A new approach to the generalization of Darbo's fixed point problem by using simulation functions with application to integral equations

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Abstract. We investigate the existence of fixed points of self-mappings via simulation functions and measure of noncompactness. We use different classes of additional functions to get some general contractive inequalities. As an application of our main conclusions, we survey the existence of a solution for a class of integral equations under some new conditions. An example will be given to support our results.

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1. Introduction and Preliminaries

The research in fixed point theory is one of the most investigated topics in metric spaces' theory. Indeed, the method of proofs still very descendent from the original one in the contraction principle is a simple and powerful tool to approach various problems in nonlinear analysis. Here, we focus our attention on the concept of simulation function, recently introduced in Khojasteh-Shukla-Radenović [10] which is a way to unify and extend certain existing fixed point results in the literature. Based on this paper and the subsequent one of Argoubi-Samet-Vetro [4] (where a slight modification of the definition of simulation function is proposed), many authors established new existence results in generalized metric spaces and solved fixed-point problems involving integral equations, differential equations, variational inequalities and special classes of operators. Such a kind of problems were investigated in [12, 13, 20, 22, 24]. Here, we prove the existence of fixed points of certain mappings via simulation functions and measure of noncompactness. We shall also indicate that several results in the literature can be derived from our main theorems. For some recent works in this direction see [1, 6, 9, 17, 19, 21]. For the best proximity point problem, see [2, 11].

We construct our results on the following basic concepts.

Definition 1.1 ([10]). A function $\sigma : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is said to be simulation function if it fulfills: $(\sigma_1) \ \sigma(0, 0) = 0$:

 $(\sigma_2) \sigma(t, u) < u - t$ for all t, u > 0;

(σ_3) if $\{t_n\}$, $\{u_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \to \infty} t_n = \lim_{n \to \infty} u_n > 0$, then $\lim_{n \to \infty} \sup \sigma(t_n, u_n) < 0$.

 $n \rightarrow \infty$

Let Σ^s be the collection of all simulation functions $\sigma : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$. On account of the property (σ_2) , we conclude that

$$\sigma(t,t) < 0$$
 for all $t > 0$.

Example. Let $\sigma : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping such that $\sigma(t, u) = \frac{u}{2} - t$ for all $t, u \in [0, \infty)$. It is obvious that σ is a simulation function. For more examples of simulation functions see [10].

Suppose (X, d) is a metric space, T is a self-mapping on X and $\sigma \in \Sigma^s$. We say that T is a Σ^s -contraction with respect to σ (see [10]), if

$$\sigma(d(Tx, Ty), d(x, y)) \ge 0, \quad \text{ for all } x, y \in X.$$

For all distinct $x, y \in X$, by (σ_2) , we have $d(Tx, Ty) \neq d(x, y)$. Thus, we conclude that whenever a Σ^s -contraction T (in a metric space) has a fixed point, then it is necessarily unique.

Theorem 1.2. Every Σ^s -contraction on a complete metric space has a unique fixed point.

We also need the following class of operators given in [3] by Altun and Turkoglu.

Definition 1.3. Let $F([0,\infty))$ be the class of all functions $f:[0,\infty) \to [0,\infty]$. Then by Θ we denote the class of all operators

$$O(\bullet; \cdot) : F([0, \infty)) \to F([0, \infty)), \text{ by } f \mapsto O(f; \cdot)$$

satisfying the following conditions:

- (i) O(f;t) > 0 for t > 0 and O(f;0) = 0;
- (ii) $O(f;t) \leq O(f,s)$ for $t \leq s$;
- (iii) $\lim_{n \to \infty} O(f; t_n) = O(f; \lim_{n \to \infty} t_n);$

(iv) $O(f; \max\{t, s\}) = \max\{O(f; t), O(f; s)\}$ for some $f \in F([0, \infty))$.

Definition 1.4. Let X and Y be normed linear spaces and K be a subset of X. A mapping $T: K \to Y$ is said to be a compact operator if T is continuous and maps bounded sets into relatively compact sets.

Here, we recall the well-known generalization of Schauder's fixed point theorem.

Theorem 1.5. Let K be a nonempty, bounded, closed and convex subset of a Banach space X and $T: K \to K$ be a compact operator. Then T has a fixed point.

An improved version of Theorem 1.5 was presented by Darbo [8] using a notion of *measure of noncompactness*.

Definition 1.6 (Kuratowski, 1930). Let (X, d) be a metric space and Σ^b be the family of all nonempty and bounded subsets of X. The function $\alpha : \Sigma^b \to [0, \infty)$ defined as

 $\alpha(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by finitely many sets with diameter } \le \varepsilon\},\$

for every $B \in \Sigma^b$, is called the Kuratowski measure of noncompactness.

