MULTIPLE SOLUTIONS WITH SIGN INFORMATION FOR SEMILINEAR NEUMANN PROBLEMS WITH CONVECTION

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ABSTRACT. We consider a semilinear Neumann problem with convection. We assume that the drift coefficient is indefinite. Using the theory of nonlinear operators of monotone type, together with truncation and comparison techniques and flow invariance arguments, we prove a multiplicity theorem producing three nontrivial smooth solutions (positive, negative and nodal).

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following semilinear Neumann problem with convection

$$(1) \qquad -\Delta u(z) + \xi(z)u(z) = r(z)|\nabla u(z)| + f(z, u(z)) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0.$$

For the potential function $\xi(z)$ we make the following hypothesis

$H(\xi)$: $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$, $\xi \not\equiv 0$.

For the drift coefficient $r \in L^\infty(\Omega)$, the precise conditions are given just before equation (3).

In this problem the potential function $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$, $\xi \not\equiv 0$. The drift coefficient $r \in L^\infty(\Omega)$ and in general is sign changing. The perturbation $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$ $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$ $x \to f(z, x)$ is continuous). We assume that $f(z, \cdot)$ exhibits linear growth near $\pm\infty$.

Our aim is to prove a multiplicity theorem for problem (1) providing sign information for all the solutions. So, we prove a three solutions theorem, producing two solutions of constant sign (one positive and the other negative) and a third solution which is nodal (sign changing). The presence of the drift term $r(z)|\nabla u|$ makes problem (1) nonvariational. Hence our approach is topological based on the theory of nonlinear operators of monotone type, combined with appropriate truncation and comparison techniques and flow invariance arguments.

In the past, the works on the subject either produced only positive solutions or nontrivial solutions with no sign information, primarly for Dirichlet problems. We mention the semilinear works of Amann-Crandall [1], de Figueiredo-Girardi-Matzeu [6], Gasiński-Papageorgiou [8], Girardi-Matzeu [9], Matzeu-Servadei [15], Papageorgiou-Rădulescu-Repovš [20], Yan [25] and the nonlinear works of Bai [2], Bai-Gasiński-Papageorgiou [3], Faraci-Motreanu-Puglisi [5], Papageorgiou-Vetro-Vetro [21], Ruiz [23]. The only work proving the existence of nodal solutions is the recent one by Liu-Shi-Wei [13] for semilinear Dirichlet problems. They consider a problem with a superlinear reaction and

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the gradient term can not be decoupled from the perturbation. The reaction is a C^1 function on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$. Their approach uses the Nehari manifold.

2. Mathematical Background - Auxiliary Results

Let X be a reflexive Banach space. By X^* we denote the topological dual of X and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X). An operator $V : X \to X^*$ is said to be "pseudomonotone", if it satisfies the following condition:

$$u_n \xrightarrow{w} u \text{ in X }, \; V(u_n) \xrightarrow{w} u^* \text{ in } X^* \;\text{ and } \limsup_{n\to+\infty} \langle V(u_n), u_n - u \rangle \leq 0$$
$$\text{imply that } \quad u^* = V(u) \quad \text{ and } \quad \langle V(u_n), u_n \rangle \to \langle V(u), u \rangle.$$

This class of pseudomonotone operators extends the class of maximal monotone ones. Indeed, an everywhere defined maximal monotone map $A : X \to X^*$ is pseudomonotone (see Gasiński-Papageorgiou [7], Proposition 3.2.51, p. 334). Similarly to maximal monotone maps, pseudomonotone operators have remarkable surjectivity properties. Recall that a map $V : X \to X^*$ is said to be "coercive", if the following property holds

$$\frac{\langle V(u), u \rangle}{\|u\|} \to +\infty \quad \text{as } \|u\| \to +\infty, \, u \in X.$$

The following theorem demonstrates the surjectivity properties of pseudomonotone operators (see Gasiński-Papageorgiou [7], Theorem 3.2.52, p. 336).

Theorem 1. *If $V : X \to X^*$ is pseudomonotone, coercive, then $V(\cdot)$ is surjective.*

Let Y be a Banach space. By an "order cone", we understand a nonempty, closed, convex set $P \subseteq Y$ such that $\lambda P \subseteq P$ for all $\lambda \geq 0$ and $P \cap (-P) = \{0\}$. The cone induces an order $\leq$ on Y by

$$\text{for all } y, v \in Y \;\; y \leq v \text{ if and only if } v - y \in P.$$

We write $\dot{P} = P \setminus \{0\}$ and if $\operatorname{int} P \neq \emptyset$, then we say that P is a "solid cone". Given a map $K : Y \to Y$ we define:

(a) $K(\cdot)$ is "increasing" if and only if $y \leq v \;\Rightarrow\; K(y) \leq K(v)$.
(b) When P is solid $K(\cdot)$ is "strongly increasing" if and only if

$$v - y \in \dot{P} \;\; \Rightarrow K(v) - K(y) \in \operatorname{int} P.$$

We say that $K : Y \to Y$ is positively 1-homogeneous if $K(\lambda y) = \lambda K(y)$ for all $\lambda \geq 0$, all $y \in Y$. A pair $(\lambda, y) \in \mathbb{R} \times Y$ is said to be a "positive eigenpair" for $K(\cdot)$ if $\lambda > 0$ and $y \in \operatorname{int} P$.

From Marano-Papageorgiou [14] (Proposition 2.1) we have:

Proposition 1. *If Y is an ordered Banach space with a solid order cone P and $y_0 \in \operatorname{int} P$, then for every $v \in P$, we can find $\lambda_v > 0$ such that $v \leq \lambda_v y_0$.*

Let $\gamma : H^1(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\gamma(u) = \|\nabla u\|_2^2 + \int_\Omega \xi(z) u^2 dz \quad \text{for all } u \in H^1(\Omega).$$

The following inequality will be useful in what follows (see Papageorgiou-Rădulescu-Repovš [19], Lemma 2.8).

Lemma 1. *If hypotheses $H(\xi)$ hold, then there exists $c_0 > 0$ such that*

$$c_0\|u\|^2 \leq \|\nabla u\|_2^2 + \int_\Omega \xi(z)u^2dz \quad \textit{for all } u \in H^1(\Omega).$$

Hereafter by $\|\cdot\|$ we denote the norm of the Sobolev space $H^1(\Omega)$ defined by

$$\|u\| = \left[\|u\|_2^2 + \|\nabla u\|_2^2\right]^{1/2} \quad \text{for all } u \in H^1(\Omega).$$

We consider the following linear eigenvalue problem:

$$-\Delta u(z) + \xi(z)u(z) = \widehat{\lambda}u(z) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0.$$

We know that the spectrum of this eigenvalue problem is a sequence $\{\widehat{\lambda}_k\}_{k\geq 1}$ of eigenvalues such that $\widehat{\lambda}_k \to +\infty$ as $k \to +\infty$ and the corresponding eigenfunctions $\widehat{u}_k \in C^1(\overline{\Omega})$. For the first (smallest) eigenvalue $\widehat{\lambda}_1$, we have:

$$\widehat{\lambda}_1 > 0 \text{ and it is simple;}$$

$$\widehat{\lambda}_1 = \inf\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in H^1(\Omega), u \neq 0\right]. \tag{2}$$

Then the elements of the corresponding one-dimensional eigenspace have constant sign. These facts lead to the following useful lemma (see Mugnai-Papageorgiou [16], Lemma 4.11).

