification of the simple superalgebras, we get the following result.

Theorem 15: *If $A = B \oplus B^{\varphi}$ is a finite-dimensional simple superalgebra with superautomorphism φ over an algebraically closed field F of characteristic zero, then A is isomorphic to*

- $(M_{k,h}(F) \oplus \overline{M_{k,h}(F)}, ex)$ *or*
- $(Q(n) \oplus \overline{Q(n)}, ex)$.

We are left to investigate the case in which our superalgebra A with superautomorphism φ is super simple. Actually, in light of the following result, we have to consider only the case $A \cong M_{k,h}(F)$.

Proposition 16: *It is not possible to define superautomorphisms of order* ≤ 2 *on the superalgebra* $Q(n)$.

Proof. Suppose that $\varphi\colon Q(n) \to Q(n)$ is a superautomorphism.

Since φ is in particular a graded linear map, for all $a + cb \in Q(n)$, one can write

$$\varphi(a + cb) = f(a) + cg(b),$$

where f, g are linear maps on $M_n(F)$, $f = \varphi|_{M_n(F)}$ and $g\colon M_n(F) \to M_n(F)$ is such that $g(a) = b$ if $\varphi(ca) = cb$.

Clearly f is an automorphism on $M_n(F)$.

Now let us prove that $g(1)$ is a scalar matrix. We have that

$$\begin{aligned} cg(1)f(b) &= \varphi(c1)\varphi(b) = \varphi(c1b) = \varphi(cb) = \varphi(cb1) \\ &= \varphi(bc1) = \varphi(b)\varphi(c1) = f(b)cg(1) = cf(b)g(1). \end{aligned}$$

It follows that $g(1)$ commutes with $f(b)$, for any $b \in M_n(F)$. Since f is in particular surjective, $g(1)$ commutes with any element of $M_n(F)$ and so it is a scalar matrix.

Moreover, we have that $g(1)^2 = -1$. In fact,

$$1 = f(1) = \varphi(1) = \varphi(c1 \cdot c1) = -\varphi(c1)\varphi(c1) = -g(1)g(1) = -g(1)^2.$$

In conclusion, we have proved that $g(1) = \alpha 1$ with $\alpha^2 = -1$. This leads to a contradiction. In fact

$$\begin{aligned} c1 &= \varphi^2(c1) = \varphi(\varphi(c1)) = \varphi(cg(1)) = \varphi(c\alpha 1) \\ &= \alpha\varphi(c1) = \alpha cg(1) = \alpha c\alpha 1 = -c1. \end{aligned} \qquad \blacksquare$$

Finally let us assume that the φ-simple superalgebra A is isomorphic to $M_{k,h}(F)$. In case $h = 0$, $M_{k,0}(F) = M_k(F)$ is endowed with the trivial grading. Hence by Remark 2, the superautomorphisms on $M_n(F)$ coincide with the automorphisms, which are described in the second item of Theorem 12. Notice that the number of non-isomorphic automorphisms of order ≤ 2 coincides with the number of partitions of n in 2 parts which is $\lfloor \frac{n}{2} \rfloor + 1$.

We are left to consider the case $h > 0$. We start with the following lemma.

LEMMA 17: *Let $A = A_0 \oplus A_1$ be a superalgebra with non-trivial grading endowed with a superautomorphism φ of order 2.*

If A is φ-simple then either $(A_0, \varphi|_{A_0})$ is simple, as an algebra with automorphism, or $A_0 = C_1 \oplus C_2$, $A_1 = D_1 \oplus D_2$, where $(C_i, \varphi|_{C_i})$ are simple and D_i are irreducible A_0-bimodules, φ-invariant. Moreover:

- $C_2D_1 = C_1D_2 = D_1D_1 = D_2D_2 = D_1C_1 = D_2C_2 = \{0\}$.
- $D_1D_2 = C_1$, $D_2D_1 = C_2$, $C_1D_1 = D_1$, $D_1C_2 = D_1$, $C_2D_2 = D_2$, $D_2C_1 = D_2$.

Proof. The result can be proved by following word by word the proof of [13, Theorem 12]. ∎

Now we shall prove that for the algebra $M_{k,h}(F)$ the first case of Lemma 17 cannot occur.