We also mention that if we define $\chi: \Sigma^b \to [0,\infty)$ by

 $\chi(B) = \inf\{\varepsilon > 0 : B \text{ can be covered by finitely many balls with radius } \le \varepsilon\},\$

for every $B \in \Sigma^b$, then χ is said to be a *Hausdorff measure of noncompactness* which was firstly introduced in Gohberg-Goldenštein-Markus[14]; clearly it is an extension of Kuratowski measure of noncompactness. We refer to Ayerbe Toledano-Dominguez Benavides-López-Acedo[5] for more interesting information related to measures of noncompactness.

We collect some interesting properties of Kuratowski and Hausdorff measures of noncompactness as follows.

Definition 1.7. Let (X, d) be a complete metric space and Σ^b be the family of all bounded subsets of X. A function $\mu : \Sigma^b \to [0, \infty)$ is called a measure of noncompactness (MNC, for short) if it satisfies the following conditions:

(i) $\mu(A) = 0$ iff A is relatively compact;

(*ii*) $\mu(A) = \mu(\overline{A})$ for all $A \in \Sigma^b$;

(*iii*) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for all $A, B \in \Sigma^b$.

If μ is an MNC on Σ^b , then the following properties can be concluded immediately (see Ayerbe Toledano-Dominguez Benavides-López-Acedo [5]):

 (p_1) if $A \subseteq B$, then $\mu(A) \leq \mu(B)$;

 $(p_2) \ \mu(A \cap B) \leq \min\{\mu(A), \mu(B)\}$ for all $A, B \in \Sigma^b$;

 (p_3) if A is a finite set, then $\mu(A) = 0$;

 (p_4) if $\{A_n\}$ is a decreasing sequence of nonempty, bounded and closed subsets of X such that $\lim_{n \to \infty} \mu(A_n) = 0$, then $A_{\infty} := \bigcap_{n \ge 1} A_n$ is nonempty and compact.

Also, if X is a Banach space and we denote by co(A) the closed and convex hull of a set A, then

 $(p_5) \ \mu(\operatorname{co}(A)) = \mu(A) \text{ for all } A \in \Sigma^b;$ $(p_6) \ \mu(tA) = |t|\mu(A), \text{ for any number } t \text{ and } A \in \Sigma^b;$ $(p_7) \ \mu(A+B) \le \mu(A) + \mu(B), \text{ for all } A, B \in \Sigma^b.$

2. Main results

We establish the existence of at least one fixed point for self-mappings under suitable hypotheses. So, to obtain our first theorem, we use the following class of functions.

Definition 2.1. Let Ψ denote the class of all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

- (i) ψ is non-decreasing;
- (ii) ψ is continuous;
- (iii) $\psi^{-1}(0) = 0;$
- (iv) $\psi(t) < t$ for t > 0;
- (iv) $\lim_{n \to \infty} \psi^n(t) = 0$ for each $t \ge 0$.

Theorem 2.2. Let C be a nonempty, bounded, closed and convex subset of a Banach space X and let $T : C \to C$ be a continuous operator satisfying

$$\sigma(O(f;\mu(T(X)) + \varphi(\mu(T(X)))), \psi(O(f;\mu(X) + \varphi(\mu(X))))) \ge 0, \quad (2.1)$$

for any $\emptyset \neq C \subseteq X$, where μ is an arbitrary MNC, $f \in F([0,\infty)), O(\bullet; \cdot) \in \Theta, \psi \in \Psi, \varphi : [0,\infty) \to [0,\infty)$ is a continuous function. Then T has at least one fixed point in C.

Proof. We construct a sequence $\{C_n\}_{n=0}^{\infty}$ by

$$C_0 := C, \quad C_n := \operatorname{co}(T(C_{n-1})), \text{ for all } n = 1, 2, \dots$$

Now, let us prove that

$$C_{n+1} \subseteq C_n, \quad T(C_n) \subseteq C_n, \quad \text{for all } n = 1, 2, \dots$$
 (2.2)

The first inclusion will be proved via mathematical induction method. Let n = 0. Since $C_0 = C$, C is convex and closed, then we have $C_1 = co(T(C_0)) \subset C_0$. Now assume that $C_n \subset C_{n-1}$. Then $co(T(C_n)) \subset co(T(C_{n-1}))$. So we obtain that $C_{n+1} \subset C_n$. The second inclusion follows immediately from the first one

$$T(C_n) \subset \operatorname{co}(T(C_n)) = C_{n+1} \subset C_n.$$

If there exists $N \in \mathbb{N}$ such that $\mu(C_N) + \varphi(\mu(C_N)) = 0$ then $\mu(C_N) = \varphi(\mu(C_N)) = 0$ so C_N is compact and the Schauder's fixed point theorem ensures that T has a fixed point in C_N where $C_N \subset C$.