Lemma 2. *If $\vartheta \in L^\infty(\Omega)_+$ and $\vartheta(z) \leq \widehat{\lambda}_1$ for a.a. $z \in \Omega$ with strict inequality on a set of positive measure, then there exists $c_1 > 0$ such that*

$$c_1\|u\|^2 \leq \gamma(u) - \int_\Omega \vartheta(z)u^2dz \quad \textit{for all } u \in H^1(\Omega).$$

In what follows $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ is defined by

$$\langle A(u), v\rangle = \int_\Omega (\nabla u, \nabla v)_{\mathbb{R}^N}dz \quad \text{for all } u, v \in H^1(\Omega).$$

For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Given $u \in H^1(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in H^1(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

Also, if $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example, a Carathéodory function), by N_g we denote the Nemitsky operator associated with the function g, that is,

$$N_g(u)(\cdot) = g(\cdot, u(\cdot))$$

for every measurable function $u : \Omega \to \mathbb{R}$.

Also, if Y, Z are Banach spaces, a map $G : Y \to Z$ is said to be "compact" if it is continuous and maps bounded sets in Y into relatively compact sets in Z.

If $u, v \in H^1(\Omega)$ and $v \leq u$, then we define

$$[v, u] = \{w \in H^1(\Omega) : v(z) \leq w(z) \leq u(z) \text{ for a.a. } z \in \Omega\}.$$

By $\text{int}_{C^1(\overline{\Omega})}[v, u]$ we denote the interior of $[v, u]$ in the $C^1(\overline{\Omega})$-norm topology.

Finally if $\varphi \in C^1(H^1(\Omega), \mathbb{R})$, then

$$K_\varphi = \{u \in H^1(\Omega) : \varphi'(u) = 0\} \quad \text{(the critical set of } \varphi).$$

In addition to the Sobolev space $H^1(\Omega)$, in the analysis of problem (1) we will also use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with order (positive) cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This order cone is solid, that is, $\operatorname{int} C_+ \neq \emptyset$. In fact $\operatorname{int} C_+$ contains the open set $D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}$. Note that D_+ is the interior of C_+ when $C^1(\overline{\Omega})$ is furnished with the relative $C^1(\overline{\Omega})$-norm topology.

The hypotheses on the drift coefficient are:

$H(r)$: $r \in L^\infty(\Omega)$ and $\|r\|_\infty < \widehat{\lambda}_1^{1/2}$.

We start by considering the following Neumann problem:

$$-\Delta u(z) + \xi(z)u(z) = r(z)|\nabla u(z)| + h(z) \quad \text{in } \Omega, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0. \tag{3}$$

Proposition 2. *If hypotheses $H(\xi)$, $H(r)$ hold and $h \in L^\infty(\Omega)$, then problem (3) has a unique solution $K(h) \in C^1(\overline{\Omega})$; moreover, if $h \geq 0$ (resp. $h \leq 0$), $h \not\equiv 0$, then $K(h) \in D_+$ (resp. $K(h) \in -D_+$).*

Proof. Let $V : H^1(\Omega) \to H^1(\Omega)^*$ be the nonlinear operator defined by

$$\langle V(u), y\rangle = \langle A(u), y\rangle + \int_\Omega \xi(z)uydz - \int_\Omega r(z)|\nabla u|ydz \quad \text{for all } u, y \in H^1(\Omega).$$

<u>Claim 1:</u> $V(\cdot)$ is pseudomonotone.

Let $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega)$ and assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega),\ V(u_n) \xrightarrow{w} u^* \text{ in } H^1(\Omega)^*, \text{ and } \limsup_{n\to+\infty}\langle V(u_n), u_n - u\rangle \leq 0. \tag{4}$$

From (4) we have $u_n \to u$ in $L^2(\Omega)$.

Therefore we have

$$\int_\Omega \xi(z)u_n(u_n - u)dz \to 0 \text{ and } \int_\Omega r(z)|\nabla u_n|(u_n - u)dz \to 0.$$

So, from (4) we have

$$\begin{aligned} &\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0, \\ \Rightarrow\quad &\limsup_{n\to+\infty}\|\nabla u_n\|_2 \leq \|\nabla u\|_2. \end{aligned}$$

On the other hand since $u_n \xrightarrow{w} u$ in $H^1(\Omega)$, we have

$$\begin{aligned} &\|\nabla u\|_2 \leq \liminf_{n\to+\infty}\|\nabla u_n\|_2, \\ \Rightarrow\quad &\|\nabla u_n\|_2 \to \|\nabla u\|_2, \\ \Rightarrow\quad &u_n \to u \text{ in } H^1(\Omega) \text{ (by the Kadec-Klee property, see (4))}. \end{aligned} \tag{5}$$

Then it follows that

$$\begin{aligned} &V(u_n) \to V(u) \text{ in } H^1(\Omega) \text{ (see (5))}, \\ \Rightarrow\quad &u^* = V(u) \text{ and } \langle V(u_n), u_n\rangle \to \langle V(u), u\rangle. \end{aligned}$$

Hence $V(\cdot)$ is pseudomonotone and this proves Claim 1.

<u>Claim 2:</u> $V(\cdot)$ is coercive.

For every $u \in H^1(\Omega)$, we have

$$\begin{aligned}
&\langle V(u), u\rangle = \|\nabla u\|_2^2 + \int_\Omega \xi(z)u^2 dz - \int_\Omega r(z)|\nabla u|u dz,\\
&\geq \gamma(u) - \|r\|_\infty \|\nabla u\|_2 \|u\|_2\\
&\text{(see } H(r) \text{ and use the Cauchy-Schwarz inequality)}\\
&\geq \gamma(u) - \|r\|_\infty \gamma(u)^{1/2} \left(\frac{\gamma(u)}{\widehat{\lambda}_1}\right)^{1/2} \quad \text{(see (2))}\\
&= \left[1 - \frac{\|r\|_\infty}{\widehat{\lambda}_1^{1/2}}\right]\gamma(u),\\
\Rightarrow\quad & V(\cdot) \text{ is coercive (see hypothesis } H(r)).
\end{aligned}$$

This proves Claim 2.

On account of Claims 1 and 2 and invoking Theorem 1, we have that $V(\cdot)$ is surjective. Thus we can find $u \in H^1(\Omega)$ such that

$$V(u) = h \text{ in } H^1(\Omega)^*. \tag{6}$$

In fact this solution is unique. Indeed, if $v \in H^1(\Omega)$ is another such solution, then

$$\begin{aligned}
&\|\nabla(u-v)\|_2^2 + \int_\Omega \xi(z)(u-v)^2 dz - \int_\Omega r(z)[|\nabla u| - |\nabla v|](u-v)dz = 0,\\
\Rightarrow\quad & \gamma(u-v) - \|r\|_\infty \int_\Omega |\nabla(u-v)||u-v|dz \leq 0,\\
\Rightarrow\quad & \left[1 - \frac{\|r\|_\infty}{\widehat{\lambda}_1^{1/2}}\right]\gamma(u-v) \leq 0,\\
\Rightarrow\quad & u = v \quad \text{(see hypothesis } H(r) \text{ and Lemma 1).}
\end{aligned}$$

So, we have proved that the solution u of (6) is unique.

