HIGHER ORDER EVOLUTION INEQUALITIES INVOLVING LERAY-HARDY POTENTIAL SINGULAR ON THE BOUNDARY

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Abstract. We consider a higher order (in time) evolution inequality posed in the half ball, under Dirichlet type boundary conditions. The involved elliptic operator is the sum of a Laplace differential operator and a Leray-Hardy potential with a singularity located at the boundary. Using a unified approach, we establish a sharp nonexistence result for the evolution inequalities and hence for the corresponding elliptic inequalities. We also investigate the influence of a nonlinear memory term on the existence of solutions to the Dirichlet problem, without imposing any restrictions on the sign of solutions.

1. Introduction

For $N \geq 2$, let

$$B^N = \left\{x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N : |x| \leq 1\right\}.$$

We denote by B_+^N the half ball defined by

$$B_+^N = \left\{x = (x_1, x_2, \cdots, x_N) \in B^N : x_N \geq 0\right\}.$$

The boundary of B_+^N is denoted by $\partial B_+^N = \Gamma_0^N \cup \Gamma_1^N$, where

$$\Gamma_0^N = \left\{x \in B_+^N : x_N = 0\right\}$$

and

$$\Gamma_1^N = \left\{x \in B_+^N : x_N > 0,\, |x| = 1\right\}.$$

For $\mu \in \mathbb{R}$, we consider elliptic operators of the form

$$\mathcal{L}_\mu = -\Delta + \frac{\mu}{|x|^2}, \quad x \in B_+^N \backslash \{0\},$$

defined by the sum of a Laplace differential operator and a singular Leray-Hardy potential term. The Leray-Hardy potential is recognized as a key tool to study borderline situations or critical behavior in different contexts, as well as the study of existence of solutions to nonlinear problems.

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In this paper, we establish the existence and nonexistence of weak solutions to higher order evolution inequalities of the forms

$$\begin{cases} \dfrac{\partial^k u}{\partial t^k} + \mathcal{L}_\mu u \geq |x|^{-a}|u|^p \text{ in } (0,\infty)\times B_+^N\backslash\{0\}, \\ u(t,x) \geq 0 \text{ on } (0,\infty)\times \Gamma_0^N\backslash\{0\}, \\ u(t,x) \geq f(x) \text{ on } (0,\infty)\times \Gamma_1^N \end{cases} \tag{1.1}$$

and

$$\begin{cases} \dfrac{\partial^k u}{\partial t^k} + \mathcal{L}_\mu u \geq |x|^{-a} I_0^\alpha |u|^p \text{ in } (0,\infty)\times B_+^N\backslash\{0\}, \\ u(t,x) \geq 0 \text{ on } (0,\infty)\times \Gamma_0^N\backslash\{0\}, \\ u(t,x) \geq f(x) \text{ on } (0,\infty)\times \Gamma_1^N, \\ \dfrac{\partial^i u}{\partial t^i}(0,x) = u_i(x),\ 0\leq i\leq k-1 \text{ in } B_+^N\backslash\{0\}, \end{cases} \tag{1.2}$$

where $k \geq 1$ is an integer, $\mu > \frac{-N^2}{4}$, $a \in \mathbb{R}$, $p > 1$, $f \in L^1\left(\Gamma_1^N\right)$ is a nontrivial function, $u_i \in L^1_{\text{loc}}\left(B_+^N\backslash\{0\}\right)$, $\alpha > 0$ (namely, the memory term) and

$$I_0^\alpha |u|^p(t,x) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|u(s,x)|^p\,ds.$$

Here, $\Gamma(\cdot)$ denotes the Gamma function. Notice that the value $\frac{-N^2}{4}$ appears in the following Leray-Hardy inequality (see [9])

$$\int_{\mathbb{R}_+^N} |\nabla\phi|^2\,dx - \frac{N^2}{4}\int_{\mathbb{R}_+^N}\frac{\phi^2}{|x|^2}\,dx \geq 0, \quad \phi\in C_0^\infty(\mathbb{R}_+^N), \tag{1.3}$$

where $\mathbb{R}_+^N = \left\{x=(x_1,x_2,\cdots,x_N)\in\mathbb{R}^N : x_N > 0\right\}$.

We recall that the Leray-Hardy potential plays a significant role in the establishment of a Fujita exponent for nonlinear evolution problems with zero (non-zero) boundary data (namely, the nonexistence of solutions and related blow-up phenomena in a finite time). In fact, when $k=1$, $\mu=a=0$ and $u\geq 0$, problem (1.1) (with equality instead of inequality) posed in the whole space, reduces to the following equation

$$\frac{\partial u}{\partial t} - \Delta u = u^p \text{ in } (0,\infty)\times\mathbb{R}^N. \tag{1.4}$$

In his famous paper [12], Fujita proved that (1.4) admits a critical behavior in the following sense:

(a) If $1 < p < 1+\frac{2}{N}$ and $u(0,\cdot) > 0$, then (1.4) does not have any global positive solution;
(b) If $p > 1+\frac{2}{N}$ and u_0 is smaller than a small Gaussian, then (1.4) admits global positive solutions.

We say that $p_F = 1 + \frac{2}{N}$ is critical in the sense of Fujita. Later, it was shown that p_F belongs to the case (a) (see [15] for $N = 1, 2$ and [21] for any $N \geq 1$). It is interesting to mention that p_F is still critical for the following parabolic inequality

$$\frac{\partial u}{\partial t} - \Delta u \geq |u|^p \text{ in } (0, \infty) \times \mathbb{R}^N. \tag{1.5}$$

For more details about (1.5), see e.g. [22]. When $k = 2$ and $\mu = a = 0$, problem (1.1) posed in the whole space, reduces to

$$\frac{\partial^2 u}{\partial t^2} - \Delta u \geq |u|^p \text{ in } (0, \infty) \times \mathbb{R}^N. \tag{1.6}$$

Problem (1.6) was firstly studied by Kato [19]. Namely, he found another critical exponent $p_K = \frac{N+1}{N-1}$. Pohozaev and Véron [24] generalized Kato's result and pointed out the sharpness of p_K for problem (1.6). When $a = 0$, problem (1.1) posed in the whole space, reduces to

$$\frac{\partial^k u}{\partial t^k} + \mathcal{L}_\mu u \geq |u|^p \text{ in } (0, \infty) \times \mathbb{R}^N. \tag{1.7}$$

Problem (1.7) was studied by Hamidi and Laptev [14], in the case where $N \geq 3$ and $\mu \geq -\left(\frac{N-2}{2}\right)^2$. Adopting the notation

$$s^* = \frac{N-2}{2} + \sqrt{\mu + \left(\frac{N-2}{2}\right)^2}$$

and

$$s_* = -\frac{N-2}{2} + \sqrt{\mu + \left(\frac{N-2}{2}\right)^2},$$

it was shown in [14] that under suitable initial conditions, if

(i) $\mu \geq 0$ and $1 < p \leq 1 + \dfrac{2}{s^* + 2/k}$; or

(ii) $-\left(\frac{N-2}{2}\right)^2 \leq \mu < 0$ and $1 < p \leq 1 + \dfrac{2}{-s_* + 2/k}$,

then (1.7) has no nontrivial global solution.

