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Weak Solutions for a Class of Nonlocal Singular Problems Over the Nehari Manifold

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ABSTRACT

In this paper, we consider a nonlocal model of dilatant non-Newtonian fluid with a Dirichlet boundary condition. By using the Nehari manifold and fibering map methods, we obtain the existence of at least two weak solutions, with sign information. In detail, we construct two local minimizers of the energy functional associate to the main problem on proper Nehari submanifolds; in this way, we bypass the singularity to get boundedness below and coercivity of the functional. Then, we conclude that such minimizers correspond to weak solutions.

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1 | Introduction

Many physical phenomena are originated by chemical composition and the interaction of materials, as well as from the state of matter itself. In the context of non-Newtonian fluids and plasticity theory, laminar flows require mathematical models in the sense of (elliptic) problems that may be singular with respect to the unknown function. We refer to the monograph by Hernández-Mancebo [1, Chapter 4] for a wider discussion. In this direction, the basic reaction-diffusion system can be stated as

$$\varphi_t - \Delta_p \varphi - \Delta_q \varphi = f(z, \varphi),$$

where the function φ is the concentration, the nonlinearity $f(z, \varphi)$ is the reaction linked to sources and loss processes, and the operators $\Delta_p \varphi = \operatorname{div}(|\nabla \varphi|^{p(z)-2}\nabla \varphi)$ and $\Delta_q \varphi = \operatorname{div}(|\nabla \varphi|^{q(z)-2}\nabla \varphi)$ represent diffusion, and are the classical p-Laplace and q-Laplace differential operators in the case of constant exponents $p, q \in (1, +\infty)$, or, as in our study, in the case of variable exponents $1 < p(z), q(z) \in C(\overline{\Omega})$, where $\Omega \subseteq \mathbb{R}^N$. Despite the single p-Laplacian (resp., q-Laplacian) is a homogeneous differential operator, the sum of such operators gives us the nonhomogeneity of the operator $-\Delta_p - \Delta_q$, which is a starting point to define a large class of operators arising in the study of energy functionals with non-standard growth as introduced by Marcellini [2, 3] and Zhikov [4, 5] in the context of calculus of variations and elasticity theory, see also

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Marcellini [6] and the references therein. According to these investigations we know that the operator $-\Delta_p - \Delta_q$ exhibits balanced growth; hence, it is possible to analyze its behavior in the framework setting of Lebesgue and Sobolev spaces with constant exponents (resp., variable exponents); see the monograph of Rădulescu-Repovš [7] for more details and information. On the other hand, Kirchhoff-type functions lead to weighted (nonlocal) operators useful in modeling the deformation of elastic materials. We recall that Kirchhoff [8] generalized the classical D'Alembert wave equation as follows

$$\rho\varphi_{tt} - \left(a + b\int_0^L \frac{1}{2}\left|\varphi_x\right|^2 dx\right)\varphi_{xx} = 0 \tag{1}$$

under a proper set of physical parameters ρ, P_0, h, E, L (i.e., mass density, initial tension, area of the cross-section, Young modulus of the material, length of the string) with $a = P_0/h$ and $b = E/L$. Now, the Kirchhoff terms may act successfully on $-\Delta_p, -\Delta_q$, as shown, e.g., by Figueiredo-Vetro [9], who designed a Galerkin-type approximation procedure to discuss equations with gradient-dependent reaction (convection phenomena). In this paper, combining such features of the principal operator, we focus on the singular Kirchhoff-type Dirichlet problem given by

