A SHARP ESTIMATE FOR NEUMANN EIGENVALUES OF THE LAPLACE-BELTRAMI OPERATOR FOR DOMAINS IN A HEMISPHERE

RAFAEL D. BENGURIA, BARBARA BRANDOLINI, AND FRANCESCO CHIACCHIO

Abstract. Here we prove an isoperimetric inequality for the harmonic mean of the first $N-1$ non-trivial Neumann eigenvalues of the Laplace-Beltrami operator for domains contained in a hemisphere of $\mathbb{S}^N$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ and let us consider the eigenvalues of the classical Neumann-Laplacian in Ω,

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \le \mu_2(\Omega) \le ...$$

Isoperimetric inequalities for the μ_i's go back to the classical theorem of Szegő [17] and Weinberger [19]: *the ball maximizes $\mu_1(\Omega)$ among all bounded smooth domains Ω in $\mathbb{R}^N$ having the same measure*. Szegő, using conformal maps, proved it for simply connected domains in $\mathbb{R}^2$, while Weinberger introduced a method that allowed him to get this result in full generality in $\mathbb{R}^N$. His technique has been adapted in different contexts to establish isoperimetric results for combination of eigenvalues of the Laplacian with Dirichlet or Neumann boundary conditions (see e.g. [2, 5, 6, 8, 9, 11, 12, 16]). For further references see, e.g., the monographs [10, 14, 15] and the survey paper [1]. Actually, as well-known, the conformal map technique used by Szegő allows to prove the stronger inequality

$$\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} \ge \frac{2}{\mu_1(\Omega^\star)}, \tag{1}$$

again for simply connected domains in $\mathbb{R}^2$. Here and in the sequel, $\Omega^\star$ will denote the disk, or, more in general, the ball in $\mathbb{R}^N$ having the same measure as Ω. Inequality (1) is sharp since equality sign is achieved if and only if Ω is a disk. Later, in [3], the assumption of simply connectedness was removed. In the same paper the authors conjectured that an inequality analogous to (1) holds true in $\mathbb{R}^N$, namely

$$\frac{1}{\mu_1(\Omega)} + ... + \frac{1}{\mu_N(\Omega)} \ge \frac{N}{\mu_1(\Omega^\star)}.$$

Very recently, in [18] the authors made an important step toward the proof of this conjecture, by showing the following inequality

$$\frac{1}{\mu_1(\Omega)} + ... + \frac{1}{\mu_{N-1}(\Omega)} \ge \frac{N-1}{\mu_1(\Omega^\star)}.$$

The aim of this manuscript is to prove an analogous result for the Laplace-Beltrami operator with Neumann boundary conditions. Precisely, we deal with non-trivial Neumann eigenvalues of an arbitrary domain Ω contained in a hemisphere of $\mathbb{S}^N$, defined by the following boundary value problem

$$\begin{cases} -\Delta_{\mathbb{S}^N} u = \mu u & \text{in } \Omega \\ \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where ν is the unit normal to $\partial\Omega$. We still denote the eigenvalues of (2) with $\mu_i(\Omega)$ and we intend them arranged in an increasing way, that is

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq ...$$

If we denote by $\{u_i\}_i$ a sequence of orthonormal set of eigenfunctions corresponding to $\mu_i(\Omega)$, then the following variational characterization holds true

$$\mu_i(\Omega) = \min \left\{ \frac{\displaystyle\int_\Omega |\nabla \phi|^2 \, d\omega}{\displaystyle\int_\Omega \phi^2 \, d\omega} : \phi \in H^1(\Omega) \setminus \{0\}, \ \phi \in \operatorname{span}\{u_0, u_1, ..., u_{i-1}\}^\perp \right\}. \tag{3}$$

The analogous of the Szegő-Weinberger result is already known and was proved in [4]. Our main result is the following

Theorem 1.1. *With the notation as above,*

$$\sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \geq \sum_{i=1}^{N-1} \frac{1}{\mu_i(D_\gamma)} \tag{4}$$

where D_γ is a geodesic ball contained in a hemisphere of $\mathbb{S}^N$ having the same N-volume as Ω, and γ is its radius. More precisely, γ is determined by

$$|\Omega| = N\omega_N \int_0^\gamma \sin^{N-1} t \, dt,$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. Equality sign holds in (4) *if and only if Ω is a geodesic ball.*