Suppose $A_n := \mu(C_n) + \varphi(\mu(C_n)) > 0$ for each $n \in \mathbb{N}$. Now, by (2.1), we have

$$\begin{split} B_n &:= \sigma(O(f; \mu(C_{n+1})) + \varphi(\mu(C_{n+1}))), \psi(O(f; \mu(C_n) + \varphi(\mu(C_n))))) \\ &= \sigma(O(f; \mu(\operatorname{co}(T(C_n))) + \varphi(\mu(\operatorname{co}(T(C_n))))), \psi(O(f; \mu(C_n) + \varphi(\mu(C_n))))) \\ &= \sigma(O(f; \mu(T(C_n)) + \varphi(\mu(T(C_n)))), \psi(O(f; \mu(C_n) + \varphi(\mu(C_n))))) \\ &= \sigma(O(f; A_{n+1}), \psi(O(f; A_n))) \\ &\leq \psi(O(f; A_n)) - O(f; A_{n+1}). \\ & \text{So, we get} \end{split}$$

$$0 < O(f; A_{n+1}) \le \psi(O(f; A_n)) < O(f; A_n), \tag{2.3}$$

and also $0 < A_{n+1} \leq A_n$. Thus $\lim_{n \to \infty} A_n = s$ for some $s \geq 0$. If s > 0, by using the properties (*ii*) and (*iii*) of ψ and the property (*iii*) of $O(\bullet; \cdot)$, we must have

$$\lim_{n \to \infty} \psi(O(f; A_n)) = \psi(O(f; \lim_{n \to \infty} A_n)) = \psi(O(f; s)).$$

By (2.3), we obtain

$$\lim_{n \to \infty} O(f; A_n) = \lim_{n \to \infty} \psi(O(f; A_n)) = r.$$

Now, by (σ_3) , we get

$$0 \le \limsup_{n \to \infty} \sigma(O(f; A_n), \psi(O(f; A_n))) < 0,$$

which is a contradiction. This ensures that r = 0. Hence, by the property (*iii*) of $O(\bullet; \cdot)$ and the continuity of φ , we conclude that

$$\lim_{n \to \infty} O(f; A_n) = \lim_{n \to \infty} O(f; \mu(C_n) + \varphi(\mu(C_n)))$$
$$= O(f; \lim_{n \to \infty} \mu(C_n) + \varphi(\lim_{n \to \infty} \mu(C_n))).$$

It now follows from the property (i) of $O(\bullet; \cdot)$ that $\lim_{n \to \infty} \mu(C_n) = 0$. Now the conclusion follows by (2.2) and (p_4) .

In the next theorem, we use the following class of Geraghty's functions in [15] (see also [7]).

Definition 2.3. Let \triangle denote the class of all functions $\beta : [0, \infty) \rightarrow [0, 1)$ which satisfy the condition

$$t_n \to 0$$
 whenever $\beta(t_n) \to 1$.

Theorem 2.4. Let C be a nonempty, bounded, closed and convex subset of a Banach space X and let $T : C \to C$ be a continuous operator satisfying

$$\sigma(\psi(O(f;\mu(T(X)) + \varphi(\mu(T(X))))), \\ \beta(O(f;\psi(\mu(X))))\psi(O(f;\mu(X) + \varphi(\mu(X))))) \ge 0,$$
(2.4)

for any $\emptyset \neq C \subseteq X$, where μ is an arbitrary MNC, $f \in F([0,\infty))$, $O(\bullet; \cdot) \in \Theta$, $\varphi : [0,\infty) \rightarrow [0,\infty)$ is a continuous function, $\beta \in \Delta$ and $\psi \in \Psi$. Then T has at least one fixed point in C.

Proof. According to the notation used in Theorem 2.2 and by (2.4), we have

$$B_n := \sigma(\psi(O(f; \mu(C_{n+1}) + \varphi(\mu(C_{n+1})))), \\ \beta(O(f; \psi(\mu(C_n))))\psi(O(f; \mu(C_n) + \varphi(\mu(C_n))))) \\ = \sigma(O(f; \mu(T(C_n)) + \varphi(\mu(T(C_n)))), \\ \beta(O(f; \psi(\mu(C_n))))\psi(O(f; \mu(C_n) + \varphi(\mu(C_n))))) \\ = \sigma(\psi(O(f; A_{n+1})), \beta(O(f; \psi(\mu(C_n))))\psi(O(f; A_n))) \\ \leq \beta(O(f; \psi(\mu(C_n))))\psi(O(f; A_n)) - \psi(O(f; A_{n+1})).$$

So, we obtain

$$0 < \psi(O(f; A_{n+1})) \le \beta(O(f; \psi(\mu(C_n))))\psi(O(f; A_n)) < \psi(O(f; A_n)) < O(f; A_n).$$
(2.5)

