PAIRS OF SOLUTIONS FOR ROBIN PROBLEMS WITH AN INDEFINITE AND UNBOUNDED POTENTIAL, RESONANT AT ZERO AND INFINITY

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Abstract. We consider a semilinear Robin problem driven by the Laplacian plus an indefinite and unbounded potential and a Caratheodory reaction term which is resonant both at zero and $\pm\infty$. Using the Lyapunov-Schmidt reduction method and critical groups (Morse theory), we show that the problem has at least two nontrivial smooth solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following semilinear Robin problem

$$(1) \qquad \begin{cases} -\Delta u(z) + \xi(z)u(z) = f(z, u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this problem $\xi \in L^s(\Omega)$ $(s > N)$ is an indefinite (that is, sign changing) and unbounded potential. The reaction term $f(z, x)$ is a Caratheodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous). We assume that for almost all $z \in \Omega$, $f(z, \cdot)$ is linear near $\pm\infty$ and asymptotically as $x \to \pm\infty$ resonance can occur with respect to any eigenvalue of $u \to -\Delta u + \xi(z)u$ with Robin boundary condition. Also, at zero we have resonance with respect to any eigenvalue different from the one for which we have resonance at $\pm\infty$. The boundary coefficient $\beta(\cdot)$ belongs in $W^{1,\infty}(\partial\Omega)$ and we assume that $\beta(z) \geq 0$ for all $z \in \partial\Omega$. If $\beta \equiv 0$, then our problem reduces to the Neumann problem. Hence our work, here contains as a special case the Neumann problem. Under these conditions of double resonance at both zero and infinity and using the Lyapunov-Schmidt reduction method together with Morse theory (critical groups), we prove the existence of two nontrivial smooth solutions. The Lyapunov-Schmidt reduction method for semilinear elliptic problems was first developed by Amann [2] and Castro-Lazer [3].

Existence and multiplicity results for doubly resonant semilinear Dirichlet problems with zero potential (that is, $\xi \equiv 0$), were obtained by Liang-Su [15], Liu [16]. For Neumann problems with zero potential (that is, $\xi \equiv 0$), there are the works of Gasiński-Papageorgiou [10, 11], Motreanu-Motreanu-Papageorgiou [17]. Semilinear elliptic problems driven by the Laplacian plus an indefinite potential were studied recently by Gasiński-Papageorgiou [12], Kyritsi-Papageorgiou [14], Papageorgiou-Papalini [20] (Dirichlet problems), Papageorgiou-Radulescu [21, 23], Papageorgiou-Smyrlis [25]

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(Neumann problems) and Hu-Papageorgiou [13] (Robin problems with logistic reaction). We mention also the very recent nonlinear works on problems driven by the p-Lapalcian plus an indefinite potential by Mugnai-Papageorgiou [19] and Fragnelli-Mugnai-Papageorgiou [7, 8] and the semilinear works of D'Aguì-Marano-Papageorgiou [5], Papageorgiou-Radulescu [24].

2. Mathematical Background

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X,\mathbb{R})$, we say that φ satisfies the Cerami condition (the "C-condition" for short), if the following is true:

- Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $(1+\|u_n\|)\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence.

In the study of problem (1), we will use the Sobolev space $H^1(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^q(\partial\Omega)$, $1 \leq q \leq \infty$. By $\|\cdot\|$ we denote the norm of the Sobolev space $H^1(\Omega)$, defined by

$$\|u\| = \left[\|u\|_2^2 + \|\nabla u\|_2^2\right]^{1/2} \quad \text{for all } u \in H^1(\Omega).$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the boundary Lebesgue spaces $L^q(\partial\Omega)$ $(1 \leq q \leq \infty)$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

So, we understand the trace map as representing the boundary values of a Sobolev function $u \in H^1(\Omega)$. We know that γ_0 is compact into $L^q(\partial\Omega)$ with $1 \leq q < \frac{2N-2}{N-2}$ if $N \geq 3$ and into $L^q(\partial\Omega)$ with $q \geq 1$ if $N = 1, 2$. Also, we have

$$\operatorname{im}\gamma_0 = H^{\frac{1}{2},2}(\partial\Omega) \quad \text{and} \quad \ker\gamma_0 = H_0^1(\Omega).$$

In the sequel, for the sake of notational simplicity, we drop the use of the trace map γ_0. All restrictions of the Sobolev functions on $\partial\Omega$ are understood in the sense of traces. Our hypotheses on the data of problem (1), involve the spectrum of the differential operator $u \to -\Delta u + \xi(z)u$ with Robin boundary condition. So, we consider the following linear eigenvalue problem:

$$(2) \qquad \begin{cases} -\Delta u(z) + \xi(z)u(z) = \widehat{\lambda}u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem, for Neumann boundary condition, was investigated by Papageorgiou-Radulescu [21, 23], Papageorgiou-Smyrlis [25] and for the p-Laplacian by Mugnai-Papageorgiou [19]. For Robin boundary condition, it was studied by Papageorgiou-Radulescu [24] and D'Aguì-Marano-Papageorgiou [5]. Assume that $\xi \in L^s(\Omega)$ $(s > N)$ and let $\gamma : H^1(\Omega) \to \mathbb{R}$ be the C^2-functional defined by

$$\gamma(u) = \|\nabla u\|_2^2 + \int_\Omega \xi(z)u^2 dz + \int_{\partial\Omega} \beta(z)u^2 d\sigma \quad \text{for all } u \in H^1(\Omega).$$

The eigenvalue problem (2) has a smallest eigenvalue $\widehat{\lambda}_1 > -\infty$ given by

$$\widehat{\lambda}_1 = \inf\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in H^1(\Omega), u \neq 0\right]. \tag{3}$$