2. Properties of the Neumann eigenvalues and eigenfunctions of a geodesic ball

Let D_γ be a geodesic ball on $\mathbb{S}^N$ having radius γ. We think to this geodesic ball as the set of points of $\mathbb{S}^N$ with angle from the positive x_{N+1}-axis less that γ, that is a polar cap. By standard separation of variables technique, we find that the eigenvalues of (2), with $\Omega = D_\gamma$, are the eigenvalues of the following one-dimensional problems

$$\begin{cases} -\dfrac{1}{\sin^{N-1}\theta} \dfrac{d}{d\theta}\left(\sin^{N-1}\theta \dfrac{dy}{d\theta}\right) + \dfrac{l(l+N-2)}{\sin^2\theta} y = \mu_{l,k}\, y \quad \text{in } (0,\gamma) \\ \\ y(0) \text{ finite}, y'(\gamma) = 0 \end{cases}$$

with $l \in \mathbb{N}_0, k \in \mathbb{N}$. Clearly, $\mu_1(D_\gamma) = \min\{\mu_{0,2}, \mu_{1,1}\}$. In [4] the authors show that $\mu_1(D_\gamma) = \mu_{1,1}$ at least if $\gamma \leq \frac{\pi}{2}$. Hence, an eigenfunction g (assumed positive) associated to $\mu_{1,1} = \mu_1(D_\gamma)$ satisfies

$$\begin{cases} -g'' - (N-1)\cot\theta\, g' + \dfrac{N-1}{\sin^2\theta}\, g = \mu\, g \quad \text{in } (0,\gamma) \\ \\ g(0) = g'(\gamma) = 0. \end{cases} \tag{5}$$

Multiplying the equation in (5) by g and then integrating on D_γ yields

$$\mu_1(D_\gamma) = \frac{\displaystyle\int_{D_\gamma} \left[g'(\theta)^2 + (N-1)\, \frac{g(\theta)^2}{\sin^2\theta} \right] d\omega}{\displaystyle\int_{D_\gamma} g(\theta)^2 \, d\omega}. \tag{6}$$

The following properties are also proved in [4].

(1) If $0 < \gamma \le \frac{\pi}{2}$, then $g' > 0$ in $[0, \gamma)$, thus g is strictly increasing in $[0, \gamma]$.
(2) $\mu_1(D_\gamma)$ is a strictly decreasing function of γ for $0 < \gamma \le \frac{\pi}{2}$.
(3) $\mu_1(D_\gamma) > N = \mu_1(D_{\pi/2})$ for $0 < \gamma < \frac{\pi}{2}$.

We also recall that $\mu_1(D_\gamma)$ is N-fold degenerate, that is

$$\mu_1(D_\gamma) = \mu_2(D_\gamma) = ... = \mu_N(D_\gamma).$$

Now, define $G : \left[0, \frac{\pi}{2}\right] \to [0, +\infty)$ by

$$G(\theta) = \begin{cases} g(\theta) & \theta \le \gamma \\ g(\gamma) & \theta > \gamma. \end{cases} \tag{7}$$

Lemma 2.1. *The function* $\dfrac{G(\theta)}{\sin\theta}$ *is strictly decreasing in* $\left[0, \frac{\pi}{2}\right]$.

Proof. By Taylor-Frobenius expansion we have $G(\theta) = \theta - a\,\theta^3 + o(\theta^3)$, where

$$a = \frac{\mu_1(D_\gamma) - \frac{2}{3}(N-1)}{2N+4} > 0.$$

In order to get the claim it is enough to prove that

$$W(\theta) := G'(\theta) - G(\theta)\,\cot\theta < 0.$$

Using the behavior of $G(\theta)$ near $\theta = 0$ we have

$$W(\theta) = \left(\frac{1}{3} - 2a\right)\theta^2 + o(\theta^2) = \left(\frac{N - \mu_1(D_\gamma)}{N+2}\right)\theta^2 + o(\theta^2).$$