Therefore,

$$0 < O(f; A_{n+1}) \le O(f; A_n),$$

and so

 $0 < A_{n+1} \le A_n.$

Assume that $\lim_{n\to\infty} O(f; A_n) = r$ for some $r \ge 0$ and $\lim_{n\to\infty} A_n = s$ for some $s \ge 0$. If s > 0, then by (2.5) we have

$$0 < \frac{\psi(O(f; A_{n+1}))}{\psi(O(f; A_n))} \le \beta(O(f; \psi(\mu(C_n)))) < 1.$$

Using the property of $\beta \in \Delta$ and the property (*iii*) of $O(\bullet; \cdot)$ we get

$$\lim_{n \to \infty} \beta(O(f; \psi(\mu(C_n)))) = 1$$

which implies

$$\lim_{n \to \infty} O(f; \psi(\mu(C_n))) = 0$$

Since

$$\lim_{n \to \infty} O(f; \psi(\mu(C_n))) = O(f; \psi(\lim_{n \to \infty} \mu(C_n))),$$

we conclude that

$$O(f; \psi(\lim_{n \to \infty} \mu(C_n))) = 0$$

which implies

$$\lim_{n \to \infty} \mu(C_n) = 0$$

and so

 $\lim_{n \to \infty} A_n = 0,$

while $\lim_{n \to \infty} A_n = s > 0$, which is impossible.

By particularizing the choice of the simulation function and setting $\beta(t) = k$ with $k \in (0, 1)$, one can obtain the following result. This means that various existing results in the literature can be obtained as particular cases of our theorem.

Corollary 2.5. Let C be a nonempty, bounded, closed and convex subset of a Banach space X and let $T : C \to C$ be a continuous operator satisfying

$$O(f; \mu(T(X)) + \varphi(\mu(T(X)))) \leq \beta(O(f; \psi(\mu(X))))\psi(O(f; \mu(X) + \varphi(\mu(X)))),$$

for any $\emptyset \neq C \subseteq X$, where μ is an arbitrary MNC, $f \in F([0, \infty)), O(\bullet; \cdot) \in \Theta$,
 $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function, $\beta \in \Delta$ and $\psi \in \Psi$. Then T has
at least one fixed point in C.

In what follows, we use the following class of Mizoguchi-Takahashi's functions (MT-functions, for short) which was presented in [18].

Definition 2.6. Let Υ denote the class of all MT-functions $\chi: [0,\infty) \to [0,1)$ satisfying the condition

> $\limsup \chi(s) < 1 \quad \text{for all } t \in [0, \infty).$ $s \rightarrow t^+$

We note that if $\chi : [0,1) \to [0,1)$ is a non-decreasing function or a non-increasing function, then χ is an MT-function.

Definition 2.7. Let Ω denote the set of all functions $\omega : [0, \infty) \to [0, \infty)$ satisfying:

(i) ω is non-decreasing:

(ii) $\omega(t) = 0$ if and only if t = 0.

Theorem 2.8. Let C be a nonempty, bounded, closed and convex subset of a Banach space X and let $T: C \to C$ be a continuous operator satisfying

$$\begin{aligned} \sigma(\omega(O(f;\mu(T(X)) + \varphi(\mu(T(X))))), \\ \chi(O(f;\omega(\mu(X))))\omega(O(f;\mu(X) + \varphi(\mu(X))))) \geq 0. \end{aligned}$$

for any $\emptyset \neq C \subseteq X$, where μ is an arbitrary MNC, $f \in F([0,\infty)), O(\bullet; \cdot) \in \Theta$, $\varphi: [0,\infty) \to [0,\infty)$ is a continuous function, $\chi \in \Upsilon$ and $\omega \in \Omega$. Then T has at least one fixed point in C.

Proof. Using again the notation in the previous theorems, we get

$$\begin{split} B_n &:= \sigma(\omega(O(f; \mu(C_{n+1})) + \varphi(\mu(C_{n+1})))), \\ &\chi(O(f; \omega(\mu(C_n))))\omega(O(f; \mu(C_n) + \varphi(\mu(C_n))))) \\ &= \sigma(\omega(O(f; \mu(\operatorname{co}(T(C_n))) + \varphi(\mu(\operatorname{co}(T(C_n))))), \\ &\chi(O(f; \omega(\mu(C_n))))\omega(O(f; \mu(C_n) + \varphi(\mu(C_n))))) \\ &= \sigma(\omega(O(f; \mu(T(C_n)) + \varphi(\mu(T(C_n)))), \\ &\chi(O(f; \omega(\mu(C_n))))\omega(O(f; \mu(C_n) + \varphi(\mu(C_n))))) \\ &= \sigma(\omega(O(f; A_{n+1})), \chi(O(f; \omega(\mu(C_n))))\omega(O(f; A_n))) \\ &\leq \chi(O(f; \omega(\mu(C_n))))\omega(O(f; A_n)) - \omega(O(f; A_{n+1})). \end{split}$$