Then we can find $\mu > 0$ such that

$$\gamma(u) + \mu\|u\|_2^2 \geq c_0\|u\|^2 \quad \text{for all } u \in H^1(\Omega),\ \text{some } c_0 > 0 \quad \text{(see [5])}. \tag{4}$$

If we use (4) and the spectral theorem for compact self-adjoint operators, we produce the spectrum of (2), which consists of a sequence $\{\widehat{\lambda}_k\}_{k\geq 1}$ of eigenvalues such that $\widehat{\lambda}_k \to +\infty$ as $k \to +\infty$. By $E(\widehat{\lambda}_k)$ we denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k$. We have

$$E(\widehat{\lambda}_k) \subseteq C^1(\overline{\Omega}) \quad \text{(see Wang [27])}$$

and it has the unique continuation property (the UCP for short), that is, if $u \in E(\widehat{\lambda}_k)$ and $u(z) = 0$ for all z in a set of positive measure, then $u = 0$ (see Motreanu-Motreanu-Papageorgiou [18]). If $\overline{H}_m = \bigoplus_{k=1}^{m} E(\widehat{\lambda}_k)$ and $\widehat{H}_m = \overline{H}_m^{\perp} = \overline{\bigoplus_{k\geq m+1} E(\widehat{\lambda}_k)}$, then $\overline{H}_m$ is finite dimensional and we have the following orthogonal direct sum decomposition

$$H^1(\Omega) = \overline{H}_m \oplus \widehat{H}_m.$$

The higher eigenvalues $\{\widehat{\lambda}_m\}_{m\geq 2}$ have the following variational characterizations:

$$\begin{aligned}\widehat{\lambda}_m &= \inf\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in \widehat{H}_m, u \neq 0\right]\\ &= \sup\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in \overline{H}_m, u \neq 0\right], \quad m \geq 2.\end{aligned} \tag{5}$$

In both (3) and (5) the infimum (and for (5) also the supremum) is realized on the corresponding eigenspace. The first eigenvalue $\widehat{\lambda}_1 \in \mathbb{R}$ is simple and has eigenfunctions of constant sign. In fact, if $\widehat{u}_1$ denotes the L^2-normalized (that is, $\|\widehat{u}_1\|_2 = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1$, then $\widehat{u}_1(z) > 0$ for all $z \in \overline{\Omega}$. All the other eigenvalues have nodal (that is, sign changing) eigenfunctions.

Using (3) and (5) and the UCP of the eigenspaces, we have the following useful inequalities.

Proposition 1. *(a) If $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \widehat{\lambda}_k$ for a.a. $z \in \Omega$ and the inequality is strict on a set of positive measure, then there exists $c_1 > 0$ such that*

$$\gamma(u) - \int_\Omega \eta(z)u^2 dz \geq c_1\|u\|^2 \quad \textit{for all } u \in \widehat{H}_k.$$

(b) If $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \widehat{\lambda}_k$ for a.a. $z \in \Omega$ and the inequality is strict on a set of positive measure, then there exists $c_2 > 0$ such that

$$\gamma(u) - \int_\Omega \eta(z)u^2 dz \leq -c_2\|u\|^2 \quad \textit{for all } u \in \overline{H}_k.$$

Finally, let us recall some basic definitions and facts from Morse theory (critical groups).

Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets:

$$\varphi^c = \{u \in X : \varphi(u) \leq c\},$$

$$K_\varphi = \{u \in X : \varphi'(u) = 0\},$$
$$K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

Let (Y_1, Y_2) be a pair of spaces such that $Y_2 \subseteq Y_1 \subseteq X$. For every $k \in \mathbb{N}$, by $H_k(Y_1, Y_2)$ we denote the k^{th}-relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Let $u \in K_\varphi^c$ be isolated. Then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the isolating neighborhood U.

Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and $-\infty < \inf \varphi(K_\varphi)$, then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0 \text{ and with } c < \inf \varphi(K_\varphi).$$

Using the second deformation theorem (see Gasiński-Papageorgiou [9], p. 628), we see that this definition of critical groups of φ at infinity is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We define the following expressions:

$$M(t, u) = \sum_{k \geq 0} \text{rank } C_k(\varphi, u) t^k \quad \text{for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi,$$
$$P(t, \infty) = \sum_{k \geq 0} \text{rank } C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}.$$

The "Morse relation" is the following equality:

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \quad \text{for all } t \in \mathbb{R},$$

where $Q(t) = \sum_{k \geq 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients β_k.

If $X = H$ is a Hilbert space, $\varphi \in C^2(H, \mathbb{R})$ and $u \in K_\varphi$, then we say that u is "nondegenerate", if $\varphi''(u) \in \mathcal{L}(H, H)$ is invertible. Also, the Morse index m of u is defined to be the supremum of the dimensions of the vector subspaces of H on which $\varphi''(u)$ is negative definite. If $u \in K_\varphi$ is isolated and nondegenerate, then

$$C_k(\varphi, u) = \delta_{k,m} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \quad (m \text{ is the Morse index of } u).$$

Here by $\delta_{k,m}$ we denote the Kronecker symbol defined by

$$\delta_{m,k} = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \neq m. \end{cases}$$

3. Multiplicity Theorem

In this section we prove the existence of two nontrivial smooth solutions for problem (1). The hypotheses on the data of problem (1) are the following:

$H(\xi)$: $\xi \in L^s(\Omega)$ with $s > N$.

$H(\beta)$: $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Remark 1. The case $\beta \equiv 0$ is possible and corresponds to the Neumann problem.

