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Multiple solutions for a discrete boundary value problem involving the p-Laplacian

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Abstract

Multiple solutions for a discrete boundary value problem involving the p-Laplacian are established. Our approach is based on critical point theory.

Keywords: Difference equations; Discrete boundary value problems; Multiple solutions; p-Laplacian; Critical points theory

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1. Introduction

The aim of this paper is to look for the existence of multiple solutions to the following problem

$$(P_\lambda)\begin{cases}-\Delta(\phi_p(\Delta u(k-1))) = \lambda f(k,u(k)), & k\in[1,T],\\ u(0)=u(T+1)=0,\end{cases}$$

where, T is a fixed positive integer, $[1,T]$ is the discrete interval $\{1,\dots,T\}$, λ is a positive real parameter, $\Delta u(k)=u(k+1)-u(k)$ is the forward difference operator, $\phi_p(s)=|s|^{p-2}s$, $1<p<+\infty$ and $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Our approach is based on the variational framework developed, in [1], by R. P. Agarval, K. Perera and D. O'Regan to study problem (P_1); for $p=2$, see also [2]. While the authors of the papers mentioned focus on the nonlinearity f, here we work with its primitive $F(\cdot,t)=\int_0^t f(\cdot,\xi)\mathrm{d}\xi$. More precisely, in order to establish the existence of at least three solutions to (P_λ) (Theorems 3.1 and 3.3), we point out a suitable relationship between the behavior of F with a precise bounded interval of parameters λ. The existence of nonnegative solutions (Theorem 3.2) is chiefly obtained by using a useful consequence of the *strong comparison principle* given in [1] (Lemma 2.1). However, we also obtain at least two positive solutions (Theorem 3.4) even if λ belongs to a precise half-line and f chances sign. To achieve our goal, we give a finite dimensional version of Theorem 2.1 of [3], which is our main tool for investigating (P_λ)

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(Theorem 2.1). For more details on the subject treated here we refer the reader to [4,5] and the references given therein. For completeness, we also mention closely related results given in [6] and in [4] (for $p = 2$) where the existence of at least one solution to (P_λ), for each λ lying in a suitable interval, has been proved by fixed point theory. Section 2 is devoted to auxiliary results and variational framework. The main results are contained in Section 3.

2. Auxiliary results and variational framework

Let X be a finite dimensional real Banach space and let $J_\lambda : X \to \mathbb{R}$ be a functional satisfying the following structural hypothesis:

(Λ) $J_\lambda(u) := \Phi(u) + \lambda\Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \to \mathbb{R}$ are two functionals of class C^1 on X with Φ coercive, i.e. $\lim_{\|u\|\to\infty} \Phi(u) = \infty$, and λ is a positive parameter.

Further, for each $r > \inf_X \Phi$, put

$$\varphi_1(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\Psi(u) - \inf\limits_{\Phi^{-1}(]-\infty,r])} \Psi}{r - \Phi(u)},$$

$$\varphi_2(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \sup_{v\in\Phi^{-1}([r,+\infty[)} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)},$$

an immediately consequence of Theorem 2.1 of [3] is as follows.

Theorem 2.1. *Assume that:*

(a_1) *there exist $r > \inf_X \Phi$ such that $\varphi_1(r) < \varphi_2(r)$;*

(a_2) *for each $\lambda \in]1/\varphi_2(r), 1/\varphi_1(r)[$ one has $\lim_{\|u\|\to\infty} J_\lambda(u) = +\infty$.*

Then, for each $\lambda \in]1/\varphi_2(r), 1/\varphi_1(r)[$, J_λ has at least three critical points.

For the reader's convenience we recall a consequence of *strong comparison principle* [2, Lemma 2.3] which we will use in the sequel in obtaining nonnegative as well as positive solutions to (P_λ), *i.e.* $u(k) > 0$ *for each* $k \in [1, T]$.

Lemma 2.1. *If*

$$-\Delta(\phi_p(\Delta u(k-1))) \geq 0, \quad k \in [1, T],$$
$$u(0) \geq 0, \quad u(T+1) \geq 0,$$

then either u is positive or $u \equiv 0$.