Therefore, we have

$$\omega(O(f; A_{n+1})) \le \chi(O(f; \omega(\mu(C_n)))) \omega(O(f; A_n)) \le \omega(O(f; A_n)).$$
(2.6)
So, we get

$$\lim_{n \to \infty} \omega(O(f; A_n)) = v, \tag{2.7}$$

for some $v \ge 0$. If v > 0, since $\chi \in \Upsilon$, we have $\limsup \chi(O(f;t)) < 1$ and $t \rightarrow v^+$ $\chi(O(f;v)) < 1$, then there exists $\delta \in [0,1), \varepsilon > 0$ such that $\chi(O(f;t)) < \delta$ for all $t \in [v, v + \varepsilon)$. By (2.7), take N > 0 such that

$$v \le \omega(O(f; A_n)) < v + \varepsilon \text{ for all } n \ge N.$$

Then, by (2.6) we have

$$\omega(O(f; A_{n+1})) \le \chi(O(f; \omega(\mu(C_n)))) \omega(O(f; A_n)) \le \delta \omega(O(f; A_n)), \quad (2.8)$$

for all $n \ge N$. Passing to the limit in (2.8) we have $v \le \delta v$, which means v = 0 and so

$$\lim_{n \to \infty} \omega(O(f; A_n)) = 0.$$

Since $\{O(f; \omega(A_n))\}$ is a non-increasing sequence and ω is non-decreasing, then by the property (*ii*) of $O(\bullet; \cdot)$, we have that $\{A_n\}$ is also a non-increasing sequence of positive numbers. So $\lim_{n\to\infty} A_n = u$ for some $u \ge 0$. Since ω is nondecreasing and by the properties (*ii*) - (*iii*) of $O(\bullet; \cdot)$, we have

$$\omega(O(f; A_n)) \ge \omega(O(f; u)),$$

and hence

$$0 = \lim_{n \to \infty} \omega(O(f; A_n)) \ge \omega(O(f; u)),$$

which implies that u = 0. The rest of proof is similar to the proofs of previous theorems, so we omit the details.

3. Application to integral equations

We prove the existence of at least one solution for the integral equation

$$u(t) = g(t, u(t)) + \int_0^t G(t, s, u(s)) ds, \quad t \in [0, \infty),$$
(3.1)

where $G : [0, \infty) \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $g : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are continuous (see also Hussain-Kanwal-Mitrović-Radenović [16], Vetro-Vetro [23]).

By $BC([0,\infty))$ we denote the space of all bounded and continuous real functions on $[0,\infty)$. Also, we use the norm $||u|| = \sup\{|u(t)| : t \in [0,\infty)\}$. Moreover, let $\kappa \in [0,\infty)$ and C be a nonempty, bounded, closed and convex subset of $BC([0,\infty))$. For $u \in C$ and $\varepsilon > 0$, let $\omega^{\kappa}(u,\varepsilon)$ the modulus of continuity of u on $[0,\kappa]$, that is,

$$\omega^{\kappa}(u,\varepsilon) = \sup\{|u(t) - u(s)| : \varepsilon \ge |t - s|, \, t, s \in [0,\kappa]\}.$$

Set $\omega^{\kappa}(C,\varepsilon) = \sup\{\omega^{\kappa}(u,\varepsilon) : u \in C\}$, $\omega_0^{\kappa}(C) = \lim_{\varepsilon \to 0} \omega^{\kappa}(C,\varepsilon)$ and $\omega_0(C) = \lim_{\kappa \to \infty} \omega_0^{\kappa}(C)$. Fixed $t \in [0,\infty)$, we get $C(t) = \{u(t) : u \in C\}$ and hence we consider the measure of noncompactness μ on the family of all nonempty bounded, closed and convex subsets of $BC([0,\infty))$, say $B(BC([0,\infty)))$, as follows

$$\mu(C) = \omega_0(C) + \limsup_{t \to \infty} \operatorname{diam} C(t), \qquad (3.2)$$

where diam $C(t) = \sup\{|u(t) - v(t)| : u, v \in C\}$. Next, we define the operator T on $BC([0, \infty))$ by

$$(Tu)(t) = g(t, u(t)) + \int_0^t G(t, s, u(s))ds, \text{ for all } t \in [0, \infty), \ u \in BC([0, \infty)).$$
(3.3)

So, the problem of existence of a solution to (3.1) is equivalent to the problem of existence of a fixed point to (3.3). We establish the following result.

