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A Perturbed Cauchy Viscoelastic Problem in an Exterior Domain

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Abstract: A Cauchy viscoelastic problem perturbed by an inverse-square potential, and posed in an exterior domain of $\mathbb{R}^N$, is considered under a Dirichlet boundary condition. Using nonlinear capacity estimates specifically adapted to the non-local nature of the problem, the potential function and the boundary condition, we establish sufficient conditions for the nonexistence of weak solutions.

Keywords: viscoelastic problem; hardy potential; exterior domain; nonexistence

MSC: 46F25; 46G05; 46N20; 35A01

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1. Introduction

Let B_1 be the open unit ball of $\mathbb{R}^N$ ($N \geq 2$). In this paper, we study the following perturbed viscoelastic equation:

$$u_{tt} - L_\lambda u + \frac{\delta}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L_\lambda u(s,x)\, ds + u_t = |u|^p, \quad t > 0,\, x \in \mathbb{R}^N \backslash B_1. \tag{1}$$

In this problem, the elliptic operator $L_\lambda = \Delta - \frac{\lambda}{|x|^2}$ for $\lambda \in \mathbb{R}$, is the Laplacian with Hardy perturbation, $u = u(t,x)$, $\Gamma(\cdot)$ is the gamma function, and we assume the following hypotheses on the data

$$\lambda \geq -\left(\frac{N-2}{2}\right)^2,\ \delta \neq 0,\ \alpha > 0,\ p > 1.$$

We are interested in the range of values of N, λ (see also the parameter λ_N in Equation (8)) and p for which Equation (1) does not admit weak solutions, under the Dirichlet boundary condition

$$u(t,x) = w(x), \quad t > 0,\, x \in \partial B_1, \tag{2}$$

where $w \in L^1(\partial B_1)$, and imposing the initial conditions

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^N \backslash B_1. \tag{3}$$

We shall discuss separately the cases $w = 0$ and $w \neq 0$. We now briefly recall some known results related to perturbed viscoelastic Equation (1). When $\delta = \lambda = 0$, Equation (1) reduces to the semilinear damped wave equation

$$u_{tt} - \Delta u + u_t = |u|^p, \quad t > 0,\, x \in \mathbb{R}^N \backslash B_1. \tag{4}$$

Ogawa-Takeda [1] investigated Equation (4) under the boundary condition (2) with $w = 0$. Hence, for compactly supported initial data, they showed that there is a non-negative global solution whenever $1 < p < 1 + \frac{2}{N}$. The approach used in [1] is based on the Kaplan–Fujita method (see the works of Kaplan [2] and Fujita [3] for more information). Later, Fino-Ibrahim-Wehbe [4] proved that the value $p = 1 + \frac{2}{N}$ belongs to the blow-up case. In a recent paper, Jleli-Samet [5] considered Equation (4) under the boundary condition (2) in the case when w is a non-negative nontrivial function. Hence, they obtained the following results:

(i) If $N = 2$, then for all $p > 1$, Equation (4) under the boundary condition (2) admits no global weak solution;
(ii) If $N \geq 3$, then for all $1 < p < 1 + \frac{2}{N-2}$, Equation (4) under the boundary condition (2) admits no global weak solution;
(iii) If $N \geq 3$ and $p > 1 + \frac{2}{N-2}$, then Equation (4) under the boundary condition (2) admits suitable solutions for some $w > 0$.

When dealing with problem (4) posed in the whole space $\mathbb{R}^N$, we mention the works of Kirane-Qafsaoui [6] (m-iterated Laplacian equation, $m \geq 1$), Todorova-Yordanov [7] and Zhang [8] (global existence, blow-up and asymptotic behavior of global solutions); see also the references therein.

On the other hand, the issue of nonexistence and blow-up in finite time for viscoelastic wave equations of the form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s,x)\,ds + h(u_t) = f(u), \quad t > 0,\, x \in \Omega \tag{5}$$

is present in many publications. We mention the works of Haraux-Zuazua [9] (hyperbolic problems), Kafini-Messaoudi [10,11] (nonlinear viscoelastic system and equation, respectively), and Messaoudi [12] (blow-up of solutions with negative initial energy). For instance, in [11], the authors investigated (5) in $\Omega = \mathbb{R}^N$ with $h(u_t) = u_t$ and $f(u) = |u|^{p-1}u$. Namely, under a certain condition on the kernel function g, it was shown that, if $1 \leq p < \frac{N}{N-2}$, $N \geq 3$; or $p \geq 1$, $N = 1, 2$, and

$$\int_0^t g(s)\,ds < \frac{2p-2}{2p-1},$$

then for any initial data $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ with compact support, satisfying

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{p+1}\|u_0\|_{p+1}^{p+1} \leq 0,$$
$$\int_{\mathbb{R}^N} u_0(x)u_1(x)\,dx \geq 0,$$

the corresponding solution blows up in finite time. We point out that the approach in [11] is based on the energy method.