In [17], we considered problem (1.1) with $k = 2$ and $a = 0$, posed in the exterior domain $\mathbb{R}^N \backslash B_1$, under an inhomogeneous Robin-type boundary condition, where B_1 is the unit ball. Namely, we investigated the existence and nonexistence of weak solutions to

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \mathcal{L}_\mu u \geq |u|^p & \text{in } (0, \infty) \times \mathbb{R}^N \backslash B_1, \\[2mm] \alpha \dfrac{\partial u}{\partial \nu}(t, x) + \beta u(t, x) \geq f(x) & \text{on } (0, \infty) \times \partial B_1, \end{cases} \tag{1.8}$$

where $N \geq 2$, $\mu \geq -\left(\frac{N-2}{2}\right)^2$, $\alpha, \beta \geq 0$ and $(\alpha, \beta) \neq (0, 0)$. In the case $\mu = -\left(\frac{N-2}{2}\right)^2$, we proved that

(i) if $N = 2$ and $\int_{\partial B_1} f(x)\, d\sigma > 0$, then for all $p > 1$, (1.8) admits no weak solution;

(ii) if $N \geq 3$ and $\int_{\partial B_1} f(x)\, d\sigma > 0$, then for all

$$1 < p < 1 + \frac{4}{N-2},$$

(1.8) admits no weak solution;

(iii) if $N \geq 3$ and

$$p > 1 + \frac{4}{N-2},$$

then (1.8) admits solutions for some $f > 0$.

In the case $\mu > -\left(\frac{N-2}{2}\right)^2$, we shown that

(i) if $\int_{\partial B_1} f(x)\, d\sigma > 0$, then for all

$$1 < p < 1 + \frac{4}{N-2+2\sqrt{\left(\frac{N-2}{2}\right)^2 + \mu}},$$

(1.8) admits no weak solution;

(iii) if

$$p > 1 + \frac{4}{N-2+2\sqrt{\left(\frac{N-2}{2}\right)^2 + \mu}},$$

then (1.8) admits solutions for some $f > 0$.

For additional results related to evolution equations and inequalities in exterior domains of $\mathbb{R}^N$, see e.g. [16, 18, 25, 28].

The study of parabolic equations with Leray-Hardy potential in a bounded domain of $\mathbb{R}^N$ has been considered by some authors. For instance, Abdellaoui et al. [3] considered parabolic equations of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{L}_\mu u = u^p + f \quad \text{in } (0,\infty)\times\Omega,\ u \geq 0, \\ u(t,x) = 0 \quad \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{1.9}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded regular domain containing the origin, $p > 1$, $\mu < 0$, and $u_0 \geq 0, f \geq 0$ belong to a suitable class of functions. Namely, it was shown the existence of a critical exponent $p_+(\mu)$ such that for $p \geq p_+(\mu)$, there is no distributional solution to (1.9), while for $p < p_+(\mu)$, and under some additional conditions on the data, (1.9) admits solutions. Notice that in [3], the positivity of u is essential in the proof of the obtained results. Moreover, in this reference, the authors used the comparison principle for the heat equation, which cannot be applied for our problems when $k \geq 2$. For other contributions related to the study of parabolic equations and inequalities with Leray-Hardy potential in a bounded domain, see e.g. [4, 5, 7, 13, 26]. For the study of elliptic equations involving Leray-Hardy potential, see e.g. [1, 2, 6, 8, 9, 10, 11].

Notice that in the limit case $\alpha \to 0^+$, problem (1.2) reduces to problem (1.1). Our aim for considering problem (1.2) is to study the influence of the parameter α on the critical behavior of problem (1.1).

A feature of our results is that we do not impose any restrictions on the sign of solutions. To the best of our knowledge, the study of higher order (in time) evolution inequalities with Leray-Hardy potential in the half ball has not previously considered in the literature, even in the parabolic case with nonnegative solutions.

Before stating our main results, we need to introduce the notions of weak solutions to the considered problems (namely, problems (1.1) and (1.2)). We consider the following sets

$$\Omega = (0,\infty) \times B_+^N \backslash \{0\},\ \Gamma_0 = (0,\infty) \times \Gamma_0^N \backslash \{0\},\ \Gamma_1 = (0,\infty) \times \Gamma_1^N,$$

and hence $\Gamma_i \subset \Omega$, $i = 0, 1$. The appropriate setting to introduce the definition of weak solution to (1.1) requires the following functional space (namely, the test function space Φ).

Definition 1.1. A function $\varphi = \varphi(t, x)$ belongs to Φ, if the following conditions are satisfied:

(i) $\varphi \in C_{t,x}^{k,2}(\Omega)$, $\varphi \geq 0$;
(ii) $\mathrm{supp}(\varphi) \subset\subset \Omega$;
(iii) $\varphi|_{\Gamma_i} = 0$, $i = 0, 1$;
(iv) $\dfrac{\partial \varphi}{\partial \nu_i}|_{\Gamma_i} \leq 0$, $i = 0, 1$, where ν_i is the outward unit normal vector on Γ_i^N.

Hence, using standard integration by parts, we define weak solutions to (1.1) as follows.

Definition 1.2. We say that $u \in L^p_{\mathrm{loc}}(\Omega)$ is a weak solution to (1.1) if

$$\int_\Omega |x|^{-a}|u|^p \varphi\, dx\, dt - \int_{\Gamma_1} f(x) \frac{\partial \varphi}{\partial \nu_1}\, d\sigma\, dt \leq (-1)^k \int_\Omega u \frac{\partial^k \varphi}{\partial t^k}\, dx\, dt + \int_\Omega u \mathcal{L}_\mu \varphi\, dx\, dt \quad (1.10)$$

for every $\varphi \in \Phi$.

In order to define weak solutions to (1.2), we need to recall some basic properties on fractional calculus. For more details, see e.g. [20].

Let $T > 0$ be fixed, then for $\alpha > 0$ and $g \in L^1([0, T])$, we consider the integral operators

$$I_0^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds$$

and

$$I_T^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} g(s)\, ds.$$

The operators I_0^α and I_T^α are called respectively the left-sided and right-sided Riemann-Liouville fractional integrals of order α.

If $g, h \in C([0, T])$, then we have the following equality (see [20, Lemma 2.7])

$$\int_0^T I_0^\alpha g(t)\, h(t)\, dt = \int_0^T g(t)\, I_T^\alpha h(t)\, dt. \quad (1.11)$$

Now, we consider the following sets

$$\Omega_T = [0,T] \times B_+^N \backslash \{0\},\ \Gamma_{0,T} = [0,T] \times \Gamma_0^N \backslash \{0\},\ \Gamma_{1,T} = [0,T] \times \Gamma_1^N,$$

and introduce a second functional space (namely, the test function space Ψ_T).

Definition 1.3. A function $\psi = \psi(t,x)$ belongs to Ψ_T, if the following conditions are satisfied:

(i) $\psi \in C^{k,2}_{t,x}(\Omega_T)$, $\psi \geq 0$;
(ii) $\mathrm{supp}(\psi) \subset\subset \Omega_T$;
(iii) $\psi|_{\Gamma_{i,T}} = 0$, $i = 0, 1$;
(iv) $\frac{\partial \psi}{\partial \nu_i}|_{\Gamma_{i,T}} \leq 0$, $i = 0, 1$;
(v) $\frac{\partial^i \psi}{\partial t^i}(T, \cdot) = 0$, $0 \leq i \leq k-1$.

Using the equality (1.11), we can define weak solution to (1.2) as follows.