From (6) we have

$$\langle A(u), y\rangle + \int_\Omega \xi(z)uy dz = \int_\Omega r(z)|\nabla u| y dz + \int_\Omega hy dz \quad \text{for all } y \in H^1(\Omega), \tag{7}$$

$$\Rightarrow\quad -\Delta u(z) + \xi(z)u(z) = r(z)|\nabla u(z)| + h(z) \quad \text{for a.a. } z \in \Omega, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0. \tag{8}$$

(see Papageorgiou-Rădulescu [18]).

From (8) and Theorem 4.1 of Winkert [24], we have $u \in L^\infty(\Omega)$.

So, we can apply Theorem 2 of Lieberman [12] and have

$$u = K(h) \in C^{1,\alpha}(\overline{\Omega}) \quad \text{for some } \alpha \in (0,1).$$

If $h \geq 0$, $h \neq 0$, then $u = K(h) \neq 0$ and from (7) with $y = -u^- \in H^1(\Omega)$, we obtain

$$\begin{aligned}
&\|\nabla u^-\|_2^2 + \int_\Omega \xi(z)(u^-)^2 dz + \int_\Omega r(z)|\nabla u^-|u^- dz = \int_\Omega h(-u^-)dz \leq 0,\\
\Rightarrow\quad & \gamma(u^-) - \|r\|_\infty \|\nabla u^-\|_2 \|u^-\|_2 \leq 0,\\
\Rightarrow\quad & \left[1 - \frac{\|r\|_\infty}{\widehat{\lambda}_1^{1/2}}\right]\gamma(u^-) \leq 0 \quad \text{(see (2))}
\end{aligned}$$

$$\Rightarrow \quad c_1\|u^-\|^2 \leq 0 \quad \text{for some } c_1 > 0 \text{ (see hypothesis } H(r) \text{ and Lemma 1)},$$
$$\Rightarrow \quad u \geq 0,\ u \neq 0.$$

Therefore $u = K(h) \in C_+ \setminus \{0\}$. From (8) we have

$$\Delta u(z) \leq \|r\|_\infty |\nabla u(z)| + \xi(z)u(z) \quad \text{for a.a. } z \in \Omega \text{ (recall that } h \geq 0).$$

Applying Theorem 5.4.1, p. 111 of Pucci-Serrin [22] (the nonlinear strong maximum principle), we infer that

$$u(z) > 0 \quad \text{for all } z \in \Omega.$$

Then the nonlinear boundary point theorem (see Pucci-Serrin [22], Theorem 5.5.1, p. 120), implies that $u \in D_+ \subseteq \operatorname{int} C_+$.

In a similar fashion, we show that if $h \leq 0$, $h \neq 0$, then $K(h) \in -D_+ \subseteq -\operatorname{int} C_+$. □

We will need the following comparison result.

Proposition 3. *If hypotheses $H(\xi)$, $H(r)$ hold, $h_1, h_2 \in L^\infty(\Omega)$, $h_1(z) \leq h_2(z)$ for a.a. $z \in \Omega$, and $u_1 = K(h_1)$, $u_2 = K(h_2) \in C^1(\overline{\Omega})$ (see Proposition 2), then $u_1 \leq u_2$.*

Proof. We have

$$\begin{aligned}
&\|\nabla(u_1 - u_2)^+\|_2^2 + \int_\Omega \xi(z)[(u_1-u_2)^+]^2 dz - \int_\Omega r(z)[|\nabla u_1| - |\nabla u_2|](u_1-u_2)^+ dz \\
&= \int_\Omega (h_1 - h_2)(u_1-u_2)^+ dz \leq 0, \\
\Rightarrow \quad & \gamma((u_1-u_2)^+) - \int_\Omega \|r\|_\infty |\nabla(u_1-u_2)^+||(u_1-u_2)^+| dz \leq 0, \\
\Rightarrow \quad & \left[1 - \frac{\|r\|_\infty}{\widehat{\lambda}_1^{1/2}}\right] \gamma((u_1-u_2)^+) \leq 0, \\
\Rightarrow \quad & u_1 \leq u_2 \quad \text{(see hypothesis } H(r) \text{ and Lemma 1)}.
\end{aligned}$$

□

We consider the solution map $K : C_+ \to C_+$. The next proposition states the main properties of this map.

Proposition 4. *If hypotheses $H(\xi)$, $H(r)$ hold, then $K(\cdot)$ is increasing, compact and positively 1-homogeneous.*

Proof. From Proposition 3 we see that $K(\cdot)$ is increasing. Also, it is clear that $K(\cdot)$ is positively 1-homogeneous. It remains to show that $K(\cdot)$ is compact. First we show that $K(\cdot)$ is continuous. So, let $h_n \to h$ in C_+ and set $u_n = K(h_n)$, $n \in \mathbb{N}$ and $u = K(h)$. We have

$$\langle A(u_n), y\rangle + \int_\Omega \xi(z) u_n y dz = \int_\Omega r(z)|\nabla u_n| y dz + \int_\Omega h_n y dz \tag{9}$$

for all $y \in H^1(\Omega)$, all $n \in \mathbb{N}$. Choose $y = u_n$. Then

$$\begin{aligned}
\gamma(u_n) &= \int_\Omega r(z)|\nabla u_n| u_n dz + \int_\Omega h_n u_n dz \\
&\leq \|r\|_\infty \|\nabla u_n\|_2 \|u_n\|_2 + \|h_n\|_2 \|u_n\|_2 \quad \text{for all } n \in \mathbb{N},
\end{aligned}$$

$$\Rightarrow \quad \left[1 - \frac{\|r\|_\infty}{\widehat{\lambda}_1^{1/2}}\right]\gamma(u_n) \le c_2\|u_n\| \quad \text{for some } c_2 > 0, \text{ all } n \in \mathbb{N},$$

$$\Rightarrow \quad \|u_n\| \le c_3 \quad \text{for some } c_3 > 0, \text{ all } n \in \mathbb{N} \text{ (using hypothesis } H(r) \text{ and Lemma 1).}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \to u \text{ in } L^2(\Omega). \tag{10}$$

In (9) we choose $y = u_n - u$ and pass to the limit as $n \to +\infty$. Using (10), we obtain

$$\lim\langle A(u_n), u_n - u\rangle = 0,$$
$$\Rightarrow \quad \|\nabla u_n\|_2 \to \|\nabla u\|_2,$$
$$\Rightarrow \quad u_n \to u \text{ in } H^1(\Omega) \text{ (by the Kadec-Klee property, see (10)).} \tag{11}$$

Passing to the limit as $n \to +\infty$ in (9) and using (11), we have

$$\langle A(u), y\rangle + \int_\Omega \xi(z)uydz = \int_\Omega r(z)|\nabla u|ydz + \int_\Omega hydz \text{ for all } y \in H^1(\Omega),$$
$$\Rightarrow \quad u = K(h),$$
$$\Rightarrow \quad u_n = K(h_n) \to K(h) = u \text{ in } H^1(\Omega).$$

From Winkert [24], we have

$$\|u_n\|_\infty \le c_4 \quad \text{for some } c_4 > 0, \text{ all } n \in \mathbb{N}.$$

So, Theorem 2 of Lieberman [12] implies that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad 0 < \alpha < 1 \text{ and } \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \le c_5 \text{ for some } c_5 > 0, \text{ all } n \in \mathbb{N}. \tag{12}$$

Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, from (12) and (11), we have for the initial sequence that

$$u_n \to u \text{ in } C^1(\overline{\Omega}),$$
$$\Rightarrow \quad K(\cdot) \text{ is continuous.}$$

Moreover, from the above argument it is clear that if $B \subseteq C_+$ is bounded, then $K(B) \subseteq C_+$ is relatively compact. Therefore $K(\cdot)$ is a compact map. □

Remark 1. The same can be said for the map $K : (-C_+) \to (-C_+)$.

