CONVERGENCE FOR VARYING MEASURES

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ABSTRACT. Some limit theorems of the type

$$\int_\Omega f_n \, dm_n \to \int_\Omega f \, dm$$

are presented for scalar, (vector), (multi)-valued sequences of m_n-integrable functions f_n. The convergences obtained, in the vector and multivalued settings, are in the weak or in the strong sense.

1. Introduction

Many problems in measure theory and its applications deal with sequences of measures $(m_n)_n$ converging in some sense rather than with a single measure m. Convergence results have significant applications to various fields of pure and applied sciences. Examples of areas of applications include stochastic processes, statistics, control and game theories, transportation problems, neural networks, signal and image processing (see, for example, [2, 10, 11, 15, 17, 19, 22, 27, 34]).

In particular, for the last applications, recently, multifunctions have been applied because the discretization of a continuous signal or image is affected by quantization errors ([23, 26]) and its numerical discretization can be viewed as an approximation by means of a suitable sequence of multifunctions $(\Gamma_n)_n$ (as happens in case of scalar functions [26]) which converges to a (multi)-signal Γ corresponding to the original signal. Obviously, since the signals are discontinuous ([1, 4]) suitable convergence notions are needed.

In the present paper we continue the research started in [14–17, 21, 25, 28, 36] for the scalar case and we provide sufficient conditions in order to obtain some kind of Vitali's convergence theorems for a sequence of (multi)functions $(f_n)_n$ integrable with respect to a sequence of measures $(m_n)_n$. In particular we consider the asympotic properties

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of $(\int_\Omega f_n dm_n)_n$ with respect to setwise and in total variation convergences of the measures in a arbitrary measurable spaces $(\Omega, \mathcal{A})$.

The paper is organized as follows: in Section 2, we consider the case of the scalar integrands and an analogous of the Vitali's classic convergence result is obtained for finite and non negative measures in Theorem 2.11 using the uniform absolute continuity of the involved integrals and the setwise convergence of measures. We compare also our results with the existing ones in literature and we extend our result to signed measures in Corollary 2.13. In Section 3 we consider multifunctions taking values in the hyperspace of non empty, weakly compact and convex subsets of a Banach space and limit results for the Pettis integral are provided both in the weak sense (Theorem 3.3) and in the strong sense making use of the Hausdorff metric, by means of the convergence in total variation of measures and the scalar equi-convergence in measure of the sequence of multifunctions (Theorem 3.5).
In Subsection 3.1 we get the analogous results when the integrands are vector valued functions. In this setting we obtain the converge both in the weak sense (Theorem 3.7) and in the strong sense (Theorem 3.9). Finally in subsection 3.2 we consider the McShane integration and in this setting we obtain a convergence result for the vector case (Theorem 3.11). If Γ_n's are McShane integrable multifunction and $i : cb(X) \to \ell^\infty(B_{X^*})$ is the Rådström embedding, then the vector functions $i \circ \Gamma_n$'s are also McShane integrable and viceversa. So the assumptions for multifunctions can be then translated for $i \circ \Gamma_n$'s and we can get the convergence from the vector case. Unfortunately, in general the Rådström embedding of a Pettis integrable multifunction, even weakly compact valued, is not necessary Pettis integrable. So the McShane integrability is essential for this reverse investigation.

2. The scalar case for integrands

Let $(\Omega, \mathcal{A})$ be a measurable space and let $\mathcal{M}(\Omega)$ br the vector space of *finite real-valued measures* on $(\Omega, \mathcal{A})$. $\mathcal{M}_+(\Omega)$ denotes the cone of non-negative members of $\mathcal{M}(\Omega)$. Let $|m|$ be the total variation of a measure m. By the symbol $m \ll \nu$ we denote the usual absolutely continuity of m with respect to ν. We recall that

- A sequence $(m_n)_n \subset \mathcal{M}(\Omega)$ is *setwise convergent* to $m \in \mathcal{M}(\Omega)$ ($m_n \stackrel{s}{\to} m$) if $\lim_n m_n(A) = m(A)$ for every $A \in \mathcal{A}$ ([21, Section 2.1], [17, Definition 2.3]).
- A sequence $(m_n)_n \subset \mathcal{M}(\Omega)$ *converges in total variation* to m ($m_n \stackrel{tv}{\to} m$) if $|m - m_n|(\Omega) \to 0$. Then $(m_n)_n$ is convergent to m uniformly on Σ, ([25, Section 2]).

- A sequence $(m_n)_n \subset \mathcal{M}(\Omega)$ is bounded if $\sup_n |m_n|(\Omega) < \infty$.

Remark 2.1. Since simple functions are dense in the space of bounded measurable functions, a sequence $(m_n)_n$ of measures setwise converges to m if and only if

$$\int_\Omega f dm_n \to \int_\Omega f dm, \quad \text{for all bounded measurable} \quad f : \Omega \to \mathbb{R}.$$

See also ([28]).

If m is a finite signed measures m, we denote by $m^\pm$ its positive and negative parts respectively. We recall that every finite signed measure has finite total variation. Moreover we observe that

Remark 2.2. Let $(m_n)_n, m$ be measures in $\mathcal{M}(\Omega)$. If the sequences $(m_n^+)_n$ and $(m_n^-)_n$ are setwise convergent to m^+ and m^- respectively, then $(m_n)_n$ is setwise convergent to m. Unfortunately, the reverse implication fails in general. In fact, if the reverse implication were valid, then we would have the convergence $m - m_n \stackrel{s}{\to} 0$ and hence $(m - m_n)^\pm \stackrel{s}{\to} 0$. Consequently $|m - m_n|(\Omega) \to 0$, and this is false. As a counterexample one can take a sequence $(f_n)_n$ of functions in $L^1(\mu)$ that is weakly convergent to $f \in L^1(\mu)$ but not strongly. Then take $m_n(E) := \int_E f_n \, d\mu$ and $m(E) = \int_E f \, d\mu$. The convergence $|m - m_n|(\Omega) \to 0$ means $\int_E |f - f_n| \, d\mu \to 0$, which contradicts the assumption that $(f_n)_n$ is not convergent in the norm topology of $L^1(\mu)$.

Question 2.3. *What is the relation between convergence in variation and setwise convergence of $(m_n^\pm)_n$ to $m^\pm$?*

In general $(m_n - m)^\pm \neq m_n^\pm - m^\pm$. Using the Jordan decomposition of a measure and the triangular inequality we have that

$$\begin{aligned} m_n^+(E) - m^+(E) &= \frac{1}{2}\Big(|m_n|(E) + m_n(E) - |m|(E) - m(E)\Big) = \\ &= \frac{1}{2}(|m_n| - |m|)(E) + \frac{1}{2}(m_n - m)(E) \leq \\ &\leq \frac{1}{2}\Bigg|\, |m_n| - |m| \,\Bigg|(E) + \frac{1}{2}(m_n - m)(E) \leq \\ &\leq \frac{1}{2}|m_n - m|(E) + \frac{1}{2}(m_n - m)(E) = (m_n - m)^+(E). \end{aligned}$$

Moreover

$$\begin{aligned} 0 \le \left| m_n^+(E) - m^+(E) \right| &= \frac{1}{2}\Big| |m_n|(E) + m_n(E) - |m|(E) - m(E) \Big| \le \\ &\le \frac{1}{2}\Big| |m_n|(E) - |m|(E) \Big| + \frac{1}{2}\Big| m_n(E) - m(E) \Big| \le \\ &\le \frac{1}{2}\Big| \, |m_n| - |m| \, \Big|(E) + \frac{1}{2}|m_n - m|(E) \le \\ &\le \frac{1}{2}|m_n - m|(E) + \frac{1}{2}|m_n - m|(E) = |m_n - m|(E). \end{aligned}$$

Analogously we could prove the inequality with $(m_n - m)^-$ and so the convergence in (total) variation is stronger that the setwise convergence of $(m_n^{\pm})_n$ to $m^{\pm}$.