Now we recall some references in the literature on evolution equations and inequalities perturbed by the Hardy potential $\frac{\lambda}{|x|^2}$. We refer to the works of Abdellaoui-Miri-Peral-Touaoula [13] (p-Laplacian equation), Abdellaoui-Peral-Primo [14,15] (Laplacian equations), Jleli-Samet-Vetro [16] (inhomogeneous wave inequalities) and again the work of Haraux-Zuazua [9]. However, to the best of our knowledge, problems of type (1) have not been previously studied in the literature. The motivation to consider Equation (1) originates from the idea to combine the effects of viscoelastic behavior and singular Hardy potential into a single wave equation. Referring to a physical context, viscoelastic materials (i.e., polymers) exhibit both the behavior of a liquid (viscous case) and of a solid (elastic case). For instance, first a suitable tension produces some elastic deformation, then (time-dependent) viscous stress occurs, hence there are material properties leading to so-called memory effects. Now, the degree of viscoelasticity can be controlled by a parameter varying in an appropriate range (see also Chapter 7 of Mills-Jenkins-Kukureka [17]). From a mathematical perspective, the effects of memory are linked to the kernel function in the integral term of the equation (i.e., the function g in (5)), hence it is interesting to show the behavior of solutions to classes of viscoelastic wave equations, under minimal (or specific) assumptions on g. We mention, for example, the works of Cavalcanti et al. [18] (using the multiplier method together with a lemma about convergent and divergent series for establishing the uniform decay of the energy of the solution) and Wu [19] (using the perturbed energy technique for establishing the uniform decay of the energy of the solution to the system of viscoelastic wave equations). On the other hand, the singular Hardy potential is recognized as a suitable prototype to analyze the critical behavior of different nonlinear problems in physics, hence in dealing with the existence and stability of solutions (for more details and information, we refer to the comprehensive book of Alonso-de Diego [20]). It makes more sense to study how the parametric Hardy potential $\frac{\lambda}{|x|^2}$ for $\lambda \in \mathbb{R}$, affects instantaneous and complete blow-up of solutions to (1) (i.e., nonexistence phenomenon).

In order to define weak solutions to (1) under conditions (2) and (3), we recall below some notions from fractional calculus (see the comprehensive book of Kilbas-Srivastava-Trujillo [21] for more details), hence we fix notation.

Let $T > 0$ be fixed. Given $f \in L^1([0,T])$ and $\beta > 0$, the left-sided and right-sided Riemann–Liouville fractional integrals of order β of f, are defined, respectively, by

$$I_0^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\, ds$$

and

$$I_T^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} f(s)\, ds$$

for almost everywhere $t \in [0,T]$. The following property can be found in Kilbas-Srivastava-Trujillo ([21], Lemma 2.7).

Lemma 1. *Let $\beta > 0$, $m, q \geq 1$, and $\frac{1}{m} + \frac{1}{q} \leq 1 + \beta$ ($m \neq 1$ and $q \neq 1$ if $\frac{1}{m} + \frac{1}{q} = 1 + \beta$). If $f \in L^m([0,T])$ and $g \in L^q([0,T])$, then*

$$\int_0^T g(t) I_0^\beta f(t)\, dt = \int_0^T f(t) I_T^\beta g(t)\, dt.$$

Let $F : [0,T] \times \mathbb{R}^N \backslash B_1 \to \mathbb{R}$ be a given function. The left-sided and right-sided Riemann–Liouville fractional integrals of order $\beta > 0$ of F with respect to the time-variable t, are denoted, respectively, by $I_0^\beta F$ and $I_T^\beta F$, namely we have

$$I_0^\beta F(t,x) = I_0^\beta F(\cdot,x)(t)$$

and

$$I_T^\beta F(t,x) = I_T^\beta F(\cdot,x)(t).$$

Using the above notations, the nonlocal term in Equation (1) can be written in the form

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L_\lambda u(s,x)\, ds = I_0^\alpha L_\lambda u(t,x). \tag{6}$$

For some contributions related to the applications of fractional derivative operators in diffusion processes, we mention the works of Hurtado-Salvatierra [22], Jleli [23], Nashine et al. [24], Villagran et al. [25], and the references therein. Precisely, [22] deals with non-local diffusion problems driven by the fractional p-Laplacian differential operator in the Heisenberg group. The approach is based on the theory of monotone operators and pullback attractors. In [23], the author investigates the existence of solutions to a Dirichlet problem for the Kohn Laplacian on the Heisenberg group too, using partial ordering methods. In [24], the authors study generalized fractional integral equations, using fixed-point arguments in Banach spaces. In [25], the authors investigate stability for a system of wave equations. They establish well-posedness and polynomial stability using the semigroup theory and certain sharp results. For all $T > 0$, we denote

$$Q_T = [0,T] \times \mathbb{R}^N \backslash B_1, \quad \Gamma_T = [0,T] \times \partial B_1,$$

hence, $\Gamma_T \subset Q_T$. Let Φ_T be the set of functions $\varphi = \varphi(t,x)$ satisfying the following properties:
(P_1) $\varphi \in C^2(Q_T)$, $\mathrm{supp}_x(\varphi) \subset\subset \mathbb{R}^N \backslash B_1$, $\varphi \geq 0$;
(P_2) $\varphi(T,\cdot) = \varphi_t(T,\cdot) = 0$;
(P_3) $\varphi_{|\Gamma_T} = 0$.

Using standard integrations by parts, together with Lemma 1 and (6), we define weak solutions to problem (1)–(3) as follows.