Now we consider the following parametric Neumann problem

$$-\Delta u(z) + \xi(z)u(z) = r(z)|\nabla u(z)| + \widetilde{\lambda}u(z) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0, \, \widetilde{\lambda} > 0. \tag{13}$$

Proposition 5. *If hypotheses $H(\xi)$, $H(r)$ hold, then problem* (13) *has a unique solution (eigenpair) $(\widetilde{\lambda}_1, \widetilde{u}_1)$ with $\widetilde{\lambda}_1 > 0$, $\widetilde{u}_1 \in \operatorname{int} C_+$, $\|\widetilde{u}_1\|_2 = 1$.*

Proof. If $\widetilde{\lambda} > 0$ and u solve (13), then

$$u = K(\widetilde{\lambda}u) = \widetilde{\lambda}K(u) \quad \text{(see Proposition 4).}$$

Also, from Proposition 2 we know that if $u \neq 0$, then $K(u) \in D_+$. So, by Proposition 1, we can find $c_6 > 0$ such that $c_6 u \le K(u)$. Hence, we can apply Theorem 3.1(3) of Chang [4] (see also p. 544) and infer that problem (13) has a unique solution (eigenpair) $(\widetilde{\lambda}_1, \widetilde{u}_1)$ with $\widetilde{\lambda}_1 > 0$, $\widetilde{u}_1 \in \operatorname{int} C_+$, $\|\widetilde{u}_1\|_2 = 1$. □

Remark 2. In a similar fashion, we can produce a unique solution pair $(\widetilde{\lambda}_1^*, \widetilde{v}_1)$ of (13) such that

$$\widetilde{\lambda}_1^* > 0,\ \widetilde{v}_1 \in -\mathrm{int}\, C_+,\ \|\widetilde{v}_1\|_2 = 1.$$

3. Solutions of Constant Sign

In this section we show the existence of positive and negative smooth solutions for problem (1). Moreover, we show that there exist extremal constant sign solutions, that is, a smallest positive solution and a biggest negative solution.

We impose the following conditions on the perturbation $f(z,x)$.

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \le a(z)[1+|x|]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$;

(ii) there exists a function $\vartheta \in L^\infty(\Omega)_+$ such that if $\xi_0 = 1 - \dfrac{\|r\|_\infty}{\widehat{\lambda}_1^{1/2}} > 0$ (see hypothesis $H(r)$), then

$$\vartheta(z) \le \widehat{\lambda}_1 \xi_0 \text{ for a.a. } z \in \Omega,\ \vartheta \not\equiv \widehat{\lambda}_1 \xi_0,$$
$$\limsup_{x \to \pm\infty} \frac{f(z,x)}{x} \le \vartheta(z) \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exists $\delta > 0$ such that

$$f(z,x) \ge \widetilde{\lambda}_1 x \text{ for a.a. } z \in \Omega, \text{ all } x \in [0,\delta],$$
$$f(z,x) \le \widetilde{\lambda}_1^* x \text{ for a.a. } z \in \Omega, \text{ all } x \in [-\delta,0].$$

Proposition 6. *If hypotheses $H(\xi)$, $H(r)$, $H(f)$ hold, then problem* (1) *admits two nontrivial solutions of constant sign $\widehat{u} \in D_+$ and $\widehat{v} \in -D_+$.*

Proof. Let $(\widetilde{\lambda}_1, \widetilde{u}_1)$ be the solution pair of problem (13) produced in Proposition 5. We know that $\widetilde{u}_1 \in D_+$. So, we can find $t \in (0,1)$ small such that

$$t\widetilde{u}_1(z) \in (0,\delta] \quad \text{for all } z \in \overline{\Omega}. \tag{14}$$

We set $\widetilde{u}_* = t\widetilde{u}_1 \in D_+$. We have

$$\begin{aligned} -\Delta \widetilde{u}_*(z) + \xi(z)\widetilde{u}_*(z) &= r(z)|\nabla \widetilde{u}_*(z)| + \widetilde{\lambda}_1 \widetilde{u}_*(z) \\ &\le r(z)|\nabla \widetilde{u}_*(z)| + f(z, \widetilde{u}_*(z)) \quad \text{for a.a. } z \in \Omega \\ &\qquad \text{(see (14) and hypothesis } H(f)\,(iii)). \end{aligned} \tag{15}$$

Using $\widetilde{u}_* \in D_+$ we introduce truncations of the perturbation $f(z,\cdot)$ and of the drift term. So, let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by

$$f_0(z,x) = \begin{cases} f(z, \widetilde{u}_*(z)) & \text{if } x \le \widetilde{u}_*(z), \\ f(z,x) & \text{if } \widetilde{u}_*(z) < x. \end{cases} \tag{16}$$

Also let $T : H^1(\Omega) \to L^2(\Omega)$ be the nonlinear operator defined by

$$T(u)(z) = \begin{cases} r(z)|\nabla \widetilde{u}_*(z)| & \text{if } u(z) \le \widetilde{u}_*(z), \\ r(z)|\nabla u(z)| & \text{if } \widetilde{u}_*(z) < u(z). \end{cases} \tag{17}$$

Evidently $T(\cdot)$ is continuous.

We consider the following nonlinear Neumann problem

$$-\Delta u(z) + \xi(z)u(z) = T(u)(z) + f_0(z, u(z)) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0. \tag{18}$$

We introduce the nonlinear operator $G : H^1(\Omega) \to H^1(\Omega)^*$ defined by

$$\langle G(u), y\rangle = \langle A(u), y\rangle + \int_\Omega \xi(z)uydz - \int_\Omega T(u)ydz - \int_\Omega f_0(z, u)ydz \text{ for all } y \in H^1(\Omega).$$

As in the proof of Proposition 2 (see Claim 1 in that proof), we can check that

$$G(\cdot) \text{ is pseudomonotone.} \tag{19}$$

Hypotheses $H(f)\,(i)$, (ii) and (16) imply that given $\varepsilon > 0$, we can find $c_7 = c_7(\varepsilon) > 0$ such that

$$f_0(z, x)x \leq [\vartheta(z) + \varepsilon]x^2 + c_7 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{20}$$

Let $u \in H^1(\Omega)$. We have

$$\langle G(u), u\rangle = \gamma(u) - \int_\Omega T(u)udz - \int_\Omega f_0(z, u)udz. \tag{21}$$