$$\begin{cases} -\left(M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\operatorname{div}(|\nabla\varphi|^{p(z)-2}\nabla\varphi) + M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\operatorname{div}(|\nabla\varphi|^{q(z)-2}\nabla\varphi)\right) \\ +\theta\left(|\varphi|^{p(z)-2}\varphi + |\varphi|^{q(z)-2}\varphi\right) = \dfrac{f(z)}{\varphi^{\alpha(z)}} + \lambda|\varphi|^{\beta(z)-2}\varphi \quad \text{in } \Omega, \\ \varphi\Big|_{\partial\Omega} = 0,\ \theta > 0, \lambda > 0, \end{cases} \tag{P_0}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $0 < M \in C^1(0,\infty)$, $0 < \alpha(z) \in C(\overline{\Omega})$ and $1 < \beta(z) \in C(\overline{\Omega})$. In our problem, the nonlocal operators $-M\left(\int_\Omega \frac{|\nabla\cdot|^{p(z)}}{p(z)}dz\right)\Delta_p$ and $-M\left(\int_\Omega \frac{|\nabla\cdot|^{q(z)}}{p(z)}dz\right)\Delta_q$ aim to extend the Kirchhoff term in Equation (1) (i.e., the case $p = q = 2$) to the more general case of variable exponents $1 < p(z), q(z) \in C(\overline{\Omega})$; hence, we recall that the interest for nonlocal problems was revived by the approach introduced by Lions [10, 11] (the so-called Lions' representation theorem) to deal with evolutionary problems arising in physical phenomena. On the left-hand side, we also have a parametric power term $\varphi \to \theta\left(|\varphi|^{p(z)-2}\varphi + |\varphi|^{q(z)-2}\varphi\right)$, and in the right-hand side the reaction exhibits the combined effects of a singular term $\varphi \to f(z)\varphi^{-\alpha(z)}$ with weight $f(z) \in L^\infty(\Omega)$ and of a parametric concave term $\varphi \to \lambda|\varphi|^{\beta(z)-2}\varphi$. Some necessary links among the exponents and constants involved in our model are assumed as follows. Denote $p_- := \inf_{z\in\overline{\Omega}} p(z) \leq p(z) \leq p_+ := \sup_{x\in\overline{\Omega}} p(z)$, $q_- := \inf_{z\in\overline{\Omega}} q(z) < q(z) < q_+ := \sup_{z\in\overline{\Omega}} q(z)$, and $\theta < \beta_- := \inf_{z\in\overline{\Omega}} \beta(z) < \beta(z) < \beta_+ := \sup_{z\in\overline{\Omega}} \beta(z)$, then we take

$$q_- \leq q_+ < p_- \leq p_+ < \beta_- \leq \beta_+ < p^*(z) := \frac{Np(z)}{N - p(z)} \text{ for } z \in \overline{\Omega},$$

where we assume $p(z) < N$ for all $z \in \overline{\Omega}$ (see (2)). We note that the existence of weak solutions to singular problems has been largely investigated by variational and topological approaches. For a nice survey, we refer to the book by Papageorgiou-Rădulescu-Repovš [12], where the authors collect basic constructive methods, as well as their applications to physical systems. Singular equations were examined by Papageorgiou-Qin-Rădulescu [13] for non-autonomous differential operators of $p\&q$-type in the case of constant exponents. Their reaction exhibits the competing effects of a singular term, a concave term, and a Carathéodory perturbation. In the past, Papageorgiou-Winkert [14] focused also on the similar equations (again in the case of constant exponents), assuming a superlinear non-negative perturbation. We recall that in the case when the perturbation is non-negative, one can easily obtain a lower solution for the problem, and this solution permits one to bypass the singularity; hence, the authors can retrieve C^1 energy functionals associate to the problems under investigation. This strategy is used in Reference [14] to conclude a bifurcation-type theorem for existence and multiplicity of weak solutions; hence, the finding is extended to the case of variable exponents in the work of Gasinski-Papageorgiou [15]. In detail, the authors of References [14, 15] combine variational tools, together with truncation and comparison methods. Among the available approaches, the Nehari manifold method is one of the most useful tools in establishing weak solutions to boundary value problems by using the critical points of energy functionals. Briefly, denoting by $\mathcal{J}_\lambda(\varphi)$ the Euler energy functional associate to problem (P_0), we know that all critical points of the functional have to be in the set $\{\varphi : \langle \mathcal{J}'_\lambda(\varphi), \varphi\rangle = 0\}$ (see Section 3 for more details). This set is exactly the Nehari manifold of $\mathcal{J}_\lambda(\varphi)$ (see the pioneering paper by Nehari [16]). Hence, showing that the functional is bounded below on suitable Nehari submanifolds (i.e., we do not need that the functional is bounded below in the whole space where it is posed), one can conclude the existence of critical points by employing a minimization procedure (see again [16], and the monograph by Willem [17]). Following this strategy, one first shows that the Nehari manifold is not empty, then constructs minimizing sequences converging to minimizers on the Nehari submanifolds, and then concludes that such minimizers correspond to