Definition 1. *Let $u_0, u_1 \in L^1_{\mathrm{loc}}(\mathbb{R}^N \backslash B_1)$ and $w \in L^1(\partial B_1)$. We say that*

$$u \in L^p_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^N \backslash B_1)$$

is a weak solution to (1)–(3) if

$$\begin{aligned}
&\int_{Q_T} |u|^p \varphi \, dx\, dt + \int_{\mathbb{R}^N \backslash B_1} u_0(x)(\varphi(0,x) - \varphi_t(0,x))\, dx \\
&+ \int_{\mathbb{R}^N \backslash B_1} u_1(x)\varphi(0,x)\, dx - \int_{\Gamma_T} \left(\frac{\partial \varphi}{\partial \nu} - \delta \frac{\partial (I^\alpha_T \varphi)}{\partial \nu} \right) w(x)\, dS_x\, dt \\
&= \int_{Q_T} u(\varphi_{tt} - L_\lambda \varphi + \delta L_\lambda (I^\alpha_T \varphi) - \varphi_t)\, dx\, dt
\end{aligned} \tag{7}$$

for every $T > 0$ and $\varphi \in \Phi_T$, where ν denotes the outward unit normal vector on ∂B_1, relative to $\mathbb{R}^N \backslash B_1$.

For $\lambda \geq -\left(\frac{N-2}{2}\right)^2$, we introduce the parameter

$$\lambda_N = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda} \tag{8}$$

and the truncation function

$$H_\lambda(x) = \begin{cases} |x|^{\lambda_N}\left(1 - |x|^{2-N-2\lambda_N}\right) & \text{if} \quad \lambda > -\left(\frac{N-2}{2}\right)^2, \\ |x|^{\lambda_N} \ln|x| & \text{if} \quad \lambda = -\left(\frac{N-2}{2}\right)^2. \end{cases} \tag{9}$$

Our aim in this work is to establish sufficient conditions for the nonexistence of weak solutions to problem (1)–(3). Therefore, we need to find ways to deal with the nonlocal nature of the problem, the elliptic operator L_λ and the boundary condition (2). We come up with an approach based on nonlinear capacity estimates specifically adapted to our needs.

The rest of the paper is organized as follows. In Section 2, we obtain some preliminary estimates. Namely, we first prove an a priori estimate for problem (1)–(3), then we construct a family of test functions belonging to the functional space Φ_T, and provide some useful estimates involving such functions. In Section 3, we provide the proofs of Theorems 1 and 2.

2. Preliminaries

In this section, we give the mathematical background necessary to establish our results. Here, the symbols C, C_i denote always generic positive constants, which are independent of the scaling parameters T, R and the solution u. Their values could be changed from one line to another. First, we impose the following hypotheses on the data: $N \geq 2$, $\lambda \geq -\left(\frac{N-2}{2}\right)^2$, $\alpha > 0$, $\delta \neq 0$, $p > 1$, $u_0, u_1 \in L^1_{\mathrm{loc}}(\mathbb{R}^N \backslash B_1)$ and $w \in L^1(\partial B_1)$.

2.1. A Priori Estimate

For $T > 0$ and $\varphi \in \phi_T$, we introduce the integral terms

$$J_1(\varphi) = \int_{\mathrm{supp}(\varphi)} \varphi^{\frac{-1}{p-1}} |\varphi_t|^{\frac{p}{p-1}} \, dx \, dt, \tag{10}$$

$$J_2(\varphi) = \int_{\mathrm{supp}(\varphi)} \varphi^{\frac{-1}{p-1}} |\varphi_{tt}|^{\frac{p}{p-1}} \, dx \, dt, \tag{11}$$

$$J_3(\varphi) = \int_{\mathrm{supp}(\varphi)} \varphi^{\frac{-1}{p-1}} |L_\lambda \varphi|^{\frac{p}{p-1}} \, dx \, dt, \tag{12}$$

$$J_4(\varphi) = \int_{\mathrm{supp}(\varphi)} \varphi^{\frac{-1}{p-1}} |L_\lambda (I^\alpha_T \varphi)|^{\frac{p}{p-1}} \, dx \, dt. \tag{13}$$

Hence, we establish the following a priori estimate.

Lemma 2. *If $u \in L^p_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^N \backslash B_1)$ is a weak solution to problem (1)–(3), then we have the estimate*

$$\begin{aligned} &\int_{\mathbb{R}^N \backslash B_1} u_0(x)(\varphi(0,x) - \varphi_t(0,x)) \, dx + \int_{\mathbb{R}^N \backslash B_1} u_1(x)\varphi(0,x) \, dx \\ &- \int_{\Gamma_T} \left(\frac{\partial \varphi}{\partial \nu} - \delta \frac{\partial (I^\alpha_T \varphi)}{\partial \nu} \right) w(x) \, dS_x \, dt \leq C \sum_{i=1}^{4} J_i(\varphi) \end{aligned} \tag{14}$$

for every $T > 0$ and $\varphi \in \Phi_T$, provided that $J_i(\varphi) < \infty$, $j = 1,2,3,4$.

Proof. Let $u \in L^p_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^N \backslash B_1)$ be a weak solution to problem (1)–(3). For $T > 0$ and $\varphi \in \Phi_T$ satisfying the conditions $J_i(\varphi) < \infty$, $j = 1,2,3,4$, using (7) (i.e., Definition 1), we obtain

$$\begin{aligned} &\int_{Q_T} |u|^p \varphi \, dx \, dt + \int_{\mathbb{R}^N \backslash B_1} u_0(x)(\varphi(0,x) - \varphi_t(0,x)) \, dx \\ &+ \int_{\mathbb{R}^N \backslash B_1} u_1(x)\varphi(0,x) \, dx - \int_{\Gamma_T} \left(\frac{\partial \varphi}{\partial \nu} - \delta \frac{\partial (I^\alpha_T \varphi)}{\partial \nu} \right) w(x) \, dS_x \, dt \\ &\leq \int_{Q_T} |u||\varphi_t| \, dx \, dt + \int_{Q_T} |u||\varphi_{tt}| \, dx \, dt \\ &+ \int_{Q_T} |u||L_\lambda \varphi| \, dx \, dt + |\delta| \int_{Q_T} |u||L_\lambda (I^\alpha_T \varphi)| \, dx \, dt. \end{aligned} \tag{15}$$