Definition 1.4. We say that $u \in L^p_{\mathrm{loc}}([0,\infty) \times B^N_+ \backslash \{0\})$ is a weak solution to (1.2) if

$$\int_{\Omega_T} |x|^{-a}|u|^p I^\alpha_T \psi \, dx\, dt - \int_{\Gamma_{1,T}} f(x) \frac{\partial \psi}{\partial \nu_1} \, d\sigma \, dt + \sum_{i=0}^{k-1} (-1)^i \int_{B_1^c} u_{k-i-1}(x) \frac{\partial^i \psi}{\partial t^i}(0,x)\, dx$$
$$\leq (-1)^k \int_{\Omega_T} u \frac{\partial^k \psi}{\partial t^k} \, dx\, dt + \int_{\Omega_T} u \mathcal{L}_\mu \psi \, dx\, dt \tag{1.12}$$

for all $T > 0$ and $\psi \in \Psi_T$.

For $\mu > \frac{-N^2}{4}$, let us introduce the parameter

$$\tau = -\frac{N}{2} + \sqrt{\mu + \left(\frac{N}{2}\right)^2}, \tag{1.13}$$

and, for all $f \in L^1(\Gamma^N_1)$, we set

$$\mathcal{I}_f = \int_{\Gamma^N_1} f(x) x_N \, d\sigma. \tag{1.14}$$

We also denote by $L^{1,+}(\Gamma^N_1)$ the functional space given by

$$L^{1,+}(\Gamma^N_1) = \left\{ f \in L^1(\Gamma^N_1) : \mathcal{I}_f > 0 \right\}.$$

Our main results for problem (1.1) are stated in the following theorem.

Theorem 1.5. *Let $k \geq 1$, $N \geq 2$, $\mu > \frac{-N^2}{4}$, $a \in \mathbb{R}$ and $p > 1$. We distinguish the following cases:*

(I) *Let $f \in L^{1,+}(\Gamma^N_1)$. If*

$$(\tau + 1)p < a + \tau - 1, \tag{1.15}$$

then (1.1) admits no weak solution.

(II) *If*

$$(\tau + 1)p > a + \tau - 1, \tag{1.16}$$

then (1.1) admits nonnegative (stationary) solutions for some $f \in L^{1,+}(\Gamma^N_1)$.

Remark 1.6. Theorem 1.5 leaves open the issue of existence and nonexistence in the critical case:

$$(\tau + 1)p = a + \tau - 1.$$

Remark 1.7. We point out the following facts:

(i) Let $\mu = 1 - N$. In this case, $\tau = -1$ and (1.15) reduces to $a > 2$.

(ii) Let $\mu > 1 - N$. In this case, $\tau + 1 > 0$ and (1.15) reduces to

$$a > 2,\ 1 < p < 1 + \frac{a-2}{\tau+1}.$$

(iii) Let $\frac{-N^2}{4} < \mu < 1 - N$ and $N \geq 3$. In this case, $\tau + 1 < 0$ and (1.15) reduces to

$$p > \max\left\{1, 1 + \frac{a-2}{\tau+1}\right\} = \left\{\begin{array}{ll} 1 & \text{if} \quad a \geq 2, \\ 1 + \frac{a-2}{\tau+1} & \text{if} \quad a < 2. \end{array}\right.$$

(iv) Condition (1.15) is independent on k.

Clearly, Theorem 1.5 yields existence and nonexistence results for the corresponding elliptic problem

$$\left\{\begin{array}{l} \mathcal{L}_\mu u \geq |x|^{-a}|u|^p \quad \text{in } B_+^N\backslash\{0\}, \\ \\ u(x) \geq 0 \quad \text{on } \Gamma_0^N\backslash\{0\}, \\ \\ u(x) \geq f(x) \quad \text{on } \Gamma_1^N. \end{array}\right. \tag{1.17}$$

Corollary 1.8. Let $N \geq 2$, $\mu > \frac{-N^2}{4}$, $a \in \mathbb{R}$ and $p > 1$. We distinguish the following cases:

(I) Let $f \in L^{1,+}(\Gamma_1^N)$. If (1.15) holds, then (1.17) admits no weak solution.
(II) If (1.16) holds, then (1.17) admits nonnegative solutions for some $f \in L^{1,+}(\Gamma_1^N)$.

Our main result for problem (1.2) is stated in the following theorem.

Theorem 1.9. *Let $k \geq 1$, $N \geq 2$, $\mu > \frac{-N^2}{4}$, $a \in \mathbb{R}$, $p > 1$ and $u_i \in L^1_{\text{loc}}(B_+^N\backslash\{0\})$, $u_i \geq 0$, for all $0 \leq i \leq k-1$. If $f \in L^{1,+}(\Gamma_1^N)$, then for all $\alpha > 0$, (1.2) admits no weak solution.*

Remark 1.10. We point out the following facts:

(i) By Theorem 1.9, we deduce that for any $\alpha > 0$, the Fujita critical exponent of (1.2) is equal to ∞.
(ii) By Theorem 1.5 (see also Remark 1.7), we deduce that, if

$$\mu = 1 - N,\ a > 2; \quad \text{or} \quad \frac{-N^2}{4} < \mu < 1 - N,\ a \geq 2,$$

then the memory term (namely, the parameter $\alpha > 0$) has no effect on the critical behavior of problem (1.1).
(iii) By Theorem 1.5 (see also Remark 1.7), we deduce that, if

$$\mu > 1 - N,\ a > 2,$$

then the Fujita critical exponent of problem (1.2) jumps form $1 + \frac{a-2}{\tau+1}$ (the Fujita critical exponent of problem (1.1)) to ∞.

The proofs of Theorem 1.5 (I) and Theorem 1.9 rely on nonlinear capacity estimates specifically adapted to the domain B_+^N, the operator $\mathcal{L}_\mu$ and the considered boundary conditions. The existence result given by Theorem 1.5 (II) is established by the construction of explicit solutions.

The rest of the paper is organized as follows. In Section 2, we first construct two families of test functions belonging respectively to the test function spaces Φ and Ψ_T. Next, we establish some useful integral estimates involving the constructed test functions. Section 3 is devoted to the proofs of Theorems 1.5 and 1.9. A short Section 4 concludes the paper.

2. Preliminaries

Throughout this paper, the letter C denotes always a generic positive constant whose value is unimportant and may vary at different occurrences.

Let $k \geq 1$, $N \geq 2$, $a \in \mathbb{R}$, $p > 1$, $\alpha > 0$ and $\mu > \frac{-N^2}{4}$. We introduce the nonnegative function ϑ defined by

$$\vartheta(x) = x_N |x|^\tau \left(|x|^{-N-2\tau} - 1\right), \quad x \in B_+^N \backslash \{0\}, \tag{2.1}$$

where τ is given by (1.13). It is not difficult to show that ϑ is a solution to

$$\begin{cases} \mathcal{L}_\mu \vartheta = 0 \ \text{ in } B_+^N \backslash \{0\}, \\ \\ \vartheta = 0 \ \text{ on } \Gamma_0^N \backslash \{0\} \cup \Gamma_1^N. \end{cases} \tag{2.2}$$

In the sequel, we need two cut-off functions $\eta, \xi \in C^\infty([0,\infty))$ satisfying the following conditions

$$\eta \geq 0, \ \operatorname{supp}(\eta) \subset\subset (0,1) \tag{2.3}$$

and

$$0 \leq \xi \leq 1, \ \xi(s) = 0 \text{ if } 0 \leq s \leq \frac{1}{2}, \ \xi(s) = 1 \text{ if } s \geq 1. \tag{2.4}$$