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weak solutions. This leads to multiplicity results in the sense of Lin-Tang [18] and Szulkin-Weth [19]. Further, the study can be enriched by exploiting the connection between the Nehari manifold and the so-called fibering map denoted by $\psi_\varphi(t)$ (i.e., a map $t \to J_\lambda(t\varphi)$, $t > 0$). The related fibering method is due to the work by Pohozaev [20] and permits one to conclude the existence of solutions under appropriate assumptions on the functional. We also refer to Zhou-Ge [21] and the references therein, for some applications of the fibering method to problems driven by the operator $-\Delta_p$ (both constant and variable exponent, but without singular reaction). Among papers on singular problems in a bounded domain, we mention Papageorgiou-Vetro-Vetro [22] (constant exponents, critical term and locally defined (with respect to the second variable) perturbation), [23] (variable exponents, concave term and sign-changing perturbation). For non-singular problems using the Nehari manifold approach, we cite Ramos Quoirin-Silva [24] (minimization procedure for the functional up to a critical parameter λ^*), and the very recent work of Leite-Ramos Quoirin-Silva [25] renewing the attention towards the Nehari method. The authors precisely established the existence of ground-state critical points to the energy functionals associate to several classes of p-Laplace and Kirchhoff type equations. Finally, we mention the recent work of Bobkov-Tanaka [26], focusing on the multiplicity of solutions to the Dirichlet eigenvalue problem for the operator $-\Delta_p - \Delta_q$ (constant exponents), i.e.,

$$-\Delta_p\varphi - \Delta_q\varphi = \theta_1|\varphi|^{p-2}\varphi + \theta_2|\varphi|^{q-2}\varphi, \quad \varphi\Big|_{\partial\Omega} = 0,\ \theta_1, \theta_2 > 0.$$

Closer to our work here is the recent paper by Avic [27], where the author considered the single p-Laplace differential operator (in the case of variable exponent) plus a power term, and the reaction presents only the singular term. The author discussed the existence and uniqueness of solutions by designing a strategy based on the Nehari manifold method. Our aim in this paper is to extend the existence result [27, Theorem 3.10] to the operator $-\Delta_p - \Delta_q$, also adding a multiplicity part. We remark that such an operator extends the applicability of the operator $-\Delta_p$ in biophysics and plasma physics (see also Cherfils-Ilyasov [28]). Throughout the paper, we assume the following conditions on the data:

- (A_α) $\alpha(z) : \overline{\Omega} \to (0,1)$ is a continuous function;
- (A_f) $f \in L^\infty(\Omega)$ is positive a.e. in Ω;
- (A_m) $M : (0,\infty) \to (0,\infty)$ is a C^1 function such that there exist $\varpi_1 \geq \varpi_0 > 0$ and $2 < a_2 \leq a_1$ with $a_1 p_+ < \beta_-$ satisfying

$$\varpi_0 t^{a_2-2} \leq M'(t) \leq \varpi_1 t^{a_1-2} \quad \text{for } t > 0;$$

- (A_M) there exists $m_0 \in \mathbb{R}$ with $0 < m_0 < 1$ such that

$$tM'(t) \leq \frac{m_0}{1-m_0} M(t) \text{ for } t > 0, \quad \text{where } \frac{p_+}{1-m_0} < \beta_-.$$

Remark 1. Condition (A_α) means that we consider the case of weak singularity. Assume condition (A_m) and let $\mathcal{M}(t) = \int_0^t M(s)ds$, then condition (A_M) gives us the inequality $\mathcal{M}(t) \geq (1-m_0)M(t)t$ for $t > 0$.

Hence, the main result of the paper is the following existence and multiplicity theorem.

Theorem 1. *Assume that hypotheses $(A_\alpha), (A_f), (A_m), (A_M)$ hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (P_0) has two nontrivial non-negative solutions.*

The novelty of our study is in considering the effects on $-\Delta_p - \Delta_q$ of two nonlocal Kirchhoff terms having a growth dictated by conditions (A_m) and (A_M), together with a singular term of the type $f(z)\varphi^{-\alpha(z)}$ satisfying the conditions (A_α) and (A_f), all in the context of variable exponents $1 < p(z), q(z) \in C(\overline{\Omega})$ (non-Newtonian dilatant fluids).

Briefly, as mentioned above, we underline the connection between variable exponent spaces and functionals with non-standard growth and coercivity features (see again [2–5]). It is known that non-standard variational problems are essential models for so-called electrorheological fluids, see the monograph of Růžička [29]. Such fluids exhibit mechanical properties which may change with the space variable $z \in \overline{\Omega}$, for instance, this is the case of adjustments to viscosity by a magnetic field. From the perspective of modeling, it means we can better understand and control the properties of such fluids by choosing the right functions $p(z), q(z) \in C(\overline{\Omega})$ in place of constants p, q. Differently from the constant exponents, the variable ones also permit us to modulate the growth, introducing smoothness (as for a logarithmic term) or properly designing decay properties on the boundary and at infinity, see Diening-Harjulehto-Hästö-Růžička [30, Section 4.1].