Theorem 3.1. Let T be the self-operator on $BC([0,\infty))$ in (3.3). If:

- (i) the function $t \to g(t,0)$ is a member of the space $BC([0,\infty))$;
- (ii) there exists $\alpha \in [1, +\infty)$ such that, for each $t \in [0, \infty)$ and for all $u, v \in \mathbb{R}$, we have $|u v| \ge 2e^{\alpha}|g(t, u) g(t, v)|$;
- (iii) there are continuous $c_0, c_1 : [0, \infty) \to [0, \infty)$ with $\lim_{t \to \infty} c_0(t) \int_0^t c_1(s) ds = 0$ and $c_0(t)c_1(s) \ge |G(t, s, u)|$ for all $t, s \in [0, \infty)$ such that $t \ge s$, and for each $u \in \mathbb{R}$;
- (iv) there exists a positive r_0 such that $(e^{\alpha} 1)r_0 \ge e^{\alpha}m$, where m is given by $m = \sup_{t\ge 0} \{|g(t,0)| + c_0(t) \int_0^t c_1(s)ds\},\$

then T admits a fixed point in $BC([0,\infty))$.

Proof. We prove that T is well-defined on $B(r_0) = \{u \in BC([0,\infty)) : ||u|| \le r_0\}$. From (3.3), by the assumptions on g and G, we have immediately that Tu is continuous for all $u \in BC([0,\infty))$. Moreover, we get

$$\begin{aligned} |(Tu)(t)| &= |g(t, u(t)) - g(t, 0) + g(t, 0) + \int_0^t G(t, s, u(s))ds| \\ &\leq |g(t, u(t)) - g(t, 0)| + |g(t, 0)| + |\int_0^t G(t, s, u(s))ds| \\ &\leq \frac{1}{2e^{\alpha}}|u(t)| + |g(t, 0)| + c_0(t)\int_0^t c_1(s)ds \leq \frac{1}{2e^{\alpha}}|u(t)| + m, \end{aligned}$$

where m is as in (iv). So, we have $2||Tu|| \le e^{-\alpha}||u|| + 2m$, which means that T maps $B(r_0)$ into $B(r_0)$.

Now, we show that T is continuous on $B(r_0)$. We fix $\varepsilon > 0$ so that, for $u, v \in B(r_0)$ with $||u - v|| \le \varepsilon$, we have

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\leq \frac{1}{2e^{\alpha}} |u(t) - v(t)| + \int_{0}^{t} |G(t, s, u(s)) - G(t, s, v(s))| ds \\ &\leq \frac{1}{2e^{\alpha}} |u(t) - v(t)| + \int_{0}^{t} (|G(t, s, u(s))| + |G(t, s, v(s))|) ds \\ &\leq \frac{1}{2e^{\alpha}} |u(t) - v(t)| + 2c_{0}(t) \int_{0}^{t} c_{1}(s) ds, \end{aligned}$$
(3.4)

for all $t \in (0, \infty)$. So, by (iii), there is $\kappa > 0$ such that

$$2c_0(t) \int_0^t c_1(s)ds \le \varepsilon \quad \text{for all } t \ge \kappa.$$
(3.5)

From (3.4) and (3.5), we obtain

$$|(Tu)(t) - (Tv)(t)| \le 2\varepsilon \quad \text{for all } t \ge \kappa.$$
(3.6)

For the modulus of continuity, we set

$$\begin{split} \omega^{\kappa}(G,\varepsilon) &= \sup\{|G(t,s,u) - G(t,s,v)| : t, s \in [0,\kappa], u, v \in [-r_0,r_0], |u-v| \leq \varepsilon\}.\\ \text{As } G(t,s,u) \text{ is uniformly continuous on } [0,\kappa] \times [0,\kappa] \times [-r_0,r_0], \text{ we get} \end{split}$$

$$\lim_{\varepsilon \to 0} \omega^{\kappa}(G, \varepsilon) = 0.$$

By (3.4), for an arbitrarily fixed $t \in [0, \kappa]$, we deduce that

$$|(Tu)(t) - (Tv)(t)| \le \varepsilon + \int_0^t \omega^{\kappa}(G,\varepsilon) ds = \varepsilon + \kappa \omega^{\kappa}(G,\varepsilon).$$

Consequently, from (3.6) and the above facts on $\omega^{\kappa}(G,\varepsilon)$, we conclude that T is continuous on $B(r_0)$.

