Nonlinear Anal. RWA 94 (2027) 104737

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Boundary regularity for quasiminima of double-phase problems on metric spaces

Antonella Nastasi [a], Cintia Pacchiano Camacho [b,*]

[a] Department of Engineering, University of Palermo, Viale delle Scienze, 90128, Palermo, Italy

[b] Instituto de Matemáticas, Unidad Cuernavaca, Universidad Nacional Autónoma de México, Av. Universidad, 62210, Cuernavaca, Morelos, Mexico

ARTICLE INFO

2020 MSC:
49Q20
49N60
31C45
35J60
46E35

Keywords:
Quasiminimizers
Double-phase problems
Hölder continuity
Boundary regularity
Metric measure spaces

ABSTRACT

We use variational methods to prove pointwise estimates near a boundary point for quasiminima of integrals of p, q-Laplace type, whose integrand switches between p and q growth rate. This study is conducted in the general setting of metric measure spaces equipped with a doubling measure and supporting a Poincaré inequality, the obtained results are new even in the Euclidean case. The proofs rely on a careful phase analysis and estimates in the intrinsic geometries and they are based on De Giorgi method to obtain a sufficient condition for Hölder continuity at a boundary point for (p, q)-quasiminimizers.

1. Introduction

This paper aims to obtain boundary regularity of quasiminima for the double-phase integral defined on a complete metric measure space (X, d, μ) supporting a weak $(1, p)$-Poincaré inequality and equipped with a metric d and a doubling measure μ satisfying an upper Q-Ahlfors regularity condition.
Let Ω be an open and bounded subset of X. The double-phase integral is given by

$$\int_\Omega H(x, g_u)\,\mathrm{d}\mu = \int_\Omega \left(g_u^p + a(x) g_u^q\right)\mathrm{d}\mu, \tag{1.1}$$

where g_u is the minimal p-weak upper gradient of u and $1 < p < q$, with $p < Q$ and Q being the generalization of the Euclidean concept of space dimension (see Section 2). The modulating coefficient function $a(\cdot) \geq 0$ is assumed to meet certain standard regularity conditions, which are detailed in (2.3) below. The integral (1.1) shows an energy density that alternates between two types of degenerate behavior, depending on the modulating coefficient $a(\cdot)$, which dictates the phase. Specifically, when $a(x) = 0$, the variational integral (1.1) simplifies to the classical p-growth problem. In contrast, when $a(x) \geq c > 0$, it corresponds to the (p, q)-problem. Additionally, the ratio of exponents q/p is subjected to the condition

$$\frac{q}{p} \leq 1 + \frac{\alpha}{Q}, \tag{1.2}$$

* Corresponding author.
E-mail addresses: antonella.nastasi@unipa.it (A. Nastasi), cintia.pacchiano@im.unam.mx (C. Pacchiano Camacho).

https://doi.org/10.1016/j.nonrwa.2026.104737
Received 2 February 2026; Received in revised form 4 August 2026; Accepted 7 August 2026
Available online 17 August 2026

where $0 < \alpha \leq 1$ and $Q = \log_2 C_D$, with C_D being the doubling constant of the measure. This condition is known to be sharp for ensuring regularity of minima already in the Euclidean setting, in this direction see the recent contributions of Baroni-Colombo-Mingione [1] and Colombo-Mingione [8,9].

The main advantage of the notion of quasiminima of (1.1) is that it simultaneously covers a wide class of problems where the variational integrand $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions and

$$\lambda H(x,z) \leq F(x,u,z) \leq \Lambda H(x,z), \quad 0 < \lambda < \Lambda < \infty,$$

for every $x \in \Omega$ and $u, z \in \mathbb{R}$. Originally developed by Giaquinta and Giusti [19,20], quasiminima offer a flexible framework that extends beyond traditional differentiable structures, thus providing a unified treatment of variational integrals, elliptic equations and systems, obstacle problems and quasiregular mappings. When defined over a metric measure space, quasiminima allow the analysis of variational problems in settings where classical derivatives may not be defined, relying instead on notions such as minimal weak upper gradients to understand how functions vary along paths. Those notions permit to overcome the difficulties due to working on general metric spaces, which provide a relaxed geometric setting and lack a smooth structure. This adaptability has opened the application of regularity theory to non-Euclidean spaces and various areas of analysis, such as weighted Sobolev spaces, calculus on Riemannian manifolds, and potential theory on graphs, to name a few, see Björn-Björn [3], Franchi-Hajłasz-Koskela [18], Hajłasz [23], Heinonen-Koskela-Shanmugalingam-Tyson [25], Kilpeläinen-Kinnunen-Martio [31], Liu-Zhou-Shanmugalingam [38], Masson-Parviainen [47], Shanmugalingam [57] and references therein. The study of variational problems on metric measure spaces originated from independent proofs by Grigor'yan [22] and Saloff-Coste [56], which show that, on Riemannian manifolds, the doubling property and the Poincaré inequality are equivalent to a specific Harnack-type inequality for solutions of the heat equation. However, our focus changes from Riemannian manifolds to more general spaces.

Recently, the study of Sobolev spaces without a differentiable structure, along with the variational theory of different energy functionals in metric measure spaces, has captured the interest of researchers. For example, Cheeger [7] studied differentiability properties, while Kinnunen-Martio [34] focused on aspects of potential theory. Additionally, Kinnunen-Shanmugalingam [36] established local properties of quasiminima for the p-energy integral. Björn-Björn-Shanmugalingam [4] explored the Dirichlet problem for p-harmonic functions. Furthermore, Kinnunen-Marola-Martio [33] proved the Harnack principle. Recently, for the double-phase problems, Kinnunen-Nastasi-Pacchiano Camacho [35] established higher integrability results for the gradient of quasiminima and Nastasi-Pacchiano Camacho [54] derived other types of local regularity results. On the other hand, the regularity theory of nonlinear parabolic problems in the metric space context has been developed and studied in Herán [26,27], Ivert-Marola-Masson [29], Marola-Masson [45], Masson-Miranda Jr.-Paronetto-Parviainen [46] and Masson-Siljander [48]. We also report some recent contributions concerning the study of the fractional p-Laplacian in the context of general metric measure spaces, see Capogna-Kline-Korte-Shanmugalingam-Snipes [6].

Double-phase integrals have been introduced by Zhikov in the context of Homogenization and Lavrentiev's phenomenon [59–61]. Regularity theory for double-phase integrals has been extensively developed, starting from the seminal works of Marcellini [41–43]. In the Euclidean context, Colombo-Mingione [8,9], Cupini-Marcellini-Mascolo [10], De Filippis-Mingione [11], Esposito-Leonetti-Mingione [17], Marcellini-Nastasi-Pacchiano Camacho [44], Mingione [52], Ok [55] considered classes of nonuniformly elliptic problems and proved different regularity properties. For example, Hölder regularity results for certain class of double-phase problems with non-standard growth conditions can be found in Düzgün-Marcellini-Vespri [14,15], Eleuteri [16], Harjulehto-Hästö-Toivanen [24], Di Benedetto-Gianazza-Vespri [12]. Moreover, Di Benedetto-Trudinger [13] and Baroni-Colombo-Mingione [1] independently obtained Harnack inequalities for double-phase problems. See also Kinnunen-Lehrbäck-Vähäkangas [32] and Marcellini [39,40] for other results and literature reviews.

However, the analysis of boundary regularity, especially in metric measure spaces, remains relatively underdeveloped. Existing works include boundary regularity results for minimizers and quasiminima in the Euclidean case, such as the Wiener-type condition proved by Ziemer [62], see also Tachikawa [58] for double-phase functionals and Irving-Koch [28] for (p,q)-growth functionals. For the generalization to the metric setting, we have the works of Björn [2] and Nastasi-Pacchiano Camacho [53]. Further contributions include boundary oscillation estimates by Björn-MacManus-Shanmugalingam [5] for p-harmonic functions and p-energy minimizers. Despite these advances, the boundary behavior of double-phase problems remains a relatively open problem. Addressing this gap, the present work studies quasiminima of (1.1) in the Newtonian–Sobolev space $N^{1,1}(X)$, with $H(\cdot, g_u) \in L^1(X)$, for X a general metric measure space. The lack of classical derivatives in metric measure spaces requires variational methods that focus on the modulus of the gradient, therefore avoiding the need for smoothness.

To the best of our knowledge, boundary regularity for double-phase problems remains largely unexplored, even in the Euclidean setting. Motivated by recent developments, such as the works of Tachikawa [58] and Björn [2], we aim to extend regularity results up to the boundary within the general framework of metric measure spaces, where no smooth structure is assumed. A central aspect of our approach is the analysis of functionals with non-standard growth, particularly the transition between the p- and (p,q)-phases governed by the coefficient function $a(x)$. A key novelty of this work is the introduction of a Maz'ya-type estimate for frozen functionals (Theorem 4), inspired by classical pointwise capacitary estimates due to Maz'ya [49,50]. As far as we know, such a result has not been previously established, even in the Euclidean case. This estimate enables us to generalize local regularity results, such as those in De Filippis-Mingione [11], to boundary points.

The present work is organized as follows: we first prove the local boundedness of quasiminima (Section 3), then examine the behavior of frozen functionals (Section 4), and finally derive a pointwise boundary estimate (Proposition 3, Section 5). This leads to the main result of the paper: a sufficient condition for the Hölder continuity of quasiminima at boundary points (Theorem 6, Section 6).

Throughout, we aim to provide detailed and self-contained arguments that contribute to the regularity theory of variational integrals under double-phase growth.

2. Preliminaries

This section collects definitions and results needed when working in the general context of metric measure spaces. Further details can be found in the book by Björn-Björn [3]. Let (X, d, μ) be a complete metric measure space with a metric d and a Borel regular measure μ. The measure μ is assumed to be doubling, meaning that there exists a constant $C_D \geq 1$ such that, for every ball B_r in X,

$$0 < \mu(B_{2r}) \leq C_D \mu(B_r) < \infty. \tag{2.1}$$

We denote by $B_r = B(x, r) = \{x \in X : d(y, x) < r\}$ an open ball centered in $x \in X$ and with radius $0 < r < \infty$. The following lemma introduces Q, which is a generalization of the usual concept of dimension in metric spaces supporting a doubling measure. Indeed, in the Euclidean n-space with the Lebesgue measure, we have $Q = n$.

Lemma 1. *([3], Lemma 3.3) Let (X, d, μ) be a metric measure space with a doubling measure μ. Then*

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq C\left(\frac{r}{R}\right)^Q, \tag{2.2}$$

for every $0 < r \leq R < \infty$, $x \in X$ and $y \in B(x, R)$. Here $Q = \log_2 C_D$ and $C = C_D^{-2}$.

Note that if (2.2) holds with some Q, then it holds also with all exponents bigger or equal than Q. We refer the reader to Sections 3.1 and 4.4 of [3] for a comprehensive overview of the role and properties of Q.

Let $a : X \to [0, \infty)$ be the coefficient function in (1.1) and let δ_μ be a quasi-distance defined by

$$\delta_\mu(x, y) = \big(\mu(B(x, d(x, y))) + \mu(B(y, d(x, y)))\big)^{1/Q}, \quad x, y \in X,\ x \neq y,$$

with $Q = \log_2 C_D$ as in (2.2) and $\delta_\mu(x, x) = 0$. We suppose that there exists $0 < \alpha \leq 1$ such that

$$[a]_\alpha = \sup_{x,y\in X, x\neq y} \frac{|a(x) - a(y)|}{\delta_\mu(x, y)^\alpha} < \infty. \tag{2.3}$$

In the next remark, we point out that under certain regularity conditions on the measure, the quasi-distance δ_μ and the usual distance d are equivalent.

Remark 1. Let Q be the fixed exponent in (2.2). A measure is called *Ahlfors–David regular* if there exist constants $0 < C_L \leq C_U < \infty$ such that

$$C_L r^Q \leq \mu(B(x, r)) \leq C_U r^Q,$$

for every $x \in X$ and every $r > 0$. In this case, the quasi-distance δ_μ and the metric distance d are equivalent, namely,

$$\delta_\mu(x, y) \approx d(x, y), \qquad x, y \in X.$$

This way, the condition $[a]_\alpha < \infty$ is equivalent to the usual Hölder continuity of a with exponent α.

Throughout this manuscript, we assume that μ is *upper* Q-Ahlfors regular, meaning, there exists a constant $C_U > 0$ such that

$$\mu(B(x, r)) \leq C_U r^Q \qquad \text{for every } x \in X \text{ and } r > 0. \tag{2.4}$$

This assumption is essential in the analysis of the transition between the p- and (p, q)-phases, it is used in the proof of the almost standard Caccioppoli inequality (Lemma 4) and, therefore, in the subsequent phase analysis.

Upper Q-Ahlfors regularity is a fairly general assumption and is satisfied by many classes of metric measure spaces, including self-similar fractals, metric measure spaces with controlled curvature, uniformly rectifiable sets, and Carnot groups, to name a few.

Moreover, although we only assume the upper estimate (2.4), together with the doubling estimate (2.2) it gives a corresponding lower bound on the measures relevant to this paper. Indeed, since Ω is bounded, all balls considered have radii bounded by a fixed multiple of $\operatorname{diam}(\Omega)$. Hence, for every ball $B(y, r) \subset B(x, R)$ with R fixed, Lemma 1 gives

$$\mu(B(y, r)) \geq C_D^{-2}\left(\frac{r}{R}\right)^Q \mu(B(x, R)) = c\, r^Q,$$

where $c = c(C_D, R, \mu(B(x, R))) > 0$. Therefore, on the bounded scales considered throughout this work, the measure satisfies both upper and lower Q-dimensional estimates, that is, it is locally Ahlfors regular.

We discuss the notion of upper gradient as a way to generalize the modulus of the gradient in the Euclidean case to the metric setting.

Definition 1. Let $1 \leq p < \infty$. A nonnegative Borel function g is said to be an upper gradient of function $u : X \to [-\infty, \infty]$ if, for all paths γ connecting x and y, we have

$$|u(x) - u(y)| \leq \int_\gamma g \,\mathrm{d}s,$$

whenever $u(x)$ and $u(y)$ are both finite and $\int_\gamma g \,\mathrm{d}s = \infty$ otherwise. Here x and y are the endpoints of γ. Moreover, if a nonnegative measurable function g satisfies the inequality above for p-almost every path, that is, with the exception of a path family of zero p-modulus, then g is called a p-weak upper gradient of u.