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This makes the variable exponent spaces more challenging both for pure mathematics and applied mathematics, because boundary conditions and behavior near the boundary are crucial to ensure the existence (and uniqueness or multiplicity) of solutions to differential equations and can represent physical constraints useful to regularize and approximate solutions (e.g., in the context of Galerkin-type methods, see again [9]).

Our strategy makes use of the Nehari manifold method to design a suitable geometry of the energy functional associate to problem (P_0) and is based on critical point results. Hence, the proof of the theorem is obtained by adapting the classical approach to deduce the existence of at least two critical points of the functional $\mathcal{J}_\lambda(\varphi)$, which are exactly the two weak solutions of problem (P_0). The main difficulties to overcome are due to the presence of the singular term and the necessity to prove that $\mathcal{J}_\lambda(\varphi)$ can be bounded below, developing suitable estimates of integral terms.

The paper is organized as follows. In Section 2, we fix notation and collect mathematical background about variable Lebesgue and Sobolev spaces, while in Section 3, we introduce the fibering map $\psi_\varphi(t)$ corresponding to $\mathcal{J}_\lambda(\varphi)$ and show the link between them. Then we prove that $\mathcal{J}_\lambda(\varphi)$ is coercive and bounded below on the Nehari manifold (Lemma 1). Working on the appropriate submanifolds, we first show that there exists a value of parameter λ, say $\tilde{\lambda}_0 > 0$, such that for all $\lambda \in (0, \tilde{\lambda}_0)$ the fibering map admits two critical points in Lemma 2, then we show that certain local minimizers of $\mathcal{J}_\lambda(\varphi)$ on the Nehari manifold are exactly critical points of $\mathcal{J}_\lambda(\varphi)$ in Lemma 3. Finally, we obtain an extremal parameter λ_0^* such that for any $\lambda \in (0, \lambda_0^*)$, the submanifold $\{\varphi \in \mathcal{A} : \psi_\varphi''(1) = 0\}$ is empty (see Lemma 4). Section 4 contains Lemmas 5 and 6, which are the two crucial steps for the proof of Theorem 1. In the first lemma we show that corresponding to the value $\lambda^* = \min\{\tilde{\lambda}_0, \lambda_0^*\}$, the functional $\mathcal{J}_\lambda(\varphi)$ admits a minimizer on the submanifold $\{\varphi \in \mathcal{A} : \psi_\varphi''(1) > 0\}$, for every $\lambda \in (0, \lambda^*)$, in the second lemma the similar claim is concluded with respect to the submanifold $\{\varphi \in \mathcal{A} : \psi_\varphi''(1) < 0\}$. Indeed, combining Lemmas 5 and 6, in view of Lemma 3, we derive the existence of two weak solutions to problem (P_0).

2 | Preliminaries

The analysis of the problem (P_0) is based on the variable Lebesgue and Sobolev spaces. For a comprehensive survey we refer again to [7] but also to the previous monographs by Cruz-Uribe-Fiorenza [31] and Diening-Harjulehto-Hästö-Růžička [30].

For $r(z) \in C(\overline{\Omega})$, we consider the set

$$C_+(\overline{\Omega}) = \left\{ r(z) \in C(\overline{\Omega}) \ : \ 1 < r(z) \text{ for all } z \in \overline{\Omega} \right\}.$$

Let $\mathcal{G}(\Omega)$ be the space of all measurable functions $\varphi : \Omega \to \mathbb{R}$. Given $r(z) \in C_+(\overline{\Omega})$, the variable Lebesgue space $L^{r(z)}(\Omega)$ is given by

$$L^{r(z)}(\Omega) = \left\{ \varphi \in \mathcal{G}(\Omega) \ : \ \rho_r(\varphi) = \int_\Omega |\varphi|^{r(z)} dz < +\infty \right\},$$

then this space is endowed with the Luxemburg norm

$$\|\varphi\|_{r(z)} = \inf \left[\zeta > 0 \ : \ \int_\Omega \left(\frac{|\varphi|}{\zeta} \right)^{r(z)} dz \le 1 \right]$$

is a separable and uniformly convex (thus reflexive, see Fan-Zhao [32]) Banach space. Let $r' \in C_+(\overline{\Omega})$ be the conjugate variable exponent to $r(z)$; i.e., $r(z)$ and $r'(z)$ are linked as follows