Moreover, for $T > 0$ and sufficiently large ℓ, we introduce the functions

$$\eta_T(t) = \eta\left(\frac{t}{T}\right)^\ell, \quad t \geq 0, \tag{2.5}$$

$$\gamma_T(t) = T^{-\ell}(T-t)^\ell, \quad 0 \leq t \leq T. \tag{2.6}$$

For $R > 1$, we define the following function

$$\xi_R(x) = \vartheta(x)\xi(R|x|)^\ell, \quad x \in B_+^N \backslash \{0\}, \tag{2.7}$$

that is,

$$\xi_R(x) = \begin{cases} 0 & \text{if } \ x_N \geq 0, \ 0 < |x| \leq \frac{1}{2R}, \\ \\ \vartheta(x)\xi(R|x|)^\ell & \text{if } \ x_N \geq 0, \ \frac{1}{2R} \leq |x| \leq \frac{1}{R}, \\ \\ \vartheta(x) & \text{if } \ x_N \geq 0, \ \frac{1}{R} \leq |x| \leq 1. \end{cases} \tag{2.8}$$

On this basis, we consider two test functions of the form

$$\varphi(t,x) = \eta_T(t)\xi_R(x), \quad (t,x) \in \Omega \tag{2.9}$$

and

$$\psi(t,x) = \gamma_T(t)\xi_R(x), \quad (t,x) \in \Omega_T. \tag{2.10}$$

In the following lemma, we show that $\varphi \in \Phi$.

Lemma 2.1. *The function φ defined by (2.9) belongs to the test function space Φ.*

Proof. Clearly, for sufficiently large ℓ, we have that $\eta_T \in C^k((0,\infty))$ and $\xi_R \in C^2(B_+^N\backslash\{0\})$, which imply that $\varphi \in C_{t,x}^{k,2}(\Omega)$. Moreover, combining the information in formulas (2.3), (2.4), (2.5) and (2.8), we deduce that

$$\operatorname{supp}(\varphi) = [c_1T, c_2T] \times \left\{x \in \mathbb{R}^N : x_N \geq 0, \frac{1}{2R} \leq |x| \leq 1\right\},$$

for some $0 < c_1 < c_2 < 1$. On the other hand, by (2.2) we note that

$$\varphi\big|_{\Gamma_i} = 0, \quad i = 0, 1.$$

So, we have just to show that

$$\frac{\partial \varphi}{\partial \nu_i}(t,x) \leq 0, \quad (t,x) \in \Gamma_i,\, i = 0,1. \tag{2.11}$$

We first prove that the inequality in (2.11) is true in the case where $i = 0$. In fact, by (2.1) and (2.7), we get

$$\begin{aligned}\frac{\partial \xi_R}{\partial x_N}(x) &= \frac{\partial \vartheta}{\partial x_N}(x)\xi(R|x|)^\ell + \vartheta(x)\frac{\partial}{\partial x_N}\left[\xi(R|x|)^\ell\right] \\ &= |x|^\tau\left(|x|^{-N-2\tau} - 1\right)\xi(R|x|)^\ell + x_N\frac{\partial}{\partial x_N}\left(|x|^\tau\left(|x|^{-N-2\tau} - 1\right)\right)\xi(R|x|)^\ell \\ &\quad + x_N|x|^\tau\left(|x|^{-N-2\tau} - 1\right)\frac{\partial}{\partial x_N}\left[\xi(R|x|)^\ell\right].\end{aligned}$$

Hence, for $x \in \Gamma_0^N\backslash\{0\}$, we obtain that

$$\frac{\partial \xi_R}{\partial \nu_0}(x) = -\frac{\partial \xi_R}{\partial x_N}(x) = -|x|^\tau\left(|x|^{-N-2\tau} - 1\right)\xi(R|x|)^\ell,$$

which implies, by using (2.3) together with (2.5) and (2.9), that

$$\begin{aligned}\frac{\partial \varphi}{\partial \nu_0}(t,x) &= \eta_T(t)\frac{\partial \xi_R}{\partial \nu_0}(x) \\ &= -|x|^\tau\left(|x|^{-N-2\tau} - 1\right)\xi(R|x|)^\ell\eta_T(t) \leq 0\end{aligned}$$

for all $(t,x) \in \Gamma_0$. This proves the inequality in (2.11) in the case where $i = 0$. On the other hand, by (2.1), (2.4) and (2.8), for sufficiently small $\varepsilon > 0$ and $x \in B_+^N\backslash\{0\}$ with $1 - \varepsilon < |x| < 1$, we have

$$\begin{aligned}\nabla\xi_R(x) &= \nabla\vartheta(x) \\ &= |x|^\tau\left(|x|^{-N-2\tau} - 1\right)\nabla x_N + x_N\left((-N-\tau)|x|^{-N-\tau-1} - \tau|x|^{\tau-1}\right)\frac{x}{|x|},\end{aligned}$$

which implies that for all $x \in \Gamma_1^N$, the following is the case

$$\begin{aligned}\frac{\partial \xi_R}{\partial \nu_1}(x) &= \nabla\xi_R(x) \cdot \frac{x}{|x|} \\ &= -(2\tau + N)x_N,\end{aligned}$$

where "$\cdot$" denotes as usual the inner product in $\mathbb{R}^N$. Hence, involving the test function φ defined in (2.9), we deduce that

$$\frac{\partial \varphi}{\partial \nu_1}(t,x) = -(2\tau+N)x_N \eta_T(t) \leq 0 \tag{2.12}$$

for all $(t,x) \in \Gamma_1$, which proves the inequality in (2.11) also in the case where $i = 1$. □

Next lemma is devoted to characterize the test function ψ defined by (2.10) in the sense that we show that $\psi \in \Psi_T$.

Lemma 2.2. *For all $T > 0$, the function ψ defined by (2.10) belongs to the test function space Ψ_T.*

Proof. For sufficiently large values of ℓ, we have that $\gamma_T \in C^k([0,T])$ and $\xi_R \in C^2(B_+^N \backslash \{0\})$, which imply that $\psi \in C^{k,2}_{t,x}(\Omega_T)$. Moreover, we have

$$\operatorname{supp}(\psi) = [0,T] \times \left\{ x \in \mathbb{R}^N : x_N \geq 0, \frac{1}{2R} \leq |x| \leq 1 \right\}$$

and (by (2.2)) we conclude that

$$\psi\big|_{\Gamma_i} = 0, \quad i = 0,1.$$

On the other hand, for all $0 \leq i \leq k-1$, we have $\dfrac{d^i \gamma_T}{dt^i}(T) = 0$, which implies that $\dfrac{\partial^i \psi}{\partial t^i}(T,\cdot) = 0$. Finally, proceeding as in the proof of Lemma 2.1, we obtain the conditions

$$\frac{\partial \psi}{\partial \nu_0}(t,x) = -|x|^\tau \left(|x|^{-N-2\tau} - 1\right) \xi(R|x|)^\ell \gamma_T(t) \leq 0, \quad (t,x) \in \Gamma_0$$

and

$$\frac{\partial \psi}{\partial \nu_1}(t,x) = -(2\tau+N)x_N \gamma_T(t) \leq 0, \quad (t,x) \in \Gamma_1. \tag{2.13}$$

□

In the following, we shall give some integral estimates involving the functions φ and ψ defined by (2.9) and (2.10), respectively.

2.1. Estimates involving φ.

The first estimate follows immediately from (2.3) and (2.5), and hence we do not give a proof of the lemma.

Lemma 2.3. *For all $T > 0$ and sufficiently large ℓ, there holds*

$$\int_{\operatorname{supp}(\eta_T)} \eta_T(t)^{\frac{-1}{p-1}} \left| \frac{d^k \eta_T(t)}{dt^k} \right|^{\frac{p}{p-1}} dt \leq C T^{1-\frac{kp}{p-1}}.$$

The second estimate mainly incorporates the effects of the truncation function ξ_R in the integral term.