Remark 2.4. Observe that if $\nu_n = m_n - m,$ with $m_n, m \in \mathcal{M}_+(\Omega)$ for every $n \in \mathbb{N}$, then $\nu_n^+ \le m_m$ and $\nu_n^- \le m$. In fact, if (P_n, N_n) is a Hahn decomposition for ν_n then, for every $E \in \mathcal{A}$ it is

$$\begin{aligned} \nu_n^+(E) &= \nu_n(E \cap P_n) = m_m(E \cap P_n) - m(E \cap P_n) \le \qquad (1) \\ &\le m_n(E \cap P_n) \le m_n(E); \\ \nu_n^-(E) &= -\nu_n(E \cap N_n) = m(E \cap N_n) - m_n(E \cap N_n) \le \\ &\le m(E \cap N_n) \le m(E). \end{aligned}$$

If f is a non-negative function integrable with respect m and m_n for every $n \in \mathbb{N}$ then, for every $E \in \mathcal{A}$

$$\int_E f d\nu_n^+ \le \int_E f dm_n, \qquad \int_E f d\nu_n^- \le \int_E f dm.$$

(This is true for simple functions s and then we apply with $0 \le s \le f$.) So, if $f \in L^1(m_n) \cap L^1(m)$, then $f \in L^1(|\nu_n|)$. Moreover, since

$$\int_E f d(m_n - m) = \int_E f dm_n - \int_E f dm,$$

we have that, by formulas (1)

$$\begin{aligned} \left| \int_E f d\nu_n \right| &= \left| \int_E f dm_n - \int_E f dm \right| = \left| \int_E f d\nu_n^+ - \int_E f d\nu_n^- \right| \qquad (2) \\ &\le \int_E |f| d\nu_n^+ + \int_E |f| d\nu_n^- = \int_E |f| d|\nu_n| \le \\ &\le \int_E |f| dm_n + \int_E |f| dm < +\infty. \end{aligned}$$

□

From now on we assume that all the measures we consider belong to $\mathcal{M}_+(\Omega)$ unless otherwise specified.

Let's start by proving an analogue of Vitali's classic theorem for varying measures. We can have different versions of it depending on the assumptions we use on $(f_n)_n$ and on the varying measures $(m_n)_n$.

Definition 2.5. Let $(m_n)_n$ be a sequence of measures. We say that:

(u.a.c.) a sequence of measurable functions $(f_n)_n : \varOmega \to \mathbb{R}$ has *uniformly absolutely continuous* (m_n)*-integrals on* $\varOmega$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $n \in \mathbb{N}$

$$(A \in \mathcal{A} \quad \text{and} \quad m_n(A) < \delta) \quad \Longrightarrow \quad \int_A |f_n|\, dm_n < \varepsilon. \tag{3}$$

If $f_n = f$ for all $n \in \mathbb{N}$, then we say that f has *uniformly absolutely continuous* (m_n)*-integrals on* $\varOmega$.
If $m_n = m$ for all $n \in \mathbb{N}$, then we say that $f_n's$ have *uniformly absolutely continuous* m*-integrals on* $\varOmega$.

(u.i.) a sequence of measurable functions $(f_n)_n : \varOmega \to \mathbb{R}$ is *uniformly* (m_n)*-integrable on* $\varOmega$, if

$$\lim_{\alpha \to +\infty} \sup_n \int_{\{|f_n| > \alpha\}} |f_n|\, dm_n \ = 0. \tag{4}$$

If $f_n = f$ for all $n \in \mathbb{N}$, then we say that f is *uniformly* (m_n)*-integrable on* $\varOmega$.
If $m_n = m$ for all $n \in \mathbb{N}$, then we say that $f_n's$ are *uniformly* m*-integrable on* $\varOmega$.

It is obvious that if a sequence $(f_n)_n$ of measurable fuctions is uniformly bounded, then it is uniformly $(m_n)_n$-integrable for an arbitrary $(m_n)_n$ such that $\sup_n m_n(\varOmega) < +\infty$.

The following result is contained in Serfozo's paper [36, Lemma 2.5 (i) $\Longleftrightarrow$ (iii)] where the author gives a different proof using Markov's inequality and the tight $(m_n)_n$-integrability condition of $(f_n)_n$. Moreover, in [36], the uniform absolute continuity is given in a slightly different form, but Serfozo's and our definitions are equivalent.

Proposition 2.6. *Let* $(m_n)_n$ *be a bounded sequence of measures and* $(f_n)_n$ *be a sequence of real valued measurable functions on* $\varOmega$*. Then, the sequence* $(f_n)_n$ *is uniformly* (m_n)*-integrable on* $\varOmega$ *if and only if it has uniformly absolutely continuous* (m_n)*-integrals and*

$$\sup_n \int_{\varOmega} |f_n|\, dm_n < +\infty\,. \tag{5}$$

Proof. Assume that $(f_n)_n$ is uniformly (m_n)-integrable on Ω. Let $\varepsilon > 0$ be fixed and $\alpha_0 > 0$ be such that $\sup_n \int_{\{|f_n|>\alpha_0\}} |f_n|\, dm_n < \varepsilon/2$. Let $\delta = \varepsilon/2\alpha_0$. If $m_n(A) < \delta$, then

$$\begin{aligned}\int_A |f_n|\, dm_n &= \int_{A\cap\{|f_n|\le\alpha_0\}} |f_n|\, dm_n + \int_{A\cap\{|f_n|>\alpha_0\}} |f_n|\, dm_n \\ &\le \alpha_0 m_n(A) + \varepsilon/2 < \varepsilon\,.\end{aligned}$$

Let $K := \sup_n m_n(\Omega) < +\infty$. The inequality (5) is a consequence of

$$\sup_n \int_\Omega |f_n|\, dm_n \le \alpha_0 \sup_n m_n(\{|f_n| \le \alpha_0\}) + \varepsilon/2 \le K\alpha_0 + \varepsilon/2.$$

Assume now the uniform absolute continuity of $(f_n)_n$ and the validity of (5). Let $a := \sup_n \int_\Omega |f_n|\, dm_n < +\infty$. We have

$$a \ge \sup_n \int_{\{|f_n|>\alpha\}} |f_n|\, dm_n \ge \alpha \sup_n m_n\{|f_n| > \alpha\}$$

and so $\lim_{\alpha\to\infty} \sup_n m_n(\{|f_n| > \alpha\}) = 0$. If ε and δ are as in (3), then there exists α_0 such that for all $\alpha > \alpha_0$ $\sup_n m_n(\{|f_n| > \alpha\}) < \delta$. Since the sequence has uniformly absolutely continuous (m_n)-integrals, for all $\alpha > \alpha_0$ we get the inequality

$$\sup_n \int_{\{|f_n|>\alpha\}} |f_n|\, dm_m < \varepsilon$$

and that proves the required uniform (m_n)-integrability. □

Before proceeding we would observe that the boundedness of $(m_n)_n$ is used only in the if part of the previous proof. Moreover we want to highlight that it implies the tightly $(m_n)_n$-integrability of the sequence $(f_n)_n$ as observed in [36, Formula (2.5) pag 283].

As a consequence of the previous result we obtain

Corollary 2.7. *Let $(m_n)_n$ be a bounded sequence of measures. A measurable function $f : \Omega \to \mathbb{R}$ is uniformly (m_n)-integrable on Ω if and only if f has uniformly absolutely continuous (m_n)-integrals and*

$$\sup_n \int_\Omega |f|\, dm_n < +\infty\,. \tag{6}$$

Corollary 2.8. *If the sequence $(m_n)_n$ is setwise convergent to m, then a measurable function $f : \Omega \to \mathbb{R}$ has uniformly absolutely continuous (m_n)-integrals if and only if f is uniformly (m_n)-integrable on Ω.*

Moreover,

$$\sup_n \int_\Omega |f|\, dm_n < +\infty\,. \tag{7}$$

Proof. Assume that f has uniformly absolutely continuous (m_n)-integrals and $(m_n)_n$ is setwise convergent to m. Given $\varepsilon > 0$ let $\delta > 0$ be such that (3) is fulfilled. For each $a > 0$ let $E_a := \{t \in \Omega : |f(t)| > a\}$. There exists a_0 with $m(E_{a_0}) < \delta/2$. In fact if we suppose by absurd that such a_0 does not exist, then we can construct a sequence of positive numbers $(b_n)_n \uparrow +\infty$ for which $m(\bigcap_n E_{b_n}) = \lim_n m(E_{b_n}) \geq \delta/2$, since $E_{b_{n+1}} \subseteq E_{b_n}$ for every $n \in \mathbb{N}$.
So the set $\bigcap_n E_{b_n} = \{t \in \Omega : f(t) = +\infty\}$ has positive measure which is in contradiction with the hypothesis $f : \Omega \to \mathbb{R}$. Moreover, due to the setwise convergence of the measures, there exists $n_0(a_0) \in \mathbb{N}$ such that $\sup_{n \geq n_0} m_n(E_{a_0}) < \delta/2$.
Then, analogously, for each $i \leq n_0$, there exists $a_i > 0$ with $m_i(E_{a_i}) < \delta/2$. Let $\overline{a} = \max\{a_1, \ldots, a_{n_0}, a_0\}$.
So, by the monotonicity, $\sup_n \int_{E_a} |f|\, dm_n < \varepsilon$ for every $a \geq \overline{a}$. The inequality (7) follows from Corollary 2.7. □

Remark 2.9. If we assume that f is $\overline{\mathbb{R}}$ valued then we obtain the inequality (7) of Corollary 2.8 under the additional hypothesis $f \in L^1(m)$, or $m\{|f| = +\infty\} = 0$.