Considering each one of the integrals on the right hand side of (15) separately, by means of Young's inequality, we first obtain

$$\begin{aligned} \int_{Q_T} |u||\varphi_t| \, dx \, dt = & \int_{Q_T} \left(|u|\varphi^{\frac{1}{p}} \right) \left(\varphi^{\frac{-1}{p}} |\varphi_t| \right) dx \, dt \\ \leq & \frac{1}{4} \int_{Q_T} |u|^p \varphi \, dx \, dt + C J_1(\varphi), \end{aligned} \tag{16}$$

and similarly we obtain the other bounds

$$\int_{Q_T} |u||\varphi_{tt}| \, dx \, dt \leq \frac{1}{4} \int_{Q_T} |u|^p \varphi \, dx \, dt + C J_2(\varphi), \tag{17}$$

$$\int_{Q_T} |u||L_\lambda \varphi| \, dx \, dt \leq \frac{1}{4} \int_{Q_T} |u|^p \varphi \, dx \, dt + C J_3(\varphi), \tag{18}$$

$$|\delta| \int_{Q_T} |u||L_\lambda (I^\alpha_T \varphi)| \, dx \, dt \leq \frac{1}{4} \int_{Q_T} |u|^p \varphi \, dx \, dt + C J_4(\varphi). \tag{19}$$

After combining together inequalities (16)–(19) with the principal inequality (15), we obtain the desired estimate in (14). □

2.2. Construction of a Family of Functions Belonging to Φ_T

Let $T > 0$. For sufficiently large ℓ, we introduce the function

$$\iota_T(t) = T^{-\ell}(T-t)^{\ell}, \quad 0 \leq t \leq T. \tag{20}$$

Next, for sufficiently large R, we consider a family of cut-off functions $\{\xi_R\}_R$ fulfilling the following properties:

$$\begin{aligned} &0 \leq \xi_R \leq 1, \quad \xi_R \in C^\infty(\mathbb{R}^N \backslash B_1), \\ &\xi_R = 1 \text{ if } 1 \leq |x| \leq R, \quad \xi_R = 0 \text{ if } |x| \geq 2R, \end{aligned} \tag{21}$$

but also

$$|\nabla \xi_R| \leq C R^{-1}, \quad |\Delta \xi_R| \leq C R^{-2}. \tag{22}$$

We now introduce a test function of the form

$$\varphi(t,x) = \iota_T(t)\psi_R(x), \quad (t,x) \in Q_T, \tag{23}$$

where

$$\psi_R(x) = H_\lambda(x)\xi_R^\ell(x), \quad x \in \mathbb{R}^N \backslash B_1 \tag{24}$$

and $H_\lambda(\cdot)$ is the truncation in (9).

We immediately find that the test function in (23) possesses the properties (P_1)–(P_3), hence $\varphi \in \Phi_T$.

Lemma 3. *For sufficiently large T, R and ℓ, the function φ defined by* (23) *belongs to Φ_T.*

Proof. Property (P_1) follows immediately from (9) and (20)–(24). Moreover, by (9) and (20), one has

$$\iota_T(T) = \iota_T'(T) = 0, \quad H_{\lambda|\partial B_1} = 0,$$

which shows by (23) that properties (P_2) and (P_3) are satisfied too. □

2.3. Estimates of $J_i(\varphi)$

Now we consider again the integral terms (10)–(13) to establish their bounds from above. For sufficiently large T, R and ℓ, let φ be the test function defined by (23). The first result holds for $J_i(\varphi)$, with $j = 1, 2$, and is given in the following lemma.

Lemma 4. *For $i = 1, 2$ and λ_N given in* (8)*, we have*

$$J_i(\varphi) \leq C T^{1-\frac{ip}{p-1}} R^{\lambda_N + N} \ln R. \tag{25}$$

Proof. Starting from the formulas (10) and (11), and involving the cut-off function (21) and the test function (23) (recall (24) too), for $i = 1, 2$, we obtain

$$J_i(\varphi) = \left(\int_0^T \iota_T^{\frac{-1}{p-1}}(t)|\iota_T^{(i)}(t)|^{\frac{p}{p-1}}\,dt\right)\left(\int_{1<|x|<2R} H_\lambda(x)\xi_R^\ell(x)\,dx\right), \tag{26}$$

where $\iota_T^{(i)} = \dfrac{d^i \iota_T}{dt^i}$. On the other hand, by (20), for all $0 < t < T$ one has

$$|\iota_T^{(i)}(t)| = C T^{-\ell}(T-t)^{\ell-i},$$

which yields

$$\iota_T^{\frac{-1}{p-1}}(t)|\iota_T^{(i)}(t)|^{\frac{p}{p-1}} = C T^{-\ell}(T-t)^{\ell-\frac{ip}{p-1}}.$$

Integrating over $(0,T)$ this equation, we obtain

$$\int_0^T \iota_T^{\frac{-1}{p-1}}(t)|\iota_T^{(i)}(t)|^{\frac{p}{p-1}}\,dt = CT^{1-\frac{ip}{p-1}}. \tag{27}$$

Moreover, we use the truncation (9) together with the appropriate property in (21) (namely $0 \le \xi_R \le 1$) to deduce that

$$H_\lambda(x)\xi_R^\ell(x) \le H_\lambda(x) \le |x|^{\lambda_N}\ln|x|, \quad 1<|x|<2R.$$

Integrating over $(1,2R)$, we obtain (notice that $\lambda_N + N - 1 > 0$)

$$\begin{aligned}\int_{1<|x|<2R} H_\lambda(x)\xi_R^\ell(x)\,dx &\le \int_{1<|x|<2R}|x|^{\lambda_N}\ln|x| \\ &\le \ln R\int_{r=1}^{2R} r^{\lambda_N+N-1}\,dr \\ &\le C\ln R\,R^{\lambda_N+N-1}R \\ &= CR^{\lambda_N+N}\ln R.\end{aligned} \tag{28}$$

Combining (26)–(28), we conclude the estimate (25). □

The next result follows by elementary calculations, hence we avoid the proof of this lemma.