We note that if u has an upper gradient $g \in L^p(X)$, there exists a unique minimal p-weak upper gradient $g_u \in L^p(X)$ with $g_u \le g$ μ-almost everywhere for all p-weak upper gradients $g \in L^p(X)$ of u, see [[3], Theorem 2.5].

Let $\|u\|_{N^{1,p}(X)} = \|u\|_{L^p(X)} + \|g_u\|_{L^p(X)}$. Consider the collection of functions $u \in L^p(X)$ with an upper gradient $g \in L^p(X)$ and let

$$\widetilde{N}^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}.$$

The Newtonian space is defined by

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

The corresponding local Newtonian space is defined by $u \in N^{1,p}_{\mathrm{loc}}(X)$ if $u \in N^{1,p}(\Omega')$ for all $\Omega' \Subset X$, see [[3], Proposition 2.29], where $\Omega' \Subset X$ means that $\overline{\Omega'}$ is a compact subset of X.

In order to give a definition of quasiminimizers of the energy integral, we need a Newtonian space with zero boundary values. We first report the definition of p-capacity.

Definition 2 ([2], Definition 2.4). The p-capacity of a set $E \subset X$ is

$$C_p(E) = \inf_u \|u\|^p_{N^{1,p}},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \ge 1$ on E. We say that a property holds p-quasieverywhere (p-q.e.) if the set of points for which it does not hold has p-capacity zero.

We recall that functions in the Newtonian space $N^{1,p}(X)$ are identified up to sets of p-capacity zero. More precisely,

$$\|u - v\|_{N^{1,p}(X)} = 0 \quad \Longleftrightarrow \quad u = v \quad \text{p-q.e.}$$

Thus, the notion of p-capacity provides the appropriate way of distinguishing representatives of Newtonian functions. Moreover, Corollary 3.3 in [57] together with the lattice property of $N^{1,p}(X)$ implies that if $u, v \in N^{1,p}(X)$ and $u \le v$ μ-almost everywhere, then

$$u \le v \qquad \text{p-q.e.}$$

Let Ω be an open subset of X. We define $N^{1,p}_0(\Omega)$ as the set of functions $u \in N^{1,p}(X)$ that are zero on $X \setminus \Omega$ p-q.e. The space $N^{1,p}_0(\Omega)$ is equipped with the norm $\|\cdot\|_{N^{1,p}}$. Note also that if $C_p(X \setminus \Omega) = 0$, then $N^{1,p}_0(\Omega) = N^{1,p}(X)$. We shall therefore always assume that $C_p(X \setminus \Omega) > 0$.

We assume that X supports the following Poincaré inequality.

Definition 3. Let $1 \le p < \infty$. A metric measure space (X, d, μ) supports a weak $(1, p)$-Poincaré inequality if there exist a constant C_{PI} and a dilation factor $\lambda \ge 1$ such that

$$\fint_{B_r} |u - u_{B_r}|\,\mathrm{d}\mu \le C_{PI}\, r \left(\fint_{B_{\lambda r}} g_u^p \,\mathrm{d}\mu\right)^{\frac{1}{p}},$$

for every ball B_r in X and for every $u \in L^1_{\mathrm{loc}}(X)$, where we denote the integral average with

$$u_{B_r} = \fint_{B_r} u\,\mathrm{d}\mu = \frac{1}{\mu(B_r)} \int_{B_r} u\,\mathrm{d}\mu.$$

As shown in [[30], Theorem 1.0.1] by Keith and Zhong, see also [[3], Theorem 4.30], the Poincaré inequality is a self-improving property.

Theorem 1. *Let (X, d, μ) be a complete metric measure space with a doubling measure μ and a weak $(1, p)$-Poincaré inequality with $p > 1$. Then there exists $\varepsilon > 0$ such that X supports a weak $(1, s)$-Poincaré inequality for every $s > p - \varepsilon$. Here, ε and the constants associated with the $(1, s)$-Poincaré inequality depend only on C_D, C_{PI} and p.*

The following result shows that the Poincaré inequality implies a Sobolev–Poincaré inequality, see [[3], Theorem 4.21 and Corollary 4.26].

Theorem 2. *Assume that μ is a doubling measure and X supports a weak $(1, p)$-Poincaré inequality and let $Q = \log_2 C_D$ be as in (2.2). Let $1 \le p^* := \frac{Qp}{Q-p}$ for $1 \le p < Q$ and $1 \le p^* < \infty$ for $Q \le p < \infty$. Then X supports a weak (p^*, p)-Poincaré inequality, that is, there exist a constant C=$C(C_D, C_{PI}, p)$ such that*

$$\left(\fint_{B_r} |u - u_{B_r}|^{p^*}\,\mathrm{d}\mu\right)^{\frac{1}{p^*}} \le C r \left(\fint_{B_{2\lambda r}} g_u^p\,\mathrm{d}\mu\right)^{\frac{1}{p}},$$

for every ball B_r in X and every $u \in L^1_{\mathrm{loc}}(X)$.

Remark 2. By the Hölder inequality we see that a weak (p^*, p)-Poincaré inequality implies the same inequality for smaller values of p^*. Meaning that X will then support a weak (t, p)-Poincaré inequality for all $1 < t < p^*$.

Remark 3. Throughout the paper we choose exponents p and q, where $p < Q$, with Q given in (2.2), and satisfying the closeness condition (1.2), that is

$$\frac{q}{p} \leq 1 + \frac{\alpha}{Q},$$

where $0 < \alpha \leq 1$.

Since

$$1 + \frac{\alpha}{Q} < \frac{Q}{Q-p} = \frac{p^*}{p},$$

it follows that

$$q < p^*.$$

Therefore, by continuity of the Sobolev exponent, there exists $s < p$ such that

$$q < s^* := \frac{Qs}{Q-s}.$$

Consequently, by the self-improving property of Poincaré inequalities, see Theorem 1, there exists $s = s(C_{PI}, C_D, p, q)$, with $1 < s < p < q < s^*$, such that X supports a $(1, s)$-Poincaré inequality.

Definition 4. ([2], Definition 6.13) Let $E \subset \Omega$. Then we define the variational capacity

$$\mathrm{cap}_p(E, \Omega) = \inf_u \int_\Omega g_u^p \mathrm{d}\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(\Omega)$ such that $u \geq 1$ on E.

The variational capacity is also known as the relative capacity. We define the infimum to be ∞ in the absence of any admissible functions u.

The next result shows that the capacities cap_p and C_p are essentially equivalent. Furthermore, it provides explicit estimates for the variational capacity, especially in the case of balls. A detailed proof is available in [3].

Proposition 1. *Assume that μ is a doubling measure with constant C_D and X supports a weak $(1,p)$-Poincaré inequality. Then there exists $C > 0$ such that if $E \subset B_r = B(x_0, r)$ with $0 < r < \frac{diam(X)}{8}$, then*

$$\frac{\mu(E)}{Cr^p} \leq cap_p(E, B_{2r}) \leq \frac{C_D \mu(B_r)}{r^p}.$$

and

$$\frac{C_p(E)}{C(1+r^p)} \leq cap_p(E, B_{2r}) \leq 2^{p-1}\left(1 + \frac{1}{r^p}\right) C_p(E).$$

In particular, for bounded sets E, $C_p(E) = 0$ if and only $cap_p(E, B_{2r}) = 0$ for some ball B_r containing E.

Remark 4. Notice that for $s_2 \leq s_1$ and a small radius $r \leq 1$, we have that if $u \in N_0^{1,s_1}(B_r)$ is such that $u \geq 1$ on E, it is then clear that $u \in N_0^{1,s_2}(B_r)$. Therefore by Hölder's inequality we obtain

$$\begin{aligned} \mathrm{cap}_{s_2}(E, B_r) &\leq \int_{B_r} g_u^{s_2} \, \mathrm{d}\mu \\ &\leq \left(\int_{B_r} g_u^{s_1} \, \mathrm{d}\mu \right)^{\frac{s_2}{s_1}} \mu(B_r)^{\frac{s_1 - s_2}{s_1}}. \end{aligned}$$

So,

$$\mu(B_r)^{\frac{s_2 - s_1}{s_1}} \mathrm{cap}_{s_2}(E, B_r) \leq \left(\int_{B_r} g_u^{s_1} \, \mathrm{d}\mu \right)^{\frac{s_2}{s_1}}.$$

Therefore,

$$\left(\mu(B_r)^{\frac{s_2 - s_1}{s_1}} \mathrm{cap}_{s_2}(E, B_r) \right)^{\frac{s_1}{s_2}} \leq \int_{B_r} g_u^{s_1} \, \mathrm{d}\mu, \qquad \text{for all } u \in N_0^{1,s_2}(B_r) \text{ with } u \geq 1 \text{ on } E.$$

Meaning,

$$\mu(B_r)^{\frac{s_2 - s_1}{s_1}} \mathrm{cap}_{s_2}(E, B_r) \leq \mathrm{cap}_{s_1}^{s_2/s_1}(E, B_r), \qquad \text{for } s_2 \leq s_1.$$

Furthermore, we have

$$\frac{1}{\mathrm{cap}_{s_1}(E, B_r)} \leq \left(\mu(B_r)^{\frac{s_1 - s_2}{s_1}} \frac{1}{\mathrm{cap}_{s_2}(E, B_r)} \right)^{\frac{s_1}{s_2}}.$$

Finally, since we are asking for the measure μ to be upper Q-Alhfors regular, by inequality (2.4) and the assumption $r \le 1$, we conclude

$$\frac{1}{\mathrm{cap}_{s_1}(E, B_r)} \le \frac{C}{\mathrm{cap}_{s_2}^{s_1/s_2}(E, B_r)}, \tag{2.5}$$

where C=$C(C_U, s_1, s_2)$.

The following proposition is a capacity version of the Sobolev-Poincaré inequality, also referred to as Maz'ya type estimate. The proof is a straightforward generalization of the Euclidean case and can be found in [2].

Proposition 2. *([2], Proposition 3.2)Let X be a doubling metric measure space supporting a weak $(1, s)$-Poincaré inequality. Then there exists C and $\lambda \ge 1$ such that for all balls B_r in X, $u \in N^{1,s}(X)$ and $S = \{x \in B_{\frac{r}{2}} : u(x) = 0\}$, then*

$$\left(\fint_{B_r} |u|^t \,\mathrm{d}\mu\right)^{\frac{1}{t}} \le \left(\frac{C}{\mathrm{cap}_s(S, B_r)} \int_{B_{\lambda r}} g^s \,\mathrm{d}\mu\right)^{\frac{1}{s}}, \tag{2.6}$$

for C depending on C_{PI} and $1 < t < s^*$.

Throughout this paper, we consider a complete metric measure space (X, d, μ) equipped with a metric d and a doubling Borel regular measure μ. Moreover, we assume that X supports a weak $(1, p)$-Poincaré inequality.

From now on and without further notice, we fix $s = s(C_{PI}, C_D, p, q)$ such that $1 < s < p < q < s^*$ and for which X also admits a weak $(1, s)$-Poincaré inequality. Such s is given by Remark 3 and will be used in various of our results.

Finally, we let $\Omega \subset X$ be a nonempty bounded open set satisfying

$$C_p(X \setminus \Omega) > 0.$$

We note that positive constants are denoted by C and the dependencies on parameters are listed in the parentheses. We denote

$$C(\mathrm{data}) = C(C_D, C_{PI}, \lambda, p, q, \alpha, [a]_\alpha).$$

3. Energy estimates and local boundedness

In this section, our main interest focuses on proving the so-called Caccioppoli inequality, Lemma 2, and local boundedness, Corollary 1, for quasiminima of the double-phase integral (1.1). We begin by defining quasiminima with boundary values.

Definition 5. Let $w \in N^{1,1}(X)$ with $H(\cdot, g_w) \in L^1(X)$. A function $u \in N^{1,1}(X)$ with $H(\cdot, g_u) \in L^1(X)$ is a quasiminimizer on Ω with boundary data w if $u - w \in N_0^{1,1}(\Omega)$ and there exists a constant $K \ge 1$ such that

$$\int_{\{u \ne v\}} H(x, g_u) \,\mathrm{d}\mu \le K \int_{\{u \ne v\}} H(x, g_v) \,\mathrm{d}\mu,$$

for every function $v \in N^{1,1}(\Omega)$ with $u - v \in N_0^{1,1}(\Omega)$.

Lemma 2. *(Caccioppoli inequality) Assume that $u \in N^{1,1}(X)$ with $H(\cdot, g_u) \in L^1(X)$ is a quasiminimizer in Ω with boundary data $w \in N^{1,1}(X)$, $H(\cdot, g_w) \in L^1(X)$. Let $x_0 \in X$, $B(x_0, r) = B_r$ for every $r > 0$, and $S_{k,r} = \{x \in B_r : u(x) > k\}$. Then, for $0 < r < R$ and $k \ge \operatorname{ess\,sup}_{B_R} w$ there exists $C = C(K, q) > 0$ such that the following inequality*

$$\int_{S_{k,r}} H(x, g_u) \,\mathrm{d}\mu \le C \int_{B_R} H\left(x, \frac{(u-k)_+}{R-r}\right) \mathrm{d}\mu \tag{3.1}$$

is satisfied, where $(u - k)_+ = \max\{u - k, 0\}$.

Proof. The proof is a modification of the one originally given for local quasiminima of double-phase functionals in [[35], Lemma 2.9], which we furthermore specialize to the case of quasiminimizer $u \in N^{1,1}(X)$ with boundary data as in Definition 5.

Let η be a $(R - r)^{-1}$-Lipschitz cutoff function such that $0 \le \eta \le 1$, $\eta = 1$ on B_r and $\eta = 0$ in $X \setminus B_R$. Let $v = u - \eta(u - k)_+$. Then, as $u = w \le k$ on $B_R \setminus \Omega$ and $\eta = 0$ outside of B_R, we have $u - v \in N_0^{1,1}(\Omega)$. Also, $v = (1 - \eta)(u - k)_+ + k$ on $S_{k,R}$ and $v = u$ outside $S_{k,R}$.