Finally, in order to give the variational formulation of problem (P_λ), on the T-dimensional Banach space

$$W := \{u : [0, T+1] \to \mathbb{R} : u(0) = u(T+1) = 0\},$$

equipped with the norm

$$\|u\| := \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^p\right)^{1/p}$$

we define the functional $J_\lambda : W \to \mathbb{R}$ by putting, for every $u \in W$,

$$J_\lambda(u) := \sum_{k=1}^{T+1} \left[\frac{1}{p}|\Delta u(k-1)|^p - \lambda F(k, u(k))\right],$$

where, $F(k, t) := \int_0^t f(k, \xi)\mathrm{d}\xi$ for every $(k, t) \in [1, T] \times \mathbb{R}$. An easy computation ensures that J_λ turns out to be of class C^1 on W with

$$J'_\lambda(u)(v) = \sum_{k=1}^{T+1} \left[\phi_p\left(\Delta u(k-1)\right)\Delta v(k-1) - \lambda f(k, u(k))v(k)\right], \quad v \in W.$$

Therefore, taking into account that, for every $u, v \in W$, we have

$$-\sum_{k=1}^{T+1} \Delta(\phi_p(\Delta u(k-1)))v(k) = \sum_{k=1}^{T+1} \phi_p\,(\Delta u(k-1))\,\Delta v(k-1). \tag{1}$$

It is clear that the critical points of J_λ are exactly the solutions of problem (P_λ).

3. Results

Let c and d be two positive constants, we write

$$\Theta(c) := \frac{\sum_{k=1}^{T} \sup_{|\xi|\le c} F(k,\xi)}{c^p} \quad \text{and} \quad \Gamma(d) := \frac{\sum_{k=1}^{T} \left[F(k,d) - \sup_{|\xi|\le c} F(k,\xi) \right]}{d^p}.$$

We now give the following theorem.

Theorem 3.1. *Assume that there exist four positive constants a, c, d and s with $c < d$ and $s < p$ such that*

(b$_1$) $\Theta(c) < \left(\frac{2}{T+1}\right)^{p-1} \Gamma(d)$;
(b$_2$) $F(k,\xi) \le a(1+|\xi|^s)$ *for all* $(k,\xi) \in [1,T] \times \mathbb{R}$.

Then, for every $\lambda \in \left]\frac{2}{p\Gamma(d)}, \frac{2^p}{p\Theta(c)(T+1)^{p-1}}\right[$, *problem* (P_λ) *admits at least three solutions.*

Proof. In applying Theorem 2.1, choice $X = W$, for every $u \in W$, put

$$\Phi(u) := \sum_{k=1}^{T+1} \left(\frac{1}{p} |\Delta u(k-1)|^p \right), \qquad \Psi(u) := -\sum_{k=1}^{T} F(k, u(k)),$$

and, for each $\lambda > 0$, $J_\lambda(u) := \Phi(u) + \lambda\Psi(u)$. Clearly, J_λ satisfies condition (Λ) and taking,

$$r = \frac{(2c)^p}{p(T+1)^{p-1}},$$

we claim that $\varphi_1(r) < \varphi_2(r)$. To this end, we observe that for every $u \in W$, there exists $j \in [1,T]$ such that $u(j) = \max_{k\in[1,T]} |u(k)|$. Therefore, taking in account that $u(0) = u(T+1) = 0$, an easy computation ensures that

$$u(j) \le \frac{1}{2} \sum_{k=1}^{T+1} |u(k+1) - u(k)|,$$

and by using the discrete Hölder inequality, one has

$$\max_{k\in[1,T]} |u(k)| \le \frac{(T+1)^{(p-1)/p}}{2} \|u\|. \tag{2}$$

From this follows that

$$\begin{aligned} \varphi_1(r) &= \inf_{\|u\|<(pr)^{1/p}} \frac{\Psi(u) - \inf_{\|u\|\le(pr)^{1/p}} \Psi}{r - \frac{\|u\|^p}{p}} \le \frac{-\inf_{\|u\|\le(pr)^{1/p}} \Psi}{r} \\ &= \frac{\sup_{\|u\|\le(pr)^{1/p}} \sum_{k=1}^{T} F(k,u(k))}{r} \le \frac{p(T+1)^{p-1}}{2^p} \Theta(c). \end{aligned} \tag{3}$$

Further, since one has $c < \left(\frac{T+1}{2}\right)^{(p-1)/p} d$, it results that $\|v\| > (pr)^{1/p}$ where,

$$v(k) = \begin{cases} d & \text{if } k \in [1,T], \\ 0 & \text{otherwise.} \end{cases}$$

Hence, bearing in mind that $\Gamma(d) > 0$, by (b_1) and (2), we get

$$\begin{aligned}\varphi_2(r) &\geq p \inf_{\|u\|<(pr)^{1/p}} \frac{\sum_{k=1}^{T} F(k,d) - \sum_{k=1}^{T} \sup_{|\xi|\leq c} F(k,\xi)}{2d^p - \|u\|^p} \\ &> \frac{p(T+1)^{p-1}}{2^p}\Theta(c). \qquad (4)\end{aligned}$$