H: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) there exist $m \in \mathbb{N}$ and $\eta \in L^\infty(\Omega)$ such that
- $\diamond$ $\eta(z) \le \widehat{\lambda}_{m+1}$ for a.a. $z \in \Omega$,
- $\diamond$ the above inequality is strict on a set of positive measure,
- $\diamond$ $(f(z,x) - f(z,y))(x-y) \le \eta(z)(x-y)^2$ for a.a. $z \in \Omega$, all $x, y \in \mathbb{R}$;

(ii) $\widehat{\lambda}_m \le \liminf\limits_{x\to\pm\infty} \frac{f(z,x)}{x}$ uniformly for a.a. $z \in \Omega$ and if $F(z,x) = \int_0^x f(z,s)ds$, then
$$\lim_{x\to\pm\infty} [f(z,x)x - 2F(z,x)] = -\infty \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) there exist $l \in \mathbb{N}$, $l \neq m$ and $\delta > 0$ such that
$$\widehat{\lambda}_l x^2 \le f(z,x)x \le \widehat{\lambda}_{l+1} x^2 \text{ for a.a. } z \in \Omega, \text{ all } |x| \le \delta.$$

Remark 2. Hypothesis $H(ii)$ implies that asymptotically at $\pm\infty$ we can have resonance with respect to any eigenvalue of the differential operator. In fact hypotheses $H(i), (ii)$ imply that
$$\widehat{\lambda}_m \le \liminf_{x\to\pm\infty} \frac{f(z,x)}{x} \le \limsup_{x\to\pm\infty} \frac{f(z,x)}{x} \le \eta(z) \quad \text{uniformly for a.a. } z \in \Omega.$$

Hypothesis $H(iii)$ implies that at zero, we can have resonance with respect to both endpoints of any spectral interval distinct from $[\widehat{\lambda}_m, \widehat{\lambda}_{m+1}]$ (double resonance).

A simple function satisfying the above conditions is the following. For the sake of simplicity we drop the z-dependence
$$f(x) = \begin{cases} \widehat{\lambda}_l x + cx^3 & \text{if } |x| < 1, \\ \widehat{\lambda}_m x - \dfrac{\widehat{\lambda}_m - (\widehat{\lambda}_l + c)}{x^3} & \text{if } 1 < |x|, \end{cases} \quad \text{with } m > l,\ c \le \widehat{\lambda}_{l+1} - \widehat{\lambda}_l,\ \widehat{\lambda}_m > \widehat{\lambda}_l + c.$$

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by
$$\varphi(u) = \frac{1}{2}\gamma(u) - \int_\Omega F(z, u(z))dz \quad \text{for all } u \in H^1(\Omega).$$
Evidently $\varphi \in C^1(H^1(\Omega))$. In what follows, we set
$$Y = \overline{H}_m = \bigoplus_{i=1}^m E(\widehat{\lambda}_i) \quad \text{and} \quad V = \widehat{H}_m = \overline{\bigoplus_{i\ge m+1} E(\widehat{\lambda}_i)} = Y^\perp.$$
We have the following orthogonal direct sum decomposition
$$H^1(\Omega) = Y \oplus V.$$
So, every $u \in H^1(\Omega)$ can be written in a unique way as
$$u = y + v \quad \text{with } y \in Y, v \in V.$$

Proposition 2. *If the hypotheses $H(\xi)$, $H(\beta)$, H hold, then there exists a continuous map $\vartheta : Y \to V$ such that*
$$\varphi(y + \vartheta(y)) = \inf[\varphi(y+v) : v \in V] \quad \textit{for all } y \in Y.$$

Proof. We fix $y \in Y$ and consider the C^1-functional $\varphi_y : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_y(u) = \varphi(y+u) \quad \text{for all } u \in H^1(\Omega).$$

By $i_V : V \to H^1(\Omega)$ we denote the inclusion map. We define

$$\widehat{\varphi}_y = \varphi_y \circ i_V$$

From the chain rule we have

$$\widehat{\varphi}_y' = p_{V^*} \circ \varphi_y', \tag{6}$$

with p_{V^*} being the orthogonal projection of $H^1(\Omega)^*$ onto V^*. By $\langle \cdot, \cdot \rangle_V$ we denote the duality brackets for the pair (V^*, V). For $v_1, v_2 \in V$ we have

$$\begin{aligned}
&\langle \widehat{\varphi}_y'(v_1) - \widehat{\varphi}_y'(v_2), v_1 - v_2 \rangle_V \\
=&\langle {\varphi_y}'(v_1) - \varphi_y'(v_2), v_1 - v_2 \rangle \quad \text{(see (6))} \\
=&\|\nabla(v_1 - v_2)\|_2^2 + \int_\Omega \xi(z)(v_1 - v_2)^2 dz + \int_{\partial\Omega} \beta(z)(v_1 - v_2)^2 d\sigma \\
&- \int_\Omega (f(z, v_1) - f(z, v_2))(v_1 - v_2) dz \\
\geq&\gamma(v_1 - v_2) - \int_\Omega \eta(z)(v_1 - v_2)^2 dz \quad \text{(see hypothesis } H(i)) \\
\geq&c_3\|v_1 - v_2\|^2 \quad \text{for some } c_3 > 0 \quad \text{(see Proposition 1)}, \qquad (7) \\
\Rightarrow&\widehat{\varphi}_y' \text{ is strongly monotone, hence } \widehat{\varphi}_y \text{ is strictly convex.}
\end{aligned}$$

We have

$$\begin{aligned}
\langle \widehat{\varphi}_y'(v), v \rangle_V &= \langle \widehat{\varphi}_y'(v) - \widehat{\varphi}_y'(0), v \rangle_V + \langle \widehat{\varphi}_y'(0), v \rangle_V \\
&\geq c_3\|v\|^2 - c_4\|v\| \quad \text{for some } c_4 > 0 \quad \text{(see (7))}, \qquad (8) \\
&\Rightarrow \widehat{\varphi}_y' \text{ is coercive.}
\end{aligned}$$