Lemma 2.4. *For sufficiently large ℓ and R, there holds*

$$\int_{\operatorname{supp}(\xi_R)} |x|^{\frac{a}{p-1}} \xi_R(x)\, dx \leq C \left(\ln R + R^{\tau-1-\frac{a}{p-1}} \right). \tag{2.14}$$

Proof. Starting from the definition of the function ϑ in (2.1) and involving the condition (2.4) and the function ξ_R defined by (2.7), we obtain the following chain of inequalities

$$\begin{aligned}
&\int_{\mathrm{supp}(\xi_R)} |x|^{\frac{a}{p-1}}\xi_R(x)\,dx \\
&= \int_{\frac{1}{2R}<|x|<1,\,x_N>0} |x|^{\frac{a}{p-1}} x_N |x|^{\tau}\left(|x|^{-N-2\tau}-1\right)\xi(R|x|)^{\ell}\,dx \\
&\le \int_{\frac{1}{2R}<|x|<1,\,x_N>0} x_N |x|^{\tau+\frac{a}{p-1}}\left(|x|^{-N-2\tau}-1\right)\,dx \\
&\le \int_{\frac{1}{2R}<|x|<1} |x|^{-\tau+1-N+\frac{a}{p-1}}\,dx \\
&= C\int_{r=\frac{1}{2R}}^{1} r^{\frac{a}{p-1}-\tau}\,dr \\
&= \begin{cases} C & \text{if } (\tau-1)(p-1)<a, \\ C\ln R & \text{if } (\tau-1)(p-1)=a, \\ CR^{\tau-1-\frac{a}{p-1}} & \text{if } (\tau-1)(p-1)>a. \end{cases}
\end{aligned}$$

It follows that the estimate (2.14) is established. □

Next estimate relies on the test function φ defined by (2.9) and its k-th derivative in time variable.

Lemma 2.5. *For all $T>0$ and sufficiently large ℓ and R, there holds*

$$\int_{\mathrm{supp}(\varphi)} |x|^{\frac{a}{p-1}}\varphi^{\frac{-1}{p-1}}\left|\frac{\partial^k\varphi}{\partial t^k}\right|^{\frac{p}{p-1}}\,dx\,dt \le CT^{1-\frac{kp}{p-1}}\left(\ln R + R^{\tau-1-\frac{a}{p-1}}\right), \tag{2.15}$$

where φ is the function defined by (2.9).

Proof. By the definition of φ in (2.9), we get that

$$\begin{aligned}
&\int_{\mathrm{supp}(\varphi)} |x|^{\frac{a}{p-1}}\varphi^{\frac{-1}{p-1}}\left|\frac{\partial^k\varphi}{\partial t^k}\right|^{\frac{p}{p-1}}\,dx\,dt \\
&= \left(\int_{\mathrm{supp}(\eta_T)} \eta_T(t)^{\frac{-1}{p-1}}\left|\frac{d^k\eta_T(t)}{dt^k}\right|^{\frac{p}{p-1}}\,dt\right)\left(\int_{\mathrm{supp}(\xi_R)} |x|^{\frac{a}{p-1}}\xi_R(x)\,dx\right).
\end{aligned}$$

Hence, we can use Lemmas 2.3 and 2.4 to obtain the estimate (2.15). □

Now, we consider the elliptic operator $\mathcal{L}_\mu$ and construct the following estimate.

Lemma 2.6. *For sufficiently large ℓ and R, there holds*

$$\int_{\mathrm{supp}(\xi_R)} |x|^{\frac{a}{p-1}}\xi_R(x)^{\frac{-1}{p-1}}\left|\mathcal{L}_\mu\xi_R(x)\right|^{\frac{p}{p-1}}\,dx \le CR^{\frac{(\tau+1)p+1-a-\tau}{p-1}}. \tag{2.16}$$

Proof. By the definition of ξ_R (recall (2.7)), for $x \in \mathrm{supp}(\xi_R)$ we obtain the representation formula

$$\begin{aligned}
-\mathcal{L}_\mu \xi_R(x) &= \Delta\left(\vartheta(x)\xi(R|x|)^\ell\right) - \frac{\mu}{|x|^2}\vartheta(x)\xi(R|x|)^\ell \\
&= \xi(R|x|)^\ell \Delta\vartheta(x) + \vartheta(x)\Delta\left[\xi(R|x|)^\ell\right] + 2\nabla\vartheta(x)\cdot\nabla\left[\xi(R|x|)^\ell\right] \\
&\quad - \frac{\mu}{|x|^2}\vartheta(x)\xi(R|x|)^\ell \\
&= -\xi(R|x|)^\ell \mathcal{L}_\mu\vartheta(x) + \vartheta(x)\Delta\left[\xi(R|x|)^\ell\right] + 2\nabla\vartheta(x)\cdot\nabla\left[\xi(R|x|)^\ell\right],
\end{aligned}$$

which implies by (2.2) that

$$-\mathcal{L}_\mu \xi_R(x) = \vartheta(x)\Delta\left[\xi(R|x|)^\ell\right] + 2\nabla\vartheta(x)\cdot\nabla\left[\xi(R|x|)^\ell\right]. \tag{2.17}$$

Hence, by condition (2.4), we deduce that

$$\begin{aligned}
&\int_{\mathrm{supp}(\xi_R)} |x|^{\frac{a}{p-1}} \xi_R(x)^{\frac{-1}{p-1}} \left|\mathcal{L}_\mu \xi_R(x)\right|^{\frac{p}{p-1}}\,dx \\
&= \int_{\frac{1}{2R}<|x|<\frac{1}{R},\, x_N>0} |x|^{\frac{a}{p-1}} \xi_R(x)^{\frac{-1}{p-1}} \left|\mathcal{L}_\mu \xi_R(x)\right|^{\frac{p}{p-1}}\,dx.
\end{aligned} \tag{2.18}$$

On the other hand, the same condition (2.4), for $\frac{1}{2R} < |x| < \frac{1}{R}$, $x_N > 0$, gives us

$$\left|\Delta\left[\xi(R|x|)^\ell\right]\right| \le CR^2 \xi(R|x|)^{\ell-2},$$

which implies by (2.1) that

$$\begin{aligned}
\vartheta(x)\left|\Delta\left[\xi(R|x|)^\ell\right]\right| &\le CR^2 x_N |x|^\tau \left(|x|^{-N-2\tau} - 1\right)\xi(R|x|)^{\ell-2} \\
&\le CR^{N+\tau+2} x_N \xi(R|x|)^{\ell-2}.
\end{aligned} \tag{2.19}$$

Moreover, we have

$$\begin{aligned}
&\nabla\vartheta(x)\cdot\nabla\left[\xi(R|x|)^\ell\right] \\
&= \left[|x|^\tau\left(|x|^{-N-2\tau} - 1\right)e_N + x_N\left((-N-\tau)|x|^{-N-\tau-1} - \tau|x|^{\tau-1}\right)\frac{x}{|x|}\right] \\
&\quad \cdot 2\ell R\xi\left(R|x|\right)^{\ell-1}\xi'\left(R|x|\right)\frac{x}{|x|} \\
&= 2\ell R\xi\left(R|x|\right)^{\ell-1}\xi'\left(R|x|\right)x_N|x|^{\tau-1}\left[(1-N-\tau)|x|^{-N-2\tau} - 1 - \tau\right],
\end{aligned}$$

where $e_N = (0, \cdots, 0, 1) \in \mathbb{R}^N$, which implies (since $0 \le \xi \le 1$) that

$$\left|\nabla\vartheta(x)\cdot\nabla\left[\xi(R|x|)^\ell\right]\right| \le CR^{N+\tau+2} x_N \xi\left(R|x|\right)^{\ell-2}. \tag{2.20}$$