By the Leibniz rule for the upper gradients [[3], Lemma 2.18], we have

$$g_v \le (u - k)_+ g_\eta + (1 - \eta) g_u + g_u \chi_{X \setminus S_{k,R}}.$$

Since u is a quasiminimizer, by Definition 5 we obtain

$$\begin{aligned}
\int_{S_{k,r}} H(x, g_u) \,\mathrm{d}\mu &\le \int_{\{u \ne v\}} H(x, g_u) \,\mathrm{d}\mu \le K \int_{\{u \ne v\}} H(x, g_v) \,\mathrm{d}\mu \\
&\le K \left(\int_{S_{k,R}} H(x, (u - k)_+ g_\eta + (1 - \eta) g_u) \,\mathrm{d}\mu \right) \\
&\le 2^q K \left(\int_{B_R} H\left(x, \frac{(u - k)_+}{R - r}\right) \mathrm{d}\mu + \int_{S_{k,R} \setminus S_{k,r}} H(x, g_u) \,\mathrm{d}\mu \right).
\end{aligned} \tag{3.2}$$

By adding $K2^q \int_{S_{k,r}} H(x, g_u)\,\mathrm{d}\mu$ to the both sides of (3.2), we get

$$(1+K2^q)\int_{S_{k,r}} H(x,g_u)\,\mathrm{d}\mu \le K2^q\left(\int_{B_R} H\left(x, \frac{(u-k)_+}{R-r}\right)\mathrm{d}\mu + \int_{S_{k,R}} H(x,g_u)\,\mathrm{d}\mu\right).$$

This implies

$$\begin{aligned}\int_{S_{k,r}} H(x,g_u)\,\mathrm{d}\mu &\le \theta\left(\int_{B_R} H\left(x, \frac{(u-k)_+}{R-r}\right)\mathrm{d}\mu + \int_{S_{k,R}} H(x,g_u)\,\mathrm{d}\mu\right)\\ &\le (R-r)^{-p}\int_{B_R}(u-k)_+^p\,\mathrm{d}\mu + (R-r)^{-q}\int_{B_R} a(x)(u-k)_+^q\,\mathrm{d}\mu + \theta\int_{S_{k,R}} H(x,g_u)\,\mathrm{d}\mu,\end{aligned}$$

with $\theta = \frac{K2^q}{1+K2^q} < 1$. We apply a standard iteration lemma, see [[21], Lemma 6.1], to obtain

$$\int_{S_{k,r}} H(x,g_u)\,\mathrm{d}\mu \le C\int_{B_R} H\left(x, \frac{(u-k)_+}{R-r}\right)\mathrm{d}\mu,$$

where C=$C(q,K)$. ☐

Remark 5. Notice that if u is a quasiminimizer then $-u$ is also a quasiminimizer. Therefore, by Lemma 2, we get that $-u$ satisfies (3.1).

Remark 6. The condition $k \ge \operatorname{ess\,sup}_{B_R} w$ is crucial in the formulation of Lemma 2. If $\operatorname{ess\,sup}_{B_R} w < \infty$, then we may choose k large enough so that the Caccioppoli inequality (3.1) holds. On the other hand, if $\operatorname{ess\,sup}_{B_R} w = \infty$, there is no finite truncation level k satisfying this requirement, and consequently the inequality becomes void. In particular, the subsequent iteration scheme leading to local boundedness cannot be carried out. This is consistent with the fact that when the boundary data is unbounded above, no boundedness property of the quasiminimizer u can be expected.

The goal now is to prove that a quasiminimizer, as Definition 5, is locally bounded. We first prove an auxiliary lemma. We note that a version of these results was first proved for local quasminimizers in [54]. The main difference is that the results proven in the present work hold for any ball in X and not only for compactly contained balls in Ω.

Lemma 3. *Let $u \in N^{1,1}(X)$ with $H(\cdot, g_u) \in L^1(X)$, $x_0 \in X$, $B(x_0,R) = B_R$, $0 < \frac{R}{2} < \rho < s \le R \le \min\{1, \frac{\operatorname{diam}(X)}{6}\}$, and concentric balls $B_\rho \subset B_s \subseteq B_R$. Assume u satisfies the double-phase Caccioppoli inequality (3.1) for all $k \ge k^*$ and $0 < r < R$. Then, for any positive numbers $k^* \le h < k$, there exists a constant $C = C(\mathrm{data}, \|u\|_{N^{1,p}(B_R)})$, and an exponent $0 < \theta = \theta(data)$, such that*

$$\int_{S_{k,\rho}} H(x,u-k)\mathrm{d}\mu \le \frac{C}{(s-\rho)^{q-p}}\left(\frac{\mu(S_{k,s})}{\mu(B_s)}\right)^\theta \int_{S_{h,s}} H\left(x, \frac{u-h}{s-\rho}\right)\mathrm{d}\mu.$$

Proof. Let $0 < \frac{R}{2} < \rho < s \le R \le \min\{1, \frac{\operatorname{diam}(X)}{6}\}$. We define $t = (s+\rho)/2$ to simplify notation. Notice that, by definition, $\rho < t < s$. By the double-phase Caccioppoli inequality (3.1), for $k \ge k^*$ we have

$$\begin{aligned}\int_{B_t} H(x, g_{(u-k)_+})\mathrm{d}\mu &\le C\int_{B_s} H\left(x, \frac{(u-k)_+}{s-t}\right)\mathrm{d}\mu\\ &\le C2^q\int_{B_s} H\left(x, \frac{(u-k)_+}{s-\rho}\right)\mathrm{d}\mu\\ &= C\int_{B_s} H\left(x, \frac{(u-k)_+}{s-\rho}\right)\mathrm{d}\mu,\end{aligned}\tag{3.3}$$

where C=$C(K,q)$. Let τ be a $\frac{C}{s-\rho}$-Lipschitz cutoff function so that $0 \le \tau \le 1$, $\tau = 1$ on B_ρ and the support of τ is contained in B_t. Let $w = \tau(u-k)_+ \in N_0^{1,1}(B_t)$. By Leibniz rule, [[3], Lemma 2.18], we have

$$g_w \le g_{(u-k)_+}\tau + (u-k)_+ g_\tau \le g_{(u-k)_+} + \frac{C}{s-\rho}(u-k)_+, \qquad \text{on } B_t.$$

Using inequality (3.3), we get

$$\begin{aligned}\int_{B_t} H(x,g_w)\mathrm{d}\mu &\le 2^{q-1}\int_{B_t} H(x, g_{(u-k)_+})\mathrm{d}\mu + 2^{q-1}C\int_{B_t} H\left(x, \frac{(u-k)_+}{s-\rho}\right)\mathrm{d}\mu\\ &\le 2^{q-1}C\int_{B_s} H\left(x, \frac{(u-k)_+}{s-\rho}\right)\mathrm{d}\mu + 2^{q-1}C\int_{B_s} H\left(x, \frac{(u-k)_+}{s-\rho}\right)\mathrm{d}\mu\\ &= C\int_{B_s} H\left(x, \frac{(u-k)_+}{s-\rho}\right)\mathrm{d}\mu.\end{aligned}\tag{3.4}$$

By Hölder inequality, the doubling property, the definitions of t and w and [[54], Lemma 4], there is a constant C=$C(\mathrm{data})$ and exponents $0 < d_2 < 1 \le d_1 < \infty$, such that

$$⨍_{B_\rho} H\big(x,(u-k)_+\big)\mathrm{d}\mu \le ⨍_{B_\rho} H\left(x, \frac{(u-k)_+}{t}\right)\mathrm{d}\mu$$

$$
\begin{aligned}
&\le \left(\fint_{B_\rho} H\left(x, \frac{(u-k)_+}{t}\right)^{d_1} \mathrm{d}\mu \right)^{\frac{1}{d_1}} \\
&= \left(\fint_{B_\rho} H\left(x, \frac{\tau(u-k)_+}{t}\right)^{d_1} \mathrm{d}\mu \right)^{\frac{1}{d_1}} \\
&= \left(\fint_{B_\rho} H\left(x, \frac{w}{t}\right)^{d_1} \mathrm{d}\mu \right)^{\frac{1}{d_1}} \\
&\le \left(\frac{\mu(B_t)}{\mu(B_\rho)} \right)^{\frac{1}{d_1}} \left(\fint_{B_t} H\left(x, \frac{w}{t}\right)^{d_1} \mathrm{d}\mu \right)^{\frac{1}{d_1}} \\
&\le C\left(1 + \|g_w\|_{L^p(B_t)}^{q-p} \mu(B_t)^{\frac{\alpha}{Q}-\frac{q-p}{p}}\right) \left(\fint_{B_t} H(x, g_w)^{d_2} \mathrm{d}\mu \right)^{\frac{1}{d_2}}.
\end{aligned} \tag{3.5}
$$

In the last inequality, we used the doubling property to estimate

$$
\left(\frac{\mu(B_t)}{\mu(B_\rho)} \right)^{\frac{1}{d_1}} \le C, \tag{3.6}
$$

where C depends on the exponent Q in (2.2). By Hölder inequality, we have

$$
\begin{aligned}
\left(\fint_{B_t} H(x, g_w)^{d_2} \mathrm{d}\mu \right)^{\frac{1}{d_2}} &= \mu(B_t)^{-\frac{1}{d_2}} \left(\int_{S_{k,t}} H(x, g_w)^{d_2} \mathrm{d}\mu \right)^{\frac{1}{d_2}} \\
&\le \mu(B_t)^{-\frac{1}{d_2}} \mu(S_{k,t})^{\frac{1}{d_2}-1} \int_{S_{k,t}} H(x, g_w) \mathrm{d}\mu.
\end{aligned} \tag{3.7}
$$

By (3.5), (3.7) and (3.4), we get

$$
\begin{aligned}
&\int_{B_\rho} H(x, (u-k)_+) \mathrm{d}\mu \\
&\le C\left(1 + \|g_w\|_{L^p(B_t)}^{q-p} \mu(B_t)^{\frac{\alpha}{Q}-\frac{q-p}{p}}\right) \left(\frac{\mu(S_{k,t})}{\mu(B_t)} \right)^{\frac{1}{d_2}-1} \int_{S_{k,t}} H(x, g_w) \mathrm{d}\mu \\
&\le C\left(1 + \|g_w\|_{L^p(B_t)}^{q-p} \mu(B_R)^{\frac{\alpha}{Q}-\frac{q-p}{p}}\right) \left(\frac{\mu(S_{k,t})}{\mu(B_t)} \right)^{\frac{1}{d_2}-1} \int_{B_s} H\left(x, \frac{(u-k)_+}{s-\rho}\right) \mathrm{d}\mu.
\end{aligned} \tag{3.8}
$$

On the other hand, notice that

$$
\begin{aligned}
\|g_w\|_{L^p(B_t)}^p &= \int_{B_t} g_w^p \mathrm{d}\mu \\
&\le \int_{B_t} \left(g_{(u-k)_+} + \frac{(u-k)_+}{s-\rho} \right)^p \mathrm{d}\mu \\
&\le 2^{p-1} \left(\int_{B_t} g_{(u-k)_+}^p \mathrm{d}\mu + \int_{B_t} \left(\frac{(u-k)_+}{s-\rho} \right)^p \mathrm{d}\mu \right) \\
&\le C\left(\|g_u\|_{L^p(B_t)}^p + \frac{\|u\|_{L^p(B_t)}^p}{(s-\rho)^p} \right) \\
&\le \frac{C}{(s-\rho)^p} \left(\|g_u\|_{L^p(B_t)}^p + \|u\|_{L^p(B_t)}^p \right).
\end{aligned}
$$

Since $u \in N^{1,1}(X)$, $H(\cdot, g_u) \in L^1(X)$ and the nature of the double-phase functional, then $u \in N^{1,p}_{\mathrm{loc}}(X)$. Therefore,

$$
\begin{aligned}
\|g_w\|_{L^p(B_t)}^{q-p} &\le \frac{C}{(s-\rho)^{q-p}} \left(\|g_u\|_{L^p(B_t)}^p + \|u\|_{L^p(B_t)}^p \right)^{\frac{q-p}{p}} \\
&\le \frac{C}{(s-\rho)^{q-p}} \left(\|g_u\|_{L^p(B_t)} + \|u\|_{L^p(B_t)} \right)^{q-p} \\
&= \frac{C}{(s-\rho)^{q-p}} \|u\|_{N^{1,p}(B_t)}^{q-p} \\
&\le \frac{C}{(s-\rho)^{q-p}} \|u\|_{N^{1,p}(B_R)}^{q-p},
\end{aligned}
$$

since by definition $t \le R$. By (3.8), the assumption that our measure is upper Q-Alhfors regular (2.4), $R \le 1$ and the last inequality, we then obtain

$$
\int_{B_\rho} H(x, (u-k)_+) \mathrm{d}\mu
$$

$$
\begin{aligned}
&\le \frac{C}{(s-\rho)^{q-p}}\left(1+\|u\|^{q-p}_{N^{1,p}(B_R)}\mu(B_R)^{\frac{\alpha}{Q}-\frac{q-p}{p}}\right)\left(\frac{\mu(S_{k,t})}{\mu(B_t)}\right)^{\frac{1}{d_2}-1}\int_{B_s}H\left(x,\frac{(u-k)_+}{s-\rho}\right)\mathrm{d}\mu\\
&\le \frac{C}{(s-\rho)^{q-p}}\left(1+\|u\|^{q-p}_{N^{1,p}(B_R)}\right)\left(\frac{\mu(S_{k,t})}{\mu(B_t)}\right)^{\frac{1}{d_2}-1}\int_{B_s}H\left(x,\frac{(u-k)_+}{s-\rho}\right)\mathrm{d}\mu\\
&= \frac{C}{(s-\rho)^{q-p}}\left(\frac{\mu(S_{k,t})}{\mu(B_t)}\right)^{\frac{1}{d_2}-1}\int_{B_s}H\left(x,\frac{(u-k)_+}{s-\rho}\right)\mathrm{d}\mu,
\end{aligned}
\tag{3.9}
$$

where $C{=}C(\text{data}, C_U, \|u\|_{N^{1,p}(B_R)})$.