PROPOSITION 18: *If $A = M_{k,h}(F)$ is a superalgebra with non-trivial grading endowed with a superautomorphism φ of order 2, then $(A_0, \varphi|_{A_0})$ cannot be simple as an algebra with automorphism.*

Proof. Suppose by absurd that $A_0 = M_k(F) \oplus M_h(F)$ is simple as an algebra with automorphism. Since $M_k(F) \oplus M_h(F)$ is not simple (as an algebra), by Theorem 12, we get that $k = h$ and, up to isomorphism,

$$(A_0, \varphi = \varphi|_{A_0}) = (M_k(F) \oplus M_k(F), \varphi),$$

where $(a, b)^\varphi = (b^\varphi, a^\varphi)$. Now let us consider the following elements:

$$a_{11} = \sum_{i=1}^{k} e_{ii}, \quad a_{12} = \sum_{i=1}^{k} e_{ik+i}, \quad a_{21} = \sum_{i=1}^{k} e_{k+ii}, \quad a_{22} = \sum_{i=k+1}^{2k} e_{ii},$$

where the e_{ij}'s are the elementary matrices. We have that

$$A_0 = M_k(F)a_{11} \oplus M_k(F)a_{22}, \quad A_1 = M_k(F)a_{12} \oplus M_k(F)a_{21}$$

and $a_{11}^\varphi = a_{22}$, $a_{22}^\varphi = a_{11}$.

Hence $a_{12}^\varphi = (a_{11}a_{12}a_{22})^\varphi = a_{22}a_{12}^\varphi a_{11}$ and so $a_{12}^\varphi = ea_{21}$, for some $e \in M_k(F)$. Analogously, $a_{21}^\varphi = e'a_{12}$, for some $e' \in M_k(F)$. Moreover, for any $b \in M_k(F)$, we have that

$$\begin{aligned} eb^\varphi a_{21} &= ea_{21}b^\varphi a_{11} = ((a_{12})(ba_{22}))^\varphi \\ &= (ba_{12})^\varphi = ((ba_{11})a_{12})^\varphi = b^\varphi a_{22}ea_{21} = b^\varphi ea_{21}. \end{aligned}$$

It follows that $e \in Z(M_k(F)) \cong F$. In the same way it is possible to show that $e' \in Z(M_k(F)) \cong F$.

Since φ has order 2, we have that $a_{12} = (a_{12}^{\varphi})^{\varphi} = (ea_{21})^{\varphi} = ea_{21}^{\varphi} = ee'a_{12}$ and so $ee' = 1$.

But, $a_{22} = a_{11}^{\varphi} = (a_{12}a_{21})^{\varphi} = -a_{12}^{\varphi}a_{21}^{\varphi} = -ee'a_{22}$ and so $ee' = -1$, a contradiction. ∎

By taking into account Lemma 17 and Proposition 18, we are able to prove the following theorem.

Theorem 19: *If $M_{k,h}(F)$, $h > 0$, is endowed with a superautomorphism of order 2, then it is isomorphic to $(M_{k,h}(F), \varphi)$, where*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\varphi} = \begin{pmatrix} PAP & PBQ \\ -QCP & QDQ \end{pmatrix}$$

with

$$P = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix}, \quad Q = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix},$$

and $I_{k_1}, I_{k_2}, I_{h_1}, I_{h_2}$, are the identity matrices of orders k_1, k_2, h_1, h_2, respectively, $k = k_1 + k_2$, $h = h_1 + h_2$, $k_1 \geq k_2$ and $h_1 \geq h_2$.

Proof. According to Proposition 18 we have that $(M_{k,h}(F))_0$ is not simple as an algebra with automorphism. Clearly

$$(M_{k,h}(F))_0 = M_k(F) \oplus M_h(F) \quad \text{and} \quad (M_{k,h}(F))_1 = M_{k\times h}(F) \oplus M_{h\times k}(F).$$

Now, by Lemma 17 we have that $M_k(F)$ and $M_h(F)$ are simple, as algebras with automorphism. Hence there exists $P \in M_k(F)$ with $P^2 = I_k$ such that

$$\varphi|_{M_k(F)}(A) = PAP, \quad A \in M_k(F).$$

Analogously, there exists $Q \in M_h(F)$ with $Q^2 = I_h$ such that

$$\varphi|_{M_h(F)}(D) = QDQ, \quad D \in M_h(F).$$

On the other hand, according to Lemma 17, we have that $M_{k\times h}(F)$ and $M_{h\times k}(F)$ are φ-invariant. Now, if we take a matrix units e_{ij} with $i \in \{1, \dots, k\}$ and $j \in \{k+1, \dots, k+h\}$, we have that, for some $\alpha \in F$,

$$\varphi(e_{ij}) = \varphi(e_{ii}e_{ij}e_{jj}) = \varphi(e_{ii})\varphi(e_{ij})\varphi(e_{jj}) = P[e_{ii}P\varphi(e_{ij})Qe_{jj}]Q = \alpha Pe_{ij}Q.$$