From (17) we have

$$\begin{aligned} \int_\Omega T(u)udz &= \int_{\{u\leq\widetilde{u}_*\}} r(z)|\nabla\widetilde{u}_*|udz + \int_{\{\widetilde{u}_*<u\}} r(z)|\nabla u|udz \\ &\leq c_8\|u\| + \frac{\|r\|_\infty}{\widehat{\lambda}_1^{1/2}}\gamma(u) \quad \text{for some } c_8 > 0. \end{aligned} \tag{22}$$

Moreover, using (20) we obtain

$$\int_\Omega f_0(z, u)udz \leq \int_\Omega [\vartheta(z) + \varepsilon]u^2dz + c_9\|u\| \quad \text{for some } c_9 > 0, \text{ all } u \in H^1(\Omega). \tag{23}$$

Returning to (21) and using (22) and (23), we obtain

$$\begin{aligned} \langle G(u), u\rangle &\geq \xi_0\gamma(u) - \int_\Omega \vartheta(z)u^2dz - \varepsilon\|u\|^2 - c_{10}\|u\| \quad \text{with } c_{10} = c_8 + c_9 > 0 \\ &\geq c_{11}\|u\|^2 - c_{10}\|u\| \quad \text{for some } c_{11} > 0, \text{ all } u \in H^1(\Omega) \\ &\qquad \text{(use Lemma 2 and choose } \varepsilon > 0 \text{ small)}, \end{aligned}$$

$$\Rightarrow \quad G(\cdot) \text{ is coercive.} \tag{24}$$

Then (19) and (24) permit the use of Theorem 1. So, we can find $\widehat{u} \in H^1(\Omega)$ such that

$$\begin{aligned} &G(\widehat{u}) = 0, \\ \Rightarrow \quad &\langle A(\widehat{u}), y\rangle + \int_\Omega \xi(z)\widehat{u}ydz = \int_\Omega T(\widehat{u})ydz + \int_\Omega f_0(z, \widehat{u})ydz \quad \text{for all } y \in H^1(\Omega). \end{aligned} \tag{25}$$

In (24) we choose $y = (\widetilde{u}_* - \widehat{u})^+ \in H^1(\Omega)$. Using (16) and (17), we obtain

$$\begin{aligned} &\langle A(\widehat{u}), (\widetilde{u}_* - \widehat{u})^+\rangle + \int_\Omega \xi(z)\widehat{u}(\widetilde{u}_* - \widehat{u})^+dz \\ &= \int_\Omega r(z)|\nabla\widetilde{u}_*|(\widetilde{u}_* - \widehat{u})^+dz + \int_\Omega f(z, \widetilde{u}_*)(\widetilde{u}_* - \widehat{u})^+dz \end{aligned}$$

$$
\begin{aligned}
&\geq \langle A(\widetilde{u}_*), (\widetilde{u}_* - \widehat{u})^+\rangle + \int_\Omega \xi(z)\widetilde{u}_*(\widetilde{u}_* - \widehat{u})^+ dz \quad \text{(see (15))},\\
\Rightarrow\quad & \gamma((\widetilde{u}_* - \widehat{u})^+) \leq 0,\\
\Rightarrow\quad & \widetilde{u}_* \leq \widehat{u} \quad \text{(see Lemma 1)}.
\end{aligned}
$$

On account of (16), (17), (24), (25) we have

$$
\begin{aligned}
& -\Delta\widehat{u}(z) + \xi(z)\widehat{u}(z) = r(z)|\nabla\widehat{u}(z)| + f(z,\widehat{u}(z)) \quad \text{for a.a. } z\in\Omega, \quad \left.\frac{\partial\widehat{u}}{\partial n}\right|_{\partial\Omega} = 0,\\
\Rightarrow\quad & \widehat{u} \in D_+ \subseteq \operatorname{int} C_+ \text{ is a positive solution of problem (1).}
\end{aligned}
$$

Similarly using the solution pair $(\widetilde{\lambda}_1^*, \widetilde{v}_1)$ and the second part of hypothesis $H(f)\,(iii)$ we produce $\widehat{v} \in -D_+ \subseteq -\operatorname{int} C_+$ a negative solution of (1) such that $\widehat{v} \leq \widetilde{v}_*$. ☐

Let S_+ (resp. S_-) be the set of positive solutions $\widehat{u}$ (resp. of negative solutions $\widehat{v}$) of problem (1) such that $\widetilde{u}_* \leq \widehat{u}$ (resp. $\widehat{v} \leq \widetilde{v}_*$). We have seen in Proposition 6 that

$$\emptyset \neq S_+ \subseteq D_+ \text{ and } \emptyset \neq S_- \subseteq -D_+.$$

Next we show the existence of extremal constant sign solutions for problem (1) (that is, the existence of a minimal positive solution and of a maximal negative solution).

Proposition 7. *If hypotheses $H(\xi)$, $H(r)$, $H(f)$ hold, then there exist $\widehat{u}_+ \in S_+$ and $\widehat{v}_- \in S_-$ such that*

$$
\begin{aligned}
&\widehat{u}_+ \leq \widehat{u} \quad \textit{for all } \widehat{u} \in S_+ \subseteq D_+,\\
&\widehat{v} \leq \widehat{v}_- \quad \textit{for all } \widehat{v} \in S_- \subseteq -D_+.
\end{aligned}
$$

Proof. On account of Theorem 1 of Le [11], we have that the solution set S_+ is downward directed (that is, if $\widehat{u}_1, \widehat{u}_2 \in S_+$, then we can find $\widehat{u} \in S_+$ such that $\widehat{u} \leq \widehat{u}_1$, $\widehat{u} \leq \widehat{u}_2$). Invoking Lemma 3.10, p. 178, of Hu-Papageorgiou [10], we can find a decreasing sequence $\{\widehat{u}_n\}_{n\geq 1} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n\geq 1} \widehat{u}_n,\ \widetilde{u}_* \leq \widehat{u}_n \quad \text{for all } n \in \mathbb{N}.$$

We have

$$\langle A(\widehat{u}_n), y\rangle + \int_\Omega \xi(z)\widehat{u}_n y dz = \int_\Omega r(z)|\nabla\widehat{u}_n| y dz + \int_\Omega f(z,\widehat{u}_n) y dz \quad \text{for all } y \in H^1(\Omega), \text{ all } n\in\mathbb{N}. \tag{26}$$

Choosing $y = \widehat{u}_n$ in (26) and since $\widetilde{u}_* \leq \widehat{u}_n \leq \widehat{u}_1 \in D_+$ for all $n \in \mathbb{N}$, we infer that

$$\{\widehat{u}_n\}_{n\geq 1} \subseteq H^1(\Omega) \text{ is bounded.}$$

So, we may assume that

$$\widehat{u}_n \xrightarrow{w} \widehat{u}_+ \text{ in } H^1(\Omega) \text{ and } \widehat{u}_n \to \widehat{u}_+ \text{ in } L^2(\Omega). \tag{27}$$