Next, we prove that T has a fixed point in $B(r_0)$. In fact, let C be an arbitrary nonempty subset of $B(r_0)$, fix $\varepsilon > 0$ and $\kappa > 0$, and choose arbitrarily $t, s \in [0, \kappa]$ such that $|t - s| \leq \varepsilon$. Without loss of generality we assume that s < t. So, for $u \in C$, we have

$$\begin{aligned} |(Tu)(t) - (Tu)(s)| \\ &\leq |g(t, u(t)) - g(s, u(s))| + |\int_0^t G(t, z, u(z))dz - \int_0^s G(s, z, u(z))dz| \\ &\leq |g(t, u(t)) - g(s, u(t))| + |g(s, u(t)) - g(s, u(s))| \\ &+ \int_0^t |G(t, z, u(z)) - G(s, z, u(z))|dz + \int_s^t |G(s, z, u(z))|dz \\ &\leq \omega_1^\kappa(g, \varepsilon) + \frac{\omega^\kappa(u, \varepsilon)}{2e^\alpha} + \int_0^t \omega_1^\kappa(G, \varepsilon)dz + c_0(s) \int_s^t c_1(z)dz \\ &\leq \omega_1^\kappa(g, \varepsilon) + \frac{\omega^\kappa(u, \varepsilon)}{2e^\alpha} + \kappa\omega_1^\kappa(g, \varepsilon) + \varepsilon \sup\{c_0(s)c_1(t) : t, s \in [0, \kappa]\}, \quad (3.7) \end{aligned}$$

where $\omega_1^{\kappa}(g,\varepsilon) = \sup\{|g(t,u) - g(s,u)| : t, s \in [0,\kappa], u \in [-r_0,r_0], |t-s| \le \varepsilon\}$ and $\omega_1^{\kappa}(G,\varepsilon) = \sup\{|G(t,z,u) - G(s,z,u)| : t, s, z \in [0,\kappa], u \in [-r_0,r_0], |t-s| \le \varepsilon\}.$

From the uniform continuity of g on $[0, \kappa] \times [-r_0, r_0]$ and G on $[0, \kappa] \times [0, \kappa] \times [-r_0, r_0]$, we infer that $\lim_{\varepsilon \to 0} \omega_1^{\kappa}(g, \varepsilon) = 0$ and $\lim_{\varepsilon \to 0} \omega_1^{\kappa}(G, \varepsilon) = 0$. Since c_0, c_1 are two continuous functions on $[0, \infty)$, by (iii) we deduce $\sup\{c_0(s)c_1(t): t, s \in [0, \kappa]\}$ is finite. All these remarks and (3.7) imply that $\omega_0^{\kappa}(T(C)) \leq \lim_{\varepsilon \to 0} \frac{\omega^{\kappa}(C, \varepsilon)}{2e^{\alpha}}$. So, $\omega_0^{\kappa}(T(C)) \leq \frac{\omega_0^{\kappa}(C)}{2e^{\alpha}}$ and hence

$$\omega_0(T(C)) \le \frac{\omega_0(C)}{2e^{\alpha}}.$$
(3.8)

Next we choose two arbitrary functions $u, v \in C$ so that, for $t \in [0, \infty)$, we have

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| \\ &\leq |g(t, u(t)) - g(t, v(t))| + \int_0^t |G(t, s, u(s))| ds + \int_0^t |G(t, s, v(s))| ds \\ &\leq \frac{1}{2e^\alpha} |u(t) - v(t)| + 2c_0(t) \int_0^t c_1(s) ds. \end{aligned}$$

Starting from the above inequality, using the notion of diameter of a set, we deduce that diam $(T(C))(t) \leq \frac{1}{2e^{\alpha}} \operatorname{diam} C(t) + 2c_0(t) \int_0^t c_1(s) ds$, and

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so we get

$$\limsup_{t \to \infty} \operatorname{diam}(T(C))(t) \le \frac{1}{2e^{\alpha}} \limsup_{t \to \infty} \operatorname{diam}(C)(t).$$
(3.9)

By (3.2), (3.8) and (3.9), we deduce that

$$\mu(T(C)) \le \frac{1}{2e^{\alpha}}\mu(C),$$

which means that (2.1) is satisfied with $\sigma : [0, \infty) \times [0, \infty) \to \mathbb{R}$ given as $\sigma(v, s) = \frac{s}{2} - v$ for all $s, v \in [0, \infty), \psi : [0, \infty) \to [0, \infty)$ given as $\psi(s) = e^{-\alpha s} s$ for all $s \in [0, \infty)$ and $\alpha \in [1, \infty), \varphi : [0, \infty) \to [0, \infty)$ given as $\varphi(s) = 0$ for all $s \in [0, \infty)$, and $O(f; \cdot) : [0, \infty) \to [0, \infty)$ given as the identity function on $[0, \infty)$.

So, all the assumptions of Theorem 2.2 hold true and hence T has a fixed point in $B(r_0)$, which is a solution to (3.1) in $BC([0,\infty))$.

Finally, we give an illustrative example.