$$\frac{1}{r(z)} + \frac{1}{r'(z)} = 1 \text{ for all } z \in \overline{\Omega},$$

which immediately gives us $r'(z) = \frac{r(z)}{r(z)-1}$ for all $z \in \overline{\Omega}$. Further the topological dual space of $L^{r(z)}(\Omega)$ can be stated by

$$L^{r(z)}(\Omega)^* = L^{r'(z)}(\Omega).$$

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It is well-known that the following Hölder-type inequality

$$\int_\Omega |uv| dz \leq \left[\frac{1}{r_-} + \frac{1}{r'_-}\right] \|u\|_{r(z)} \|v\|_{r'(z)}$$

holds for all $u \in L^{r(z)}(\Omega)$, all $v \in L^{r'(z)}(\Omega)$.

About the basic embedding result in variable exponent Lebesgue spaces, we recall that if $r_1, r_2 \in C_+(\overline{\Omega})$ and $r_1(z) \leq r_2(z)$ for all $z \in \overline{\Omega}$, then we have

$$L^{r_2(z)}(\Omega) \hookrightarrow L^{r_1(z)}(\Omega) \text{ continuously.}$$

Furthermore, the norm $\|\varphi\|_{r(z)}$ and the modular function $\rho_r(\varphi) = \int_\Omega |\varphi|^{r(z)} dz$ are linked each other by the following proposition.

Proposition 1 ([32]). *Suppose $r \in C_+(\overline{\Omega})$, $\varphi \in L^{r(z)}(\Omega)$, and $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq L^{r(z)}(\Omega)$, then we have*:

a. $\|\varphi\|_{r(z)} < 1$ $(resp. = 1, > 1) \Leftrightarrow \rho_r(\varphi) < 1 (resp. = 1, > 1)$;
b. $\|\varphi\|_{r(z)} < 1 \Rightarrow \|\varphi\|^{r_+}_{r(z)} \leq \rho_r(\varphi) \leq \|\varphi\|^{r_-}_{r(z)}$;
c. $\|\varphi\|_{r(z)} > 1 \Rightarrow \|\varphi\|^{r_-}_{r(z)} \leq \rho_r(\varphi) \leq \|\varphi\|^{r_+}_{r(z)}$;
d. $\|\varphi_n\|_{r(z)} \to 0$ $(resp. \to +\infty) \Leftrightarrow \rho_r(\varphi_n) \to 0$ $(resp. \to +\infty)$.

Now, we denote by $W^{1,r(z)}(\Omega)$ the variable Sobolev space, i.e.,

$$W^{1,r(z)}(\Omega) = \left\{\varphi \in L^{r(z)}(\Omega) \ : \ |\nabla\varphi| \in L^{r(z)}(\Omega)\right\},$$

and endow it with the norm

$$\|\varphi\|_{1,r(z)} = \|\varphi\|_{r(z)} + \|\nabla\varphi\|_{r(z)} \text{ for all } \varphi \in W^{1,r(z)}(\Omega),$$

where $\|\nabla\varphi\|_{r(z)} = \||\nabla\varphi|\|_{r(z)}$, being $\nabla\varphi$ the weak gradient of φ. In what follows, we denote with $W^{1,r(z)}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $W^{1,r(z)}(\Omega)$. We recall that the spaces $W^{1,r(z)}(\Omega)$ and $W^{1,r(z)}_0(\Omega)$ are separable and reflexive Banach spaces. In addition we remark that the norms $\|\varphi\|_{1,r(z)}$ and $\|\nabla\varphi\|_{r(z)}$ are equivalent in the Sobolev space $W^{1,r(z)}_0(\Omega)$. This is a consequence of the Poincaré inequality, namely, there exists a positive constant depending on the bounded domain Ω, say $c := c(\Omega) > 0$, such that

$$\|\varphi\|_{r(z)} \leq c \ \|\nabla\varphi\|_{r(z)} \text{ for all } \varphi \in W^{1,r(z)}_0(\Omega).$$

So, we can use the norm $\|\varphi\| = \|\nabla\varphi\|_{r(z)}$ for all $\varphi \in W^{1,r(z)}_0(\Omega)$. Furthermore, we denote by $r^*(z)$ the critical Sobolev variable exponent corresponding to $r \in C_+(\overline{\Omega})$, namely

$$r^*(z) = \begin{cases} \dfrac{Nr(z)}{N - r(z)} & \text{if } r(z) < N, \\ +\infty & \text{if } N \leq r(z), \end{cases} \tag{2}$$

and have a variable exponent version of the Sobolev embedding theorem, which we state in the following proposition.