In (26) we choose $y = \widehat{u}_n - \widehat{u}_+$, pass to the limit as $n \to +\infty$ and use (27) and the Kadec-Klee property of Hilbert spaces. We obtain

$$\widehat{u}_n \to \widehat{u}_+ \text{ in } H^1(\Omega). \tag{28}$$

So, if in (26) we pass to the limit as $n \to +\infty$ and use (28), then we infer that

$$\widehat{u}_+ \in S_+ \subseteq D_+ \text{ and } \widehat{u}_+ \leq \widehat{u} \text{ for all } \widehat{u} \in S_+.$$

Similarly, using the fact that S_- is upward directed (that is, if $\widehat{v}_1, \widehat{v}_2 \in S_-$, then we can find $\widehat{v} \in S_-$ such that $\widehat{v}_1 \leq \widehat{v}$, $\widehat{v}_2 \leq \widehat{v}$), we can produce a solution $\widehat{v}_- \in S_- \subseteq -D_+$ such that $\widehat{v} \leq \widehat{v}_-$ for all $\widehat{v} \in S_-$. ∎

4. Nodal Solutions

In this section, using the extremal constant sign solutions from Proposition 7 and the flow invariance argument from Papageorgiou-Papalini [17], we will establish the existence of a nodal (sign-changing) solution. To this end we employ the method of the frozen variable followed by an iteration process. We need to strengthen the conditions on the perturbation $f(z,x)$.

$H(f)'$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that $f(z,0) = 0$ for a.a. $z \in \Omega$, hypotheses $H(f)'\,(i)$, (ii), (iii) are the same as the corresponding hypotheses $H(f)\,(i)$, (ii), (iii) and

(iv) $|f(z,x) - f(z,y)| \leq k(z)|x-y|$ for a.a. $z \in \Omega$, all $x, y \in \mathbb{R}$, with $k \in L^\infty(\Omega)$, $\|k\|_\infty + \|r\|_\infty < 1$ and there exists $\widehat{\xi} > 0$ such that for a.a. $z \in \Omega$, the function

$$x \to f(z,x) + \widehat{\xi} x$$

is nondecreasing.

Proposition 8. *If hypotheses $H(\xi)$, $H(r)$, $H(f)'$ hold, then problem* (1) *admits a nodal solution $\widehat{y} \in C^1(\overline{\Omega})$.*

Proof. As we already mentioned we will employ the "frozen variable method". We freeze the gradient term. So, let $v \in C^1(\overline{\Omega})$ and consider the C^1-functional $\varphi_v : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_v(u) = \frac{1}{2}\gamma(u) - \int_\Omega r(z)|\nabla v| u dz - \int_\Omega F(z,u) dz \quad \text{for all } u \in H^1(\Omega),$$

where $F(z,x) = \int_0^x f(z,s)ds$. We introduce the following inner product on $H^1(\Omega)$

$$(u,y)_* = \int_\Omega (\nabla u, \nabla y)_{\mathbb{R}^N} dz + \int_\Omega \xi(z) u y dz \quad \text{for all } u, y \in H^1(\Omega).$$

By $\|\cdot\|_*$ we denote the corresponding norm on $H^1(\Omega)$. On account of Lemma 1 and hypothesis $H(r)$, we have that $\|\cdot\|_*$ and $\|\cdot\|$ are equivalent norms on $H^1(\Omega)$. The map $u \to A(u) + \xi(z)u$ belongs in $\mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ and it is monotone, coercive (see Lemma 1), thus it is surjective (see Gasiński-Papageorgiou [7], Corollary 3.2.32, p. 320). So, we can define

$$L = (A + \xi(z)I)^{-1}.$$

By the Banach Theorem, we have $L \in \mathcal{L}(H^1(\Omega)^*, H^1(\Omega))$. Let

$$E_v = L \circ (r(z)|\nabla v| I + N_f).$$

Evidently $E_v : H^1(\Omega) \to H^1(\Omega)$ is continuous and standard regularity theory (see Lieberman [12], Theorem 2) implies that

$$E_v(H^1(\Omega)) \subseteq C^1(\overline{\Omega}) \text{ and } L \in \mathcal{L}(L^2(\Omega), C^1(\overline{\Omega})). \tag{29}$$

<u>Claim 1:</u> $E_v(\cdot)$ is compact and strongly increasing.

The compactness of $E_v(\cdot)$ follows from the Sobolev embedding theorem. Also, let $u, y \in H^1(\Omega)$, $u \neq y$, $y \leq u$. We set

$$w = E_v(u) \text{ and } \eta = E_v(y).$$

We have

$$\begin{aligned}
&\Delta(\eta - w)(z) + [\xi(z) + \widehat{\xi}](w - \eta)(z)\\
&= f(z, u(z)) + \widehat{\xi}u(z) - (f(z, y(z)) + \widehat{\xi}y(z))\\
&\geq 0 \quad \text{for a.a. } z \in \Omega \text{ (see hypothesis } H(f)'\,(iv) \text{ and recall that } y \leq u). \qquad (30)
\end{aligned}$$

Acting on (30) with $(\eta - w)^+ \in H^1(\Omega)$, via Green's identity, we obtain

$$\begin{aligned}
&\gamma((\eta - w)^+) + \widehat{\xi}\,\|(\eta - w)^+\|_2^2 \leq 0,\\
\Rightarrow\quad &\eta \leq w.
\end{aligned}$$

From (30) it follows that

$$\begin{aligned}
&\Delta(w - \eta)(z) \leq [\|\xi\|_\infty + \widehat{\xi}](w - \eta)(z) \quad \text{for a.a. } z \in \Omega,\\
\Rightarrow\quad &w - \eta \in D_+ \quad \text{(by the maximum principle)},\\
\Rightarrow\quad &E_v \text{ is strongly increasing.}
\end{aligned}$$

This proves Claim 1.

Let $\nabla\varphi_v$ denote the gradient of $\varphi_v \in C^1(H^1(\Omega), \mathbb{R})$, that is,

$$(\nabla\varphi_v(u), y)_{H^1(\Omega)} = \langle \varphi_v'(u), y\rangle \quad \text{for all } u, y \in H^1(\Omega).$$

<u>Claim 2:</u> $\nabla\varphi_v = I - E_v$.

For $u, y \in H^1(\Omega)$ we have

$$\begin{aligned}
\langle \varphi_v'(u), y\rangle &= \langle A(u), y\rangle + \int_\Omega \xi(z)uydz - \int_\Omega r(z)|\nabla v|ydz - \int_\Omega f(z,u)ydz\\
&= (u, y)_* - \int_\Omega r(z)|\nabla v|ydz - \int_\Omega f(z,u)ydz. \qquad (31)
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_\Omega [r(z)|\nabla v| + f(z,u)]ydz\\
&= \langle L^{-1}(L(r(z)|\nabla v| + N_f(u))), y\rangle\\
&= \int_\Omega (\nabla E_v(u), \nabla y)_{\mathbb{R}}dz + \int_\Omega \xi(z)E_v(u)ydz\\
&= (E_v(u), y)_*. \qquad (32)
\end{aligned}$$

Returning to (31) and using (32), we obtain

$$\begin{aligned}
&\langle \varphi_v'(u), y\rangle = (u - E_v(u), y)_* \quad \text{for all } y \in H^1(\Omega),\\
\Rightarrow\quad &\nabla\varphi_v = I - E_v.
\end{aligned}$$

This proves Claim 2.