The first part of the following result is contained in [36, Lemmata 2.2 and 2.5, Theorem 2,4]. For the convenience of the reader we prefer to give here a direct proof.

Proposition 2.10. *Let m and $(m_n)_n$ be measures such that the sequence $(m_n)_n$ is setwise convergent to m. Moreover let $f : \Omega \to \mathbb{R}$ have uniformly absolutely continuous (m_n)-integrals on Ω. Then $f \in L^1(m)$ and for all $A \in \mathcal{A}$*

$$\lim_n \int_A f\, dm_n = \int_A f\, dm. \tag{8}$$

Proof. By Corollary 2.8 we have that $\sup_n \int_\Omega |f| dm_n < +\infty$. The equality (8) holds for simple functions. So if $0 \leq f_k \nearrow |f|$ are simple, then

$$\int_\Omega f_k dm \leq \liminf_n \int_\Omega |f| dm_n \leq \sup_n \int_\Omega |f| dm_n \overset{Cor. 2.8}{<} +\infty\,. \tag{9}$$

Now we apply the Lebesgue Monotone Convergence Theorem and obtain that $f \in L^1(m)$.
We will show now that (8) holds. It is sufficient to prove it for Ω. Let

$\varepsilon > 0$ be fixed. By the hypothesis and Corollary 2.8 there exists α_ε such that

$$\sup\left\{\int_{\{|f|>\alpha_\varepsilon\}} |f|dm, \int_{\{|f|>\alpha_\varepsilon\}} |f|dm_n,\ n \in \mathbb{N}\right\} < \varepsilon. \tag{10}$$

Moreover by the classical Dominated Convergence Theorem for varying measures (see e.g. [35] Ch.11 Proposition 18) if $A \in \mathcal{A}$ is fixed, then there exists n_0 such that for every $n > n_0$

$$\left|\int_{\{|f|\le\alpha_\varepsilon\}} f\, dm - \int_{\{|f|\le\alpha_\varepsilon\}} f\, dm_n\right| < \varepsilon. \tag{11}$$

Then by (10) and (11) for $n > n_0$ we have

$$\begin{aligned}\left|\int_\Omega f\, dm_n - \int_\Omega f\, dm\right| &\le \left|\int_{\{|f|\le\alpha_\varepsilon\}} f dm_n - \int_{\{|f|\le\alpha_\varepsilon\}} f dm\right| + \\ &+ \left|\int_{\{|f|>\alpha_\varepsilon\}} f dm\right| + \left|\int_{\{|f|>\alpha_\varepsilon\}} f dm_n\right| < 3\varepsilon,\end{aligned}$$

and the thesis follows.

□

The next convergence result is obtained also in [36, Theorems 2.7 and 2.8] using the thightness and the uniform $(m_n)_n$-integrability of f and $(f_n)_n$, and later extended in [17, Corollary 5.3], in which the above hyphoteses on f are omitted. We give here a proof involving the uniform absolute continuity.

Theorem 2.11. *Let $f, f_n : \Omega \to \mathbb{R}$ be measurable functions and let m and $(m_n)_n$, be measures. Suppose that*

(2.11.i) *$(f_n)_n$ has uniformly absolutely continuous (m_n)-integrals on Ω;*
(2.11.ii) *$f_n(t) \to f(t)$, in m-measure, as $n \to \infty$;*
(2.11.iii) *f has uniformly absolutely continuous (m_n)-integrals on Ω;*
(2.11.iv) *$(m_n)_n$ is setwise convergent to m.*

Then, for all $A \in \mathcal{A}$,

$$\lim_n \int_A f_n dm_n = \int_A f dm. \tag{12}$$

Proof. From (2.11.ii) there exists a subsequence of $(f_n)_n$ which converges m-a.e. to f, for simplicity we denote it again $(f_n)_n$. It is sufficient to prove the equality (12) for $A = \Omega$. By Proposition 2.10 $f \in L^1(m)$. Fix $\varepsilon > 0$. Let $\delta := \min\{\varepsilon, \delta(\varepsilon/6), \delta_f(\varepsilon/6)\} > 0$, where $\delta(\varepsilon/6)$ satisfies (3) with $\varepsilon/6$ and $\delta_f(\varepsilon/6)$ is that of the absolute continuity of $\int f dm$ corresponding again to $\varepsilon/6$.
By (2.11.ii) and the Egoroff's Theorem, we can find a set $E \in \mathcal{A}$ such

that $f_n \to f$ uniformly on E^c and $m(E) < \delta/2$.
Taking into account (2.11.iv) let now $N_0 \in \mathbb{N}$ be such that

$$|m_n(E) - m(E)| < \frac{\delta}{2} \qquad \text{and} \quad |m_n(E^c) - m(E^c)| < 1, \tag{13}$$

for every $n > N_0$. Moreover, since the convergence is uniform on E^c, let $N_1 \in \mathbb{N}$ be such that

$$|f_n(t) - f(t)| < \frac{\varepsilon}{6} \cdot \frac{1}{m(E^c) + 1}, \tag{14}$$

for every $t \in E^c$ and for every $n > N_1$. Then, for every $n > N_1$,

$$\int_{E^c} |f_n - f| dm < \frac{\varepsilon}{6}. \tag{15}$$

Therefore by (13) and (14) we obtain, for every for $n > \max\{N_0, N_1\}$,

$$\int_{E^c} |f_n - f| dm_n < \frac{\varepsilon}{6} \cdot \frac{m_n(E^c)}{m(E^c) + 1} < \frac{\varepsilon}{6} \cdot \frac{m(E^c) + 1}{m(E^c) + 1} = \frac{\varepsilon}{6}. \tag{16}$$

Since, $m(E) < \delta/2$, by (13) we have also $m_n(E) < \delta$ for every for $n > N_0$. Then by (2.11.i), for every $n > N_0$ we get

$$\sup\left\{ \int_E |f_n| dm_n, \ \int_E |f| \, dm \right\} < \frac{\varepsilon}{6}\,. \tag{17}$$

Now taking into account hypothesis (2.11.iv) and Proposition 2.10, let N_2 be such that

$$\left| \int_\Omega f dm - \int_\Omega f dm_n \right| < \frac{\varepsilon}{2} \tag{18}$$

for every for $n > N_2$. Therefore by (15–18), for $n > \max\{N_0, N_1, N_2\}$ we infer

$$\begin{aligned}
& \left| \int_\Omega f dm - \int_\Omega f_n dm_n \right| \leq \left| \int_\Omega (f_n - f) dm_n \right| + \left| \int_\Omega f dm - \int_\Omega f dm_n \right| \\
\leq \ & \left| \int_{E^c} (f_n - f) dm_n \right| + \int_E |f_n| dm_n + \int_E |f| dm_n + \left| \int_\Omega f dm - \int_\Omega f dm_n \right| \\
\leq \ & \int_{E^c} |f_n - f| dm_n + \int_E |f_n| dm_n + \int_E |f| dm + \left| \int_\Omega f dm - \int_\Omega f dm_n \right| \\
< \ & \frac{\varepsilon}{2} + \left| \int_\Omega f dm - \int_\Omega f dm_n \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This implies that equality (12) is valid for the initial sequence because if, absurdly, a subsequence existed in which it is not valid, there would be a contradiction. □