Let $r \in \{1,\dots,k\}$ and $s \in \{k+1,\dots,k+h\}$. As before, we get that $\varphi(e_{rs}) = \beta P e_{rs} Q$, for some $\beta \in F$. Next we prove that $\alpha = \beta$. In fact

$$\begin{aligned}\alpha P e_{ij} Q = \varphi(e_{ij}) &= \varphi(e_{ir}e_{rs}e_{sj}) = \varphi(e_{ir})\varphi(e_{rs})\varphi(e_{sj}) \\ &= (Pe_{ir}P)(\beta P e_{rs} Q)(Qe_{sj}Q) = \beta P e_{ij} Q.\end{aligned}$$

Since $\varphi^2 = \mathrm{id}$, we have that

$$e_{ij} = \varphi^2(e_{ij}) = \varphi(\alpha P e_{ij} Q) = \alpha^2 e_{ij}$$

and so $\alpha = \pm 1$. Now, with the same argument, we get that $\varphi(e_{ji}) = -\alpha Q e_{ji} P$. Obviously, we may assume $\alpha = 1$. Hence

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\varphi} = \begin{pmatrix} PAP & PBQ \\ -QCP & QDQ \end{pmatrix}.$$

By Theorem 12 we know that $(M_k(F), \varphi|_{M_k(F)})$ is isomorphic to $(M_{k_1,k_2}(F), \psi)$ for some $k = k_1 + k_2$, $k_1 \geq k_2$. Hence, without loss of generality we may assume that P is similar to

$$P' = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix},$$

i.e., $P' = LPL^{-1}$, for some matrix L. Analogously $(M_h(F), \varphi|_{M_h(F)})$ is isomorphic to $(M_{h_1,h_2}(F), \rho)$ for some $h = h_1 + h_2$, $h_1 \geq h_2$. Hence $M^{-1}QM = Q'$ where

$$Q' = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix}.$$

Then $(M_{k,h}(F), \varphi)$ is isomorphic to $(M_{k,h}(F), \sigma)$ where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\sigma} = \begin{pmatrix} P'AP' & P'BQ' \\ -Q'CP' & Q'DQ' \end{pmatrix}.$$

In fact the map $f\colon (M_{k,h}(F), \varphi) \longrightarrow (M_{k,h}(F), \sigma)$ defined by

$$f(\begin{pmatrix} A & B \\ C & D \end{pmatrix}) = \begin{pmatrix} LAL^{-1} & LBM^{-1} \\ MCL^{-1} & MDM^{-1} \end{pmatrix}$$

is an isomorphism of superalgebras with superautomorphism. ∎

The results of this section can be summarized in the following theorem, giving the classification of simple superalgebra with superautomorphism.

Theorem 20: *Let A be a finite-dimensional simple superalgebra with superautomorphism of order ≤ 2 over an algebraically closed field F of characteristic zero. Then A is isomorphic to one of the following:*

1. *$M_{k,h}(F)$, with superautomorphism φ defined as*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\varphi} = \begin{pmatrix} PAP & PBQ \\ -QCP & QDQ \end{pmatrix},$$

   where

$$P = \begin{pmatrix} I_{k_1} & 0 \\ 0 & -I_{k_2} \end{pmatrix}, \quad Q = \begin{pmatrix} I_{h_1} & 0 \\ 0 & -I_{h_2} \end{pmatrix},$$

   with $I_{k_1}, I_{k_2}, I_{h_1}, I_{h_2}$, are the identity matrices of orders k_1, k_2, h_1, h_2, respectively, $k = k_1 + k_2$, $h = h_1 + h_2$, $k_1 \geq k_2$ and $h_1 \geq h_2$.
2. *$M_{k,h}(F) \oplus \overline{M_{k,h}(F)}$ with the exchange superautomorphism ex.*
3. *$Q(n) \oplus \overline{Q(n)}$ with the exchange superautomorphism ex.*

6. Superalgebras with superautomorphism of polynomial growth

In this section we shall characterize the varieties of superalgebras with superautomorphism of polynomial growth generated by finite-dimensional algebras.