Property (3) implies that $W(\theta) < 0$ close to 0. We also know that $W(\gamma) < 0$. Assume by contradiction that $W(\theta)$ attained a positive maximum at a point $\tilde{\theta} \in (0, \gamma)$. Hence

$$W(\tilde{\theta}) > 0, \quad W'(\tilde{\theta}) = G''(\tilde{\theta}) - G'(\tilde{\theta})\,\cot\tilde{\theta} + \frac{G(\tilde{\theta})}{\sin^2\tilde{\theta}} = 0.$$

Using the equation in (5) we gain

$$N\left[G'(\tilde{\theta})\,\cot\tilde{\theta} - \frac{G(\tilde{\theta})}{\sin^2\tilde{\theta}}\right] = -\mu_1(D_\gamma)\,G(\tilde{\theta}),$$

that is

$$N\left[W(\tilde{\theta})\,\cot\tilde{\theta} - G(\tilde{\theta})\right] = -\mu_1(D_\gamma)\,G(\tilde{\theta}).$$

Since we are assuming that $W(\tilde{\theta}) > 0$, property (3) immediately gives a contradiction.

□

3. Some mathematical tools needed for the proof of Theorem 1.1

For the proof of our main result, Theorem 1.1, it is convenient to parametrize the points of Ω in terms of the coordinates of their stereographic projection (see, for example, [7, 13]). For a point $P \in \Omega$, we denote by P' its stereographic projection from the South Pole S onto the "equator" (as illustrated in Figure 1).

For P' we use cartesian coordinates $(x_1, x_2, ..., x_N, 0)$. We also use $s = \sqrt{\sum_{i=1}^N x_i^2}$, the euclidean distance from P' to the origin O. As used we denote by θ the azimuthal angle, i. e. the angle

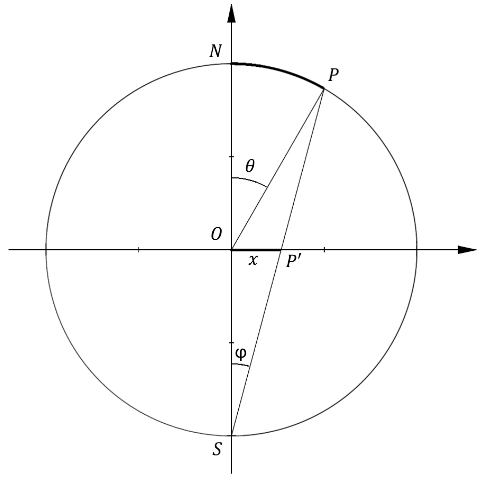

FIGURE 1. Stereographic coordinates

between ON and OP, where N stands for the North Pole. Moreover we denote by φ the angle between SN and SP. It is clear that $\theta = 2\varphi$ and $\tan\varphi = s$. Hence,

$$\theta = 2\arctan s \tag{8}$$

from which we immediately get

$$\frac{d\theta}{ds} = \frac{2}{1+s^2} = p(s), \tag{9}$$

the conformal factor associated to the differential structure on $\mathbb{S}^N$. In terms of the conformal factor p we can write

$$\nabla_{\mathbb{S}^N} = \frac{1}{p}\nabla_{\mathbb{R}^N},$$

where $\nabla_{\mathbb{R}^N}$ is the standard gradient on the equator. We also have

$$-\Delta_{\mathbb{S}^N} = -p^{-N}\mathrm{div}\left(p^{N-2}\nabla_{\mathbb{R}^N} u\right).$$

Finally, from the figure (or directly from (8) and (9)) we also have that

$$\sin\theta = p \cdot s. \tag{10}$$

In the sequel we also need to compute $\theta_{,i} := \dfrac{\partial\theta}{\partial x_i}$. Using (9), the definition of s and the chain rule we have

$$\theta_{,i} = \frac{\partial\theta}{\partial s}\cdot s_{,i} = p\,\frac{x_i}{s}, \quad i = 1, ..., N,$$

and

$$\sum_{i=1}^{N} \theta_{,i}^2 = p^2. \tag{11}$$

With the notation introduced above, we define

$$\Phi_i(x) = G(\theta)\frac{x_i}{s}, \quad i = 1, ..., N, \tag{12}$$

where $G(\theta)$ is defined in (7). In order to use Φ_i as test function in (3), we need the following orthogonality conditions