Remark 2.12. In light of the quoted result in [17] the question if the setwise convergence of m_n to m, the convergence in m-measure of f_n to f and the uniformly absolutely continuous (m_n)-integrability on Ω of f_n permit to obtain the uniformly absolutely continuous (m_n)-integrability on Ω of the function f arises spontaneously.
A partial positive answer can be given at least under the additional hypothesis that $m_n \leq m$ for each n (this means that the measures m_n are dominated by m but no monotonicity is required to the sequence $(m_n)_n$). Indeed let assume conditions (2.11.i), (2.11.ii), (2.11.iv) and $m_n \leq m$ for each n, then fix $\varepsilon > 0$, let $\delta > 0$ be such that (3) is satisfied and let $A \in \mathcal{A}$ with $m_n(A) < \delta$. By the Fatou Lemma for converging measures (see [35, p.231]) we have

$$\int_A |f| \, dm_n \leq \int_A |f| \, dm \leq \liminf_n \int_A |f_n| \, dm_n < \varepsilon.$$

So f has uniformly absolutely continuous (m_n)-integrals and taking into account Proposition 2.4, condition (2.11.iii) follows.
The same holds if f is measurable and bounded thanks to the setwise convergence. What happens if f is unbounded is unknown.
Comparing the Feinberg-Kasyanov-Liang result [17, Corollary 5.3] with Theorem 2.11 we can observe also that the hypotheses assumed in the quoted paper imply that $\sup_n \int_\Omega |f_n| dm_n < +\infty$ which is not assumed in our theorem; instead of it we require the condition (2.11.iii).

If we consider signed measures in $\mathcal{M}(\Omega)$ we get

Corollary 2.13. *Let $f, f_n : \Omega \to \mathbb{R}$ be measurable functions and let m and $(m_n)_n$, be measures on $\mathcal{M}(\Omega)$. Suppose that*

(2.13.i) *$(f_n)_n$ has uniformly absolutely continuous $(|m_n|)$-integrals on Ω;*
(2.13.ii) *$f_n(t) \to f(t)$, in $|m|$-measure as $n \to \infty$;*
(2.13.iii) *f is uniformly $(m_n^{\pm})$-integrable on Ω;*
(2.13.iv) *$(m_n^{\pm})_n$ is setwise convergent to $m^{\pm}$.*

Then, for all $A \in \mathcal{A}$,

$$\lim_n \int_A f_n dm_n = \int_A f dm. \tag{19}$$

Proof. It is enough to apply Remark 2.2 and Theorem 2.11 to the pair $(m_n^{\pm}, m^{\pm})$. □

3. The multivalued case for integrands

Let X be a Banach space with dual X^* and let B_{X^*} be the unit ball of X^*. The symbol $c(X)$ stands for the collection of all nonempty closed convex subsets of X and $cwk(X)$ (resp. $cb(X)$) denotes the family of all weakly compact (resp. bounded) members of $c(X)$. For every $C \in c(X)$ the *support function of* C is denoted by $s(\cdot, C)$ and defined on X^* by $s(x^*, C) = \sup\{\langle x^*, x\rangle :\ x \in C\}$, for each x^*. If $C, D \in cwk(X)$ (resp. $cb(X)$) then $d_H(C, D) := \sup_{||x^*|| \leq 1} |s(x^*, C) - s(x^*, D)|$ is the Hausdorff metric on the hyperspace $cwk(X)$ (resp. $cb(X)$).

Let us recall some fact on multifunctions. Any map $\Gamma : \Omega \to c(X)$ is called a *multifunction*. A multifunction Γ is said to be *scalarly measurable* if for every $x^* \in X^*$, the function $t \to s(x^*, \Gamma(t))$ is measurable; Γ is said to be *scalarly integrable* if for each $x^* \in X^*$, the function $t \to s(x^*, \Gamma(t))$ is integrable.

The multifunction Γ is said to be *Pettis integrable* in $cwk(X)$ with respect to a measure m if Γ is scalarly integrable with respect to m and for every $A \in \mathcal{A}$, there exists $M_\Gamma(A) \in cwk(X)$ such that

$$s(x^*, M_\Gamma(A)) = \int_A s(x^*, \Gamma)\, dm \text{ for all } x^* \in X^*.$$

We set $\int_A \Gamma\, dm := M_\Gamma(A)$. For what concerns the multivalued integrability in $cwk(X)$ we refer, for example, to [6, 7, 13, 31]. If Γ is single-valued we obtain the well known definition for the vector functions. As regards Pettis integrability we refere to [30–33].

Definition 3.1. *Let $(m_n)_n$ be a sequence of measures. Moreover, for each $n \in \mathbb{N}$, let $\Gamma_n : \Omega \to cwk(X)$ be a multifunction scalarly integrable with respect to m_n. We say that the sequence $(\Gamma_n)_n$ has uniformly absolutely continuous scalar (m_n)-integrals on Ω if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $n \in \mathbb{N}$ and $A \in \mathcal{A}$*

$$m_n(A) < \delta \ \Rightarrow\ \sup\left\{\int_A |s(x^*, \Gamma_n)| dm_n \colon \|x^*\| \leq 1\right\} < \varepsilon. \tag{20}$$

If $\Gamma_n = \Gamma$ for every $n \in \mathbb{N}$ we obtain the uniformly absolutely continuous scalar (m_n)-integrability of Γ on Ω.

We recall that the space X is said to be *weakly compactly generated* (WCG) if it contains a weakly compact subset that is linearly dense in X (see for example [31]).

Theorem 3.2. *Let $\Gamma, \Gamma_n : \Omega \to cwk(X)$, $n \in \mathbb{N}$, be scalarly measurable multifunctions. Morover let $(m_n)_n$, m be measures. Assume that*

- (3.2.j) *the sequence $(\Gamma_n)_n$ has uniformly absolutely continuous scalar (m_n)-integrals on Ω;*
- (3.2.jj) *$(m_n)_n$ setwise converges to m;*
- (3.2.jjj) *each multifunction Γ_n is Pettis integrable with respect to m_n;*

(3.2.jv) Γ *is scalarly integrable with respect to* m.

If for every $A \in \mathcal{A}$ *and every* $x^* \in X^*$

$$\lim_n \int_A s(x^*, \Gamma_n)\, dm_n = \int_A s(x^*, \Gamma)\, dm \tag{21}$$

then Γ *is Pettis integrable in* $cwk(X)$ *with respect to* m.

Proof. At first we show that the operator $T_\Gamma : X^* \to L^1(m)$, defined as $T_\Gamma(x^*) = s(x^*, \Gamma)$ is bounded. Since the function Γ is scalarly integrable with respect to m, Γ is Dunford-integrable in $cw^*k(X^{**})$, where in X^{**} we consider the w^*-topology, and for every $A \in \mathcal{A}$ there exists $M_\Gamma^D(A) \in cw^*k(X^{**})$ such that, for every $x^* \in X^*$,

$$s(x^*, M_\Gamma^D(A)) = \int_A s(x^*, \Gamma) dm, \tag{22}$$

see for example [8, Theorem 3.2].
Moreover the set $\{s(x^*, \Gamma) : \|x^*\| \leq 1\}$ is bounded in $L^1(m)$. Indeed it follows by (22)

$$\int_\Omega |s(x^*, \Gamma)| dm \leq 2 \sup_{A \in \mathcal{A}} \left| \int_A s(x^*, \Gamma) dm \right| = 2 \sup_{A \in \mathcal{A}} |s(x^*, M_\Gamma^D(A))| < \infty,$$

where the last equality follows from the fact that for each $x^* \in X^*$, $s(x^*, M_\Gamma^D(\cdot))$ is a scalar measure. Hence, by the Banach–Steinhaus Theorem the set $\bigcup_{A \in \mathcal{A}} M_\Gamma^D(A) \subset X^{**}$ is bounded and then

$$\sup_{\|x^*\| \leq 1} \int_\Omega |s(x^*, \Gamma)| dm \leq 2 \sup \left\{ \|x\| : x \in \bigcup_{A \in \mathcal{A}} M_\Gamma^D(A) \right\} < \infty.$$