Let $A = A_0 \oplus A_1$ be a superalgebra. We say that A is endowed with the trivial superautomorphism φ if $A_1 = 0$ and φ is the identity map.

Lemma 21: *Let A be a finite-dimensional superalgebra with superautomorphism over an algebraically closed field F of characteristic zero and suppose that $F \oplus F, UT_2, UT_2^{\mathrm{sup}}, UT_2^{\mathrm{gr}}, UT_2^{\mathrm{gr,sup}} \notin \mathrm{var}^{\varphi}(A)$. Then $A = B + J(A)$, where $B \cong F \oplus \cdots \oplus F$ is endowed with trivial (induced) superautomorphism.*

Proof. By Theorem 9

$$A = A_1 \oplus \cdots \oplus A_k + J,$$

where $A_1, \ldots, A_k$ are finite-dimensional φ-simple superalgebras and J is the Jacobson radical of A. According to Theorem 20, we have to consider just 4 cases.

Case 1. $A_i \cong (M_{k,0}(F), \varphi), k > 1$.

In this case, as stated in item (1) of Theorem 20, the superautomorphism φ is uniquely determined by the decomposition $k = k_1 + k_2$. Consider the subalgebra $C = \langle a, b, c\rangle$ of A_i generated by the elements

$$a = e_{11}, \quad b = e_{1,k_1+k_2}, \quad c = e_{k_1+k_2,k_1+k_2}.$$

It is clearly a graded subalgebra with (induced) superautomorphism isomorphic to UT_2 in case $k_2 = 0$ and to UT_2^{sup} in case $k_2 > 0$. Hence UT_2 or UT_2^{sup} belongs to $\text{var}^{\varphi}(A_i) \subseteq \text{var}^{\varphi}(A)$, a contradiction.

Case 2. $A_i \cong (M_{k,h}(F), \varphi)$, $h > 0$.

Consider the subalgebra $C = \langle a, b, c\rangle$, where

$$a = e_{11}, \quad b = e_{k+hk+h}, \quad c = e_{1k+h}.$$

Clearly C is a graded subalgebra with superautomorphism. By Theorem 20, since $\varphi(e_{1k+k}) = \pm e_{1,k+h}$, we get that C is isomorphic (as a superalgebra with superautomorphism) to UT_2^{gr} or $UT_2^{\text{gr,sup}}$, via the map f given by

$$f(a) = e_{11}, \quad f(b) = e_{22}, \quad f(c) = e_{12}.$$

This implies that UT_2^{gr} or $UT_2^{\text{gr,sup}}$ belongs to $\text{var}^{\varphi}(A_i) \subseteq \text{var}^{\varphi}(A)$, a contradiction.

Cases 3 and 4. $A_i \cong (B \oplus \bar{B}, ex)$, where B is $M_{k,h}(F)$ or $Q(n)$.

Consider the subalgebra $C = \langle a, b\rangle$ of A_i generated by the elements

$$a = (e_{11}, 0) \quad \text{and} \quad b = (0, e_{11}).$$

The linear map $f \colon C \to F \oplus F$ given by

$$f(a) = (1, 0) \quad \text{and} \quad f(b) = (0, 1)$$

is an isomorphism of superalgebras with superautomorphism. Hence $F \oplus F \in \text{var}^{\varphi}(A_i) \subseteq \text{var}^{\varphi}(A)$, a contradiction.

Hence, for every i we must have $A_i \cong M_{1,0}(F) = F$ with trivial superautomorphism and this completes the proof. ∎

Lemma 22: *Let $A = A_1 \oplus \cdots \oplus A_m + J$ be a finite-dimensional superalgebra with superautomorphism over an algebraically closed field F of characteristic zero, where for every $i = 1, \ldots, m$, $A_i \cong F$ is endowed with the trivial superautomorphism. If $UT_2, UT_2^{\text{sup}}, UT_2^{\text{gr}}, UT_2^{\text{gr,sup}} \notin \text{var}^{\varphi}(A)$ then $A_i J A_k = 0$, for all $1 \leq i, k \leq m$, $i \neq k$.*

Proof. Suppose that there exist $i, k \in \{1, \dots, m\}$, $i \neq k$, such that $A_i J A_k \neq 0$. Then there exist elements $a \in A_i, b \in A_k$, $j \in J$ such that $ajb \neq 0$ with $a^2 = a = a^\varphi$, $b^2 = b = b^\varphi$ and $|a| = |b| = 0$. Without loss of generality we may assume that j is a homogeneous element either symmetric or skew.