We consider the negative gradient flow generated by φ_v, that is, we consider the following abstract Cauchy problem

$$\frac{d\sigma(t, x_0)}{dt} = -\nabla\varphi_v(\sigma(t, x_0)) \quad t \geq 0,\ \sigma(0, x_0) = x_0. \qquad (33)$$

Note that E_v is Lipschitz continuous, hence so is $\nabla\varphi_v$ (see Claim 2). So, problem (33) has a unique global flow $\sigma(t, x_0)$, $t \geq 0$ (see Gasiński-Papageorgiou [7], Theorem

5.1.22, p. 618). We have

$$\frac{d\sigma(t,x_0)}{dt}+\sigma(t,x_0)=E_v(\sigma(t,x_0))\quad t\geq 0,\ \sigma(0,x_0)=x_0\quad \text{(see Claim 2)},$$
$$\Rightarrow\quad \sigma(t,x_0)=e^{-t}x_0+\int_0^t e^{-(t-s)}E_v(\sigma(t,x_0))ds\quad \text{for all } t\geq 0.$$

From (29) and Claim 1, we have

$$\sigma(t,C^1(\overline{\Omega}))\subseteq C^1(\overline{\Omega})\quad \text{for all } t\geq 0,$$
$$\sigma(t,x_0)\in \operatorname{int} C_+\quad \text{for all } t\geq 0,\ \text{all } x_0\in C_+\setminus\{0\}.$$

We introduce the following two sets

$$D_1^v=\{x_0\in C^1(\overline{\Omega}):\ \text{there exists } t_0>0 \text{ such that } \sigma(t,x_0)\in \operatorname{int}_{C^1(\overline{\Omega})}[v_-,u_+] \text{ for all } t\geq t_0\},$$
$$D_2^v=\{x_0\in C^1(\overline{\Omega}):\ \text{there exists } t^*>0 \text{ such that } \sigma(t,x_0)\in \operatorname{int} C_+ \text{ for all } t\geq t^*\}.$$

In principle the flow and the above sets depend on the frozen variable $v\in C^1(\overline{\Omega})$. However, if $v,v'\in C^1(\overline{\Omega})$ are close in the $C^1(\overline{\Omega})$-norm, then so are the flows $\sigma_v(t,x_0)$ and $\sigma_{v'}(t,x_0)$ in the $C^1(\overline{\Omega})$-norm uniformly in $t>0$. To see this suppose that $\|v-v'\|_{C^1(\overline{\Omega})}<\delta$. Then recalling that $\mathcal{L}(L^2(\Omega),C^1(\overline{\Omega}))$, (see (29)), we have

$$\begin{aligned}&\|\sigma_v(t,x_0)-\sigma_{v'}(t,x_0)\|_{C^1(\overline{\Omega})}\\ &\leq \int_0^t e^{-(t-s)}\|L\|_{\mathcal{L}}\|r\|_\infty\|v-v'\|_{C^1(\overline{\Omega})}ds\\ &+\int_0^t e^{-(t-s)}\|L\|_{\mathcal{L}}\|k\|_\infty\|\sigma_v(s,x_0)-\sigma_{v'}(s,x_0)\|_{C^1(\overline{\Omega})}ds.\end{aligned}$$

Invoking Gronwall's inequality, we obtain

$$\|\sigma_v(t,x_0)-\sigma_{v'}(t,x_0)\|_{C^1(\overline{\Omega})}\leq \delta\|L\|^2_{\mathcal{L}}\|r\|_\infty\|k\|_\infty.$$

So, from the definition of the sets D_1^v, D_2^v we see that we can find $\delta>0$ small such that $D_1^v=D_1^{v'}$ and $D_2^v=D_2^{v'}$ if $\|v-v'\|_{C^1(\overline{\Omega})}<\delta$.

Evidently both sets are open, positively σ-invariant, $0\in D_1$, $0\in\partial D_2$, $C_+\setminus\{0\}\subseteq D_2$. Moreover, the sets ∂D_1, ∂D_2 are positively σ-invariant (see Claim III in the proof of Theorem 2 of Papageorgiou-Papalini [17]). In addition Claim IV in that same proof of [17], says that there exists

$$\widehat{y}_v\in\partial D_1\cap\partial D_2\cap K_{\varphi_v}\subseteq C^1(\overline{\Omega}). \tag{34}$$

Since there is no canonical way to choose this critical point of φ_v we proceed via an iteration process. So, let $v_0\in H^1(\Omega)$ and $v_n=\widehat{y}_{v_{n-1}}\in C^1(\overline{\Omega})$ for all $n\in\mathbb{N}$. We have

$$A(v_n)+\xi(z)v_n=r(z)|\nabla v_{n-1}|+N_f(v_n) \text{ in } H^1(\Omega)^*,$$
$$A(v_{n+1})+\xi(z)v_{n+1}=r(z)|\nabla v_n|+N_f(v_{n+1}) \text{ in } H^1(\Omega)^*,\ n\in\mathbb{N}.$$

So, we obtain

$$\begin{aligned}&\langle A(v_n-v_{n+1}),v_n-v_{n+1}\rangle+\int_\Omega\xi(z)(v_n-v_{n+1})^2dz\\ &=\int_\Omega r(z)[|\nabla v_{n-1}|-|\nabla v_n|](v_n-v_{n+1})dz+\int_\Omega(f(z,v_n)-f(z,v_{n+1}))(v_n-v_{n+1})dz\end{aligned}$$

$$\Rightarrow \quad \gamma(v_n - v_{n+1}) \leq \|r\|_\infty \int_\Omega |\nabla(v_{n-1} - \nabla v_n)||v_n - v_{n+1}|dz + \int_\Omega k(z)|v_n - v_{n+1}|^2 dz$$

(see hypothesis $H(f)'(iv)$),

$$\Rightarrow \quad (1 - \|k\|_\infty)\gamma(v_n - v_{n+1}) \leq \|r\|_\infty \gamma(v_{n-1} - v_n)^{1/2}\gamma(v_n - v_{n+1})^{1/2},$$

$$\Rightarrow \quad \gamma(v_n - v_{n+1})^{1/2} \leq \frac{\|r\|_\infty}{1 - \|k\|_\infty}\gamma(v_{n-1} - v_n)^{1/2} \quad \text{for all } n \in \mathbb{N},$$

$$\Rightarrow \quad \{v_n\}_{n\geq 1} \subseteq H^1(\Omega) \text{ is Cauchy (see hypothesis } H(f)'(iv)).$$

It follows that

$$v_n \to \widehat{y} \text{ in } H^1(\Omega),$$

$$v_n \to \widehat{y} \text{ in } C^1(\overline{\Omega}) \quad \text{(by regularity theory, see also [8], p. 1464 or [19], p. 578)},$$

$$\Rightarrow \quad \widehat{y} \in \partial D_1 \cap \partial D_2 \quad \text{(see (34))},$$

$$\Rightarrow \quad \widehat{y} \neq 0 \quad \text{(recall that } 0 \in D_1 \text{ and } D_1 \text{ is open)}.$$

Also since $\widehat{y} \in \partial D_2$, we see that $\widehat{y} \notin [\operatorname{int} C_+ \cup (-\operatorname{int} C_+)]$. Moreover, $\widehat{y}$ is a solution of (1) and so by the nonlinear strong maximum principle of Pucci-Serrin [22] (pp. 111, 120), $\widehat{y}$ can not have fixed sign or otherwise we should have $\widehat{y} \in \operatorname{int} C_+ \cup (-\operatorname{int} C_+)$, a contradiction. Therefore $\widehat{y} \in C^1(\overline{\Omega})$ is nodal. ☐

So, we can state the following multiplicity theorem for problem (1).