Furthermore, we observe that $\mu(S_{k,t}) \le \mu(S_{k,s})$ and $\frac{1}{d_2}-1>0$. Thus, by (3.9) and the doubling property of the measure (as we did in (3.6)) we get

$$
\int_{B_\rho} H\left(x,(u-k)_+\right)\mathrm{d}\mu \le \frac{C}{(s-\rho)^{q-p}}\left(\frac{\mu(S_{k,s})}{\mu(B_s)}\right)^{\frac{1}{d_2}-1}\int_{B_s}H\left(x,\frac{(u-k)_+}{s-\rho}\right)\mathrm{d}\mu. \tag{3.10}
$$

with $C=C(\text{data}, C_U, \|u\|_{N^{1,p}(B_R)})$.

Let $k^* \le h < k$, then $(u-k)_+ \le (u-h)_+$. Therefore, we have

$$
\int_{S_{k,\rho}} H(x,u-k)\mathrm{d}\mu \le \frac{C}{(s-\rho)^{q-p}}\left(\frac{\mu(S_{k,s})}{\mu(B_s)}\right)^{\frac{1}{d_2}-1}\int_{S_{h,s}}H\left(x,\frac{u-h}{s-\rho}\right)\mathrm{d}\mu.
$$

That is

$$
\int_{S_{k,\rho}} H(x,u-k)\mathrm{d}\mu \le \frac{C}{(s-\rho)^{q-p}}\left(\frac{\mu(S_{k,s})}{\mu(B_s)}\right)^{\theta}\int_{S_{h,s}}H\left(x,\frac{u-h}{s-\rho}\right)\mathrm{d}\mu,
$$

where $\theta = \frac{1}{d_2}-1>0$. □

As a consequence of the previous lemma, we obtain the following weak Harnack inequality type result.

Theorem 3. *Let X be a doubling metric measure space admitting a weak $(1,p)$-Poincaré inequality. Then, there exists $C>0$, $C{=}C(\text{data}, C_U, R, \|u\|_{N^{1,p}(B_R)})$, such that if $u\in N^{1,1}(X)$ with $H(\cdot,g_u)\in L^1(X)$ and the condition (3.1) holds for all $k\ge k^*$ and $0<r<R<\min\{1,\frac{\operatorname{diam}(X)}{6}\}$, then for all $k_0\ge k^*$*

$$
\operatorname*{ess\,sup}_{B_{R/2}} u \le k_0 + C\left(\fint_{B_R} H\left(x,(u-k_0)_+\right)\mathrm{d}\mu\right)^{\frac{1}{p}}. \tag{3.11}
$$

Proof. This proof is similar to the one of [[54], Theorem 3], with few modifications. Let $k_0\ge k^*$. For $n\in\mathbb{N}\cup\{0\}$, let $\rho_n=\frac{R}{2}\left(1+\frac{1}{2^n}\right)\le R$ and $k_n=k_0+d\left(1-\frac{1}{2^n}\right)$, where $d>0$ will be chosen later. Then, $\rho_0=R$, $\rho_n\searrow\frac{R}{2}$ and $k_n\nearrow k_0+d$. We apply Lemma 3 with $\rho=\rho_{i+1}$, $R=\rho_i$, $k=k_{i+1}$ and $h=k_i$ and we get

$$
\begin{aligned}
\int_{S_{k_{i+1},\rho_{i+1}}} H(x,u-k_{i+1})\mathrm{d}\mu &\le \frac{C}{(\rho_i-\rho_{i+1})^{q-p}}\left(\frac{\mu(S_{k_{i+1},\rho_i})}{\mu(B_R)}\right)^{\theta}\int_{S_{k_i,\rho_i}}H\left(x,\frac{u-k_i}{\rho_i-\rho_{i+1}}\right)\mathrm{d}\mu\\
&= \frac{C}{(R2^{-i}2^{-2})^{q-p}}\left(\frac{\mu(S_{k_{i+1},\rho_i})}{\mu(B_R)}\right)^{\theta}\int_{S_{k_i,\rho_i}}H\left(x,\frac{u-k_i}{R2^{-i}2^{-2}}\right)\mathrm{d}\mu\\
&= \frac{C2^{i(2q-p)}}{R^{q-p}}\left(\frac{\mu(S_{k_{i+1},\rho_i})}{\mu(B_R)}\right)^{\theta}\int_{S_{k_i,\rho_i}}H\left(x,\frac{u-k_i}{R}\right)\mathrm{d}\mu\\
&\le \frac{C4^{iq}}{R^{2q-p}}\left(\frac{\mu(S_{k_{i+1},\rho_i})}{\mu(B_R)}\right)^{\theta}\int_{S_{k_i,\rho_i}}H(x,u-k_i)\mathrm{d}\mu\\
&\le \frac{C4^{iq}}{R^{2q}}\left(\frac{\mu(S_{k_{i+1},\rho_i})}{\mu(B_R)}\right)^{\theta}\int_{S_{k_i,\rho_i}}H(x,u-k_i)\mathrm{d}\mu.
\end{aligned}
\tag{3.12}
$$

We observe that

$$
\begin{aligned}
d^{-p}(k_{i+1}-k_i)^p\mu(S_{k_{i+1},\rho_i}) &= d^{-p}\int_{S_{k_{i+1},\rho_i}}(k_{i+1}-k_i)^p\mathrm{d}\mu\\
&\le d^{-p}\int_{S_{k_{i+1},\rho_i}}H(x,u-k_i)\mathrm{d}\mu\\
&\le d^{-p}\int_{S_{k_i,\rho_i}}H(x,u-k_i)\mathrm{d}\mu.
\end{aligned}
$$

So, we have

$$\psi_i = d^{-p} \int_{S_{k_i,\rho_i}} H(x, u - k_i) \mathrm{d}\mu \geq d^{-p} (k_{i+1} - k_i)^p \mu(S_{k_{i+1},\rho_i}).$$

This implies

$$\mu(S_{k_{i+1},\rho_i}) \leq \psi_i d^p (k_{i+1} - k_i)^{-p}. \tag{3.13}$$

Therefore, by (3.12) and (3.13) we obtain

$$\begin{aligned} d^p \psi_{i+1} &\leq \frac{C 4^{iq}}{R^{2q}} \left(\frac{\mu(S_{k_{i+1},\rho_i})}{\mu(B_R)} \right)^\theta d^p \psi_i \\ &\leq \frac{C 4^{iq}}{R^{2q}} \left(\psi_i d^p (k_{i+1} - k_i)^{-p} \right)^\theta d^p \psi_i \mu(B_R)^{-\theta} \\ &= \frac{C 4^{iq}}{R^{2q}} (d 2^{-1-i})^{-p\theta} d^{p\theta} \psi_i^{1+\theta} d^p \mu(B_R)^{-\theta} \\ &\leq \frac{C 4^{(1+\theta)qi}}{R^{2q}} \psi_i^{1+\theta} d^p \mu(B_R)^{-\theta}. \end{aligned}$$

That is,

$$\psi_{i+1} \leq \frac{C 4^{(1+\theta)qi}}{R^{2q}} \psi_i^{1+\theta} \mu(B_R)^{-\theta}, \tag{3.14}$$

for every $i \geq 0$, where $C{=}C(\mathrm{data}, C_U, \|u\|_{N^{1,p}(B_R)})$. By using a standard iteration lemma [[21], Lemma 7.1] we get

$$\lim_{i\to\infty} \psi_i = \lim_{i\to\infty} d^{-p} \int_{S_{k_i,\rho_i}} H(x, u - k_i) \mathrm{d}\mu = 0,$$

provided that $d = C R^{\frac{-2q}{p\theta}} \left(\fint_{B_R} H(x, (u - k_0)_+) \mathrm{d}\mu \right)^{\frac{1}{p}} > 0$. As a consequence,

$$\int_{B_{\frac{R}{2}}} H(x, (u - (k_0 + d))_+ \mathrm{d}\mu = 0$$

and so

$$u \leq k_0 + d \quad \text{almost everywhere in } B_{\frac{R}{2}} \text{ and for all } k_0 \geq k^*. \tag{3.15}$$

We conclude that

$$\operatorname*{ess\,sup}_{B_{\frac{R}{2}}} u \leq k_0 + d = k_0 + C \left(\fint_{B_R} H\left(x, (u - k_0)_+\right) \mathrm{d}\mu \right)^{\frac{1}{p}},$$

where $C{=}C(\mathrm{data}, C_U, R, \|u\|_{N^{1,p}(B_R)})$. □

As a corollary of Theorem 3, we obtain local boundedness for quasiminima of (1.1). We emphasize that these results are a generalization of those initially obtained in [54], since here the conclusions are valid for any ball contained in X and not only for balls compactly contained in the domain Ω.

Corollary 1. *Let $u \in N^{1,1}(X)$ with $H(\cdot, g_u) \in L^1(X)$. Assume u satisfies the double-phase Caccioppoli inequality (3.1) for all $k \geq k^*$ and $0 < r < R < \min\{1, \frac{\mathrm{diam}(X)}{6}\}$. Then,*

$$\|u\|_{L^\infty(B_{R/2})} < \infty$$

holds. In particular, if u is a quasiminimizer in Ω with boundary data $w \in N^{1,1}(X)$, $H(\cdot, g_w) \in L^1(X)$, then u is locally bounded.

Proof. Since u satisfies the double-phase Caccioppoli inequality (3.1), then by Theorem 3 and, in particular, by inequality (3.15), we obtain the desired result.

In case u is a quasiminimizer, by Lemma 2, u satisfies the double-phase Caccioppoli inequality (3.1) for all $k \geq \operatorname{ess\,sup}_{B(x_0,R)} w$. Therefore, the result follows as before. □

Once we have that quasiminima are locally bounded in any ball $B_r \subset X$, we can prove the following almost standard Caccioppoli's inequality. This states that, in the p-regime case, double-phase quasiminima satisfy a Caccioppoli type inequality analogous to the one satisfied for quasiminima of functionals with just p-growth plus some extra controllable terms. This result was proved in [54] for local quasiminima of double-phase problems. In the Euclidean case this result has been proven by Colombo-Mingione [9].

Lemma 4. *(Almost standard Caccioppoli's inequality) Assume that $u \in N^{1,1}(X)$ with $H(\cdot, g_u) \in L^1(X)$ satisfies the double-phase Caccioppoli inequality (3.1) for all $k \geq \operatorname{ess\,sup}_{B(x_0,R)} w$ and $0 < r < R < \min\{1, \frac{\mathrm{diam}(X)}{6}\}$. Furthermore, assume that*

$$\sup_{B(x_0,R)} a(x) \leq C[a]_\alpha \mu(B(x_0, R))^{\frac{\alpha}{Q}}, \tag{3.16}$$

holds for $x_0 \in X$, $0 \le R \le 1$. *Then there exists* $C = C(C_U, p, q, \alpha, Q, [a]_\alpha, \|u\|_{L^\infty(B(x_0,R))}, K) > 0$ *such that for any choice of concentric balls* $B(x_0, s) \subset B(x_0, t) \subset B(x_0, R) \subset X$, *with* $0 < t < s \le R \le 1$ *and* $2\|u\|_{L^\infty(B(x_0,R))} \ge |k| \ge k \ge \operatorname{ess\,sup}_{B(x_0,R)} w$, *the following inequality*

$$\int_{B(x_0,t)} g^p_{(u-k)_+}\, d\mu \le C\left(\left(\frac{R}{s-t}\right)^q \int_{B(x_0,s)} \left|\frac{(u-k)_+}{R}\right|^p d\mu\right) \tag{3.17}$$

is satisfied, where $(u-k)_+ = \max\{u-k, 0\}$. *In particular, if* u *is a quasiminimizer in* Ω *with boundary data* $w \in N^{1,1}(X)$, $H(\cdot, g_w) \in L^1(X)$, *then* (3.17) *holds as well.*

Proof. For this proof, we treat the case when u is a quasiminimizer with boundary values w. By Lemma 2, u satisfies the double-phase Caccioppoli's inequality (3.1) for all $k \ge \operatorname{ess\,sup}_{B(x_0,R)} w$. Therefore, we have

$$\begin{aligned}
\int_{B(x_0,t)} g^p_{(u-k)_+}\, d\mu &\le \int_{B(x_0,t)} H(x, g_{(u-k)_+})\, d\mu \le C \int_{B(x_0,s)} H\left(x, \frac{(u-k)_+}{s-t}\right) d\mu \\
&= C\left(\int_{B(x_0,s)} \left|\frac{(u-k)_+}{s-t}\right|^p d\mu + \int_{B(x_0,s)} a(x)\left|\frac{(u-k)_+}{s-t}\right|^q d\mu\right) \\
&= C\Bigg(\left(\frac{R}{s-t}\right)^p \int_{B(x_0,s)} \left|\frac{(u-k)_+}{R}\right|^p d\mu \\
&\qquad + \left(\frac{R}{s-t}\right)^q \int_{B(x_0,s)} a(x)\left|\frac{(u-k)_+}{R}\right|^q d\mu\Bigg),
\end{aligned} \tag{3.18}$$

here $C{=}C(K, q)$.

Now, we estimate the integrand function in the last term of the previous inequality. Using (3.16) and (2.4), we get

$$\begin{aligned}
a(x)\left|\frac{(u-k)_+}{R}\right|^q &\le C[a]_\alpha \frac{\mu(B(x_0,R))^{\frac{\alpha}{Q}}}{R^q} \|u\|^{q-p}_{L^\infty(B(x_0,s))} |(u-k)_+|^p \\
&\le C[a]_\alpha \frac{C_U R^\alpha}{R^q} \|u\|^{q-p}_{L^\infty(B(x_0,R))} |(u-k)_+|^p \\
&= \frac{C[a]_\alpha}{R^{q-\alpha}} \|u\|^{q-p}_{L^\infty(B(x_0,R))} |(u-k)_+|^p \\
&\le \frac{C[a]_\alpha}{R^p} \|u\|^{q-p}_{L^\infty(B(x_0,R))} |(u-k)_+|^p = C\left|\frac{(u-k)_+}{R}\right|^p,
\end{aligned}$$

where $C{=}C(C_U, p, q, \alpha, Q, [a]_\alpha, \|u\|_{L^\infty(B(x_0,R))})$. The last inequality holds true by (1.2) and Remark 3. Thus, (3.18) becomes

$$\begin{aligned}
\int_{B(x_0,t)} g^p_{(u-k)_+}\, d\mu &\le C\left(\left(\frac{R}{s-t}\right)^p \int_{B(x_0,s)} \left|\frac{(u-k)_+}{R}\right|^p d\mu + C\left(\frac{R}{s-t}\right)^q \int_{B(x_0,s)} \left|\frac{(u-k)_+}{R}\right|^p d\mu\right) \\
&\le C\left(\frac{R}{s-t}\right)^q \int_{B(x_0,s)} \left|\frac{(u-k)_+}{R}\right|^p d\mu.
\end{aligned}$$

□

4. Frozen functionals

In this section, we collect the necessary regularity results for quasiminima of the so-called frozen functionals used in subsequent sections. We consider functionals of the type

$$\int_\Omega H_0(g_u)\,d\mu = \int_\Omega (g_u^p + a_0 g_u^q)\,d\mu, \tag{4.1}$$

where $a_0 \ge 0$ is a constant. Frozen functionals belong to the class of functionals introduced by the seminal work of Lieberman [37]. One of the main difficulties when doing the careful analysis of the different phases of the double-phase functional can be solved by using frozen functionals theory, which is based on the fact that under appropriate conditions a quasiminimizer u of functional (1.1) can be seen as a quasiminimizer of a certain frozen functional.