Combining (3) and (4), it is clear that the above claim is proved. Now, owing to (b_2) and again by (2), for every $u \in W$ and $\lambda > 0$, we have

$$\begin{aligned}J_\lambda(u) &\geq \frac{\|u\|^p}{p} - \lambda\sum_{k=1}^{T} a(1+|u(k)|^s) \\ &\geq \frac{\|u\|^p}{p} - a\lambda T\left[\frac{(T+1)^{(p-1)/p}}{2}\right]^s \|u\|^s - a\lambda T,\end{aligned}$$

which clearly ensures that J_λ turns out to be coercive. So, the assumptions of Theorem 2.1 are satisfied and our conclusion follows. □

Remark 3.1. In many situations it is also important to obtain at least one solutions to (P_λ), see for instance Theorem 1.1 of [1]. In this order of ideas, a careful reading of the above proof reveals that condition (b_2) ensures the existence of at least one solution to (P_λ) for every $\lambda > 0$. Whereas, for $c > 0$, arguing again as above, but taking into account the proof of Theorem 2.1, we can show that, for every $\lambda \in \left]0, \frac{2^p}{p\Theta(c)(T+1)^{p-1}}\right[$, the same conclusion still holds without any additional assumption.

Let $h : \mathbb{R} \to \mathbb{R}$, $q : [1,T] \to \mathbb{R}$ be two nonnegative functions with $Q := \sum_{k=1}^{T} q(k) > 0$. Put, for every $t \in \mathbb{R}$, $H(t) := \int_0^t h(\xi)\mathrm{d}\xi$, a simple consequence of the previous result is the following

Theorem 3.2. *Assume that there exist four positive constants ρ, c, d and s with $c < d$ and $s < p$ such that*

(b_3) $\frac{H(c)}{c^p} < \frac{2^{p-1}}{2^{p-1}+(T+1)^{p-1}}\frac{H(d)}{d^p}$;

(b_2') $H(t) \leq \rho(1+|t|^s)$, *for all* $t \in \mathbb{R}$.

Then, for each $\lambda \in \left]\frac{2}{pQ}\frac{2^{p-1}+(T+1)^{p-1}}{(T+1)^{p-1}}\frac{d^p}{H(d)}, \frac{2^p}{pQ(T+1)^{p-1}}\frac{c^p}{H(c)}\right[$, *the problem*

$$\begin{cases}-\Delta(\phi_p(\Delta u(k-1))) = \lambda q(k)h(u(k)), & k \in [1,T],\\ u(0) = u(T+1) = 0,\end{cases}$$

admits at least three nonnegative solutions.

Proof. Taking into account Lemma 2.1 as well as that (b_3) implies

$$\frac{H(c)}{c^p} < \left(\frac{2}{T+1}\right)^{p-1}\frac{H(d)-H(c)}{d^p}.$$

We see at once that our conclusion follows form Theorem 3.1 by choosing $f(k,t) = q(k)h(t)$ for each $(k,t) \in [1,T]\times\mathbb{R}$ and $a = \rho Q$. □

Remark 3.2. We explicitly observe that, if $h(0) > 0$, then in Theorem 3.2 we can replace the word *nonnegative* with *positive*.

Example 3.1. Write, for each $k \in [1,T]$, $q(k) = k$ and let $h : \mathbb{R} \to \mathbb{R}$ be defined by putting

$$h(t) = \begin{cases}e^t, & \text{if } t \leq 12;\\ e^{12}, & \text{if } t > 12.\end{cases}$$

By choosing for instance $\rho = e^{12}$, $c = 1$, $d = 12$, $s = 1$ and $p = 3$, the assumptions of Theorem 3.2 are satisfied. Therefore, for each $\lambda \in \left]\frac{164}{10^6}, \frac{344}{10^6}\right[$, the problem

$$\begin{cases} -\Delta(|\Delta u(k-1)|\Delta u(k-1)) = \lambda k h(u(k)), & k \in [1, T], \\ u(0) = u(10) = 0, \end{cases}$$

has at least three positive solutions.

Now we discuss the case $s = p$ inside the growth condition (b_2).

Theorem 3.3. *Assume that there exist three positive constants a, c, d with $c < d$ such that* (b_1) *holds and in addition suppose that*

(b_4) $F(k, \xi) \le a(1 + |\xi|^p)$ *for all* $(k, \xi) \in [1, T] \times \mathbb{R}$, *with* $a < \frac{2^{p-1}}{T(T+1)^{p-1}}\Gamma(d)$.