Therefore the operator T_Γ is bounded.
Now fix $\varepsilon > 0$ and $x^* \in B_{X^*}$. By (3.2.j) there exists $\delta > 0$ satisfying (20). Let $E \in \mathcal{A}$ be such that $m(E) < \delta/2$ and set $E^+ = \{t \in E : s(x^*, \Gamma(t)) \geq 0\}$ and $E^- = \{t \in E : s(x^*, \Gamma(t)) < 0\}$. From (21) and (3.2.jj) we find $N_1 \geq N$ such that $m_n(E) < \delta$ for every $n \geq N_1$ and

$$\int_{E^+} s(x^*, \Gamma) dm < \left| \int_{E^+} s(x^*, \Gamma_{N_1}) dm_{N_1} \right| + \frac{\varepsilon}{2}$$

and

$$\left| \int_{E^-} s(x^*, \Gamma) dm \right| < \left| \int_{E^-} s(x^*, \Gamma_{N_1}) dm_{N_1} \right| + \frac{\varepsilon}{2}.$$

So, by (3.3.j), we get

$$\begin{aligned}\int_E |s(x^*,\Gamma)|dm &= \int_{E^+} s(x^*,\Gamma)dm + \left|\int_{E^-} s(x^*,\Gamma)dm\right| \\ &< \left|\int_{E^+} s(x^*,\Gamma_{N_1})dm_{N_1}\right| + \left|\int_{E^-} s(x^*,\Gamma_{N_1})dm_{N_1}\right| + \varepsilon \\ &\leq \int_E |s(x^*,\Gamma_{N_1})|dm_{N_1} + \varepsilon < 2\varepsilon\end{aligned}$$

and the scalar uniform integrability with respect to m of Γ follows. Then the operator $T_\Gamma : X^* \to L^1(m)$ is weakly compact.
Now we are proving that Γ is determined by a WCG generated subspace of X. Since, for each $n \in \mathbb{N}$, Γ_n is Pettis integrable, by [31, Teorem 2.5], let Y_n be a WCG subspace of X generated by a weakly compact convex set $W_n \subset B_{X^*}$ and determining the multifunction Γ_n. The set $\sum 2^{-n} W_n$ is a weakly compact set generating a space Y. We want to prove that Γ is determined by Y.
Let $y^* \in Y^\perp$, let $\Omega^+ = \{t : s(y^*,\Gamma(t)) \geq 0\}$ and $A_n := \{t \in \Omega^+ : s(y^*,\Gamma_n(t)) = 0\}$. Then $m_n(A_n) = m_n(\Omega^+)$. Let $A := \limsup_n A_n = \bigcap_{k=1}^\infty \bigcup_{p=k}^\infty A_p$. Then

$$\begin{aligned} m(A) &= \lim_k m(\bigcup_{p=k}^{\infty} A_n) \geq \limsup_k m(A_k) \geq \limsup_k m_k(A_k) \\ &= \limsup_k m_k(\Omega^+) = m(\Omega^+).\end{aligned}$$

It follows by equality (21) that $s(y^*,\Gamma(t)) = 0$ m-a.e. on the set Ω^+. Analogously if we denote by $\Omega^- = \{t : s(y^*,\Gamma(t)) < 0\}$ it follows that $s(y^*,\Gamma(t)) = 0$ m-a.e. on the set Ω^-. Thus, Y determines the multifunction Γ and the Pettis integrability of Γ follows by [31, Theorem 2.5]. □

As a consequence of Theorem 3.2 we obtain

Theorem 3.3. *Let $\Gamma, \Gamma_n : \Omega \to cwk(X)$, $n \in \mathbb{N}$, be scalarly measurable multifunctions. Morover let $(m_n)_n$, m be measures. Suppose that*

- (3.3.j) *the sequence $(\Gamma_n)_n$ has uniformly absolutely continuous scalar (m_n)-integrals on Ω;*
- (3.3.jj) *$s(x^*,\Gamma_n) \to s(x^*,\Gamma)$, in m-measure, for each $x^* \in X^*$;*
- (3.3.jjj) *Γ has uniformly absolutely continuous scalar (m_n)-integrals on Ω;*
- (3.3.jv) *$(m_n)_n$ setwise converges to m;*
- (3.3.v) *each multifunction Γ_n is Pettis integrable with respect to m_n.*

Then the multifunction Γ *is Pettis integrable with respect to* m *in* $cwk(X)$ *and*

$$\lim_n s\left(x^*, \int_A \Gamma_n \, dm_n\right) = s\left(x^*, \int_A \Gamma \, dm\right),$$

for every $x^* \in X^*$ *and for every* $A \in \mathcal{A}$.

Proof. First of all we observe that, by Theorem 2.11, we have that for every $A \in \mathcal{A}$

$$\lim_n \int_A s(x^*, \Gamma_n) \, dm_n = \int_A s(x^*, \Gamma) \, dm. \tag{23}$$

Moreover by Proposition 2.10 Γ is scalarly integrable, therefore the Pettis integrability of $\mathit{\Gamma}$ with respect to m is a consequence of Theorem 3.2. □

The following definition is a generalization of the notion of the scalar equi-convergence in measure for a sequence of scalarly measurable multifunctions $(\Gamma_n)_n$ (see [32, p.852] and [3] for the vector case).

Definition 3.4. *Let* $\mathit{\Gamma}, \mathit{\Gamma}_n \colon \mathit{\Omega} \to cwk(X)$ *be scalarly measurable multifunctions. We say that the sequence* $(\mathit{\Gamma}_n)_n$ *is scalarly equi-convergent in measure with respect to a sequence of measures* $(m_n)_n$ *to* $\mathit{\Gamma}$ *if, for every* $\delta > 0$,

$$\lim_n \sup_{\|x^*\| \leq 1} m_n\{t \in \mathit{\Omega} \colon |s(x^*, \mathit{\Gamma}_n(t)) - s(x^*, \mathit{\Gamma}(t))| > \delta\} = 0. \tag{24}$$

If in the Theorem 3.3 we substitute the convergence in condition (3.3.jj) with the scalar equi-convergence in measure and the setwise convergence of m_n to m with the convergence in total variation, we get a stronger result.

Theorem 3.5. *Let* $\Gamma, \Gamma_n : \mathit{\Omega} \to cwk(X)$, $n \in \mathbb{N}$, *be scalarly measurable multifunctions. Morover let* $(m_n)_n$, m, *be measures. Suppose that*

- (3.5.j) *the sequence* $(\Gamma_n)_n$ *has uniformly absolutely continuous scalar* (m_n)*-integrals on* $\mathit{\Omega}$;
- (3.5.jj)) *the sequence* $(\mathit{\Gamma}_n)_n$, $n \in \mathbb{N}$, *is scalarly equi-convergent in measure with respect to* $(m_n)_n$ *and* m *to* $\mathit{\Gamma}$;
- (3.5.jjj) Γ *has uniformly absolutely continuous scalar* (m_n) *and* m *integrals;*
- (3.5.jv) $(m_n)_n$ *is convergent to* m *in total variation;*
- (3.5.v) *each multifunction* Γ_n *is Pettis integrable with respect to* m_n.

Then the multifunction Γ *is Pettis integrable with respect to* m *in* $cwk(X)$ *and*

$$\lim_n d_H(M_{\Gamma_n}(A), M_\Gamma(A)) = 0$$

uniformly in $A \in \mathcal{A}$*, where* M_{Γ_n}*,* $(M_\Gamma) : \mathcal{A} \to cwk(X)$ *are the* m_n (m)*-Pettis integrals of the multifunction* Γ_n (Γ) *respectively.*

Proof. For every $x^* \in X^*$ and $A \in \mathcal{A}$ it is enough to apply [36, Theorem 2.8] to get

$$\lim_n \int_A s(x^*, \Gamma_n)\, dm_n = \int_A s(x^*, \Gamma)\, dm$$

and as in Theorem 3.3 the multifunction Γ is Pettis integrable with respect to m. In order to prove the convergence in the Hausdorff metric, fix $A \in \mathcal{A}$, $\varepsilon > 0$ and $\delta > 0$ that satisfy (24) also in case of $m_1 = m_2 = \ldots = m$ and $\Gamma_1 = \Gamma_2 = \ldots = \Gamma$. Fix also $0 < \eta < \varepsilon$. For each $n \in \mathbb{N}$ and $x^* \in B_{X^*}$ denote by H_{n,x^*} the set

$$H_{n,x^*} := \{t \in \Omega\colon |s(x^*, \Gamma_n(t)) - s(x^*, \Gamma(t))| > \eta\}.$$