The related theory for this notion due to Lieberman [37] can be used to obtain the needed estimates for regularity properties. For example, local regularity results were obtained in the Euclidean case in [9] and in the general context of a metric measure space in [54]. Here we prove the following generalization of Maz'ya's estimate, Proposition 2 (see also [[3], Theorem 6.21]), for frozen functionals. As far as we know this result is new even in the Euclidean case.

Theorem 4 (Maz'ya's type estimate for frozen functionals and small radii)**.** *Let* $u \in N^{1,1}(X)$ *with* $H_0(g_u) \in L^1(X)$, $B_r \subset X$ *any ball with radius* $r \le \min\{1, \frac{\operatorname{diam}(X)}{8}\}$ *and* $S = \{x \in B_{\frac{r}{2}} : u(x) = 0\}$. *Then, there exists* $C{=}C(C_D, C_{PI}, C_U, p, q) > 0$ *and exponents* $d_1 \ge 1 > d_2 > 0$, *with* $d_1 = d_1(C_D, C_{PI}, p, q)$ *and* $d_2 = d_2(C_D, C_{PI}, p, q)$, *such that the following inequality holds*

$$\left(\fint_{B_r} H_0(u)^{d_1} d\mu\right)^{\frac{1}{d_1}} \le C\left(\frac{1}{cap^{q/p}_{pd_2}(S, B_r)} \int_{B_{2\lambda r}} H_0(g_u)^{d_2} d\mu\right)^{\frac{1}{d_2}},$$

where λ is the dilation constant in the $(1,p)$-Poincaré inequality.

Proof. By splitting u into its positive and negative parts and considering them separately, we can assume that $u \geq 0$ in B_r. Recall that we are assuming that our space supports a $(1,s)$-Poincaré inequality with $1 < s < p < q < s^*$. By Remark 2 and Theorem 2, X supports a (s^*, s)-Poincaré inequality.

Let $\frac{s}{p} < d_2 < 1$, $d_2 = d_2(C_{\mathrm{PI}}, C_D, p, q)$, and $\frac{s^*}{q} \geq d_1 \geq 1$, $d_1 = d_1(C_{\mathrm{PI}}, C_D, p, q)$. By Proposition 2, the following two inequalities hold at the same time

$$\left(\fint_{B_r} |u|^{pd_1}\,\mathrm{d}\mu\right)^{\frac{d_2}{d_1}} \leq \frac{C}{\mathrm{cap}_{pd_2}(S,B_r)} \int_{B_{2\lambda r}} g_u^{pd_2}\,\mathrm{d}\mu \tag{4.2}$$

and

$$\left(\fint_{B_r} |u|^{qd_1}\,\mathrm{d}\mu\right)^{\frac{d_2}{d_1}} \leq \frac{C}{\mathrm{cap}_{qd_2}(S,B_r)} \int_{B_{2\lambda r}} g_u^{qd_2}\,\mathrm{d}\mu \tag{4.3}$$

Therefore, by (4.3) and (2.5) in Remark 4, we have

$$\begin{aligned}\left(\fint_{B_r} |u|^{qd_1}\,\mathrm{d}\mu\right)^{\frac{d_2}{d_1}} &\leq \frac{C}{\mathrm{cap}_{qd_2}(S,B_r)} \int_{B_{2\lambda r}} g_u^{qd_2}\,\mathrm{d}\mu \\ &\leq \frac{C}{\mathrm{cap}^{q/p}_{pd_2}(S,B_r)} \int_{B_{2\lambda r}} g_u^{qd_2}\,\mathrm{d}\mu,\end{aligned} \tag{4.4}$$

where C=$C(C_{\mathrm{PI}}, C_U, p, q)$. Now, by (4.2), (4.4), Proposition 1, μ being upper Q-Alhfors regular and $r \leq 1$, we obtain the next chain of inequalities

$$\begin{aligned}
\left(\fint_{B_r} H_0^{d_1}(u)\,\mathrm{d}\mu\right)^{\frac{1}{d_1}} &\leq \left(\fint_{B_r} 2^{d_1-1}\left(|u|^{pd_1} + a_0^{d_1}|u|^{qd_1}\right)\mathrm{d}\mu\right)^{\frac{1}{d_1}} \\
&\leq 2\left(\left(\fint_{B_r} |u|^{pd_1}\,\mathrm{d}\mu\right)^{\frac{1}{d_1}} + a_0\left(\fint_{B_r} |u|^{qd_1}\,\mathrm{d}\mu\right)^{\frac{1}{d_1}}\right) \\
&\leq C\left(\left(\frac{1}{\mathrm{cap}_{pd_2}(S,B_r)} \int_{B_{2\lambda r}} g_u^{pd_2}\,\mathrm{d}\mu\right)^{\frac{1}{d_2}} + \left(\frac{1}{\mathrm{cap}^{q/p}_{pd_2}(S,B_r)} \int_{B_{2\lambda r}} (a_0 g_u^q)^{d_2}\,\mathrm{d}\mu\right)^{\frac{1}{d_2}}\right) \\
&\leq C\left(\left(\left(1 + \frac{1}{\mathrm{cap}_{pd_2}(S,B_r)}\right) \int_{B_{2\lambda r}} g_u^{pd_2}\,\mathrm{d}\mu\right)^{\frac{1}{d_2}}\right. \\
&\qquad \left. + \left(\left(1 + \frac{1}{\mathrm{cap}_{pd_2}(S,B_r)}\right)^{q/p} \int_{B_{2\lambda r}} (a_0 g_u^q)^{d_2}\,\mathrm{d}\mu\right)^{\frac{1}{d_2}}\right) \\
&\leq C\left(1 + \frac{1}{\mathrm{cap}_{pd_2}(S,B_r)}\right)^{\frac{q}{pd_2}} \left(\left(\int_{B_{2\lambda r}} g_u^{pd_2}\,\mathrm{d}\mu\right)^{\frac{1}{d_2}} + \left(\int_{B_{2\lambda r}} (a_0 g_u^q)^{d_2}\,\mathrm{d}\mu\right)^{\frac{1}{d_2}}\right) \\
&\leq C\left(\left(\frac{\mathrm{cap}_{pd_2}(S,B_r) + 1}{\mathrm{cap}_{pd_2}(S,B_r)}\right)^{\frac{q}{pd_2}} \left(\int_{B_{2\lambda r}} g_u^{pd_2}\,\mathrm{d}\mu + \int_{B_{2\lambda r}} (a_0 g_u^q)^{d_2}\,\mathrm{d}\mu\right)^{\frac{1}{d_2}}\right) \\
&\leq C\left(\left(\frac{\left(\frac{C_D \mu(B_{\frac{r}{2}})}{\left(\frac{r}{2}\right)^{pd_2}}\right) + 1}{\mathrm{cap}_{pd_2}(S,B_r)}\right)^{q/p} \int_{B_{2\lambda r}} \left(g_u^{pd_2} + (a_0 g_u^q)^{d_2}\right)\mathrm{d}\mu\right)^{\frac{1}{d_2}} \\
&\leq C\left(\left(\frac{C_D C_U \left(\frac{r}{2}\right)^{Q-pd_2} + 1}{\mathrm{cap}_{pd_2}(S,B_r)}\right)^{q/p} \int_{B_{2\lambda r}} \left(g_u^{pd_2} + (a_0 g_u^q)^{d_2}\right)\mathrm{d}\mu\right)^{\frac{1}{d_2}} \\
&\leq C\left(\left(\frac{C+1}{\mathrm{cap}_{pd_2}(S,B_r)}\right)^{q/p} \int_{B_{2\lambda r}} \left(g_u^{pd_2} + (a_0 g_u^q)^{d_2}\right)\mathrm{d}\mu\right)^{\frac{1}{d_2}} \\
&\leq C\left(\frac{1}{\mathrm{cap}^{q/p}_{pd_2}(S,B_r)} \int_{B_{2\lambda r}} H_0(g_u)^{d_2}\,\mathrm{d}\mu\right)^{\frac{1}{d_2}}.
\end{aligned}$$

Therefore,

$$\left(\fint_{B_r} H_0^{d_1}(u)\,\mathrm{d}\mu\right)^{\frac{1}{d_1}} \leq C\left(\frac{1}{\mathrm{cap}^{q/p}_{pd_2}(S,B_r)} \int_{B_{2\lambda r}} H_0(g_u)^{d_2}\,\mathrm{d}\mu\right)^{\frac{1}{d_2}},$$

where C=$C(C_{\mathrm{PI}}, C_D, C_U, p, q)$. $\square$

5. Pointwise estimate on a boundary point

This section is devoted to the proof of a pointwise estimate near a boundary point which has a key role in obtaining the sufficient condition for Hölder continuity (Theorem 5, Section 6).

For a set $E \subset X$, we denote by $\overline{E}^X$ its closure in the space X. The boundary of Ω relative to X is denoted by

$$\partial_X \Omega := \overline{\Omega}^X \setminus \Omega.$$

Since X is fixed, we write simply $\partial\Omega$ for $\partial_X\Omega$.

Let $w \in N^{1,1}(X)$ with $H(\cdot, g_w) \in L^1(X)$, and such that $w - u \in N_0^{1,1}(\Omega)$. We shall use the notation

$$M(r, r_0) = \Big(\operatorname*{ess\,sup}_{B(x_0,r)} u - \operatorname*{ess\,sup}_{B(x_0,r_0)} w \Big)_+,$$

where $x_0 \in \partial\Omega$ is a boundary point, $0 < r \leq r_0$, $a_+ = \max\{a, 0\}$, and $u \in N^{1,1}(X)$, with $H(\cdot, g_u) \in L^1(X)$ is a quasiminimizer on Ω, with boundary values w. We note that by Corollary 1 this M is well defined. Let also

$$\gamma_{p,q}(t, r) = \frac{r^{-tq/p} \mu(B(x_0, r))}{\operatorname{cap}_t^{q/p}(B(x_0, r) \setminus \Omega, B(x_0, 2r))}. \tag{5.1}$$

Now we state the main result of this section, a pointwise estimate for quasiminima near a boundary point. The proof of this estimate is based on a careful analysis of the phases.

Proposition 3 (Pointwise estimate). *Let $w \in N^{1,1}(X)$ with $H(\cdot, g_w) \in L^1(X)$. Let $u \in N^{1,1}(X)$, with $H(\cdot, g_u) \in L^1(X)$ a quasiminimizer on Ω with boundary values w. Then there exist $\lambda \geq 1$, $0 < d_2 < 1$ and $C > 0$ such that, for all $x_0 \in \partial\Omega$ and $0 < 4\lambda r \leq r_0 < \min\{1, \frac{\operatorname{diam}(X)}{12\lambda}\}$, the next inequality holds*

$$M\Big(\frac{r}{2}, r_0\Big) \leq (1 - 2^{-n(r)-1}) M(4\lambda r, r_0),$$

where $n(r)$ is a sufficiently large integer such that

$$n(r) \geq C \gamma_{p,q}\Big(pd_2, \frac{r}{2}\Big)^{\frac{1}{1-d_2}}.$$

Proof. We denote $M = M(4\lambda r, r_0)$, where $x_0 \in \partial\Omega$ and $r_0 > 0$ are fixed. Without loss of generality, we can assume $0 < M < +\infty$, otherwise the proof is finished.

We define

$$k_j = \operatorname*{ess\,sup}_{B(x_0,r_0)} w + M(1 - 2^{-j}),$$

and

$$v_j = (u - k_j)_+ - (u - k_{j+1})_+.$$

Notice that in $B(x_0, 2\lambda r) \setminus \Omega$ we have $v_j = 0$, where λ is the dilation factor from the $(1, p)$-Poincaré inequality. Define $T(k, l, r) = S_{k,r} \setminus S_{l,r}$, then $g_u \chi_{T(k_j, k_{j+1}, 2\lambda r)}$ is a p-weak upper gradient of v_j in $B(x_0, 2\lambda r)$.