Example. Consider the following functional integral equation

$$u(t) = \frac{1+t^2}{2+t^2} \frac{\ln(1+|u(t)|)}{2\sqrt{e^{\alpha}} + \ln(1+|u(t)|)} + 2e^{-t} + \int_0^t \frac{\cos u(t)}{1+t^2} e^{-t} e^{s/2} ds \quad t \in [0,\infty),$$

in the space $BC([0,\infty))$.

Clearly, $g: [0,\infty) \times \mathbb{R} \to \mathbb{R}$, given by

$$g(t, u(t)) = \frac{1+t^2}{2+t^2} \frac{\ln(1+|u(t)|)}{2\sqrt{e^{\alpha}} + \ln(1+|u(t)|)} + 2e^{-t}, \quad \alpha \in [1, \infty),$$

is continuous and such that the function $t \to g(t, 0)$ is an element of $BC([0, \infty))$.

So, condition (i) of Theorem 3.1 holds.

Moreover, we have

$$\begin{split} 0 &\leq |g(t,u) - g(t,v)| \\ &= \frac{1+t^2}{2+t^2} \left| \frac{\ln(1+|u(t)|)}{2\sqrt{e^{\alpha}} + \ln(1+|u(t)|)} - \frac{\ln(1+|v(t)|)}{2\sqrt{e^{\alpha}} + \ln(1+|v(t)|)} \right| \\ &\leq 2 \left| \frac{\ln(1+|u(t)|) - \ln(1+|v(t)|)}{(2\sqrt{e^{\alpha}} + \ln(1+|u(t)|))(2\sqrt{e^{\alpha}} + \ln(1+|v(t)|))} \right| \\ &\leq \frac{1}{\sqrt{e^{\alpha}}} \left| \frac{\ln\frac{(1+|u(t)|)}{(2\sqrt{e^{\alpha}} + \ln(1+|u(t)|) + \ln(1+|v(t)|))}}{(2\sqrt{e^{\alpha}} + \ln(1+|u(t)|) + \ln(1+|v(t)|))} \right| \\ &\leq \frac{1}{\sqrt{e^{\alpha}}} \left| \frac{\ln\left(1 + \frac{(1+|u(t)| - 1 - |v(t)|)}{(1+|v(t)|)}\right)}{(2\sqrt{e^{\alpha}} + \ln(1+|u(t)|) + \ln(1+|v(t)|))} \right| \\ &\leq \frac{1}{\sqrt{e^{\alpha}}} \left| \frac{\ln\left(1 + |u(t) - |v(t)|\right)}{(2\sqrt{e^{\alpha}} + \ln(1+|u(t)| + |v(t)|))} \right| \\ &\leq \frac{1}{\sqrt{e^{\alpha}}} \left| \frac{\ln\left(1 + |u(t) - |v(t)|\right)}{(2\sqrt{e^{\alpha}} + \ln(1+|u(t) - v(t)|))} \right| \\ &= \frac{1}{\sqrt{e^{\alpha}}} \frac{\ln(1 + |u(t) - v(t)|)}{2\sqrt{e^{\alpha}} + \ln(1+|u(t) - v(t)|)} \\ &\leq \frac{1}{2e^{\alpha}} |u(t) - v(t)| \quad \text{for all } \alpha \in [1, \infty), \end{split}$$

which means that condition (ii) of Theorem 3.1 holds.

Let $c_1, c_2: [0, \infty) \to [0, \infty)$ be defined by

$$c_1(t) = e^{-t}, \quad c_2(s) = e^{s/2} \text{ for all } t, s \in [0, \infty),$$

Now,

$$|(G(t,s,u)| \le \frac{\cos u(t)}{1+t^2} e^{-t} e^{s/2} \le e^{-t} e^{s/2} \quad \text{for all } t, s \in [0,\infty).$$

Clearly,

$$\lim_{t \to \infty} e^{-t} \int_0^t e^{s/2} ds = \lim_{t \to \infty} 2e^{-t} (e^{t/2} - 1) = 0,$$

and hence the condition (iii) of Theorem 3.1 is also true.

Next,

$$m = \sup_{t \ge 0} \{ |g(t,0)| + c_0(t) \int_0^t c_1(s) ds \},$$

=
$$\sup_{t \ge 0} \{ 2e^{-t} + 2e^{-t} (2e^{t/2} - 1) \}$$

= 2.

So, if we put $r_0 = 3$, the condition (iv) of Theorem 3.1 is satisfied. Thus, all the hypotheses of Theorem 3.1 hold, which means that the operator

$$T(u(t)) = g(t, u(t)) + \int_0^t G(t, s, u(s)) ds, \quad t \in [0, \infty),$$

has a fixed point, that is a solution to (3.1).

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