Note that $\widehat{\varphi}_y'(\cdot)$ being continuous and monotone, it is maximal monotone (see Gasiński-Papageorgiou [9], p. 309). But a maximal monotone and coercive map is surjective (see Gasiński-Papageorgiou [9], p. 319). So, we can find $v_0 \in V$ such that

$$\widehat{\varphi}_y'(v_0) = 0. \tag{9}$$

From (7) it is clear that $v_0 \in V$ is unique. In fact, it is the unique minimizer of the strictly convex functional $\widehat{\varphi}_y = \varphi_y|_V$. So, we can define the map $\vartheta : Y \to V$ by setting $\vartheta(y) = v_0$. We have

$$p_{V^*}\varphi'(y + \vartheta(y)) = 0 \quad \text{and} \quad \varphi(y + \vartheta(y)) = \inf[\varphi(y + v) : v \in V]. \tag{10}$$

Next we establish the continuity properties of the map $\vartheta : Y \to V$. So, let $y_n \to y$ in Y. For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
0 &= \langle \widehat{\varphi}_{y_n}'(\vartheta(y_n)), \vartheta(y_n) \rangle_V \quad \text{(see (9))} \\
&\geq c_3\|\vartheta(y_n)\|^2 - c_4\|\vartheta(y_n)\| \quad \text{(see (8))} \\
&\Rightarrow \{\vartheta(y_n)\}_{n\geq1} \subseteq V \text{ is bounded.}
\end{aligned}$$

Passing to a suitable subsequence if necessary, we may assume that

$$\vartheta(y_n) \xrightarrow{w} \tilde{v} \text{ in } H^1(\Omega), \tilde{v} \in V. \tag{11}$$

The Sobolev embedding theorem and the compactness of the trace map imply that φ is sequentially weakly lower semicontinuous. So, from (11) we have

$$\varphi(y+\tilde{v}) \leq \liminf_{n\to+\infty} \varphi(y_n+\vartheta(y_n)). \tag{12}$$

From (10), we have

$$\begin{aligned}
&\varphi(y_n+\vartheta(y_n)) \leq \varphi(y_n+v) \quad \text{for all } n\in\mathbb{N},\ \text{all } v\in V,\\
\Rightarrow &\limsup_{n\to+\infty} \varphi(y_n+\vartheta(y_n)) \leq \varphi(y+v) \quad (\text{since } y_n\to y \text{ in } Y)\\
\Rightarrow &\varphi(y+\tilde{v}) \leq \varphi(y+v) \quad \text{for all } v\in V \text{ (see (12))}\\
\Rightarrow &\tilde{v}=\vartheta(y).
\end{aligned}$$

So, by the Urysohn criterion, for the initial sequence $\{\vartheta(y_n)\}_{n\geq 1}$ we get

$$\vartheta(y_n) \xrightarrow{w} \vartheta(y) \text{ in } H^1(\Omega).$$

For all $n\in\mathbb{N}$ we have

$$\begin{aligned}
0 &= \langle \varphi'(y_n+\vartheta(y_n)), \vartheta(y_n)-\vartheta(y)\rangle \quad \text{(see (10))}\\
&\Rightarrow \gamma(y_n+\vartheta(y_n)) \to \gamma(y+\vartheta(y))\\
&\Rightarrow \|\nabla(y_n+\vartheta(y_n))\|_2 \longrightarrow \|\nabla(y+\vartheta(y))\|_2\\
&\Rightarrow y_n+\vartheta(y_n) \longrightarrow y+\vartheta(y) \quad \text{(Kadec-Klee property)}\\
&\Rightarrow \vartheta(y_n) \ \longrightarrow \vartheta(y) \text{ in } H^1(\Omega)\\
&\Rightarrow \vartheta: Y\to V \text{ is continuous.}
\end{aligned}$$

Moreover, from (10) we have

$$\varphi(y+\vartheta(y)) = \inf[\varphi(y+v): v\in V].$$

□

We set

$$\varphi_0(y) = \varphi(y+\vartheta(y)) \quad \text{for all } y\in Y. \tag{13}$$

Using Proposition 2, we see that φ_0 is continuous. In fact, we can say more.

Proposition 3. *If hypotheses $H(\xi)$, $H(\beta)$, H hold, then $\varphi_0 \in C^1(Y,\mathbb{R})$ and $\varphi_0'(y) = p_{Y^*}\varphi'(y+\vartheta(y))$ for all $y\in Y$.*

Proof. Let $y,h\in Y$ and $t>0$. From (12) and Proposition 2, we have

$$\begin{aligned}
&\frac{1}{t}[\varphi_0(y+th)-\varphi_0(y)]\\
\leq &\frac{1}{t}[\varphi(y+th+\vartheta(y))-\varphi(y+\vartheta(y))]\\
\Rightarrow &\limsup_{t\to 0}\frac{1}{t}[\varphi_0(y+th)-\varphi_0(y)] \leq \langle\varphi'(y+\vartheta(y)),h\rangle.
\end{aligned} \tag{14}$$

Also we have

$$\begin{aligned}
&\frac{1}{t}[\varphi_0(y+th)-\varphi_0(y)]\\
\geq &\frac{1}{t}[\varphi(y+th+\vartheta(y+th))-\varphi(y+\vartheta(y+th))]
\end{aligned}$$

$$\Rightarrow \liminf_{t\to 0} \frac{1}{t}[\varphi_0(y+th)-\varphi_0(y)] \geq \langle \varphi'(y+\vartheta(y)), h\rangle \tag{15}$$

(since $\varphi \in C^1(H^1(\Omega),\mathbb{R})$ and $\vartheta(\cdot)$ is continuous, see Proposition 1).