Let C be the subalgebra of A generated by a, b, ajb which is a graded subalgebra with (induced) superautomorphism. Now let $f\colon UT_2 \to C$ be the linear map defined by

$$f(e_{11}) = a, \quad f(e_{22}) = b, \quad f(e_{12}) = ajb.$$

Clearly f is an isomorphism of ordinary algebras. Moreover, f can be regarded as an isomorphism of superalgebras with superautomorphism between C and $UT_2, UT_2^{\text{sup}}, UT_2^{\text{gr}}, UT_2^{\text{gr,sup}}$, according as j is symmetric or skew of homogeneous degree 0 or 1, respectively.

In all the four cases we reach a contradiction and the proof is complete. ∎

We are in a position to prove the following theorem characterizing the varieties of superalgebras with superautomorphism of polynomial growth.

Theorem 23: *Let A be a finite-dimensional superalgebra with superautomorphism over a field F of characteristic zero. Then the sequence $c_n^\varphi(A)$, $n = 1, 2, \dots$, is polynomially bounded if and only if*

$$UT_2, UT_2^{\text{sup}}, UT_2^{\text{gr}}, UT_2^{\text{gr,sup}}, F \oplus F \notin \text{var}^\varphi(A).$$

Proof. By Theorems 3 and 4, the algebras $F \oplus F, UT_2, UT_2^{\text{sup}}, UT_2^{\text{gr}}, UT_2^{\text{gr,sup}}$ generate varieties of exponential growth. Hence, if $c_n^\varphi(A)$ is polynomially bounded, then they cannot belong to the variety generated by A.

Conversely suppose that $UT_2, UT_2^{\text{sup}}, UT_2^{\text{gr}}, UT_2^{\text{gr,sup}}, F \oplus F \notin \text{var}^\varphi(A)$. Since we are dealing with codimensions that do not change by extending the base field, we may assume that the field F is algebraically closed. Hence, by Theorem 9 and Lemmas 21 and 22,

$$A = A_1 \oplus \dots \oplus A_m + J,$$

where for every $i = 1, \dots, m$, $A_i \cong F$ is endowed with the trivial superautomorphism and $A_i J A_k = 0$, for all $1 \leq i, k \leq m$, $i \neq k$. Hence $A_0^- \oplus A_1^+ \oplus A_1^- \subseteq J$ and, if q is the least positive integer such that $J^q = 0$, then $A_0^- \oplus A_1^+ \oplus A_1^-$ generates a nilpotent ideal of A of nilpotency index $\leq q$. This says that $c_{n_1,n_2,n_3,n_4}(A) = 0$

as soon as $n_2 + n_3 + n_4 \geq q$ (see [11, Theorem 2.2]). Hence, by (1), we get

$$c_n^{\varphi}(A) = \sum_{\substack{n_1+\cdots+n_4 \\ n_2+n_3+n_4<q}} \binom{n}{n_1, \ldots, n_4} c_{n_1,\ldots,n_4}(A). \tag{2}$$

Notice that the number of non-zero summands in (2) is bounded by q^3 and that $\binom{n}{n_1,\ldots,n_4} < n^q$ ([11, Proposition 2.2]). Hence, since $c_{r_1,r_2,r_3,r_4}(A) \leq c_n(A)$ and $c_n(A) \leq an^t$ (see [4, Theorem 7.2.14]), we get the desired conclusion. ∎

As a consequence we have the following corollaries.

Corollary 24: *The algebras $UT_2, UT_2^{\mathrm{sup}}, UT_2^{\mathrm{gr}}, UT_2^{\mathrm{gr,sup}}, F \oplus F$ are the only finite-dimensional superalgebras with superautomorphism generating varieties of almost polynomial growth.*

Corollary 25: *If A is a finite-dimensional superalgebra with superautomorphism, the sequence $c_n^{\varphi}(A)$, $n = 1, 2, \ldots$, either is polynomially bounded or grows exponentially.*

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