$$\int_\Omega \Phi_i \, u_j \, d\omega = 0, \quad i = 1, ..., N, \; j = 0, ..., i-1, \tag{13}$$

where, as we said, u_j is an eigenfunction corresponding to $\mu_j(\Omega)$. To fulfill these conditions we need a special "orientation" of the sphere $\mathbb{S}^N$. When $j = 0$, conditions (13) can be immediately deduced from Theorem 2.1 in [4] via the following identity

$$\int_\Omega \Phi_i \, d\omega = \int_\Omega G(\theta) \, \frac{x_i}{s} \, d\omega = \int_\Omega \frac{G(\theta)}{\sin\theta} \, y_i \, d\omega,$$

choosing $\tilde{G}(\theta) = \dfrac{G(\theta)}{\sin\theta}$. When $j > 0$, conditions (13) can be proved arguing in an analogous way as in the proof of Theorem 2.1 in [3].

4. Proof of Theorem 1.1

Recalling the definition of Φ_i given in (12), we get

$$(\nabla \Phi_i)_j \equiv \Phi_{i,j} = G'(\theta) \, p \, \frac{x_i x_j}{s^2} + G(\theta) \, \frac{\delta_{ij}}{s} - G(\theta) \frac{x_i x_j}{s^3}, \quad j = 1, ..., N. \tag{14}$$

Using (11), the definition of s and (14) we have

$$\frac{1}{p^2} \, |\nabla \Phi_i|^2 = G'(\theta)^2 \frac{x_i}{s^2} + G(\theta)^2 \frac{1}{s^2 p^2} - G(\theta)^2 \frac{x_i^2}{p^2 s^4}. \tag{15}$$

Hence, from (10) and (15),

$$\sum_{i=1}^N ||\nabla_{\mathbb{S}^N} \Phi_i||^2 = \frac{1}{p^2} \sum_{i=1}^N |\nabla \Phi_i|^2 = G'(\theta)^2 + G(\theta)^2 \frac{N-1}{s^2 p^2} = G'(\theta)^2 + G(\theta)^2 \frac{N-1}{\sin^2\theta}.$$

Using Φ_i as test function in the variational characterization (3) of $\mu_i(\Omega)$, and taking into account the orthogonality conditions (13), we get

$$\begin{aligned}
\int_\Omega \Phi_i^2 \, d\omega &\le \frac{1}{\mu_i(\Omega)} \int_\Omega G'(\theta)^2 \, \frac{x_i^2}{s^2} \, d\omega + \frac{1}{\mu_i(\Omega)} \int_\Omega \frac{G(\theta)^2}{\sin^2\theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega \\
&= \frac{1}{\mu_i(\Omega)} \int_{\Omega \cap D_\gamma} G'(\theta)^2 \, \frac{x_i^2}{s^2} \, d\omega + \frac{1}{\mu_i(\Omega)} \int_\Omega \frac{G(\theta)^2}{\sin^2\theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega \\
&\le \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} G'(\theta)^2 \, \frac{x_i^2}{s^2} \, d\omega + \frac{1}{\mu_i(\Omega)} \int_\Omega \frac{G(\theta)^2}{\sin^2\theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega \\
&= \frac{1}{N \, \mu_i(\Omega)} \int_{D_\gamma} G'(\theta)^2 \, d\omega + \frac{1}{\mu_i(\Omega)} \int_\Omega \frac{G(\theta)^2}{\sin^2\theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega. \qquad (16)
\end{aligned}$$

Summing over $i = 1, ..., N$ we get

$$\int_\Omega G(\theta)^2 \, d\omega \le \frac{1}{N} \sum_{i=1}^N \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} G'(\theta)^2 \, d\omega + \sum_{i=1}^N \frac{1}{\mu_i(\Omega)} \int_\Omega \frac{G(\theta)^2}{\sin^2\theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega.$$

Now notice that

$$\sum_{i=1}^N \frac{1}{\mu_i(\Omega)} \left(1 - \frac{x_i^2}{s^2}\right) - \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} = \frac{1}{\mu_N(\Omega)} - \sum_{i=1}^N \frac{1}{\mu_i(\Omega)} \, \frac{x_i^2}{s^2} \le 0,$$

which follows from $\mu_i(\Omega) \le \mu_N(\Omega)$ for all $i = 1, ..., N-1$ and the definition of s. Hence,

$$\int_\Omega G(\theta)^2 \, d\omega \le \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} G'(\theta)^2 \, d\omega + \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \int_\Omega \frac{G(\theta)^2}{\sin^2\theta} \, d\omega. \tag{17}$$