Hence, in view of (2.17) and the obtained inequalities in (2.19) and (2.20), we deduce that

$$\left|\mathcal{L}_\mu \xi_R(x)\right| \le CR^{N+\tau+2} x_N \xi\left(R|x|\right)^{\ell-2}, \quad \frac{1}{2R} < |x| < \frac{1}{R},\ x_N > 0. \tag{2.21}$$

Making use of (2.1), (2.4), (2.7), (2.18) and (2.21), we obtain the chain of inequalities

$$\begin{aligned}
&\int_{\mathrm{supp}(\xi_R)} |x|^{\frac{a}{p-1}} \xi_R(x)^{\frac{-1}{p-1}} \left|\mathcal{L}_\mu \xi_R(x)\right|^{\frac{p}{p-1}}\, dx \\
&\le C R^{\frac{(N+\tau+2)p}{p-1}} \int_{\frac{1}{2R}<|x|<\frac{1}{R},\, x_N>0} x_N |x|^{\frac{a-\tau}{p-1}} \left(|x|^{-N-2\tau}-1\right)^{\frac{-1}{p-1}} \xi\left(R|x|\right)^{\ell-\frac{2p}{p-1}}\, dx \\
&\le C R^{\frac{(N+\tau+2)p}{p-1}} \int_{\frac{1}{2R}<|x|<\frac{1}{R}} |x|^{1+\frac{a+\tau+N}{p-1}}\, dx \\
&\le C R^{\frac{(N+\tau+2)p}{p-1}} R^{-1-\frac{a+\tau+N}{p-1}-N},
\end{aligned}$$

which leads to the estimate (2.16). □

The following is the last estimate in this sub-section.

Lemma 2.7. *For all $T>0$ and sufficiently large R and ℓ, there holds*

$$\int_{\mathrm{supp}(\varphi)} |x|^{\frac{a}{p-1}} \varphi^{\frac{-1}{p-1}} \left|\mathcal{L}_\mu \varphi\right|^{\frac{p}{p-1}}\, dx\, dt \le C T R^{\frac{(\tau+1)p+1-a-\tau}{p-1}}, \tag{2.22}$$

where φ is the function defined by (2.9).

Proof. Starting from the definition of the function φ given by (2.9), we obtain that

$$\begin{aligned}
&\int_{\mathrm{supp}(\varphi)} |x|^{\frac{a}{p-1}} \varphi^{\frac{-1}{p-1}} \left|\mathcal{L}_\mu \varphi\right|^{\frac{p}{p-1}}\, dx\, dt \\
&= \left(\int_{\mathrm{supp}(\eta_T)} \eta_T(t)\, dt\right) \left(\int_{\mathrm{supp}(\xi_R)} |x|^{\frac{a}{p-1}} \xi_R(x)^{\frac{-1}{p-1}} \left|\mathcal{L}_\mu \xi_R(x)\right|^{\frac{p}{p-1}}\, dx\right).
\end{aligned} \tag{2.23}$$

On the other hand, using condition (2.3) together with the definition of the function η_T given by (2.5), we deduce that

$$\begin{aligned}
\int_{\mathrm{supp}(\eta_T)} \eta_T(t)\, dt &= \int_0^T \eta\left(\frac{t}{T}\right)^{\ell} dt \\
&= T \int_0^1 \eta(s)^{\ell}\, ds.
\end{aligned} \tag{2.24}$$

Involving Lemma 2.6 and using formulas (2.23) and (2.24), we obtain the estimate (2.22). □

2.2. Estimates involving ψ.

The following result follows from elementary calculations, and hence it is stated without a proof.

Lemma 2.8. *Let $T>0$. For sufficiently large ℓ, there holds*

$$I_T^\alpha \gamma_T(t) = \frac{\Gamma(\ell+1)}{\Gamma(\alpha+\ell+1)} T^{-\ell}(T-t)^{\ell+\alpha}, \quad 0\le t\le T,$$

where γ_T is the function defined by (2.6).

Based on the right-sided Riemann-Liouville fractional integral of order α, namely I_T^α, we establish the next estimate.

Lemma 2.9. *For all $T>0$ and sufficiently large R and ℓ, there holds*

$$\int_{\mathrm{supp}(\psi)} |x|^{\frac{a}{p-1}} |I_T^\alpha \psi|^{\frac{-1}{p-1}} \left|\frac{\partial^k \psi}{\partial t^k}\right|^{\frac{p}{p-1}} \,dx\,dt \leq C T^{1-\frac{kp+\alpha}{p-1}} \left(\ln R + R^{\tau-1-\frac{a}{p-1}}\right), \tag{2.25}$$

where ψ is the function defined by (2.10).

Proof. For the function ψ given by (2.10), we obtain

$$\begin{aligned}&\int_{\mathrm{supp}(\psi)} |x|^{\frac{a}{p-1}} |I_T^\alpha \psi|^{\frac{-1}{p-1}} \left|\frac{\partial^k \psi}{\partial t^k}\right|^{\frac{p}{p-1}} \,dx\,dt\\ &= \left(\int_0^T |I_T^\alpha \gamma_T(t)|^{\frac{-1}{p-1}} \left|\frac{d^k \gamma_T(t)}{dt^k}\right|^{\frac{p}{p-1}} \,dt\right) \left(\int_{\mathrm{supp}(\xi_R)} |x|^{\frac{a}{p-1}} \xi_R(x)\,dx\right).\end{aligned} \tag{2.26}$$

On the other hand, we have

$$\left|\frac{d^k \gamma_T(t)}{dt^k}\right| = C T^{-\ell} (T-t)^{\ell-k}, \quad 0 \leq t \leq T,$$

which implies by Lemma 2.8 that

$$|I_T^\alpha \gamma_T(t)|^{\frac{-1}{p-1}} \left|\frac{d^k \gamma_T(t)}{dt^k}\right|^{\frac{p}{p-1}} \leq C T^{-\ell} (T-t)^{\ell-\frac{kp+\alpha}{p-1}}, \quad 0 \leq t \leq T.$$

Integrating over $(0,T)$, we obtain

$$\int_0^T |I_T^\alpha \gamma_T(t)|^{\frac{-1}{p-1}} \left|\frac{d^k \gamma_T(t)}{dt^k}\right|^{\frac{p}{p-1}} \,dt \leq C T^{1-\frac{kp+\alpha}{p-1}}. \tag{2.27}$$

Thus, making use of Lemma 2.4, (2.26) and (2.27), we conclude the estimate (2.25). □

The last lemma involves the elliptic operator $\mathcal{L}_\mu$ in the integrand.

Lemma 2.10. *For all $T>0$ and sufficiently large R and ℓ, there holds*

$$\int_{\mathrm{supp}(\psi)} |x|^{\frac{a}{p-1}} |I_T^\alpha \psi|^{\frac{-1}{p-1}} |\mathcal{L}_\mu \psi|^{\frac{p}{p-1}} \,dx\,dt \leq C T^{1-\frac{\alpha}{p-1}} R^{\frac{(\tau+1)p+1-a-\tau}{p-1}}, \tag{2.28}$$

where ψ is the function defined by (2.10).