Lemma 5. *The function H_λ defined by (9) satisfies the following property:*

$$L_\lambda H_\lambda(x) = 0, \quad x \in \mathbb{R}^N \backslash B_1.$$

Now, we consider the integral term $J_3(\varphi)$ and establish the following estimate.

Lemma 6. *The following estimate holds:*

$$J_3(\varphi) \le CTR^{\lambda_N+N-\frac{2p}{p-1}}(\ln R)^{\frac{p}{p-1}}. \tag{29}$$

Proof. Starting from the formula (12), and involving the cut-off function (21) and the test function (23), we obtain

$$J_3(\varphi) = \left(\int_0^T \iota_T(t)\,dt\right)\left(\int_{1<|x|<2R}\psi_R^{\frac{-1}{p-1}}|L_\lambda\psi_R|^{\frac{p}{p-1}}\,dx\right). \tag{30}$$

On the other hand, by (20), one has

$$\int_0^T \iota_T(t)\,dt = T^{-\ell}\int_0^T (T-t)^\ell\,dt = CT. \tag{31}$$

Next, by (24), for $1<|x|<2R$, we obtain

$$\begin{aligned}L_\lambda\psi_R(x) &= L_\lambda(H_\lambda\xi_R^\ell)(x) \\ &= \Delta(H_\lambda(x)\xi_R^\ell(x)) - \frac{\lambda}{|x|^2}H_\lambda(x)\xi_R^\ell(x) \\ &= \xi_R^\ell(x)\Delta H_\lambda(x) + H_\lambda(x)\Delta(\xi_R^\ell(x)) + 2\nabla H_\lambda(x)\cdot\nabla(\xi_R^\ell(x)) \\ &\quad -\frac{\lambda}{|x|^2}H_\lambda(x)\xi_R^\ell(x) \\ &= \xi_R^\ell(x)L_\lambda H_\lambda(x) + H_\lambda(x)\Delta(\xi_R^\ell(x)) + 2\nabla H_\lambda(x)\cdot\nabla(\xi_R^\ell(x)),\end{aligned}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^N$. Then, by Lemma 5, we deduce that

$$L_\lambda\psi_R(x) = H_\lambda(x)\Delta(\xi_R^\ell(x)) + 2\nabla H_\lambda(x)\cdot\nabla(\xi_R^\ell(x)), \tag{32}$$

which implies, by (21), that

$$\int_{1<|x|<2R} \psi_R^{\frac{-1}{p-1}} |L_\lambda \psi_R|^{\frac{p}{p-1}}\,dx = \int_{R<|x|<2R} \psi_R^{\frac{-1}{p-1}} |L_\lambda \psi_R|^{\frac{p}{p-1}}\,dx. \tag{33}$$

On the other hand, by using the truncation (9), we obtain

$$\begin{aligned} &C_1 R^{\lambda_N} \le H_\lambda(x) \le C_2 R^{\lambda_N} \ln R, \\ &|\nabla H_\lambda(x)| \le C R^{\lambda_N - 1} \ln R, \quad R < |x| < 2R. \end{aligned} \tag{34}$$

Moreover, using the properties (22) together with

$$\Delta \xi_R^\ell = \ell(\ell-1)\xi_R^{\ell-2}|\nabla \xi_R|^2 + \ell \xi_R^{\ell-1}\Delta \xi_R,$$

we deduce that (recall $0 \le \xi_R \le 1$)

$$\begin{aligned} &|\Delta(\xi_R^\ell(x))| \le C R^{-2} \xi_R^{\ell-2}(x), \\ &|\nabla(\xi_R^\ell(x))| \le C R^{-1} \xi_R^{\ell-2}(x), \quad R < |x| < 2R. \end{aligned} \tag{35}$$

Hence, by Cauchy–Schwarz inequality, it follows from (32), (34) and (35) that

$$|L_\lambda \psi_R(x)| \le C R^{\lambda_N - 2} \ln R\, \xi_R^{\ell-2}(x), \quad R < |x| < 2R. \tag{36}$$

Furthermore, using (24), (34) and (36), we obtain

$$\psi_R^{\frac{-1}{p-1}}(x) |L_\lambda \psi_R(x)|^{\frac{p}{p-1}} \le C R^{\lambda_N - \frac{2p}{p-1}} (\ln R)^{\frac{p}{p-1}} \xi_R^{\ell - \frac{2p}{p-1}}(x), \quad R < |x| < 2R.$$

Integrating appropriately this inequality and using (33), we obtain

$$\begin{aligned} \int_{1<|x|<2R} \psi_R^{\frac{-1}{p-1}} |L_\lambda \psi_R|^{\frac{p}{p-1}}\,dx &\le C R^{\lambda_N - \frac{2p}{p-1}} (\ln R)^{\frac{p}{p-1}} \int_{R<|x|<2R} \xi_R^{\ell - \frac{2p}{p-1}}(x)\,dx \\ &\le C R^{N + \lambda_N - \frac{2p}{p-1}} (\ln R)^{\frac{p}{p-1}}. \end{aligned} \tag{37}$$

Finally, (29) follows from (30), (31) and (37). Hence, the estimate of $J_3(\varphi)$ is reached. □

Now, we consider the integral term $J_4(\varphi)$ and prove the following result.