If by $\langle\cdot,\cdot\rangle_Y$ we denote the duality brackets for the pair (Y^*, Y), from (14) and (15) we have

$$\langle \varphi_0'(y), h\rangle_Y = \langle \varphi'(y+\vartheta(y)), h\rangle \quad \text{for all } y, h \in Y,$$
$$\Rightarrow \varphi_0 \in C^1(Y,\mathbb{R}) \text{ and } \varphi_0'(y) = p_{Y^*}\varphi'(y+\vartheta(y)) \text{ for all } y \in Y.$$

□

Proposition 4. *If hypotheses $H(\xi)$, $H(\beta)$, H hold, then the functional φ_0 is anticoercive (that is, if $\|y\| \to +\infty$, $y \in Y$, then $\varphi_0(y) \to -\infty$).*

Proof. We argue indirectly. So, suppose we could find $\{y_n\}_{n\geq 1} \subseteq Y$ such that

$$\|y_n\| \to +\infty \quad \text{and} \quad \varphi_0(y_n) \geq -M_1 \text{ for some } M_1 > 0, \text{ all } n \in \mathbb{N}. \tag{16}$$

We have

$$-M_1 \leq \varphi_0(y_n) \leq \frac{1}{2}\gamma(y_n) - \int_\Omega F(z,y_n)dz \quad \text{for all } n \in \mathbb{N}. \tag{17}$$

(see Proposition 1 and (13)).

Let $w_n = \frac{y_n}{\|y_n\|}$, $n \in \mathbb{N}$. Then $\|w_n\| = 1$, $w_n \in Y$ for all $n \in \mathbb{N}$. Because Y is finite dimensional, we may assume that

$$w_n \to w \text{ in } H^1(\Omega), \quad w \in Y, \quad \|w\| = 1. \tag{18}$$

From (17) we have

$$-\frac{M_1}{\|y_n\|^2} \leq \frac{1}{2}\gamma(w_n) - \int_\Omega \frac{F(z,y_n)}{\|y_n\|^2}dz \quad \text{for all } n \in \mathbb{N}. \tag{19}$$

Hypothesis $H(i)$ implies that

$$F(z,x) \leq \frac{1}{2}\eta(z)x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{20}$$

On the other hand, hypotheses $H(ii), (iii)$ imply that

$$F(z,x) \geq -c_5 x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_5 > 0. \tag{21}$$

From (20) and (21), it follows that

$$\left\{\frac{F(\cdot,y_n(\cdot))}{\|y_n\|^2}\right\}_{n\geq 1} \subseteq L^1(\Omega) \text{ is uniformly integrable.}$$

So, by the Dunford-Pettis theorem and hypotheses $H(i), (ii)$, we have

$$\frac{F(\cdot,y_n(\cdot))}{\|y_n\|^2} \xrightarrow{w} \frac{1}{2}\widehat{\eta}w^2 \text{ in } L^1(\Omega), \tag{22}$$

$$\widehat{\lambda}_m \leq \widehat{\eta}(z) \leq \eta(z) \text{ for a.a. } z \in \Omega, \tag{23}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 30). So, if in (19) we pass to the limit as $n \to +\infty$ and use (18) and (22), we obtain

$$\int_\Omega \widehat{\eta}(z)w^2 dz \leq \gamma(w). \tag{24}$$

If $\widehat{\eta} \neq \widehat{\lambda}_m$, then from (24) and Proposition 1, we have

$$\begin{aligned} & 0 \leq \gamma(w) - \int_\Omega \widehat{\eta}(z) w^2 dz \leq -c_6 \|w\|^2 \quad \text{for some } c_6 > 0 \\ \Rightarrow & w = 0, \text{ which contradicts (18).} \end{aligned}$$

Next, we assume that $\widehat{\eta}(z) = \widehat{\lambda}_m$ for a.a $z \in \Omega$ (resonant case). Because of hypothesis $H(ii)$ given any $\mu > 0$, we can find $M_2 = M_2(\mu) > 0$ such that

$$f(z,x)x - 2F(z,x) \leq -\mu \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M_2. \tag{25}$$

We have

$$\frac{d}{dx}\left(\frac{F(z,x)}{|x|^2}\right) = \frac{f(z,x)x - 2F(z,x)}{|x|^2 x} \begin{cases} \leq -\frac{\mu}{x^3} \text{ for a.a. } z \in \Omega, \text{ all } x \geq M_2 \\ \geq -\frac{\mu}{|x|^2 x} \text{ for a.a. } z \in \Omega, \text{ all } x \leq -M_2 \end{cases}$$

$$\text{(see (25))} \Rightarrow \frac{F(z,v)}{|v|^2} - \frac{F(z,u)}{|u|^2} \leq \frac{\mu}{2}\left[\frac{1}{|v|^2} - \frac{1}{|u|^2}\right] \text{for a.a. } z \in \Omega, \text{ all } |v| \geq |u| \geq M_2. \tag{26}$$

Evidently hypotheses $H(i), (ii)$ imply that

$$\widehat{\lambda}_m \leq \liminf_{x \to \pm\infty} \frac{2F(z,x)}{|x|^2} \leq \limsup_{x \to \pm\infty} \frac{2F(z,x)}{|x|^2} \leq \eta(z) \quad \text{uniformly for a.a. } z \in \Omega. \tag{27}$$

So, if in (26) we pass to the limit as $|v| \to +\infty$ and use (27), then

$$\frac{\widehat{\lambda}_m}{2}|u|^2 - F(z,u) \leq -\frac{\mu}{2} \quad \text{for a.a. } z \in \Omega, \text{ all } |u| \geq M_2. \tag{28}$$

Since $\mu > 0$ is arbitrary, from (28) we infer that

$$\frac{\widehat{\lambda}_m}{2}|u|^2 - F(z,u) \to -\infty \quad \text{uniformly for a.a. } z \in \Omega, \text{ as } |u| \to +\infty. \tag{29}$$