Proof. By (2.10), we get

$$\begin{aligned}&\int_{\mathrm{supp}(\psi)} |x|^{\frac{a}{p-1}} |I_T^\alpha \psi|^{\frac{-1}{p-1}} |\mathcal{L}_\mu \psi|^{\frac{p}{p-1}} \,dx\,dt\\ &= \left(\int_0^T |I_T^\alpha \gamma_T(t)|^{\frac{-1}{p-1}} \gamma_T(t)^{\frac{p}{p-1}}(t)\,dt\right) \left(\int_{\mathrm{supp}(\xi_R)} |x|^{\frac{a}{p-1}} \xi_R(x)^{\frac{-1}{p-1}} |\mathcal{L}_\mu \xi_R(x)|^{\frac{p}{p-1}} \,dx\right).\end{aligned} \tag{2.29}$$

On the other hand, taking $k=0$ in (2.27), we obtain

$$\int_0^T |I_T^\alpha \gamma_T(t)|^{\frac{-1}{p-1}} \gamma_T(t)^{\frac{p}{p-1}}(t)\,dt \leq C T^{1-\frac{\alpha}{p-1}}. \tag{2.30}$$

Thus, making use of Lemma 2.6, (2.29) and (2.30), we establish (2.28). □

3. Proof of the main results

The main strategy of proof is based on the test functions method, which leads to self-contained and easy-to-follow proofs. We also develop the proofs by contradiction, using the estimates in the previous section.

Proof of Theorem 1.5. (I) We argue by contradiction by supposing that $u \in L^p_{\mathrm{loc}}(\Omega)$ is a weak solution to (1.1). Thus, making use of the formula (1.10) and Lemma 2.1, for sufficiently large ℓ, T and R, we deduce the following inequality

$$\int_\Omega |x|^{-a}|u|^p\varphi\,dx\,dt - \int_{\Gamma_1} f(x)\frac{\partial\varphi}{\partial\nu_1}\,d\sigma\,dt \le \int_\Omega |u|\left|\frac{\partial^k\varphi}{\partial t^k}\right|\,dx\,dt + \int_\Omega |u||\mathcal{L}_\mu\varphi|\,dx\,dt, \tag{3.1}$$

where $\varphi \in \Phi$ is the function defined by (2.9). On the other hand, using Young's inequality, we get two more inequalities in the following form

$$\begin{aligned}&\int_\Omega |u|\left|\frac{\partial^k\varphi}{\partial t^k}\right|\,dx\,dt\\ &\le \frac{1}{2}\int_\Omega |x|^{-a}|u|^p\varphi\,dx\,dt + C\int_{\mathrm{supp}(\varphi)} |x|^{\frac{a}{p-1}}\varphi^{\frac{-1}{p-1}}\left|\frac{\partial^k\varphi}{\partial t^k}\right|^{\frac{p}{p-1}}\,dx\,dt\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&\int_\Omega |u||\mathcal{L}_\mu\varphi|\,dx\,dt\\ &\le \frac{1}{2}\int_\Omega |x|^{-a}|u|^p\varphi\,dx\,dt + C\int_{\mathrm{supp}(\varphi)} |x|^{\frac{a}{p-1}}\varphi^{\frac{-1}{p-1}}|\mathcal{L}_\mu\varphi|^{\frac{p}{p-1}}\,dx\,dt.\end{aligned} \tag{3.3}$$

Then, combining the above inequalities (3.1), (3.2) and (3.3), we obtain the following inequality

$$\begin{aligned}&-\int_{\Gamma_1} f(x)\frac{\partial\varphi}{\partial\nu_1}\,d\sigma\,dt\\ &\le C\left(\int_{\mathrm{supp}(\varphi)} |x|^{\frac{a}{p-1}}\varphi^{\frac{-1}{p-1}}\left|\frac{\partial^k\varphi}{\partial t^k}\right|^{\frac{p}{p-1}}\,dx\,dt + \int_{\mathrm{supp}(\varphi)} |x|^{\frac{a}{p-1}}\varphi^{\frac{-1}{p-1}}|\mathcal{L}_\mu\varphi|^{\frac{p}{p-1}}\,dx\,dt\right).\end{aligned} \tag{3.4}$$

Moreover, by using (2.12), we deduce that

$$\begin{aligned}-\int_{\Gamma_1} f(x)\frac{\partial\varphi}{\partial\nu_1}\,d\sigma\,dt &= (2\tau+N)\int_0^\infty\int_{\Gamma_1^N} f(x)x_N\eta_T(t)\,d\sigma\,dt\\ &= (2\tau+N)\left(\int_{\mathrm{supp}(\eta_T)}\eta_T(t)\,dt\right)\mathcal{I}_f \quad \text{(recall (1.14))}.\end{aligned}$$

In view of (2.24), the following is the case (notice that $2\tau+N>0$)

$$-\int_{\Gamma_1} f(x)\frac{\partial\varphi}{\partial\nu_1}\,d\sigma\,dt = CT\mathcal{I}_f. \tag{3.5}$$

Hence, making use of Lemma 2.5, Lemma 2.7, (3.4) and (3.5), we obtain the following inequality

$$T\mathcal{I}_f \le C\left(T^{1-\frac{kp}{p-1}}\left(\ln R + R^{\tau-1-\frac{a}{p-1}}\right) + TR^{\frac{(\tau+1)p+1-a-\tau}{p-1}}\right),$$

that is,

$$\mathcal{I}_f \leq C\left(T^{-\frac{kp}{p-1}}\left(\ln R + R^{\tau-1-\frac{a}{p-1}}\right) + R^{\frac{(\tau+1)p+1-a-\tau}{p-1}}\right). \tag{3.6}$$

Taking $T = R^\zeta$, where

$$\zeta > \max\left\{0, \frac{p-1}{kp}\left(\tau - 1 - \frac{a}{p-1}\right)\right\},$$

inequality (3.6) reduces to the following one

$$\mathcal{I}_f \leq C\left(R^{-\frac{kp\zeta}{p-1}}\ln R + R^{\rho_1} + R^{\rho_2}\right), \tag{3.7}$$

where

$$\rho_1 = \tau - 1 - \frac{a}{p-1} - \frac{kp\zeta}{p-1}$$

and

$$\rho_2 = \frac{(\tau+1)p+1-a-\tau}{p-1}.$$

Observe that due to (1.15) and the above choice of ζ, one has

$$\rho_i < 0, \quad i = 1, 2.$$

Hence, passing to the limit as $R \to \infty$ in (3.7), we obtain that $\mathcal{I}_f \leq 0$, which contradicts the positivity of $\mathcal{I}_f$. Consequently, (1.1) admits no weak solution. This proves part (I) of Theorem 1.5.