Lemma 7. *The following estimate holds:*

$$J_4(\varphi) \le C T^{1 + \frac{\alpha p}{p-1}} R^{\lambda_N + N - \frac{2p}{p-1}} (\ln R)^{\frac{p}{p-1}}. \tag{38}$$

Proof. Starting from the formula (13), and using the test function (23), we obtain

$$J_4(\varphi) = \left(\int_0^T \iota_T^{\frac{-1}{p-1}}(t) |I_T^\alpha \iota_T(t)|^{\frac{p}{p-1}}\,dt \right) \left(\int_{1<|x|<2R} \psi_R^{\frac{-1}{p-1}} |L_\lambda \psi_R|^{\frac{p}{p-1}}\,dx \right). \tag{39}$$

On the other hand, by (20), for all $0 < t < T$, one has

$$\begin{aligned} I_T^\alpha \iota_T(t) &= \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \iota_T(s)\,ds \\ &= \frac{T^{-\ell}}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} (T-s)^\ell\,ds \\ &= \frac{T^{-\ell}}{\Gamma(\alpha)} \int_t^T ((T-t)-(T-s))^{\alpha-1} (T-s)^\ell\,ds \\ &= \frac{T^{-\ell}(T-t)^{\alpha-1}}{\Gamma(\alpha)} \int_t^T \left(1 - \frac{T-s}{T-t}\right)^{\alpha-1} (T-s)^\ell\,ds. \end{aligned}$$

Then, by the change of variable $\vartheta = \frac{T-s}{T-t}$, we obtain

$$\begin{aligned} I_T^\alpha \iota_T(t) &= \frac{T^{-\ell}(T-t)^{\sigma+\ell}}{\Gamma(\alpha)} \int_0^1 (1-\vartheta)^{\alpha-1} \vartheta^{(\ell+1)-1}\, d\vartheta \\ &= \frac{T^{-\ell}(T-t)^{\alpha+\ell}}{\Gamma(\alpha)} B(\alpha, \ell+1), \end{aligned} \tag{40}$$

where $B(\cdot,\cdot)$ is the Beta function. Hence, there holds

$$\iota_T^{\frac{-1}{p-1}}(t) |I_T^\alpha \iota_T(t)|^{\frac{p}{p-1}} = C T^{-\ell} (T-t)^{\ell + \frac{\alpha p}{p-1}}.$$

Integrating this equation over $(0, T)$, we obtain

$$\int_0^T \iota_T^{\frac{-1}{p-1}}(t) |I_T^\alpha \iota_T(t)|^{\frac{p}{p-1}}\, dt = C T^{1+\frac{\alpha p}{p-1}}. \tag{41}$$

Combining (37), (39) and (41), we conclude the estimate (38). □

3. Main Results

Our main results are stated in the following theorems. As already mentioned, we first consider the case $w = 0$.

Theorem 1. *Let $N \geq 2$, $\lambda \geq -\left(\frac{N-2}{2}\right)^2$, $\alpha > 0$, $\delta \neq 0$, $w = 0$ and $u_0, u_1 \in L^1_{\mathrm{loc}}(\mathbb{R}^N \backslash B_1)$. If*

$$(u_0 + u_1) H_\lambda \in L^1\left(\mathbb{R}^N \backslash B_1\right), \quad \int_{\mathbb{R}^N \backslash B_1} (u_0(x) + u_1(x)) H_\lambda(x)\, dx > 0, \tag{42}$$

and

$$1 < p < p^*(\alpha, \lambda, N) := 1 + \frac{2}{(\alpha+1)(\lambda_N + N)}, \tag{43}$$

then problem (1)–(3) admits no weak solution.

Proof. We use the contradiction argument. Namely, let us suppose that $u \in L^p_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^N \backslash B_1)$ is a weak solution to problem (1)–(3). Then, by Lemma 2 (with $w = 0$) and Lemma 3, for sufficiently large T, R and ℓ, we have

$$\int_{\mathbb{R}^N \backslash B_1} [u_0(x)(\varphi(0,x) - \varphi_t(0,x)) + u_1(x) \varphi(0,x)]\, dx \leq C \sum_{i=1}^4 J_i(\varphi), \tag{44}$$

where φ is the function defined by (23). On the other hand, by (23), for all $x \in \mathbb{R}^N \backslash B_1$, one has

$$\begin{aligned} \varphi(0,x) &= \iota_T(0) \psi_R(x) \\ &= \psi_R(x) \\ &= H_\lambda(x) \xi_R^\ell(x), \end{aligned}$$

as well as

$$\begin{aligned} \varphi_t(0,x) &= \iota_T'(0) \psi_R(x) \\ &= -\ell T^{-1} H_\lambda(x) \xi_R^\ell(x). \end{aligned}$$

Hence, we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus B_1}[u_0(x)(\varphi(0,x)-\varphi_t(0,x))+u_1(x)\varphi(0,x)]\,dx\\&=\int_{\mathbb{R}^N\setminus B_1}\Big[u_0(x)\Big(1+\ell T^{-1}\Big)+u_1(x)\Big]H_\lambda(x)\xi_R^\ell(x)\,dx.\end{aligned}\tag{45}$$