From (17), we see that for all $n \in \mathbb{N}$ we have

$$\begin{aligned} -M_1 & \leq \frac{1}{2}\gamma(y_n) - \int_\Omega F(z,y_n)dz \\ & \leq \int_\Omega \left[\frac{\widehat{\lambda}_m}{2} y_n^2 - F(z,y_n)\right] dz \quad \text{(see (5) and recall } y_n \in Y). \end{aligned} \tag{30}$$

From (18) we see that, if $\Omega_0 = \{z \in \Omega : w(z) = 0\}$, then $|\Omega \setminus \Omega_0|_N > 0$ (here by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$). Then

$$|y_n(z)| \to +\infty \quad \text{for a.a. } z \in \Omega \setminus \Omega_0. \tag{31}$$

From (29) and (31) and Fatou's lemma, we have

$$\int_\Omega \left[\frac{1}{2}\widehat{\lambda}_m y_n^2 - F(z,y_n)\right] dz \to -\infty \quad \text{as } n \to +\infty. \tag{32}$$

Comparing (30) and (32), we reach a contradiction. This proves the anticoercivity of φ_0. □

From this proposition, we infer that φ_0 satisfies the compactness-type condition (see Papageorgiou-Winkert [26]).

Corollary 1. *If hypotheses $H(\xi)$, $H(\beta)$, H hold, then the functional φ_0 satisfies the C-condition.*

The next lemma is a straightforward observation. For completeness we include its proof (see also Castro-Lazer [3]).

Lemma 1. *If hypotheses $H(\xi)$, $H(\beta)$, H hold, then $y \in K_{\varphi_0}$ if and only if $y+\vartheta(y) \in K_\varphi$.*

Proof. $\Leftarrow$: This is immediate from (6) and (10).
$\Rightarrow$: Suppose that $y \in K_{\varphi_0}$. Then

$$0 = \varphi_0'(y) = p_{V^*}\varphi'(y + \vartheta(y)) \quad \text{(see (6) and (13))}.$$

Since $H^1(\Omega)^* = Y^* \oplus V^*$, it follows that $\varphi'(y + \vartheta(y)) \in Y^*$ and so from Proposition 3 we have

$$\begin{aligned} &\langle \varphi'(y + \vartheta(y)), h\rangle = 0 \quad \text{for all } h \in Y, \\ &\Rightarrow \varphi'(y + \vartheta(y)) = 0, \\ &\Rightarrow y + \vartheta(y) \in K_\varphi. \end{aligned}$$

□

Using the above lemma, we see that we may assume that K_{φ_0} is finite. Otherwise K_φ is infinite (see Lemma 1) and so we have a whole sequence of distinct nontrivial solutions of (1) which belong in $C^1(\overline{\Omega})$ (regularity theory, see Wang [27]) and so we are done.

So we assume that K_{φ_0} is finite. Then because of Corollary 1, we can compute the critical groups of φ_0 at infinity. We do this using some ideas of Liu [16].

Proposition 5. *If hypotheses $H(\xi)$, $H(\beta)$, H hold, then*

$$C_k(\varphi_0, \infty) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ with } d_m = \dim Y = \dim \overline{H}_m.$$

Proof. In what follows for any $r > 0$, we define

$$C_r = \{y \in Y : \|y\| \geq r\}, \quad \partial C_r = \partial B_r^Y = \{y \in Y : \|y\| = r\}.$$

Let $m_0 < \inf \varphi_0(K_{\varphi_0})$. From Proposition 4 we know that φ_0 is anticoercive. So, we can find $\lambda < \tau < m_0$ and $R > r$ such that

$$C_R \subseteq \varphi_0^\lambda \subseteq C_r \subseteq \varphi_0^\tau.$$

We consider the following triples of sets

$$(Y, C_r, C_R) \quad \text{and } (Y, \varphi_0^\tau, \varphi_0^\lambda).$$

Using these two triples, we introduce the following commutative diagram of homomorphisms of relative singular homology groups

(33)

$$\begin{array}{ccccccccc} \cdots \to & H_k(C_r, C_R) & \xrightarrow{i_*} & H_k(Y, C_R) & \xrightarrow{j_*} & H_k(Y, C_r) & \xrightarrow{\partial_*} & H_{k-1}(C_r, C_R) & \to \cdots \\ & \downarrow h_*|_{C_r} & & \downarrow h_* & & \downarrow h_* & & \downarrow h_*|_{C_r} & \\ \cdots \to & H_k(\varphi_0^\tau, \varphi_0^\lambda) & \xrightarrow{\hat{i}_*} & H_k(Y, \varphi_0^\lambda) & \xrightarrow{\hat{j}_*} & H_k(Y, \varphi_0^\tau) & \xrightarrow{\hat{\partial}_*} & H_{k-1}(\varphi_0^\tau, \varphi_0^\lambda) & \to \cdots \end{array}$$

In (33) $i_*, \widehat{i}_*, j_*, \widehat{j}_*, h_*$ are the group homomorphism induced by the corresponding inclusion maps and $\partial_*, \widehat{\partial}_*$ are the boundary homomorphisms. In (33) the two horizontal lines are exact and the whole diagram is commutative (see Motreanu-Motreanu-Papageorgiou [18], p. 148).

Since $\lambda < \tau < m_0 < \inf \varphi_0(K_{\varphi_0})$, from the second deformation theorem (see Gasiński-Papageorgiou [9], p. 628), we have that φ_0^λ is a strong deformation of φ_0^τ. Hence

$$C_k(\varphi_0^\tau, \varphi_0^\lambda) = 0 \quad \text{for all } k \in \mathbb{N}_0 \tag{34}$$

(see Motreanu-Motreanu-Papageorgiou [18], p. 143).