(II) For δ and ϵ satisfying, respectively, the restrictions

$$-\tau < \delta < \min\left\{N + \tau, 1 + \frac{2-a}{p-1}\right\} \tag{3.8}$$

and

$$0 < \epsilon < \left(-\delta^2 + N\delta + \mu\right)^{\frac{1}{p-1}}, \tag{3.9}$$

consider the function $u_{\delta,\epsilon}$ defined by

$$u_{\delta,\epsilon}(x) = \epsilon x_N |x|^{-\delta}, \quad x \in B_+^N \backslash \{0\}. \tag{3.10}$$

Observe that $N + 2\tau > 0$. Moreover, by (1.16), one has

$$-\tau < 1 + \frac{2-a}{p-1}.$$

This shows that the set of values δ satisfying (3.8) is nonempty. Observe also that $-\tau$ and $N + \tau$ are the roots of the polynomial function

$$P(\delta) = -\delta^2 + N\delta + \mu.$$

Hence, for $-\tau < \delta < N + \tau$, one has the positivity condition $P(\delta) > 0$. This shows that $P(\delta)^{\frac{1}{p-1}}$ is well-defined and the set of values of ϵ satisfying (3.9) is nonempty. On the other hand, elementary calculations yield to the formula

$$\mathcal{L}_\mu u_{\delta,\epsilon}(x) = \epsilon x_N P(\delta)|x|^{-\delta-2}.$$

Then, using (3.8), (3.9) and (3.10), for all $x \in B_+^N \backslash \{0\}$, we obtain that

$$\begin{aligned}
\mathcal{L}_\mu u_{\delta,\epsilon}(x) &\geq \epsilon^p x_N |x|^{-\delta-2} \\
&= |x|^{-a} \epsilon^p x_N^p |x|^{-\delta p} \left(x_N^{1-p} |x|^{-\delta-2+\delta p+a} \right) \\
&\geq |x|^{-a} u_{\delta,\epsilon}^p(x) |x|^{\delta(p-1)+a-p-1} \\
&\geq |x|^{-a} u_{\delta,\epsilon}^p(x).
\end{aligned}$$

Hence, for any δ and ϵ satisfying respectively (3.8) and (3.9), functions of the form (3.10) are nonnegative stationary solutions to (1.1) with

$$f(x) = \epsilon x_N, \quad x \in \Gamma_1^N.$$

This proves part (II) of Theorem 1.5. □

Proof of Theorem 1.9. We also use the contradiction argument. Suppose that $u \in L^p_{\mathrm{loc}}([0,\infty) \times B_+^N \backslash \{0\})$ is a weak solution to (1.2). By using the formula (1.12) and Lemma 2.2, for sufficiently large ℓ, T and R, we deduce the following inequality

$$\begin{aligned}
&\int_{\Omega_T} |x|^{-a} |u|^p I_T^\alpha \psi \, dx\, dt - \int_{\Gamma_{1,T}} f(x) \frac{\partial \psi}{\partial \nu_1} \, d\sigma\, dt + \sum_{i=0}^{k-1} (-1)^i \int_{B_1^c} u_{k-i-1}(x) \frac{\partial^i \psi}{\partial t^i}(0,x)\, dx \\
&\leq \int_{\Omega_T} |u| \left| \frac{\partial^k \psi}{\partial t^k} \right| dx\, dt + \int_{\Omega_T} |u| |\mathcal{L}_\mu \psi| \, dx\, dt,
\end{aligned}$$

where $\psi \in \Psi_T$ is the test function defined by (2.10). Proceeding as in the proof of Theorem 1.5 (I), by means of Young's inequality, we obtain that

$$\begin{aligned}
&- \int_{\Gamma_{1,T}} f(x) \frac{\partial \psi}{\partial \nu_1} \, d\sigma\, dt + \sum_{i=0}^{k-1} (-1)^i \int_{B_1^c} u_{k-i-1}(x) \frac{\partial^i \psi}{\partial t^i}(0,x)\, dx \\
&\leq C \left(\int_{\mathrm{supp}(\psi)} |x|^{\frac{a}{p-1}} |I_T^\alpha \psi|^{\frac{-1}{p-1}} \left| \frac{\partial^k \psi}{\partial t^k} \right|^{\frac{p}{p-1}} dx\, dt + \int_{\mathrm{supp}(\psi)} |x|^{\frac{a}{p-1}} |I_T^\alpha \psi|^{\frac{-1}{p-1}} |\mathcal{L}_\mu \psi|^{\frac{p}{p-1}} \, dx\, dt \right).
\end{aligned} \tag{3.11}$$

On the other hand, using (2.13) we get that

$$\begin{aligned}
- \int_{\Gamma_{1,T}} f(x) \frac{\partial \psi}{\partial \nu_1} \, d\sigma\, dt &= (2\tau + N) \left(\int_0^T T^{-\ell} (T-t)^\ell \, dt \right) \mathcal{I}_f \\
&= CT \mathcal{I}_f.
\end{aligned} \tag{3.12}$$

Moreover, since $u_i \geq 0$ for all $0 \leq i \leq k-1$, we deduce that

$$\begin{aligned}
\sum_{i=0}^{k-1} (-1)^i \int_{B_1^c} u_{k-i-1}(x) \frac{\partial^i \psi}{\partial t^i}(0,x)\, dx &= \sum_{i=0}^{k-1} (-1)^i \frac{d^i \gamma_T(0)}{dt^i} \int_{B_1^c} u_{k-i-1}(x) \xi_R(x)\, dx \\
&= C \sum_{i=0}^{k-1} T^{-i} \int_{B_1^c} u_{k-i-1}(x) \xi_R(x)\, dx \\
&\geq 0.
\end{aligned} \tag{3.13}$$

Hence, using Lemma 2.9, Lemma 2.10, (3.11), (3.12) and (3.13), we obtain the following inequality

$$T\mathcal{I}_f \leq C\left(T^{1-\frac{kp+\alpha}{p-1}}\left(\ln R + R^{\tau-1-\frac{a}{p-1}}\right) + T^{1-\frac{\alpha}{p-1}}R^{\frac{(\tau+1)p+1-a-\tau}{p-1}}\right),$$

that is,

$$\mathcal{I}_f \leq C\left(T^{-\frac{kp+\alpha}{p-1}}\ln R + T^{-\frac{kp+\alpha}{p-1}}R^{\tau-1-\frac{a}{p-1}} + T^{-\frac{\alpha}{p-1}}R^{\frac{(\tau+1)p+1-a-\tau}{p-1}}\right). \tag{3.14}$$

Finally, taking $T = R^\theta$, where

$$\theta > \max\left\{0, \frac{p-1}{kp}\left(\tau - 1 - \frac{a}{p-1}\right), \frac{(\tau+1)p+1-a-\tau}{\alpha}\right\},$$

and passing to the limit as $R \to \infty$ in (3.14), we arrive to contradiction with the positivity condition $\mathcal{I}_f > 0$. This completes the proof of Theorem 1.9. □

4. Conclusions

The knowledge of intrinsic properties to any physical system is constrained by uncertainty relations among the constituents of the same system. Referring to quantum mechanics, the uncertainty principle (namely, Heisenberg's principle) bounds the accuracy with which the outcomes of a pair of certain measurements (for example, position and momentum of a particle) can be predicted. A variety of mathematical inequalities appeared in the literature to model this situation, showing the relevance to deal with singular potential terms and memory effects. Now, the Hardy inequality (refer to the type inequality (1.3)) can be considered as a first attempt to represent the above Heisenberg's principle. We recall that the Hardy inequality was originally given in one dimension and Leray provided a analogous inequality in the case $\mathbb{R}^3$ (refer again to (1.3)) and in the context of Navier-Stokes equations; see the recent book [23] for more information. So, (1.3) gives us a dimensional homogeneity (equivalence) condition between the inverse square potential and the gradient. This means that the Leray-Hardy potential possesses the same homogeneity of the Dirichlet integral for the Laplace differential operator (see again [23]). In such sense, this singular potential is significant enough to approach the process of finding solutions to various differential equations (also dealing with energy eigenfunctions and eigenvalues). As already mentioned in Section 1, a main feature of our results here is the fact that we study the existence of solutions without using any restrictions on the sign of solutions to problems (1.1) and (1.2). Moreover, since problem (1.2) reduces to problem (1.1) when $\alpha \to 0^+$ (recall that $\alpha > 0$ represents a memory term), then solving (1.2) means to observe the influence of this "memory" on the critical behavior of problem (1.1).

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