Using Lemmas 4, 6 and 7, together with (44) and (45), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus B_1}\Big[u_0(x)\Big(1+\ell T^{-1}\Big)+u_1(x)\Big]H_\lambda(x)\xi_R^\ell(x)\,dx\\&\leq C\left(T^{1-\frac{p}{p-1}}R^{\lambda_N+N}\ln R+T^{1+\frac{\alpha p}{p-1}}R^{\lambda_N+N-\frac{2p}{p-1}}(\ln R)^{\frac{p}{p-1}}\right).\end{aligned}\tag{46}$$

Taking $T=R^\theta$, where

$$\theta=\frac{2}{\alpha+1},\tag{47}$$

one obtains

$$T^{1-\frac{p}{p-1}}R^{\lambda_N+N}=T^{1+\frac{\alpha p}{p-1}}R^{\lambda_N+N-\frac{2p}{p-1}}=R^\kappa,$$

with

$$\kappa=\lambda_N+N-\frac{2}{(\alpha+1)(p-1)}.$$

Consequently, the inequality (46) reduces to the following one

$$\int_{\mathbb{R}^N\setminus B_1}\Big[u_0(x)\Big(1+\ell R^{-\theta}\Big)+u_1(x)\Big]H_\lambda(x)\xi_R^\ell(x)\,dx\leq CR^\kappa\Big(\ln R+(\ln R)^{\frac{p}{p-1}}\Big).\tag{48}$$

The hypotheses (42) and properties (21) of cut-off functions, together with the dominated convergence theorem, lead to

$$\begin{aligned}&\lim_{R\to\infty}\int_{\mathbb{R}^N\setminus B_1}\Big[u_0(x)\Big(1+\ell R^{-\theta}\Big)+u_1(x)\Big]H_\lambda(x)\xi_R^\ell(x)\,dx\\&=\int_{\mathbb{R}^N\setminus B_1}(u_0(x)+u_1(x))H_\lambda(x)\,dx>0.\end{aligned}\tag{49}$$

Moreover, the hypothesis (43) gives us

$$\kappa<0.\tag{50}$$

Next, passing to the limit as $R\to\infty$ in (48), by the limit in (49) and the sign condition (50), we obtain the following contradiction:

$$0<\int_{\mathbb{R}^N\setminus B_1}(u_0(x)+u_1(x))H_\lambda(x)\,dx\leq 0.$$

It follows that problems (1)–(3) admit no weak solutions. This completes the proof of Theorem 1. □

Following similar arguments of proof, in the inhomogeneous case $w\neq 0$, we conclude the following theorem.

Theorem 2. *Let* $N\geq 2$, $\lambda\geq-\left(\frac{N-2}{2}\right)^2$, $\alpha>0$, $\delta\neq 0$, $w\in L^1(\partial B_1)$ *and* $u_0,u_1\in L^1_{loc}(\mathbb{R}^N\setminus B_1)$, $u_0,u_1\geq 0$. *Assuming*

$$\delta\int_{\partial B_1}w(x)\,dS_x<0,\tag{51}$$

we have the following:

(I) *If* $\lambda_N + N - 2 = 0$, *then for all* $p > 1$, *problems* (1)–(3) *admit no weak solutions.*

(II) *If* $\lambda_N + N - 2 \neq 0$, *then for all*

$$1 < p < p^{**}(\alpha, \lambda, N) := 1 + \frac{2}{(\alpha+1)(\lambda_N + N - 2)} \tag{52}$$

problems (1)–(3) *admit no weak solutions.*

Proof. We use the contradiction argument. Let us suppose that $u \in L^p_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^N \backslash B_1)$ is a weak solution to the problem (1)–(3). Then, by Lemmas 2 and 3, for sufficiently large T, R and ℓ, we have

$$\begin{aligned} &\int_{\mathbb{R}^N \backslash B_1} u_0(x)(\varphi(0,x) - \varphi_t(0,x))\,dx + \int_{\mathbb{R}^N \backslash B_1} u_1(x)\varphi(0,x)\,dx \\ &- \int_{\Gamma_T} \left(\frac{\partial \varphi}{\partial \nu} - \delta \frac{\partial (I_T^\alpha \varphi)}{\partial \nu} \right) w(x)\,dS_x\,dt \leq C \sum_{i=1}^4 J_i(\varphi), \end{aligned}$$

where φ is the function defined by (23). On the other hand, since $u_0, u_1 \geq 0$, it follows from (45) that

$$\int_{\mathbb{R}^N \backslash B_1} u_0(x)(\varphi(0,x) - \varphi_t(0,x))\,dx + \int_{\mathbb{R}^N \backslash B_1} u_1(x)\varphi(0,x)\,dx \geq 0.$$

Then, we have

$$- \int_{\Gamma_T} \left(\frac{\partial \varphi}{\partial \nu} - \delta \frac{\partial (I_T^\alpha \varphi)}{\partial \nu} \right) w(x)\,dS_x\,dt \leq C \sum_{i=1}^4 J_i(\varphi). \tag{53}$$

Further, by using (21) and (23), we obtain

$$\begin{aligned} &- \int_{\Gamma_T} \left(\frac{\partial \varphi}{\partial \nu} - \delta \frac{\partial (I_T^\alpha \varphi)}{\partial \nu} \right) w(x)\,dS_x\,dt \\ &= \left(\int_0^T \left(I_T^\alpha \iota_T(t) - \frac{1}{\delta}\iota_T(t) \right) dt \right) \left(\int_{\partial B_1} \frac{\partial H_\lambda}{\partial \nu}(x) \delta w(x)\,dS_x \right). \end{aligned} \tag{54}$$