Also, let $h : [0,1] \times C_r \to Y$ be the deformation defined by

$$h(t,u) = (1-t)u + tR\frac{u}{\|u\|} \quad \text{for all } t \in [0,1], \text{ all } u \in C_r.$$

Then $h(1,\cdot)|_{\partial C_R} = id|_{\partial C_R}$ and so $\partial C_R = \partial B_R^Y$ is a deformation retract of C_r. On the other hand, using the radial retraction and Theorem 6.5, p. 325, of Dugundji [6], we see that ∂C_R is also a deformation retract of C_R. Therefore C_r and C_R are homotopy equivalent, which implies

$$H_k(C_r, C_R) = 0 \quad \text{for all } k \in \mathbb{N}_0 \tag{35}$$

(see Motreanu-Motreanu-Papageorgiou [18], Proposition 6.11, p. 143).

The exactness of the two horizontal lines in (33), together with (34), (35) and the rank theorem, imply that

$$0 = \operatorname{im} i_* = \ker j_*, \quad 0 = \operatorname{im} \widehat{i}_* = \ker \widehat{j}_*,$$
$$\Rightarrow j_* \text{ and } \widehat{j}_* \text{ are group isomorphisms.}$$

Invoking the "Five Lemma" (see, for example, Motreanu-Motreanu-Papageorgiou [18], Lemma 6.17, p. 145), we have that h_* is an isomorphism. So, we have

$$H_k(Y, C_r) = H_k(Y, \varphi_0^\tau) \quad \text{for all } k \in \mathbb{N}_0,$$
$$\Rightarrow H_k(Y, C_r) = C_k(\varphi_0, \infty) \quad \text{for all } k \in \mathbb{N}_0 \quad (\text{recall that } \tau < m_0). \tag{36}$$

But as we already established earlier

$$\partial C_r = \partial B_r^Y \quad \text{is a deformation retract of } C_r,$$
$$\Rightarrow H_k(Y, C_r) = H_k(Y, \partial B_r^Y) \quad \text{for all } k \in \mathbb{N}_0 \tag{37}$$

(see Motreanu-Motreanu-Papageorgiou [18], Corollary 6.15, p. 145).

Recall that Y is finite dimensional. Therefore

$$H_k(Y, \partial B_r^Y) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \text{ with } d_m = \dim Y. \tag{38}$$

(see Motreanu-Motreanu-Papageorgiou [18], Example 6.28(i), p. 149).

From (36), (37), (38), we conclude that

$$C_k(\varphi_0, \infty) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \text{ with } d_m = \dim Y = \dim \overline{H}_m.$$

□

Next we compute the critical groups of the energy functional φ at the origin.

Proposition 6. *If hypotheses $H(\xi)$, $H(\beta)$, H hold, then*

$$C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \textit{for all } k \in \mathbb{N}_0, \quad \textit{with } d_l = \dim \overline{H}_l.$$

Proof. Let $\overline{Y} = \overline{H}_l = \bigoplus_{i=1}^{l} E(\widehat{\lambda}_i)$ and $\overline{V} = \overline{Y}^{\perp} = \widehat{H}_l = \overline{\bigoplus_{i\geq l+1} E(\widehat{\lambda}_i)}$. We have the following orthogonal direct sum decomposition

$$H^1(\Omega) = \overline{Y} \oplus \overline{V}.$$

Then every $u \in H^1(\Omega)$ can be written in a unique way as

$$u = \overline{y} + \overline{v} \quad \text{ with } \overline{y} \in Y, \ \overline{v} \in \overline{V}. \tag{39}$$

Let $\lambda \in (\widehat{\lambda}_l, \widehat{\lambda}_{l+1})$ and consider the C^2-functional $\psi : H^1(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{2}\gamma(u) - \frac{\lambda}{2}\|u\|_2^2 \quad \text{ for all } u \in H^1(\Omega).$$

We consider the homotopy $h(t,u)$ defined by

$$h(t,u) = (1-t)\varphi(u) + t\psi(u) \quad \text{ for all } (t,u) \in [0,1] \times H^1(\Omega).$$

Suppose we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq H^1(\Omega) \setminus \{0\}$ such that

$$t_n \to t, \ u_n \to 0 \text{ in } H^1(\Omega) \quad \text{ and } \quad h'_u(t_n, u_n) = 0 \ \text{ for all } n \in \mathbb{N}. \tag{40}$$

Since K_φ is finite, we may assume that $t_n \neq 0$ for all $n \in \mathbb{N}$. We have

$$(1-t_n)\langle \varphi'(u_n), h\rangle + t_n\langle \psi'(u_n), h\rangle = 0 \quad \text{for all } n \in \mathbb{N}, \text{ all } h \in H^1(\Omega),$$

$$\Rightarrow \int_\Omega (\nabla u_n, \nabla h)_{\mathbb{R}^N} dz + \int_\Omega \xi(z) u_n h dz + \int_{\partial\Omega} \beta(z) u_n h d\sigma$$

$$= \int_\Omega [(1-t_n) f(z, u_n) + t_n \lambda u_n] h dz \quad \text{ for all } n \in \mathbb{N}, \text{ all } h \in H^1(\Omega), \tag{41}$$

which implies

$$\begin{cases} -\Delta u_n(z) + \xi(z) u_n(z) = (1-t_n) f(z, u_n(z)) + t_n \lambda u_n(z) & \text{for a.a. } z \in \Omega, \\ \frac{\partial u_n}{\partial n} + \beta(z) u_n = 0 & \text{on } \partial\Omega \end{cases}$$

(see Papageorgiou-Radulescu [22]).