We define

$$a_0 = \inf_{x \in B(x_0, 8\lambda r)} a(x).$$

Case 1: First assume that

$$a_0 > 2[a]_\alpha (2C_D^2 \mu(B(x_0, 8\lambda r)))^{\frac{\alpha}{Q}}, \tag{5.2}$$

Note that for every $x, y \in B(x_0, 8\lambda r)$, we have

$$\begin{aligned}
\delta_\mu(x, y) &= (\mu(B(x, d(x, y))) + \mu(B(y, d(x, y))))^{1/Q} \\
&\leq (\mu(B(x, 16\lambda r)) + \mu(B(y, 16\lambda r)))^{1/Q} \\
&\leq \big(2\mu(B(x_0, 24\lambda r))\big)^{1/Q} \leq (2C_D^2 \mu(B(x_0, 8\lambda r)))^{1/Q} \\
&= C\mu(B(x_0, 8\lambda r))^{1/Q}.
\end{aligned}$$

For every $x \in B(x_0, 8\lambda r)$, by (5.2) we obtain

$$\begin{aligned}
2a_0 &= 2a(x) - 2\big(a(x) - a_0\big) \geq a(x) + a_0 - 2\big(a(x) - a_0\big) \\
&\geq a(x) + 2[a]_\alpha (2C_D^2 \mu(B(x_0, 8\lambda r)))^{\alpha/Q} - 2\big(a(x) - a_0\big) \\
&\geq a(x) + 2\left(\sup_{\substack{y \in B(x_0, 8\lambda r) \\ x \neq y}} \frac{|a(x) - a(y)|}{\delta_\mu(x, y)^\alpha} \right) (2C_D^2 \mu(B(x_0, 8\lambda r)))^{\alpha/Q} - 2\big(a(x) - a_0\big)
\end{aligned}$$

$$
\begin{aligned}
&\ge a(x) + 2\left(\sup_{y\in B(x_0,8\lambda r)} |a(x)-a(y)|\right) - 2\big(a(x)-a_0\big)\\
&\ge a(x) + 2\left(\sup_{y\in B(x_0,8\lambda r)} (a(x)-a(y))\right) - 2\big(a(x)-a_0\big)\\
&\ge a(x) + \left(2a(x) - 2\inf_{y\in B(x_0,8\lambda r)} a(y)\right) - 2\big(a(x)-a_0\big) = a(x).
\end{aligned}
$$

On the other hand, we have $a(x) \ge \inf_{x\in B(x_0,8\lambda r)} a(x) \ge a_0$ for every $x \in B(x_0, 8\lambda r)$. This implies that

$$
a_0 \le a(x) \le 2a_0 \qquad \text{for every } x \in B(x_0, 8\lambda r). \tag{5.3}
$$

We define the Frozen functional

$$
H_0(z) = |z|^p + a_0|z|^q.
$$

By Theorem 4 with v_j, there exist $d_1 \ge 1 > d_2 > 0$, with $d_1 = d_1(C_{\mathrm{PI}}, C_D, p, q)$ and $d_2 = d_2(C_{\mathrm{PI}}, C_D, p, q)$, such that by Hölder inequality we have

$$
\begin{aligned}
\fint_{B(x_0,r)} H_0\big(v_j\big)\mathrm{d}\mu &\le \left(\fint_{B(x_0,r)} H_0\big(v_j\big)^{d_1}\mathrm{d}\mu\right)^{\frac{1}{d_1}}\\
&\le C\left(\frac{1}{\mathrm{cap}^{q/p}_{pd_2}\big(B\big(x_0,\frac{r}{2}\big)\setminus\Omega, B(x_0,r)\big)}\int_{B(x_0,2\lambda r)} H_0(g_{v_j})^{d_2}\mathrm{d}\mu\right)^{\frac{1}{d_2}}\\
&= C\left(\frac{r^{qd_2}\gamma_{p,q}\big(pd_2,\frac{r}{2}\big)}{\mu\big(B\big(x_0,\frac{r}{2}\big)\big)}\int_{B(x_0,2\lambda r)} H_0(g_{v_j})^{d_2}\mathrm{d}\mu\right)^{\frac{1}{d_2}}\\
&\le C\left(\frac{r^{qd_2}\gamma_{p,q}\big(pd_2,\frac{r}{2}\big)}{\mu\big(B\big(x_0,\frac{r}{2}\big)\big)}\int_{T(k_j,k_{j+1},2\lambda r)} H_0(g_u)^{d_2}\mathrm{d}\mu\right)^{\frac{1}{d_2}}\\
&= C\left(\frac{r^{qd_2}\gamma_{p,q}\big(pd_2,\frac{r}{2}\big)\mu(T(k_j,k_{j+1},2\lambda r))}{\mu\big(B\big(x_0,\frac{r}{2}\big)\big)}\right)^{\frac{1}{d_2}}\left(\fint_{T(k_j,k_{j+1},2\lambda r)} H_0(g_u)^{d_2}\mathrm{d}\mu\right)^{\frac{1}{d_2}}\\
&\le \frac{Cr^q\gamma_{p,q}\big(pd_2,\frac{r}{2}\big)^{\frac{1}{d_2}}\mu(T(k_j,k_{j+1},2\lambda r))^{\frac{1}{d_2}-1}}{\mu\big(B\big(x_0,\frac{r}{2}\big)\big)^{\frac{1}{d_2}}}\int_{T(k_j,k_{j+1},2\lambda r)} H_0(g_u)\mathrm{d}\mu\\
&\le C\left(\frac{r^{qd_2}\gamma_{p,q}\big(pd_2,\frac{r}{2}\big)\mu(T(k_j,k_{j+1},2\lambda r))^{1-d_2}}{\mu\big(B\big(x_0,\frac{r}{2}\big)\big)}\right)^{\frac{1}{d_2}}\int_{S_{k_j,2\lambda r}} H_0(g_u)\mathrm{d}\mu.
\end{aligned}
$$

Therefore, by the doubling property we obtain

$$
\int_{B(x_0,r)} H_0\big(v_j\big)\mathrm{d}\mu \le C\left(\frac{\mu\big(B\big(x_0,\frac{r}{2}\big)\big)}{\mu(T(k_j,k_{j+1},2\lambda r))}\right)^{1-\frac{1}{d_2}}\left(r^{qd_2}\gamma_{p,q}\left(pd_2,\frac{r}{2}\right)\right)^{\frac{1}{d_2}}\int_{S_{k_j,2\lambda r}} H_0(g_u)\mathrm{d}\mu. \tag{5.4}
$$

For the measure of the set $S_{k_{j+1},r}$ we have the following estimate

$$
\begin{aligned}
\int_{B(x_0,r)} H_0\big(v_j\big)\mathrm{d}\mu &\ge H_0\big(k_{j+1}-k_j\big)\mu(S_{k_{j+1},r})\\
&= H_0\big(2^{-j-1}M\big)\mu(S_{k_{j+1},r})\\
&= H_0\Big(\frac{M}{2^{j+1}}\Big)\mu(S_{k_{j+1},r})
\end{aligned} \tag{5.5}
$$

Now, by (5.3), Lemma 2, because the functional H_0 is increasing and $r \le 1$, we achieve

$$
\begin{aligned}
\int_{S_{k_j,2\lambda r}} H_0(g_u)\mathrm{d}\mu &\le \int_{S_{k_j,2\lambda r}} H(x,g_u)\mathrm{d}\mu \le \int_{B(x_0,2\lambda r)} H(x,g_{(u-k_j)_+})\mathrm{d}\mu\\
&\le C\int_{B(x_0,4\lambda r)} H\left(x,\frac{(u-k_j)_+}{2\lambda r}\right)\mathrm{d}\mu \le 2C\int_{B(x_0,4\lambda r)} H_0\left(\frac{(u-k_j)_+}{r}\right)\mathrm{d}\mu
\end{aligned}
$$

$$\begin{aligned}
&\le C\int_{B(x_0,4\lambda r)} H_0\left(\frac{\left(\operatorname{ess\,sup}_{B(x_0,4\lambda r)} u-k_j\right)_+}{r}\right)\mathrm{d}\mu\\
&\le C\int_{B(x_0,4\lambda r)} H_0\Big(\frac{M}{r2^j}\Big)\mathrm{d}\mu = C\mu(B(x_0,4\lambda r))H_0\Big(\frac{M}{r2^j}\Big)\\
&= C\mu(B(x_0,4\lambda r))\left(\left|\frac{M}{r2^j}\right|^p + a_0\left|\frac{M}{r2^j}\right|^q\right)\\
&\le C\mu(B(x_0,4\lambda r))\frac{1}{r^q}H_0\Big(\frac{M}{2^j}\Big).
\end{aligned}$$

So,

$$\int_{S_{k_j,2\lambda r}} H_0(g_u)\mathrm{d}\mu \le C\mu(B(x_0,4\lambda r))\frac{1}{r^q}H_0\Big(\frac{M}{2^j}\Big), \tag{5.6}$$

where $C{=}C(K,q)$.

By (5.4)–(5.6), and the doubling property of the measure, we have

$$\begin{aligned}
H_0\Big(\frac{M}{2^{j+1}}\Big)\mu(S_{j+1,r}) &\le \int_{B(x_0,r)} H_0(v_j)\mathrm{d}\mu\\
&\le C\left(\frac{\mu\left(B\left(x_0,\frac{r}{2}\right)\right)}{\mu(T(k_j,k_{j+1},2\lambda r))}\right)^{1-\frac{1}{d_2}}\left(r^{qd_2}\gamma_{p,q}\Big(pd_2,\frac{r}{2}\Big)\right)^{\frac{1}{d_2}}\int_{S_{k_j,2\lambda r}} H_0(g_u)\mathrm{d}\mu\\
&\le C\left(\frac{\mu\left(B\left(x_0,\frac{r}{2}\right)\right)}{\mu(T(k_j,k_{j+1},2\lambda r))}\right)^{1-\frac{1}{d_2}} r^q\gamma_{p,q}\Big(pd_2,\frac{r}{2}\Big)^{\frac{1}{d_2}}\mu(B(x_0,4\lambda r))\frac{1}{r^q}H_0\Big(\frac{M}{2^j}\Big)\\
&\le C\left(\frac{\mu\left(B\left(x_0,\frac{r}{2}\right)\right)}{\mu(T(k_j,k_{j+1},2\lambda r))}\right)^{1-\frac{1}{d_2}}\gamma_{p,q}\Big(pd_2,\frac{r}{2}\Big)^{\frac{1}{d_2}}\mu(B(x_0,r))H_0\Big(\frac{M}{2^j}\Big).
\end{aligned}$$

By convexity and induction, see [54] for the details, one can prove the following inequality involving H_0,

$$\frac{2^{-nq}}{q}H_0(b) \le H_0(2^{-n}b), \qquad \text{for } b\ge 0 \text{ and } n\in\mathbb{N}.$$

In particular, we get

$$\frac{1}{2^q q}H_0\Big(\frac{M}{2^j}\Big) \le H_0\Big(\frac{M}{2^{j+1}}\Big).$$

So,

$$\frac{1}{2^q q}H_0\Big(\frac{M}{2^j}\Big)\mu(S_{j+1,r}) \le C\left(\frac{\mu\left(B\left(x_0,\frac{r}{2}\right)\right)}{\mu(T(k_j,k_{j+1},2\lambda r))}\right)^{1-\frac{1}{d_2}}\gamma_{p,q}\Big(pd_2,\frac{r}{2}\Big)^{\frac{1}{d_2}}\mu(B(x_0,r))H_0\Big(\frac{M}{2^j}\Big).$$

Therefore,

$$\frac{\mu(S_{k_{j+1},r})}{\mu(B(x_0,r))} \le C\left(\frac{\mu(T(k_j,k_{j+1},2\lambda r))}{\mu\left(B\left(x_0,\frac{r}{2}\right)\right)}\right)^{\frac{1}{d_2}-1}\gamma_{p,q}\Big(pd_2,\frac{r}{2}\Big)^{\frac{1}{d_2}}, \tag{5.7}$$

where $C{=}C(C_{\mathrm{PI}},C_D,C_1,K,\lambda,p,q)$ and $0<d_2<1$, $d_2=d_2(C_{\mathrm{PI}},C_D,p,q)$.

If $n\ge j+1$ then $S_{k_{j+1},r}$ on the left-hand side of (5.7) can be replaced by $S_{k_n,r}$ and the inequality remains true. We get

$$\left(\frac{\mu(S_{k_n,r})}{\mu(B(x_0,r))}\right)^{\frac{d_2}{1-d_2}} \le C\gamma_{p,q}\Big(pd_2,\frac{r}{2}\Big)^{\frac{1}{1-d_2}}\frac{\mu(T(k_j,k_{j+1},2\lambda r))}{\mu\left(B\left(x_0,\frac{r}{2}\right)\right)},$$

summing up over $j=0,1,\dots,n-1$ and Lemma 1, yields

$$n\left(\frac{\mu(S_{k_n,r})}{\mu(B(x_0,r))}\right)^{\frac{d_2}{1-d_2}} \le C\gamma_{p,q}\Big(pd_2,\frac{r}{2}\Big)^{\frac{1}{1-d_2}}\frac{\mu(T(k_1,k_n,2\lambda r))}{\mu\left(B\left(x_0,\frac{r}{2}\right)\right)}$$

$$\begin{aligned}
&\le C\gamma_{p,q}\left(pd_2,\frac{r}{2}\right)^{\frac{1}{1-d_2}}\frac{\mu(B(x_0,2\lambda r))}{\mu\left(B\left(x_0,r\right)\right)}\\
&\le C\gamma_{p,q}\left(pd_2,\frac{r}{2}\right)^{\frac{1}{1-d_2}}\left(\frac{2\lambda r}{r}\right)^{Q}\\
&\le C\gamma_{p,q}\left(pd_2,\frac{r}{2}\right)^{\frac{1}{1-d_2}}.
\end{aligned}$$

Therefore,

$$\frac{\mu(S_{k_n,r})}{\mu(B(x_0,r))}\le\frac{C}{n^{\frac{1}{d_2}-1}}\gamma_{p,q}\left(pd_2,\frac{r}{2}\right)^{\frac{1}{d_2}},\tag{5.8}$$

with $C=C(C_{\mathrm{PI}},C_D,C_U,K,\lambda,p,q)$ and $0<d_2<1$, $d_2=d_2(C_{\mathrm{PI}},C_D,p,q)$.