Then, for every $\lambda \in \left]\frac{2}{p\Gamma(d)}, \frac{2^p}{p(T+1)^{p-1}} \min\left\{\frac{1}{\Theta(c)}, \frac{1}{aT}\right\}\right[$, *problem* (P_λ) *admits at least three solutions.*

Proof. Arguing as in the proof of Theorem 3.1, it is clear that our conclusion follows again by Theorem 2.1 if we show that J_λ turns out to be coercive. Indeed, keeping λ fixed as above, since for every $u \in W$, by (2), we get

$$|u(k)|^p \le \frac{(T+1)^{(p-1)}}{2}\|u\|^p, \quad \forall k \in [1, T],$$

(b_4) ensures that

$$J_\lambda(u) \ge \left(\frac{1}{p} - \lambda\frac{aT(T+1)^{p-1}}{2^p}\right)\|u\|^p - \lambda aT.$$

Thus, begin $\lambda < \frac{2^p}{aT(T+1)^{p-1}p}$, our claim holds and the proof is completed. □

Remark 3.3. Evidently, arguing as in the above proof, we get at least one solution to (P_1) if (b_4) holds with $a < \frac{2^p}{pT(T+1)^{(p-1)}}$.

Theorem 3.4. *Assume that there exist four positive constants a, c, d and s with $c < d$ and $s < p$ such that* (b_2) *holds and moreover, suppose that*

(b_5) $\max_{|\xi| \le c} F(k, \xi) \le 0$ *for all* $k \in [1, T]$;
(b_6) *there exists* $\bar{k} \in [1, T]$ *such that* $\int_0^d f(\bar{k}, \xi)\mathrm{d}\xi > 0$.

Then, for every $\lambda \in \left]\frac{2d^p}{p\int_0^d f(\bar{k},\xi)\mathrm{d}\xi}, +\infty\right[$, *problem* (P_λ) *admits at least two positive solutions.*

Proof. In applying Theorem 2.1, let J_λ be as above and $r = \frac{(2c)^p}{p(T+1)^{p-1}}$. For every $u \in W$ with $\|u\| \le (pr)^{1/p}$, by (b_5) one has that $\Psi(u) \ge 0$. From this, it follows that

$$\inf_{\|u\| \le (pr)^{1/p}} \Psi(u) = \Psi(0) = 0,$$

which furnishes $\varphi_1(r) = 0$. On the other hand it is easy to see that by setting

$$w(k) = \begin{cases} d & \text{if } k = \bar{k}; \\ 0 & \text{otherwise}, \end{cases}$$

one has $\|w\| \ge (pr)^{1/p}$. Hence, by (b_6), we get

$$\varphi_2(r) \ge p \inf_{\|u\| < (pr)^{1/p}} \frac{\int_0^d f(\bar{k}, \xi)\mathrm{d}\xi - \sum_{k=1}^{T} \sup_{|\xi| \le c} F(k, \xi)}{2d^p} \ge p\frac{\int_0^d f(\bar{k}, \xi)\mathrm{d}\xi}{2d^p} > 0.$$

So, since (b_5) implies $f(k, 0) = 0$, for every $k \in [1, T]$, Theorem 2.1 furnishes at least two nontrivial solutions, say u_1 and u_2, to (P_λ). Moreover, a simple computation shows that u_1 and u_2 turn out to be also two solutions of the following problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) = \lambda \widehat{f}(k, u(k)), & k \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases}$$

where, $\widehat{f} : [1, T] \times \mathbb{R} \to \mathbb{R}$ is defined by putting

$$\widehat{f}(k, t) = \begin{cases} f(k, t) & \text{if } t \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

We claim that u_1 and u_2 turn out to be positive. By contradiction, suppose that at least one, said u_1, is non-positive. Thus, we have that u_1 fulfills the following conditions

$$-\Delta(\phi_p(\Delta u_1(k-1))) = 0, \quad k \in [1, T], \qquad u_1(0) = u_1(T+1) = 0.$$

On the other hand, bearing in mind that u_1 is nontrivial and also Lemma 2.1, one has that u_1 is positive, which clearly is a contradiction. □

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References

[1] R.P. Agarwal, K. Perera, D. O'Regan, Multiple positive solutions of singular discrete p-Laplacian problems via variational methods, Adv. Difference Equations 2 (2005) 93–99.

[2] R.P. Agarwal, K. Perera, D. O'Regan, Multiple positive solutions of singular and nonsingular discrete problems via variational methods, Nonlinear Anal. 58 (2004) 69–73.

[3] D. Averna, G. Bonanno, A three critical points theorem and its applications to the ordinary Dirichlet problem, Topol. Methods Nonlinear Anal. 22 (2003) 93–103.

[4] R.P. Agarwal, D. O'Regan, P.J.Y. Wong, Positive Solutions of Differential, Difference and Integral Equations, Kluwer Academic Publishers, Dordrecht, 1999.

[5] D. Bai, Y. Xu, Nontrivial solutions of boundary value problems of second-order difference equations, J. Math. Anal. Appl. 326 (2007) 297–302.

[6] D. Jiang, J. Chu, D. O'Regan, R.P. Agarwal, Positive solutions for continuous and discrete boundary value problems to the one-dimensional p-Laplacian, Math. Inequal. Appl. 7 (2004) 523–534.