Then from Wang [27] and the Calderon-Zygmund estimates we see that there exist $\alpha \in (0,1)$ and $M_3 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq M_3 \quad \text{for all } n \in \mathbb{N}.$$

Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and using (40) we can say that

$$u_n \to 0 \text{ in } C^1(\overline{\Omega}) \quad \text{ as } n \to +\infty.$$

So, we can find $n_0 \in \mathbb{N}$ such that

$$u_n(z) \in [-\delta, \delta] \quad \text{for all } z \in \overline{\Omega}, \text{ all } n \geq n_0. \tag{42}$$

In (41) we choose $h = \overline{v}_n \in \overline{V} \subseteq H^1(\Omega)$ (see (39)). Then since $\overline{Y}$ and $\overline{V}$ are orthogonal and using hypothesis $H(iii)$ (see (42)), we have

$$\gamma(\overline{v}_n) \leq \int_\Omega [(1-t_n)\widehat{\lambda}_{l+1} + t_n\lambda]\overline{v}_n^2 dz \quad \text{ for all } n \geq n_0$$

$$\Rightarrow c_7\|\overline{v}_n\|^2 \leq 0 \quad \text{ for all } n \geq n_0, \text{ some } c_7 > 0$$

$$\text{(see Proposition 1 and recall } t_n \neq 0 \text{ for all } n \in \mathbb{N}, \ \lambda \in (\widehat{\lambda}_l, \widehat{\lambda}_{l+1})),$$

$$\Rightarrow \overline{v}_n = 0 \quad \text{ for all } n \geq n_0. \tag{43}$$

Next in (41) we choose $h = \overline{y}_n \in \overline{Y} \subseteq H^1(\Omega)$ (see (39)). Then as above we have

$$\begin{aligned} &\gamma(\overline{y}_n) \geq \int_\Omega [(1-t_n)\widehat{\lambda}_l + t_n\lambda]\overline{y}_n^2 dz \quad \text{(see (42) and hypothesis } H(iii)), \\ &\Rightarrow c_8\|\overline{y}_n\|^2 \leq 0 \quad \text{for all } n \geq n_0, \text{ some } c_8 > 0, \\ &\Rightarrow \overline{y}_n = 0 \quad \text{for all } n \geq n_0, \\ &\Rightarrow u_n = 0 \quad \text{for all } n \geq n_0 \quad \text{(see (43) and (39))}, \end{aligned}$$

a contradiction.

So, (40) can not occur. This permits the use of Theorem 5.2 of Corvellec-Hantoute [4] (the homotopy invariance of the critical groups). Hence we have

$$C_k(\varphi, 0) = C_k(\psi, 0) \quad \text{for all } k \in \mathbb{N}_0. \tag{44}$$

Since $\lambda \in (\widehat{\lambda}_l, \widehat{\lambda}_{l_H})$, $u = 0$ is a nondegenerate critical point of $\psi \in C^2(H^1(\Omega))$ with Morse index $d_l = \dim \overline{Y} = \dim \overline{H}_l$. Therefore

$$C_k(\psi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \quad \text{with } d_l = \dim \overline{Y}. \tag{45}$$

(see Motreanu-Motreanu-Papageorgiou [18], Theorem 6.51, p. 155). From (44) and (45), we conclude that

$$C_k(\varphi, 0) = \delta_{k,d_l}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

□

Since the critical groups of φ_0 at an isolated critical point y coincide with those of φ at $y + \vartheta(y)$ (see, for example, Liu [16], Lemma 2.3), we have:

Corollary 2. *If hypotheses $H(\xi)$, $H(\beta)$, H hold, then*

$$C_k(\varphi_0, 0) = \delta_{k,d_l}\mathbb{Z} \quad \textit{for all } k \in \mathbb{N}_0, \quad \textit{with } d_l = \dim \overline{Y} = \dim \overline{H}_l.$$

Now we are ready for the multiplicity theorem which produces two nontrivial smooth solutions for problem (1).

Theorem 1. *If hypotheses $H(\xi)$, $H(\beta)$, H hold, then problem* (1) *admits at least two nontrivial solutions $u_0, \widehat{u} \in C^1(\overline{\Omega})$.*

Proof. From Proposition 3 we know that the functional φ_0 is anticoercive. So, we can find $y_0 \in Y$ such that

$$\varphi_0(y_0) = \max[\varphi_0(y) : y \in Y]. \tag{46}$$

Since Y is finite dimensional, from Motreanu-Motreanu-Papageorgiou [18], Example 6.45(6) (p. 153), we have

$$C_k(\varphi_0, y_0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \ \ \text{(recall } d_m = \dim Y = \dim \overline{H}_m). \tag{47}$$

From (46) we see that

$$\begin{aligned} &y_0 \in K_{\varphi_0}, \\ &\Rightarrow y_0 + \vartheta(y_0) = u_0 \in K_\varphi \quad \text{(see Lemma 1)}, \\ &\Rightarrow u_0 \ \text{ is a solution of (1) and } u_0 \in C^1(\overline{\Omega}) \ \ \text{(see Wang [27])}. \end{aligned}$$

Since $d_l \neq d_m$ (recall $l \neq m$, see hypothesis $H(iii)$), from (47) and Corollary 2 we infer that $y_0 \neq 0$, hence $u_0 \neq 0$. From Proposition 5 we know that

$$C_k(\varphi_0, \infty) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{48}$$

Suppose $K_{\varphi_0} = \{0, y_0\}$. Then from (47), (48), Corollary 2 and the Morse relation with $t = -1$, we have $(-1)^{d_l} = 0$, a contradiction. So, there exists $\widehat{y} \in K_{\varphi_0}$, $\widehat{y} \notin \{0, y_0\}$. Then $\widehat{u} = \widehat{y} + \vartheta(\widehat{y}) \in C^1(\overline{\Omega})$ is the second nontrivial solution of (1). □

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