Since u is a quasiminimizer with boundary data w on Ω, the Caccioppoli inequality

$$\int_{S_{k,\rho_1}}H(x,g_u)\mathrm{d}\mu\le C\int_{B(x_0,\rho_2)}H\left(x,\frac{(u-k)_+}{\rho_2-\rho_1}\right)\mathrm{d}\mu,$$

holds for all $0<\rho_1<\rho_2$ and $k\ge\operatorname{ess\,sup}_{B(x_0,\rho_2)}w$. In particular, by (5.3), for any $0<\rho_1<\rho_2\le 8\lambda r$ and $k\ge\operatorname{ess\,sup}_{B(x_0,8\lambda r)}w$, we have

$$\begin{aligned}
\int_{B(x_0,\rho_1)}H_0(g_{(u-k)_+})\mathrm{d}\mu&\le\int_{S_{k,\rho_1}}H(x,g_u)\mathrm{d}\mu\\
&\le C\int_{B(x_0,\rho_2)}H\left(x,\frac{(u-k)_+}{\rho_2-\rho_1}\right)\mathrm{d}\mu\\
&\le 2C\int_{B(x_0,\rho_2)}H_0\left(\frac{(u-k)_+}{\rho_2-\rho_1}\right)\mathrm{d}\mu.
\end{aligned}$$

Furthermore, notice that $(u-k_n)$ also satisfies the previous inequality in the ball $B(x_0,8\lambda r)$. Therefore, by an application of [[54], Theorem 6] with $\Omega=B(x_0,8\lambda r)$, $u=(u-k_n)_+$ and $t_+=p$, there exist $C=C(C_{\mathrm{PI}},C_D,p,q)$ such that

$$\begin{aligned}
\operatorname*{ess\,sup}_{B\left(x_0,\frac{r}{2}\right)}(u-k_n)&\le\operatorname*{ess\,sup}_{B\left(x_0,\frac{r}{2}\right)}(u-k_n)_+\\
&\le\frac{C}{\left(1-\frac{r/2}{r}\right)^{\frac{Q}{p}}}\left(\fint_{B(x_0,r)}(u-k_n)_+^p\mathrm{d}\mu\right)^{\frac{1}{p}}\\
&\le C\left(\fint_{B(x_0,r)}(u-k_n)_+^p\mathrm{d}\mu\right)^{\frac{1}{p}}\\
&=C\left(\frac{1}{\mu(B(x_0,r))}\int_{S_{k_n,r}}(u-k_n)^p\mathrm{d}\mu\right)^{\frac{1}{p}}\\
&\le C\left(\frac{\mu(S_{k_n,r})}{\mu(B(x_0,r))}\right)^{\frac{1}{p}}\left(\operatorname*{ess\,sup}_{B(x_0,r)}u-k_n\right).
\end{aligned}$$

Therefore, by (5.8)

$$\begin{aligned}
\operatorname*{ess\,sup}_{B\left(x_0,\frac{r}{2}\right)}u&\le k_n+C\left(\frac{\mu(S_{k_n,r})}{\mu(B(x_0,r))}\right)^{\frac{1}{p}}\left(\operatorname*{ess\,sup}_{B(x_0,r)}u-k_n\right)\\
&=\operatorname*{ess\,sup}_{B(x_0,r_0)}w+M(1-2^{-n})+C\left(\frac{\mu(S_{k_n,r})}{\mu(B(x_0,r))}\right)^{\frac{1}{p}}\left(\operatorname*{ess\,sup}_{B(x_0,r)}u-\operatorname*{ess\,sup}_{B(x_0,r_0)}w-M(1-2^{-n})\right)\\
&\le\operatorname*{ess\,sup}_{B(x_0,r_0)}w+M(1-2^{-n})+C\left(\frac{\mu(S_{k_n,r})}{\mu(B(x_0,r))}\right)^{\frac{1}{p}}M2^{-n}\\
&\le\operatorname*{ess\,sup}_{B(x_0,r_0)}w+M(1-2^{-n})+C\left(\frac{1}{n^{\frac{1}{d_2}-1}}\gamma_{p,q}\left(pd_2,\frac{r}{2}\right)^{\frac{1}{d_2}}\right)^{\frac{1}{p}}M2^{-n},
\end{aligned}$$

with $C=C(C_{\mathrm{PI}},C_D,C_U,K,\lambda,p,q)$.

If $n\ge\left(2^{pd_2}\gamma_{p,q}\left(pd_2,\frac{r}{2}\right)\right)^{\frac{1}{1-d_2}}$, then $\left(\frac{1}{n^{\frac{1}{d_2}-1}}\gamma_{p,q}\left(pd_2,\frac{r}{2}\right)^{\frac{1}{d_2}}\right)^{\frac{1}{p}}\le\frac{1}{2}$. Therefore, the last term on the right-hand side of the previous inequality is at most $2^{-n-1}M$.

Thus, for

$$n \geq C\gamma_{p,q}\left(pd_2, \frac{r}{2}\right)^{\frac{1}{1-d_2}}, \tag{5.9}$$

we have

$$\begin{aligned}\operatorname*{ess\,sup}_{B\left(x_0,\frac{r}{2}\right)} u &\leq \operatorname*{ess\,sup}_{B(x_0,r_0)} w + M - 2^{-n}M + 2^{-n-1}M \\ &= \operatorname*{ess\,sup}_{B(x_0,r_0)} w + M(1-2^{-n-1}).\end{aligned}$$

Finally, we get

$$M\left(\frac{r}{2}, r\right) \leq (1-2^{-n-1})M, \tag{5.10}$$

for n satisfying (5.9). *End Case 1.*

Case 2: Next we consider the case which is complementary to (5.2), that is,

$$a_0 = \inf_{x\in B(x_0,8\lambda r)} a(x) \leq C[a]_\alpha \mu(B(x_0,8\lambda r))^{\alpha/Q}. \tag{5.11}$$

Notice that, for every $x \in B(x_0, 8\lambda r)$ and $y \in B(x_0, r)$, with $y \neq x$, we have

$$a(y) - a(x) \leq |a(x) - a(y)| = \frac{|a(x)-a(y)|}{\delta_\mu(x,y)^\alpha}\delta_\mu(x,y)^\alpha \leq [a]_\alpha \delta_\mu(x,y)^\alpha. \tag{5.12}$$

Note that, for every $x \in B(x_0, 8\lambda r)$ and $y \in B(x_0, r)$, with $y \neq x$, we have

$$\begin{aligned}\delta_\mu(x,y) &= (\mu(B(x,d(x,y))) + \mu(B(y,d(x,y))))^{1/Q} \\ &\leq (\mu(B(x,9\lambda r)) + \mu(B(y,9\lambda r)))^{1/Q} \\ &\leq \left(2\mu(B(x_0,10\lambda r))\right)^{1/Q} \leq C\mu(B(x_0,8\lambda r))^{1/Q},\end{aligned}$$

where C=$C(C_D)$. By (5.12), we get

$$a(y) \leq a(x) + C[a]_\alpha \mu(B(x_0,8\lambda r))^{\alpha/Q},$$

where C=$C(C_D, \alpha)$. By taking infimum over all $x \in B(x_0, 8\lambda r)$, we obtain

$$\begin{aligned}a(y) &\leq \inf_{x\in B(x_0,8\lambda r)} a(x) + C[a]_\alpha \mu(B(x_0,8\lambda r))^{\alpha/Q} \\ &\leq 2[a]_\alpha C(\mu(B(x_0,8\lambda r)))^{\alpha/Q} + C[a]_\alpha \mu(B(x_0,8\lambda r))^{\alpha/Q} \\ &\leq C[a]_\alpha \mu(B(x_0,8\lambda r))^{\alpha/Q},\end{aligned}$$

where C=$C(C_D, \alpha)$. By taking supremum over $y \in B(x_0, r)$, we conclude that

$$\sup_{y\in B(x_0,r)} a(y) \leq C[a]_\alpha \mu(B(x_0,8\lambda r))^{\alpha/Q}. \tag{5.13}$$

Since $1 < s < pd_2 < p$, as a consequence of Theorem 1, X supports a weak (pd_2, pd_2)-Poincaré inequality. Therefore, by Maz'ya estimate (see Proposition 2), Proposition 1, the upper Q-Ahlfors inequality (2.4), $r \leq 1$, the doubling property of μ and Hölder inequality, we get

$$\begin{aligned}\int_{B(x_0,r)} v_j^{pd_2}\,\mathrm{d}\mu &\leq \frac{C\mu(B(x_0,r))}{\operatorname{cap}_{pd_2}(B(x_0,\frac{r}{2})\setminus\Omega, B(x_0,r))}\int_{B(x_0,2\lambda r)} g_{v_j}^{pd_2}\,\mathrm{d}\mu \\ &\leq C\mu(B(x_0,r))\left(1+\frac{1}{\operatorname{cap}_{pd_2}(B(x_0,\frac{r}{2})\setminus\Omega, B(x_0,r))}\right)\int_{B(x_0,2\lambda r)} g_{v_j}^{pd_2}\,\mathrm{d}\mu \\ &\leq C\mu(B(x_0,r))\left(1+\frac{1}{\operatorname{cap}_{pd_2}(B(x_0,\frac{r}{2})\setminus\Omega, B(x_0,r))}\right)^{\frac{q}{p}}\int_{B(x_0,2\lambda r)} g_{v_j}^{pd_2}\,\mathrm{d}\mu \\ &= C\mu(B(x_0,r))\left(\frac{\operatorname{cap}_{pd_2}(B(x_0,\frac{r}{2})\setminus\Omega, B(x_0,r))+1}{\operatorname{cap}_{pd_2}(B(x_0,\frac{r}{2})\setminus\Omega, B(x_0,r))}\right)^{\frac{q}{p}}\int_{B(x_0,2\lambda r)} g_{v_j}^{pd_2}\,\mathrm{d}\mu \\ &\leq C\mu(B(x_0,r))\left(\frac{\left(\frac{C_D\mu(B(x_0,\frac{r}{2}))}{\left(\frac{r}{2}\right)^{pd_2}}\right)+1}{\operatorname{cap}_{pd_2}(B(x_0,\frac{r}{2})\setminus\Omega, B(x_0,r))}\right)^{\frac{q}{p}}\int_{B(x_0,2\lambda r)} g_{v_j}^{pd_2}\,\mathrm{d}\mu \\ &\leq C\mu(B(x_0,r))\left(\frac{C_DC_U\left(\frac{r}{2}\right)^{Q-pd_2}+1}{\operatorname{cap}_{pd_2}(B(x_0,\frac{r}{2})\setminus\Omega, B(x_0,r))}\right)^{\frac{q}{p}}\int_{B(x_0,2\lambda r)} g_{v_j}^{pd_2}\,\mathrm{d}\mu\end{aligned}$$

$$
\begin{aligned}
&\le C\mu(B(x_0,r))\left(\frac{C+1}{\mathrm{cap}_{pd_2}(B(x_0,\frac{r}{2})\setminus\Omega, B(x_0,r))}\right)^{\frac{q}{p}}\int_{B(x_0,2\lambda r)} g_{v_j}^{pd_2}\mathrm{d}\mu\\
&= \frac{C\mu(B(x_0,r))}{\mathrm{cap}_{pd_2}^{q/p}(B(x_0,\frac{r}{2})\setminus\Omega, B(x_0,r))}\int_{B(x_0,2\lambda r)} g_{v_j}^{pd_2}\mathrm{d}\mu\\
&\le C\gamma_{p,q}(pd_2,\frac{r}{2})r^{qd_2}\int_{S_{k_j,2\lambda r}} g_u^{pd_2}\chi_{T(k_j,k_{j+1},2\lambda r)}\mathrm{d}\mu\\
&\le C\gamma_{p,q}(pd_2,\frac{r}{2})r^{pd_2}\left(\int_{S_{k_j,2\lambda r}} g_u^{p}\mathrm{d}\mu\right)^{d_2}\mu(T(k_j,k_{j+1},2\lambda r))^{1-d_2},
\end{aligned}
$$

with C depending on C_{PI}, C_D and C_U, therefore

$$
\int_{B(x_0,r)} v_j^{pd_2}\mathrm{d}\mu \le C\gamma_{p,q}(pd_2,\frac{r}{2})r^{pd_2}\left(\int_{S_{k_j,2\lambda r}} g_u^{p}\mathrm{d}\mu\right)^{d_2}\mu(T(k_j,k_{j+1},2\lambda r))^{1-d_2}. \tag{5.14}
$$

For the measure of the set $S_{k_{j+1},r}$ we have the following estimate

$$
\int_{B(x_0,r)} v_j^{pd_2}\mathrm{d}\mu \ge (k_{j+1}-k_j)^{pd_2}\mu(S_{k_{j+1},r}) = \frac{M^{pd_2}}{2^{pd_2(j+1)}}\mu(S_{k_{j+1},r}). \tag{5.15}
$$

Since u is a quasiminimizer with boundary data w in Ω, by (5.13), Lemma 4, Hölder inequality and the doubling property, we have for $R=8\lambda r$, $s=2\lambda r$, $t=\lambda r$

$$
\begin{aligned}
\int_{S_{k_j,\lambda r}} g_u^p\mathrm{d}\mu &= \int_{B(x_0,\lambda r)} g_{(u-k_j)_+}^p\mathrm{d}\mu \le C\left(\left(\frac{8\lambda r}{\lambda r}\right)\int_{B(x_0,2\lambda r)}\left|\frac{(u-k_j)_+}{8\lambda r}\right|^p\mathrm{d}\mu\right)\\
&\le \frac{C}{r^p}\int_{B(x_0,2\lambda r)}(u-k_j)_+^p\mathrm{d}\mu \le \frac{C}{r^p}\int_{B(x_0,2\lambda r)}\left(\operatorname*{ess\,sup}_{B(x_0,2\lambda r)} u-k_j\right)_+^p\mathrm{d}\mu\\
&\le C\left(\frac{M}{r2^j}\right)^p\mu(B(x_0,2\lambda r)).
\end{aligned}
$$

Therefore,

$$
\int_{S_{k_j,\lambda r}} g_u^p\mathrm{d}\mu \le C\left(\frac{M}{r2^j}\right)^p\mu(B(x_0,2\lambda r)). \tag{5.16}
$$

By (5.15), (5.14) and (5.16), we obtain

$$
\begin{aligned}
\mu(S_{k_{j+1},r})\frac{M^{pd_2}}{2^{pd_2(j+1)}} &\le \int_{B(x_0,r)} v_j^{pd_2}\mathrm{d}\mu\\
&\le C\gamma_{p,q}(pd_2,\frac{r}{2})r^{pd_2}\left(\int_{S_{k_j,2\lambda r}} g_u^{pd_2}\mathrm{d}\mu\right)^{d_2}\mu(T(k_j,k_{j+1},2\lambda r))^{1-d_2}\\
&\le C\gamma_{p,q}(pd_2,\frac{r}{2})r^{pd_2}\frac{CM^{pd_2}}{r^{pd_2}2^{jpd_2}}\mu(B(x_0,2\lambda r))^{d_2}\mu(T(k_j,k_{j+1},2\lambda r))^{1-d_2}\\
&= C\gamma_{p,q}(pd_2,\frac{r}{2})\frac{CM^{pd_2}}{2^{jpd_2}}\mu(B(x_0,2\lambda r))^{d_2}\mu(T(k_j,k_{j+1},2\lambda r))^{1-d_2}.
\end{aligned}
$$

So,

$$
\frac{\mu(S_{k_{j+1},r})}{\mu(B(x_0,r))} \le C\gamma_{p,q}\left(pd_2,\frac{r}{2}\right)\left(\frac{\mu(T(k_j,k_{j+1},2\lambda r))}{\mu(B(x_0,r))}\right)^{1-d_2}. \tag{5.17}
$$