By the assumption of the scalar equi-convergence in measure with respect to m_n and m, there exists $k \in \mathbb{N}$ such that for all $n \geq k$

$$\sup_{\|x^*\| \leq 1} \max \left\{ m_n(H_{n,x^*}), m(H_{n,x^*}) \right\} < \delta.$$

Then, for all $n \geq k$ and $\|x^*\| \leq 1$

$$\begin{aligned} &\sup_{\|x^*\| \leq 1} \left| \int_A s(x^*, \Gamma_n)\, dm_n - \int_A s(x^*, \Gamma)\, dm \right| \leq \\ \leq\ &\sup_{\|x^*\| \leq 1} \left| \int_{A \cap H_{n,x^*}} s(x^*, \Gamma_n)\, dm_n - \int_{A \cap H_{n,x^*}} s(x^*, \Gamma)\, dm \right| + \\ +\ &\sup_{\|x^*\| \leq 1} \left| \int_{A \cap H^c_{n,x^*}} s(x^*, \Gamma_n)\, dm_n - \int_{A \cap H^c_{n,x^*}} s(x^*, \Gamma)\, dm \right|. \end{aligned}$$

Observe that, by (3.5.j-3.5.jjj) and formula (24), we have

$$\begin{aligned} &\sup_{\|x^*\| \leq 1} \left| \int_{A \cap H_{n,x^*}} s(x^*, \Gamma_n)\, dm_n - \int_{A \cap H_{n,x^*}} s(x^*, \Gamma)\, dm \right| \leq \\ &\sup_{\|x^*\| \leq 1} \left| \int_{A \cap H_{n,x^*}} s(x^*, \Gamma_n)\, dm_n \right| + \sup_{\|x^*\| \leq 1} \left| \int_{A \cap H_{n,x^*}} s(x^*, \Gamma)\, dm \right| \leq 2\varepsilon. \end{aligned}$$

Relatively to the second summand we apply Remark 2.4 and formula (2) and we obtain

$$\begin{aligned}
&\sup_{\|x^*\|\le 1}\left|\int_{A\cap H^c_{n,x^*}} s(x^*,\varGamma_n)\,dm_n - \int_{A\cap H^c_{n,x^*}} s(x^*,\varGamma)\,dm\right| \le \\
&\le \sup_{\|x^*\|\le 1}\left|\int_{A\cap H^c_{n,x^*}} s(x^*,\varGamma_n)\,dm_n - \int_{A\cap H^c_{n,x^*}} s(x^*,\varGamma)\,dm_n\right| + \\
&+ \sup_{\|x^*\|\le 1}\left|\int_{A\cap H^c_{n,x^*}} s(x^*,\varGamma)\,dm_n - \int_{A\cap H^c_{n,x^*}} s(x^*,\varGamma)\,dm\right| \le \\
&\le \sup_{\|x^*\|\le 1} \eta\, m_n(A\cap H^c_{n,x^*}) + \sup_{\|x^*\|\le 1}\left|\int_{A\cap H^c_{n,x^*}} s(x^*,\varGamma)\,d(m_n-m)\right| \le \\
&\le \varepsilon \sup_n m_n(\Omega) + \sup_{\|x^*\|\le 1}\int_{A\cap H^c_{n,x^*}} |s(x^*,\varGamma)| d|m_n-m|.
\end{aligned}$$

Let $\overline{m} := \sum_{k=1}^{\infty} |m_n - m|(E)\cdot(2^n(1+|m_n-m|(\Omega)))^{-1}$ be a probability measure on $\mathcal{A}$ such that $|m_n - m| \ll \overline{m}$ for every $n\in\mathbb{N}$. According to [31, Proposition 1.2] there exists a measurable function $\varphi_\Gamma : \varOmega \to [0,\infty)$ such that for each $x^*\in X^*$ the inequality $|s(x^*,\varGamma(t))| \le \varphi_\Gamma(t)\|x^*\|$ holds true $\overline{m}$-a.e. In particular the inequality holds true also $|m_n-m|$-a.e., for each $n\in\mathbb{N}$ separately.
Let $a>0$ and $F\in\mathcal{A}$ be such that $\varphi_\Gamma(t)\chi_F \le a$ $\overline{m}$-a.e. and $m(F^c) < \delta$. Since $(m_n)_n$ converges to m, there exists $\mathbb{N}\ni\tilde{k}\ge\overline{k}$ such that $m_n(F^c)<\delta$ for every $n\ge\tilde{k}$. Then, let $\mathbb{N}\ni\check{k}>\tilde{k}$ be such that $|m_n-m|(F)<\varepsilon/a$ for every $n\ge\check{k}$. If $n\ge\check{k}$, then

$$\begin{aligned}
&\sup_{\|x^*\|\le 1}\int_{\varOmega} |s(x^*,\varGamma)|\, d|m_n-m| \\
\le\ &\sup_{\|x^*\|\le 1}\int_{F} |s(x^*,\varGamma)|\, d|m_n-m| + \sup_{\|x^*\|\le 1}\int_{F^c} |s(x^*,\varGamma)\, d|m_n-m| \\
\le\ & a|m_n-m|(F) + \sup_{\|x^*\|\le 1}\int_{F^c} |s(x^*,\varGamma)|\, dm + \sup_{\|x^*\|\le 1}\int_{F^c} |s(x^*,\varGamma)|\, dm_n \le 3\varepsilon.
\end{aligned}$$

Then,

$$\begin{aligned}
d_H(M_{\Gamma_n}(A), M_\Gamma(A)) &= \sup_{\|x^*\|\le 1} |s(x^*, M_{\Gamma_n}(A)) - s(x^*, M_\Gamma(A))| \\
&\le \sup_{\|x^*\|\le 1}\left|\int_A s(x^*,\varGamma_n)\,dm_n - \int_A s(x^*,\varGamma)\,dm\right| \\
&\le 3\varepsilon + \varepsilon\sup_n m_n(\varOmega).
\end{aligned}$$

Since $\overline{k}$ is indipendent to $A \in \mathcal{A}$, then the previous convergence is uniform with respect to A. □

3.1. **The vector case for integrands.** We consider now vector valued functions which are a particular case of the multivalued one. Therefore Theorem 3.2 can be rewritten as follows:

Theorem 3.6. *Let $f, f_n : \Omega \to X$, $n \in \mathbb{N}$, be scalarly measurable functions. Morover let $(m_n)_n$, m be measures. Assume that*

(3.2.j) *the sequence $(f_n)_n$ has uniformly absolutely continuous scalar (m_n)-integrals on Ω;*
(3.2.jj) *$(m_n)_n$ setwise converges to m;*
(3.2.jjj) *each function f_n is Pettis integrable with respect to m_n;*
(3.2.jv) *f is scalarly integrable with respect to m.*

If for every $A \in \mathcal{A}$ and every $x^* \in X^*$

$$\lim_n \int_A x^* f_n \, dm_n = \int_A x^* f \, dm \tag{25}$$

then f is Pettis integrable in X with respect to m.

The following theorem is the vector valued formulation of Theorem 3.3. We present here a proof of the Pettis integrability of the limit function f using a characterization of weakly compact sets and without using the multivalued case.

Theorem 3.7. *Let $f, f_n : \Omega \to X$ be scalarly measurable functions. Moreover let $(m_n)_n$ and m be measures. Suppose that*

(3.7.j) *$(f_n)_n$ has uniformly absolutely continuous scalar (m_n)- integrals on Ω;*
(3.7.jj) *$x^* f_n(t) \to x^* f(t)$, in m-measure, for each $x^* \in X^*$;*
(3.7.jjj) *f has uniformly absolutely continuous scalar (m_n)-integrals on Ω;*
(3.7.jv) *$(m_n)_n$ is setwise convergent to m;*
(3.7.v) *each function f_n is Pettis integrable with respect to m_n.*

Then f is Pettis integrable with respect to m and

$$\lim_n \int_\Omega f_n dm_n = \int_\Omega f dm,$$

weakly in X.