Theorem 2. *If hypotheses $H(\xi)$, $H(r)$, $H(f)'$ hold, then problem* (1) *has at least three nontrivial solutions*

$$\widehat{u} \in D_+,\ \widehat{v} \in -D_+ \ \textit{and}\ \widehat{y} \in C^1(\overline{\Omega}) \ \textit{nodal.}$$

Consider the following parametric problem

$$-\Delta u(z) + \xi(z)u(z) = \lambda r(z)|\nabla u(z)| + f(z, u(z)) \quad \text{in } \Omega, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0,\ \lambda > 0. \tag{35}$$

In this case we make the following assumptions.

$H(r)'$: $r \in L^\infty(\Omega)$.

$H(f)''$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, hypotheses $H(f)''(i)$, (ii), (iii) are the same as the corresponding hypotheses $H(f)(i)$, (ii), (iii) and

(iv) $|f(z, x) - f(z, y)| \leq k(z)|x - y|$ for a.a. $z \in \Omega$, all $x, y \in \mathbb{R}$, with $k \in L^\infty(\Omega)$, $\|k\|_\infty < 1$ and there exists $\widehat{\xi} > 0$ such that for a.a. $z \in \Omega$, the function

$$x \to f(z, x) + \widehat{\xi}x$$

is nondecreasing.

Theorem 3. *If hypotheses $H(\xi)$, $H(r)'$, $H(f)''$ hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem* (35) *has at least three nontrivial solutions*

$$\widehat{u} \in D_+,\ \widehat{v} \in -D_+ \ \textit{and}\ \widehat{y} \in C^1(\overline{\Omega}) \ \textit{nodal.}$$

Remark 3. It is an open problem if we can have Theorems 2 and 3 for nonlinear equations driven by the p-Laplacian. The method of this paper seems to encounter serious difficulties in the case of p-Laplacian equations.

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References

[1] H. Amann and M. Crandall, *On some existence theorems for semilinear elliptic equations*, Indiana Univ. Math. J., **27** (1978), 779–790.

[2] Y. Bai, *Existence of solutions to nonhomogeneous Dirichlet problems with dependence on the gradient*, Electr. J. Diff. Equ., **2018**:101 (2018), 1–18.

[3] Y. Bai, L. Gasiński and N.S. Papageorgiou, *Nonlinear nonhomogeneous Robin problems with dependence on the gradient*, Bound. Value Probl., **2018**:17 (2018), 1–24

[4] K.C. Chang, *A nonlinear Krein-Rutman theorem*, J. Syst. Sci. Complex., **22** (2009), 542–554.

[5] F. Faraci, D. Motreanu and D. Puglisi, *Positive solutions of quasi-linear elliptic equations with dependence on the gradient*, Calc. Var., **54** (2015), 525–538.

[6] D. de Figueiredo, M. Girardi and M. Matzeu, *Semilinear elliptic equations with dependence on the gradient via mountain-pass techniques*, Differential Integral Equations, **17** (2004), 119–126.

[7] L. Gasiński and N.S. Papageorgiou, *Nonlinear Analysis*, Ser. Math. Anal. Appl., **9**, Chapman and Hall/CRC Press, Boca Raton, Florida (2006).

[8] L. Gasiński and N.S. Papageorgiou, *Positive solutions for nonlinear elliptic problems with dependence on the gradient*, J. Differential Equations, **263** (2017), 1451–1476.

[9] M. Girardi and M. Matzeu, *Positive and negative solutions of a quasi-linear elliptic equation by a mountain pass method and truncature techniques*, Nonlinear Anal., **59** (2004), 199–210.

[10] S. Hu and N.S. Papageorgiou, *Handbook of Multivalued Analysis. Vol. I. Theory*, Mathematics and its Applications, **419**, Kluwer Academic Publishers, Dordrecht, The Netherlands (1997).

[11] V.K. Le, *On some equivalent properties of sub- and supersolutions in second order quasilinear elliptic equations*, Hiroshima Math. J., **28** (1998), 373–380.

[12] G. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal., **12** (1988), 1203–1219.

[13] G. Liu, S. Shi and Y. Wei, *Sign-changing solutions for semilinear elliptic equations with dependence on the gradient via the Nehari method*, Mediterr. J. Math., **14** (2017), 1–12.

[14] S. Marano and N.S. Papageorgiou, *Positive solutions to a Dirichlet problem with p-Laplacian and concave-convex nonlinearity depending on a parameter*, Commun. Pure Appl. Anal., **12** (2013), 815–829.

[15] M. Matzeu and R. Servadei, *A variational approach to a class of quasilinear elliptic equations not in divergence form*, Discrete Contin. Dyn. Syst. Ser. S, **5** (2012), 819–831.

[16] D. Mugnai and N.S. Papageorgiou, *Resonant nonlinear Neumann problems with indefinite weight*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), Vol. XI (2012), 729-788.

[17] N.S. Papageorgiou and F. Papalini, *Seven solutions with sign information for sublinear equations with indefinite and unbounded potential and no symmetries*, Israel J. Math., **201** (2014), 761–796.

[18] N.S. Papageorgiou and V.D. Rădulescu, *Multiple solutions with precise sign for parametric Robin problems*, J. Differential Equations, **256** (2014), 2449–2479.

[19] N.S. Papageorgiou, V.D. Rădulescu and D. Repovš, *Positive solutions for nonlinear nonhomogeneous parametric Robin problems*, Forum Math., **30** (2018), 553–580.

[20] N.S. Papageorgiou, V.D. Rădulescu and D. Repovš, *Nonlinear elliptic inclusions with unilateral constraint and dependence on the gradient*, Appl. Math. Optim., **78** (2018), 1–23.

[21] N.S. Papageorgiou, C. Vetro and F. Vetro, *Existence of positive solutions for nonlinear Dirichlet problems with gradient dependence and arbitrary growth*, Electron. J. Qual. Theory Differ. Equ. **2018**:18 (2018), 1–9.

[22] P. Pucci and J. Serrin, *The Maximum Principle*, Birkhäuser, Basel (2007).

[23] D. Ruiz, *A priori estimates and existence of positive solutions for strongly nonlinear problems*, J. Differential Equations, **199** (2004), 96–114.

[24] P. Winkert, *L^∞-estimates for nonlinear elliptic Neumann boundary value problems*, NoDEA Nonlinear Differential Equations Appl., **17** (2010), 289–302.

[25] Z. Yan, *A note on the solvability in $W^{2,p}(\Omega)$ for the equation $-\Delta u = f(z, u, \nabla u)$*, Nonlinear Anal., **24** (1995), 1413–1416.

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