As before, if $n\ge j+1$, then (5.17) holds true when the set $S_{k_{j+1},r}$ on the left part of the inequality is replaced by $S_{k_n,r}$. Thus, by summing up over $j=0,1,\dots,n-1$ and Lemma 1

$$
\begin{aligned}
n\left(\frac{\mu(S_{k_n,r})}{\mu(B(x_0,r))}\right)^{\frac{1}{1-d_2}} &\le C\gamma_{p,q}\left(pd_2,\frac{r}{2}\right)^{\frac{1}{1-d_2}}\left(\frac{\mu(T(k_1,k_n,2\lambda r))}{\mu(B(x_0,r))}\right)\\
&\le C\gamma_{p,q}\left(pd_2,\frac{r}{2}\right)^{\frac{1}{1-d_2}}\left(\frac{\mu(B(x_0,2\lambda r))}{\mu(B(x_0,r))}\right)\\
&\le C\gamma_{p,q}\left(pd_2,\frac{r}{2}\right)^{\frac{1}{1-d_2}}\left(\frac{2\lambda r}{r}\right)^{Q}.
\end{aligned}
$$

Meaning,

$$
\frac{\mu(S_{k_n,r})}{\mu(B(x_0,r))} \le \frac{C}{n^{1-d_2}}\gamma_{p,q}\left(pd_2,\frac{r}{2}\right). \tag{5.18}
$$

Again, since u is a quasiminimizer with boundary data w in Ω, then by Lemma 2, in particular, u satisfies the Caccioppoli inequality in the ball $B(x_0, 8\lambda r)$. Since (5.13) holds, by an application of [[54], Theorem 11] with $\Omega = B(x_0, 8\lambda r)$, $u = (u - k_n)_+$ and $t_+ = p$, there exists C=$C(\mathrm{data}, C_U, \|u\|_{L^\infty(B(x_0,r))})$ such that

$$\begin{aligned}
\operatorname*{ess\,sup}_{B\left(x_0,\frac{r}{2}\right)} (u - k_n) &\le \operatorname*{ess\,sup}_{B\left(x_0,\frac{r}{2}\right)} (u - k_n)_+ \\
&\le \frac{C}{\left(1 - \frac{r/2}{r}\right)^{\frac{Q}{p}}} \left(\fint_{B(x_0,r)} (u - k_n)_+^p \,\mathrm{d}\mu \right)^{\frac{1}{p}} \\
&\le C \left(\fint_{B(x_0,r)} (u - k_n)_+^p \,\mathrm{d}\mu \right)^{\frac{1}{p}} \\
&= C \left(\frac{1}{\mu(B(x_0,r))} \int_{S_{k_n,r}} (u - k_n)^p \,\mathrm{d}\mu \right)^{\frac{1}{p}} \\
&\le C \left(\frac{\mu(S_{k_n,r})}{\mu(B(x_0,r))} \right)^{\frac{1}{p}} \left(\operatorname*{ess\,sup}_{B(x_0,r)} u - k_n \right).
\end{aligned}$$

Therefore, by (5.18)

$$\begin{aligned}
\operatorname*{ess\,sup}_{B\left(x_0,\frac{r}{2}\right)} u &\le k_n + C \left(\frac{\mu(S_{k_n,r})}{\mu(B(x_0,r))} \right)^{\frac{1}{p}} \left(\operatorname*{ess\,sup}_{B(x_0,r)} u - k_n \right) \\
&= \operatorname*{ess\,sup}_{B(x_0,r_0)} w + M(1 - 2^{-n}) + C \left(\frac{\mu(S_{k_n,r})}{\mu(B(x_0,r))} \right)^{\frac{1}{p}} \left(\operatorname*{ess\,sup}_{B(x_0,r)} u - \operatorname*{ess\,sup}_{B(x_0,r_0)} w - M(1 - 2^{-n}) \right) \\
&\le \operatorname*{ess\,sup}_{B(x_0,r_0)} w + M(1 - 2^{-n}) + C \left(\frac{\mu(S_{k_n,r})}{\mu(B(x_0,r))} \right)^{\frac{1}{p}} M 2^{-n} \\
&\le \operatorname*{ess\,sup}_{B(x_0,r_0)} w + M(1 - 2^{-n}) + C \left(\frac{1}{n^{1-d_2}} \gamma_{p,q}\left(pd_2, \frac{r}{2}\right) \right)^{\frac{1}{p}} M 2^{-n},
\end{aligned}$$

If $n \ge \left(2^p \gamma_{p,q}\left(pd_2, \frac{r}{2}\right)\right)^{\frac{1}{1-d_2}}$, then $\left(\frac{1}{n^{1-d_2}} \gamma_{p,q}\left(pd_2, \frac{r}{2}\right)\right)^{\frac{1}{p}} \le \frac{1}{2}$. Therefore, the last term on the right-hand side of the previous inequality is at most $2^{-n-1}M$.

Thus, for

$$n \ge C \gamma_{p,q}\left(pd_2, \frac{r}{2}\right)^{\frac{1}{1-d_2}}, \tag{5.19}$$

we have

$$\begin{aligned}
\operatorname*{ess\,sup}_{B\left(x_0,\frac{r}{2}\right)} u &\le \operatorname*{ess\,sup}_{B(x_0,r_0)} w + M - 2^{-n}M + 2^{-n-1}M \\
&= \operatorname*{ess\,sup}_{B(x_0,r_0)} w + M(1 - 2^{-n-1}).
\end{aligned}$$

Finally, we get

$$M\left(\frac{r}{2}, r\right) \le (1 - 2^{-n-1})M, \tag{5.20}$$

for n satisfying (5.19). *End Case 2.*

Notice that Eqs. (5.9) and (5.19) are the same, and we can uniform conditions (5.10) and (5.20).

Therefore, combining both cases, we obtain that by choosing the smallest integer such that $n \ge C\gamma_{p,q}\left(pd_2, \frac{r}{2}\right)^{\frac{1}{1-d_2}}$, we finish the proof. □

6. Sufficient condition for Hölder continuity

This last section contains the main result of the present work, Theorem 6. It is worth mentioning that after obtaining the pointwise estimate on a boundary point, see Proposition 3, the arguments used in this section are inspired by techniques from the literature, particularly [2,53], but adapted to our setting. For completeness, we provide full proofs below. We start with the following Theorem, a pointwise estimate for quasiminima near a boundary point, see [2,51,53].

Theorem 5. *Let $w \in N^{1,1}(X)$ with $H(\cdot, g_w) \in L^1(X)$. Let $u \in N^{1,1}(X)$, with $H(\cdot, g_u) \in L^1(X)$ a quasiminimizer on Ω with boundary values w. Then there exist $C_0, C_1 > 0$ such that*

$$M(\rho, r_0) \leq C_1 M(r_0, r_0) \exp\left(-\frac{1}{4}\int_\rho^{r_0} \exp\left(-C_0 \gamma_{p,q}(pd_2, r)^{\frac{1}{1-d_2}}\right)\frac{dr}{r}\right). \tag{6.1}$$

Proof. Let $M(r) = M(r, r_0)$, where $r_0 > 0$ is fixed. It is not restrictive to suppose that $0 < M(r_0) < +\infty$. We consider C and $n(r)$ as in Proposition 3, $C_0 = C \log 2$ and

$$\zeta(r) = \exp\left(-C_0 \gamma_{p,q}(pd_2, r)^{\frac{1}{1-d_2}}\right) = 2^{-n(2r)}.$$

We partition $(0, r_0) = I_1 \cup I_2$ with $I_1 \cap I_2 = \emptyset$, where

$$I_m = \bigcup_{i=1}^{+\infty} \left[(8\lambda)^{m-2i-1} r_0, (8\lambda)^{m-2i} r_0\right), \quad \text{for } m = 1, 2.$$

It follows,

$$\int_\rho^{r_0} \zeta(r)\frac{dr}{r} \leq 2\int_{(\rho, r_0)\cap I_m} \zeta(r)\frac{dr}{r}, \tag{6.2}$$

for $m = 1$ or $m = 2$. For each $i \in \mathbb{N}$, we define $r_i \in \left[(8\lambda)^{m-2i-1} r_0, (8\lambda)^{m-2i} r_0\right)$ satisfying

$$\zeta(r_i) \geq \frac{1}{(8\lambda)^{m-2i-1} r_0}\int_{(8\lambda)^{m-2i-1} r_0}^{(8\lambda)^{m-2i} r_0} \zeta(r) dr \geq \int_{(8\lambda)^{m-2i-1} r_0}^{(8\lambda)^{m-2i} r_0} \zeta(r)\frac{dr}{r}.$$

Since $\zeta(r) \leq 1$ for all r, we obtain

$$\int_{(\rho, r_0)\cap I_m} \zeta(r)\frac{dr}{r} \leq \sum_{\rho \leq r_i < \frac{r_0}{8\lambda}} \zeta(r_i) + C. \tag{6.3}$$

By Proposition 3, we get

$$\begin{aligned} M((8\lambda)^{m-2i-1} r_0) \leq M(r_i) &\leq M(8\lambda r_i)(1 - 2^{-n(2r_i)-1}) \\ &\leq M((8\lambda)^{m-2i+1} r_0)\left(1 - \frac{\zeta(r_i)}{2}\right), \quad \text{for } i \geq 1. \end{aligned}$$

By an iteration process and the trivial inequality $1 - t \leq \exp(-t)$, we deduce

$$M(\rho) \leq M(r_0) \exp\left(-\frac{1}{2}\sum_{\rho \leq r_i < \frac{r_0}{8\lambda}} \zeta(r_i)\right), \quad \text{for } 0 < \rho < r_0.$$

Furthermore,

$$-\frac{1}{2}\sum_{\rho \leq r_i < \frac{r_0}{8\lambda}} \zeta(r_i) \leq C - \frac{1}{2}\int_{(\rho, r_0)\cap I_m} \zeta(r)\frac{dr}{r} \leq C - \frac{1}{4}\int_\rho^{r_0} \zeta(r)\frac{dr}{r}.$$

Therefore,

$$\exp\left(-\frac{1}{2}\sum_{\rho \leq r_i < \frac{r_0}{8\lambda}} \zeta(r_i)\right) \leq C_1 \exp\left(-\frac{1}{4}\int_\rho^{r_0} \exp\left(-C_0 \gamma_{p,q}(pd_2, r)^{\frac{1}{1-d_2}}\right)\frac{dr}{r}\right).$$

□

Finally, Theorem 5 implies the sufficient condition for Hölder continuity of quasiminima of double-phase functionals at a boundary point.

Theorem 6. *Let $w \in N^{1,1}(X)$ with $H(x, g_w) \in L^1(X)$. Let $u \in N^{1,1}(X)$, with $H(x, g_u) \in L^1(X)$ a quasiminimizer on Ω with boundary values w. If w is a Hölder continuous function at $x_0 \in \partial\Omega$, then there exists $C_0 > 0$ such that*

$$\liminf_{\rho \to 0} \frac{1}{|\log \rho|}\int_\rho^1 \exp\left(-C_0 \gamma_{p,q}(pd_2, r)^{\frac{1}{1-d_2}}\right)\frac{dr}{r} > 0.$$

Thus u is Hölder continuous at x_0.

Proof. We notice that $-u$ is a quasiminimizer (see Remark 5). Furthermore, by hypothesis $-u - (-w) \in N_0^{1,1}(\Omega)$, so it is enough to work with the positive part $(u(x) - w(x_0))_+$. Without loss of generality, we can make $w(x_0) = 0$. Being w continuous at x_0, Corollary 1 yields to

$$M(r_0, r_0) \leq M = \operatorname*{ess\,sup}_{B(x_0, R)} u_+ < +\infty, \tag{6.4}$$

for some $R > 0$ and all $0 < r_0 < R$.

For $0 < \rho < r_0$, using Theorem 5 we obtain

$$\operatorname*{ess\,sup}_{B(x_0,\rho)} u_+ \leq \operatorname*{ess\,sup}_{B(x_0,r_0)} w_+ + M(\rho, r_0)$$
$$\leq \operatorname*{ess\,sup}_{B(x_0,r_0)} w_+ + C_1 M \exp\left(-\frac{1}{4}\int_\rho^{r_0} \exp\left(-C_0\gamma_{p,q}(pd_2, r)^{\frac{1}{1-d_2}}\right)\frac{dr}{r}\right). \tag{6.5}$$

So there are $C, h, k > 0$ satisfying $\operatorname{ess\,sup}_{B(x_0,r_0)} w_+ \leq Cr_0^h$ and, for all sufficiently small ρ and r_0,

$$\int_\rho^1 \exp\left(-C_0\gamma_{p,q}(pd_2, r)^{\frac{1}{1-d_2}}\right)\frac{dr}{r} \geq k|\log \rho|.$$

We observe that

$$\int_{r_0}^1 \exp\left(-C_0\gamma_{p,q}(pd_2, r)^{\frac{1}{1-d_2}}\right)\frac{dr}{r} \leq \int_{r_0}^1 \frac{dr}{r} = |\log r_0|,$$

for all $0 < r_0 < 1$. For sufficiently small ρ and r_0, using inequality (6.5), we derive

$$\operatorname*{ess\,sup}_{B(x_0,\rho)} u_+ \leq Cr_0^h + C_1 M \rho^{\frac{k}{4}} r_0^{-\frac{1}{4}}.$$

The proof of the Hölder continuity of u at x_0 is completed by choosing $r_0 = \rho^{k'}$ with $0 < k' < k$. □

Declaration of competing interest

The authors declare that they have no conflict of interest.

Acknowledgment

The authors would like to express their sincere gratitude to the reviewers for their valuable feedback and support in improving this study. A. Nastasi is a member of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and was partly supported by GNAMPA-INdAM Project *Regolarità per problemi ellittici e parabolici con crescite non standard*, CUP E53C22001930001.
This research was partly conducted while C. Pacchiano Camacho was at the Okinawa Institute of Science and Technology (OIST) through the Theoretical Sciences Visiting Program (TSVP). C. Pacchiano Camacho was supported by a grant from Simons Foundation International SFI-MPS-T-Institutes-00011977 JS.

Data availability

Not applicable.

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