By Lemma 2.1 we know that the function $\dfrac{G(\theta)}{\sin\theta}$ is decreasing in $(0,\gamma)$. Recalling that $|\Omega| = |D_\gamma|$, we get

$$\begin{aligned}
\int_\Omega \frac{G(\theta)^2}{\sin^2\theta} \, d\omega &= \int_{\Omega\cap D_\gamma} \frac{G(\theta)^2}{\sin^2\theta} \, d\omega + \int_{\Omega\setminus D_\gamma} \frac{G(\theta)^2}{\sin^2\theta} \, d\omega \\
&\le \int_{\Omega\cap D_\gamma} \frac{G(\theta)^2}{\sin^2\theta} \, d\omega + \frac{G(\gamma)^2}{\sin^2\gamma} |\Omega \setminus D_\gamma| \\
&= \int_{\Omega\cap D_\gamma} \frac{G(\theta)^2}{\sin^2\theta} \, d\omega + \frac{G(\gamma)^2}{\sin^2\gamma} |D_\gamma \setminus \Omega| \\
&\le \int_{\Omega\cap D_\gamma} \frac{G(\theta)^2}{\sin^2\theta} \, d\omega + \int_{D_\gamma\setminus \Omega} \frac{G(\theta)^2}{\sin^2\theta} \, d\omega \\
&= \int_{D_\gamma} \frac{g(\theta)^2}{\sin^2\theta} \, d\omega. \qquad (18)
\end{aligned}$$

On the other side, since $G(\theta)$ is non-decreasing in $\left(0, \frac{\pi}{2}\right)$, we have

$$\begin{aligned}
\int_\Omega G(\theta)^2 \, d\omega &= \int_{\Omega\cap D_\gamma} G(\theta)^2 \, d\omega + \int_{\Omega\setminus D_\gamma} G(\theta)^2 \, d\omega \\
&\ge \int_{\Omega\cap D_\gamma} G(\theta)^2 \, d\omega + G(\gamma)^2 |\Omega \setminus D_\gamma| \\
&= \int_{\Omega\cap D_\gamma} G(\theta)^2 \, d\omega + G(\gamma)^2 |D_\gamma \setminus \Omega| \\
&\ge \int_{\Omega\cap D_\gamma} G(\theta)^2 \, d\omega + \int_{D_\gamma\setminus \Omega} g(\theta)^2 \, d\omega \\
&= \int_{D_\gamma} g(\theta)^2 \, d\omega. \qquad (19)
\end{aligned}$$

Using (17), (18), (19) and the monotonicity of the sequence $\{\mu_i(\Omega)\}_i$ we have

$$\begin{aligned}
\int_{D_\gamma} g(\theta)^2 \, d\omega &\le \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} g'(\theta)^2 \, d\omega + \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} \frac{g(\theta)^2}{\sin^2\theta} \, d\omega \\
&\le \frac{1}{N-1} \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} \left[g'(\theta)^2 + (N-1) \frac{g(\theta)^2}{\sin^2\theta} \right] d\omega.
\end{aligned}$$

Finally, from (6) we conclude

$$\frac{1}{N-1} \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \ge \frac{1}{\mu_1(D_\gamma)}. \tag{20}$$

The equality sign holds in (20) if and only if Ω is a geodesic ball.

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Rafael D. Benguria, Instituto de Física, Facultad de Física, P. Universidad Católica de Chile, Casilla 306, Santiago 22, Chile

E-mail address: `rbenguri@fis.puc.cl`

Barbara Brandolini, Università degli Studi di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Complesso Monte S. Angelo - Via Cintia, 80126 Napoli, Italia.

E-mail address: `brandolini@unina.it`

Francesco Chiacchio, Università degli Studi di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Complesso Monte S. Angelo - Via Cintia, 80126 Napoli, Italia.

E-mail address: `francesco.chiacchio@unina.it`