Proof. Let $A \in \mathcal{A}$. For every $x^* \in X^*$ it is enough to apply Theorem 2.11 to get

$$\lim_n \int_A x^* f_n dm_n = \int_A x^* f dm. \tag{26}$$

Therefore to get our thesis it is enough to prove that f is Pettis integrable. By equality (26) we have that the sequence $\left(\int_\Omega f_n dm_n\right)_n$ is weakly Cauchy.
By a Grothendieck characterization of weakly compact sets ([20]) and taking into account hypothesis (3.7.v), it is enough to show that for each $(f_{n_j})_j$ and $(x^*_k)_k \subset B_{X^*}$

$$\alpha := \lim_k \lim_j \left\langle x^*_k, \int_\Omega f_{n_j} dm_{n_j} \right\rangle = \beta := \lim_j \lim_k \left\langle x^*_k, \int_\Omega f_{n_j} dm_{n_j} \right\rangle, \tag{27}$$

provided all above limits exist.
Since the function f is scalarly integrable with respect to m, f is Dunford-integrable and the set $\{x^* f : \|x^*\| \leq 1\}$ is bounded in $L^1(m)$. To see it let ν be the Dunford integral of f. Then the set $\{\nu(E) : E \in \mathcal{A}\}$ being the range of $\mathcal{A}$ in X^{**} is a bounded set (because it is weak* bounded).
If $\pi = \{E_1, ..., E_n\}$ is a partition of Ω into pairwise disjoint members of $\mathcal{A}$, and $x^* \in B_{X^*}$, then

$$\begin{aligned} \sum_{E_i \in \pi} |x^* \nu(E_i)| &= \sum_{E_i \in \pi^+} x^* \nu(E_i) - \sum_{E_i \in \pi^-} x^* \nu(E_i) = \\ &= x^* \{ \sum_{E_i \in \pi^+} \nu(E_i)\} - x^* \{ \sum_{E_i \in \pi^-} \nu(E_i)\} \leq \\ &\leq 2 \sup\{\|\nu(E)\| : \; E \in \Sigma\} \end{aligned}$$

where $\pi^+ = \{E_i : x^* \nu(E_i) \geq 0\}$ and $\pi^- = \{E_i : x^* \nu(E_i) < 0\}$, Hence, if $x^* \in B_{X^*}$, then

$$\int_\Omega |x^* f| \, dm = |x^* \nu|(\Omega) \leq 2 \sup\{\|\nu(E)\| : \; E \in \mathcal{A}\} < \infty. \tag{28}$$

At first we are proving that the sequence $(x^*_k f)_k$ has uniformly absolutely continuous integrals with respect to m on Ω.
Now let $E \in \mathcal{A}$ be such that $m(E) < \delta/2$. Set $E^+_k = \{t \in E : x^*_k f(t) > 0\}$ and $E^-_k = \{t \in E : x^*_k f(t) \leq 0\}$. Moreover let $N_k(\varepsilon) \in \mathbb{N}$ be such that for all $n > N_k(\varepsilon)$

$$\sup \left\{ \left| \int_{E^+_k} x^*_k f_n dm_n - \int_{E^+_k} x^*_k f dm \right|, \left| \int_{E^-_k} x^*_k f_n dm_n - \int_{E^-_k} x^*_k f dm \right| \right\} < \varepsilon. \tag{29}$$

Let $n_k > N_k(\varepsilon)$ be such that $m_{n_k}(E) \le m(E) + \delta/2 < \delta$. Therefore, taking into account hypothesis (3.7.j), by (29) we have

$$\begin{aligned} \int_E |x_k^* f| dm &= \left| \int_{E_k^+} x_k^* f dm \right| + \left| \int_{E_k^-} x_k^* f dm \right| \\ &\le \left| \int_{E_k^+} x_k^* f_{n_0} dm_{n_0} - \int_{E_k^+} x_k^* f dm \right| + \\ &+ \left| \int_{E_k^-} x_k^* f_{n_0} dm_{n_0} - \int_{E_k^-} x_k^* f dm \right| \\ &+ \int_{E_k^+} |x_k^* f_{n_0}| dm_{n_0} + \int_{E_k^-} |x_k^* f_{n_0}| dm_{n_0} < 4\varepsilon. \end{aligned}$$

Since the sequence $(x_k^* f)_k$ has uniformly absolutely continuous integrals with respect to m on Ω and it is bounded in $L^1(m)$, it is weakly relatively compact. So we have the existence of a subsequence $(z_k^*)_k$ of $(x_k^*)_k$ and of a real valued function $g \in L^1(m)$ such that $z_k^* f \to g$ weakly in $L^1(m)$. Mazur's Theorem yields the existence of functionals $w_k^* \in co\{z_j^* : \ j \ge k\}$ such that

$$\lim_k \int_\Omega |w_k^* f - g| dm = 0 \qquad \text{and } \lim_k w_k^* f = g, \qquad \text{m-a.e.}.$$

If w_0^* is a weak*-closter point of $(w_k^*)_k$, then $g = w_0^* f$ m-a.e.. Therefore $\alpha = \int_\Omega w_0^* f dm$. On the other hand

$$\begin{aligned} \lim_k \left\langle x_k^*, \int_\Omega f_{n_j} dm_{n_j} \right\rangle &= \lim_k \left\langle w_k^*, \int_\Omega f_{n_j} dm_{n_j} \right\rangle = \left\langle w_0^*, \int_\Omega f_{n_j} dm_{n_j} \right\rangle \\ &= \int_\Omega w_0^* f_{n_j} dm_{n_j}. \end{aligned}$$

By the hypothesis (3.7.j-3.7.jv) and by Theorem 2.11 it follows

$$\beta := \lim_j \int_\Omega w_0^* f_{n_j} dm_{n_j} = \int_\Omega w_0^* f dm.$$

So $\alpha = \beta$ and this completes the proof. □

In particular

Corollary 3.8. *Let $(m_n)_n$ be a sequence of measures that is setwise convergent to a measure m. If $f : \Omega \to X$ has uniformly absolutely continuous scalar (m_n)-integrals on Ω, then f is scalarly m-integrable. If moreover f is Pettis integrable with respect to m_n for each $n \in \mathbb{N}$,* *then f is Pettis integrable with respect to m and for every $A \in \mathcal{A}$ it is*

$$\lim_n \int_A f dm_n = \int_A f dm,$$

weakly.

Proof. The first assertion follows from Proposition 2.10. The Pettis integrability of f with respect to m follows from Theorem 3.7 when $f_n = f$ for every $n \in \mathbb{N}$. □

As in Theorem 3.5, if we consider the scalarly equi-convergence in measure and the convergence in total variation, we have

Theorem 3.9. *Let $f, f_n : \Omega \to X$ be measurable functions and let $(m_n)_n$, m be measures. Suppose that*

- (3.9.i) *$(f_n)_n$ has uniformly absolutely continuous scalar (m_n)-integrals on Ω;*
- (3.9.ii) *$(f_n)_n$ is scalarly equi-convergent in measure to f with respect to $(m_n)_n$ and m;*
- (3.9.iii) *f has uniformly absolutely continuous scalar (m_n) and m integrals;*
- (3.9.iv) *$(m_n)_n$ is convergent to m in total variation;*
- (3.9.v) *each function f_n is Pettis integrable with respect to m_n.*

Then, f is Pettis-integrable with respect to m and

$$\lim_n \left\| \int_A f_n \, dm_n - \int_A f \, dm \right\| = 0$$

uniformly with respect to $A \in \mathcal{A}$.

3.2. **The vector case for McShane integrable integrands.** Now we are going to examine the behavior of McShane integrable integrands. The McShane integral is a "gauge defined integral" and, also in the case of the generalized McShane integral introduced in 1995 by D. Fremlin in a measure space Ω, we need a topology in Ω. Briefly we recall the definition (see [18]). For simplicity we prefer to use finite partitions. So our framework Ω is a compact Radon measure space.

Let $(\Omega, \mathcal{A})$ be a compact Radon measure space space with measure m and topology $\mathcal{T}$. A *finite strict generalized McShane partition* of Ω is a family $\{(A_i, t_i)\}_{i \leq p}$ such that $A_1, ..., A_p$ is a finite disjoint cover of Ω by elements of $\mathcal{A}$ and $t_i \in \Omega, i = 1, \ldots, p$. A *gauge* Δ on Ω is a function $\Delta : \Omega \to \mathcal{T}$ such that $t \in \Delta(t)$ for every $t \in \Omega$. We say that a (finite strict generalized) McShane partition $\{(A_i, t_i)\}_{i \leq p}$ is *subordinated* to a gauge $\Delta(t)$ if $A_i \subset \Delta(t_i)$ for $i = 1, ..., p$.