Involving (20) and (40), for all $0 < t < T$, we deduce that

$$I_T^\alpha \iota_T(t) - \frac{1}{\delta}\iota_T(t) = \frac{B(\alpha, \ell+1)}{\Gamma(\alpha)} T^{-\ell}(T-t)^{\alpha+\ell} - \frac{T^{-\ell}}{\delta}(T-t)^\ell.$$

Integrating this equation over $(0, T)$, we obtain

$$\int_0^T \left(I_T^\alpha \iota_T(t) - \frac{1}{\delta}\iota_T(t) \right) dt = T^{\alpha+1} \left(\frac{B(\alpha, \ell+1)}{\Gamma(\alpha)(\alpha+\ell+1)} - \frac{1}{\delta(\ell+1)} T^{-\alpha} \right).$$

Since

$$\lim_{T \to \infty} \frac{B(\alpha, \ell+1)}{\Gamma(\alpha)(\alpha+\ell+1)} - \frac{1}{\delta(\ell+1)} T^{-\alpha} = \frac{B(\alpha, \ell+1)}{\Gamma(\alpha)(\alpha+\ell+1)} > 0,$$

then, for sufficiently large T, one has

$$\int_0^T \left(I_T^\alpha \iota_T(t) - \frac{1}{\delta}\iota_T(t) \right) dt \geq C T^{\alpha+1}. \tag{55}$$

On the other hand, with respect to the truncation function (9), for all $x \in \partial B_1$, we obtain

$$\frac{\partial H_\lambda}{\partial \nu}(x) = \begin{cases} 2 - N - 2\lambda_N < 0 & \text{if} \quad \lambda > -\left(\frac{N-2}{2}\right)^2, \\ -1 & \text{if} \quad \lambda = -\left(\frac{N-2}{2}\right)^2. \end{cases}$$

This shows that

$$\int_{\partial B_1} \frac{\partial H_\lambda}{\partial \nu}(x)\delta w(x)\, dS_x = -C\delta \int_{\partial B_1} w(x)\, dS_x. \tag{56}$$

By Lemmas 4, 6 and 7, together with formulas (53)–(56) and hypothesis (51), we obtain the following inequality:

$$T^{\alpha+1}\left(-\delta \int_{\partial B_1} w(x)\, dS_x\right) \le C\left(T^{1-\frac{p}{p-1}} R^{\lambda_N+N} \ln R + T^{1+\frac{\alpha p}{p-1}} R^{\lambda_N+N-\frac{2p}{p-1}} (\ln R)^{\frac{p}{p-1}}\right),$$

that is,

$$-\delta \int_{\partial B_1} w(x)\, dS_x \le C\left(T^{-\alpha-\frac{p}{p-1}} R^{\lambda_N+N} \ln R + T^{\frac{\alpha}{p-1}} R^{\lambda_N+N-\frac{2p}{p-1}} (\ln R)^{\frac{p}{p-1}}\right). \tag{57}$$

Taking $T = R^\theta$, where θ is given by (47), then (57) reduces to

$$-\delta \int_{\partial B_1} w(x)\, dS_x \le CR^\mu \left(\ln R + (\ln R)^{\frac{p}{p-1}}\right), \tag{58}$$

where

$$\mu = \frac{(\alpha+1)(\lambda_N+N-2)p - ((\alpha+1)(\lambda_N+N-2)+2)}{(\alpha+1)(p-1)}.$$

Observe that in the case (I), that is $\lambda_N + N - 2 = 0$, one has $\mu < 0$. Similarly, in the case (II), that is $\lambda_N + N - 2 \neq 0$ and p satisfies (52), we have $\mu < 0$. Hence, passing to the limit as $R \to \infty$ in (58), we obtain

$$-\delta \int_{\partial B_1} w(x)\, dS_x \le 0,$$

which contradicts hypothesis (51). We conclude that problem (1)–(3) admits no weak solution. This completes the proof of Theorem 2. □

4. Conclusions

In this paper, we have obtained the nonexistence of weak solutions for the viscoelastic equation (1) in the presence of both homogeneous and inhomogeneous Dirichlet boundary conditions. Then, we have constructed the proofs over corresponding estimates of integral terms in the definition of weak solutions to (1). These estimates help in the analysis of the behavior of solutions to viscoelastic equation (1) in comparison with classical (damped) wave equations of physical interest. For instance, we point out the following two facts:

(i) Referring to Theorem 1, in the limit case $\alpha \to 0^+$ we note that

$$p^*(\alpha, \lambda = 0, N) \to 1 + \frac{2}{N},$$

which is the critical exponent for Equation (4) under the boundary condition (2) with $w = 0$.

(ii) Referring to Theorem 2, in the limit case $\alpha \to 0^+$ we note that

$$p^{**}(\alpha, \lambda = 0, N) \to 1 + \frac{2}{N-2} \quad (N \ge 3),$$

which is the critical exponent for Equation (4) under the boundary condition (2) with $w \neq 0$.

Regularization and general decay of energy of solutions for different viscoelastic equations are interesting topics that can be further studied under different data and boundary conditions. We mention the work of Han-Wang [26] (positive decaying kernel function in the memory term) and Thanh Binh et al. [27] (strongly damped wave equation involving statistical discrete data).

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