A function $f : \Omega \to X$ is said to be *m-McShane* (m-(MS)) integrable on Ω with m-(MS)-integral $w \in X$ if for every $\varepsilon > 0$ there exists a gauge Δ such that for each partition $\{(A_i, t_i)\}_{i\leq p}$ subordinated to Δ, we have

$$\left\| \sum_{i=1}^{p} f(t_i)m(A_i) - w \right\| < \varepsilon. \tag{30}$$

We set $w := (MS)\int_\Omega f dm$.

Definition 3.10. Let m_n, $n = 1, 2, ...$ be measures. We say that a sequence of m_n-(MS)integrable functions $f_n : \Omega \to X$ is (m_n)-*equi-integrable on* Ω, if for every $\varepsilon > 0$ there exists a gauge Δ such that for every $n \in \mathbb{N}$

$$\left\| \sum_{i=1}^{p} f_n(t_i)m_n(A_i) - (MS)\int_\Omega f_n dm_n \right\| < \varepsilon \tag{31}$$

for each partition $\{(A_i, t_i)\}_{i\leq p}$ subordinated to Δ.

If $m_n = m$ for all $n \in \mathbb{N}$, then we have the classical condition of equi-integrability.

Theorem 3.11. *Let m and $(m_n)_n$ be measures, let $f_n : \Omega \to X$ be $m_n - (MS)$-integrable functions, $n \in \mathbb{N}$, and let $f : \Omega \to X$. Suppose that*

(3.11.i) *the sequence $(f_n)_n$ is (m_n)-equi-integrable on Ω;*
(3.11.ii) *$f_n(t) \to f(t)$, for all $t \in \Omega$;*
(3.11.iii) *$(m_n)_n$ is setwise convergent to m.*

Then, f is m-(MS)integrable and for all $A \in \mathcal{A}$,

$$\lim_n (MS)\int_A f_n dm_n = (MS)\int_A f dm. \tag{32}$$

Moreover if we substitute condition (3.11.iii) *with the convergence in total variation ($m_n \stackrel{tv}{\to} m$), then* (32) *holds uniformly in $A \in \mathcal{A}$.*

Proof. Let $A \in \mathcal{A}$ be fixed. If Δ is the gauge on Ω satisfying (3.11.i) corresponding to the value $\varepsilon > 0$, then for any $n \in \mathbb{N}$

$$\left\| \sum_{i=1}^{p} f_n(t_i)m_n(A_i \cap A) - (MS)\int_A f_n dm_n \right\| < \varepsilon \tag{33}$$

for every partition $\{(A_i, t_i)\}_{i\leq p}$ of Ω subordinated to Δ (see [18, Theorem 1N]). Since the partition is fixed the pointwise convergence of f_n

to f and the setwise convergence of m_n to m imply that

$$\lim_{n\to\infty} \sum_{i=1}^{p} f_n(t_i) m_n(A_i \cap A) = \sum_{i=1}^{p} f(t_i) m(A_i \cap A). \tag{34}$$

Choose $n_0 \in \mathbb{N}$ so that if $n, s > n_0$

$$\left\| \sum_{i=1}^{p} f_n(t_i) m_n(A_i \cap A) - \sum_{i=1}^{p} f_s(t_i) m_s(A_i \cap A) \right\| < \varepsilon.$$

Then we have

$$\begin{aligned}
&\left\| (MS) \int_A f_n dm_n - (MS) \int_A f_s dm_s \right\| \\
\leq &\left\| (MS) \int_A f_n dm_n - \sum_{i=1}^{p} f_n(t_i) m_n(A_i \cap A) \right\| + \\
+ &\left\| \sum_{i=1}^{p} f_n(t_i) m_n(A_i) - \sum_{i=1}^{p} f_s(t_i) m_s(A_i \cap A) \right\| + \\
+ &\left\| \sum_{i=1}^{p} f_s(t_i) m_s(A_i \cap A) - (MS) \int_A f_s dm_s \right\| < 3\varepsilon,
\end{aligned}$$

which shows that the sequence $\big((MS) \int_A f_n dm_n\big)_n$ is Cauchy, therefore it converges to $x_A \in X$. Then we have

$$\begin{aligned}
\left\| \sum_{i=1}^{p} f(t_i) m(A_i \cap A) - x_A \right\| &\leq \left\| \sum_{i=1}^{p} f(t_i) m(A_i \cap A) - \sum_{i=1}^{p} f_n(t_i) m_n(A_i \cap A) \right\| \\
+ \quad \left\| \sum_{i=1}^{p} f_n(t_i) m_n(A_i \cap A) - (MS) \int_A f_n dm_n \right\| &+ \left\| (MS) \int_A f_n dm_n - x_A \right\| < 3\varepsilon.
\end{aligned}$$

Therefore it follows that f is m-(MS)integrable on A and

$$\lim_n (MS) \int_A f_n dm_n = (MS) \int_A f dm.$$

Finally, if $m_n \stackrel{tv}{\to} m$, n_0 does not depend on A. Then formula (34) holds uniformly on $\mathcal{A}$ and the convergence in formula (32) is uniform. □

We now can consider the variety of multifunctions, including not only those with weakly compact values but also those with bounded closed convex values. We remember that

Definition 3.12. *A multifunction $\Gamma : \Omega \to cb(X)$ is said to be m-McShane integrable on Ω, if there exists $\varphi_\Gamma \in cb(X)$ such that for* *every $\varepsilon > 0$ there exists a gauge Δ on Ω such that for each partition $\{(A_i, t_i)\}_{i \le p}$ of Ω subordinated to Δ, we have*

$$d_H\left(\varphi_\Gamma, \sum_{i=1}^{p} \Gamma(t_i) m(A_i)\right) < \varepsilon\,. \tag{35}$$

Since the McShane integral is "gauge defined", by a Rådström embedding it is immediate to extend the previous Theorem 3.11 for vector valued functions to the McShane integrable multifunctions. A Rådström embedding theorem states that the nonempty closed convex subsets of a Banach space X can be identified with points of $\ell_\infty(B_{X^*})$ such that the embedding map $i : cb(X) \to \ell_\infty(B_{X^*})$ is additive, positively homogeneous, and isometric (see for example [5, 9, 12, 24]). This allows us to reduce the McShane integrability of multifunctions to the McShane integrability of functions by embedding $cb(X) \hookrightarrow \ell_\infty(B_{X^*})$. The key point is that $i(cb(X))$ is a closed cone, consequently, if $z \in \ell_\infty(B_{X^*})$ is the value of the integral of $i \circ \Gamma$, then there exists a set $I \in cb(X)$ with $i(I) = z$.

In the following theorem the notion of (m_n)-equi-integrability for multifunctions is analogous to that we have for functions (Definition 3.10).

Theorem 3.13. *Let m and $(m_n)_n$ be measures, let $\Gamma_n : \Omega \to cb(X)$ be $m_n - (MS)$-integrable multifunctions, $n \in \mathbb{N}$, and let $\Gamma : \Omega \to cb(X)$. Suppose that*

(3.13.i) *the sequence $(\Gamma_n)_n$ is (m_n)-equi-integrable on Ω;*
(3.13.ii) $\lim_{n\to\infty} d_H(\Gamma_n(t), \Gamma(t)) = 0$, *for each $t \in \Omega$;*
(3.13.iii) *$(m_n)_n$ is setwise convergent to m.*

Then, Γ is m-(MS)integrable and for all $A \in \mathcal{A}$,

$$\lim_n (MS)\int_A \Gamma_n\, dm_n = (MS)\int_A \Gamma\, dm. \tag{36}$$

Moreover if we substitute condition (3.13.iii) *with the convergence in total variation ($m_n \stackrel{tv}{\to} m$), then* (36) *holds uniformly in $A \in \mathcal{A}$.*

Proof. We apply Theorem 3.11 and the Rådström embedding since

$$\lim_{n\to\infty} i \circ \left((MS)\int_A \Gamma_n\, dm_n\right) = \lim_{n\to\infty} (MS)\int_A i \circ \Gamma_n\, dm_n =$$
$$(MS)\int_A i \circ \Gamma dm = i \circ \left((MS)\int_A \Gamma\, dm\right).$$

□

Conclusions Convergence results for varying measures are obtained both in weak and strong sense making use of the setwise convergence and the convergence in total variation respectively. By means of the Pettis integrability we are able to obtain the vector case as a particular case of the multivalued one. When we consider the McShane integrability we are able to pass from the vector case to the multivalued one.

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