Proposition 2 ([32]). *Let $h(z) \in C_+(\overline{\Omega})$ and $1 \leq h(z) < r^*(z)$ for all $z \in \overline{\Omega}$, then $W^{1,r(z)}(\Omega) \hookrightarrow\hookrightarrow L^{h(z)}(\Omega)$ continuously and compactly.*

Next, for the topological dual space of $W^{1,r(z)}_0(\Omega)$ we use the notation

$$W^{1,r(z)}_0(\Omega)^* = W^{-1,r'(z)}(\Omega).$$

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Thus, we introduce the convex and continuously differentiable functional $\upsilon_r \in C^1(W_0^{1,r(z)}(\Omega), \mathbb{R}^+)$ defined by $\upsilon_r(\varphi) = \int_\Omega \frac{|\nabla\varphi|^{r(z)}}{r(z)} dz$, and by $K_{\upsilon_r} = \{\varphi \in W_0^{1,r(z)}(\Omega) : \upsilon_r'(\varphi) = 0\}$ we denote the critical set of $\upsilon_r(\varphi)$ with derivative $\upsilon_r' : W_0^{1,r(z)}(\Omega) \to W_0^{1,r(z)}(\Omega)^*$ given by

$$\langle \upsilon_r'(\varphi), u\rangle = \int_\Omega |\nabla\varphi|^{r(z)-2}\nabla\varphi \cdot \nabla u dz, \quad \varphi, u \in W_0^{1,r(z)}(\Omega).$$

Here $\langle\cdot,\cdot\rangle$ is the dual pairing between $W_0^{1,r(z)}(\Omega)$ and its dual $W_0^{1,r(z)}(\Omega)^*$.

By a weak solution of problem (P_0) we mean a function $\varphi \in W_0^{1,p(z)}(\Omega)$ such that $\varphi^{-\alpha(z)}u \in L^1(\Omega)$ for all $u \in W_0^{1,p(z)}(\Omega)$ and

$$\begin{aligned} &M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)-2}\nabla\varphi\nabla u dz + M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_\Omega |\nabla\varphi|^{q(z)-2}\nabla\varphi\nabla u dz \\ &+ \theta\int_\Omega \left(|\varphi|^{p(z)-2}\varphi u + |\varphi|^{q(z)-2}\varphi u\right)dz - \int_\Omega f(z)|\varphi|^{-\alpha(z)}u dz - \lambda\int_\Omega |\varphi|^{\beta(z)-2}\varphi u dz = 0 \end{aligned}$$

for all $\varphi, u \in W_0^{1,p(z)}(\Omega)$.

3 | On the Nehari Manifold

In this section, we prove some preliminary materials, which are needed to establish our existence and multiplicity result for weak solutions to problem (P_0). So, we first introduce the Euler energy functional associated with (P_0), i.e.,

$$\begin{aligned} \mathcal{J}_\lambda(\varphi) = &\mathcal{M}\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right) + \mathcal{M}\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right) \\ &+ \theta\int_\Omega \left(\frac{|\varphi|^{p(z)}}{p(z)} + \frac{|\varphi|^{q(z)}}{q(z)}\right)dz - \int_\Omega \frac{f(z)|\varphi|^{1-\alpha(z)}}{1-\alpha(z)}dz - \lambda\int_\Omega \frac{1}{\beta(z)}|\varphi|^{\beta(z)}dz \end{aligned}$$

for all $\varphi \in W_0^{1,p(z)}(\Omega)$. Then, we denote by $\mathcal{A}$ the Nehari manifold corresponding to $\mathcal{J}_\lambda(\varphi)$, i.e.,

$$\mathcal{A} = \{\varphi \in W_0^{1,p(z)}(\Omega)\setminus\{0\} : \langle \mathcal{J}_\lambda'(\varphi), \varphi\rangle = 0\},$$

where $\langle\cdot,\cdot\rangle$ is again the duality pairing of $(W_0^{1,p(z)}(\Omega)^*, W_0^{1,p(z)}(\Omega))$. Obviously, each critical point of $\mathcal{J}_\lambda(\varphi)$ belongs to $\mathcal{A}$. We have that $\varphi \in \mathcal{A}$ if and only if

$$\begin{aligned} \langle \mathcal{J}_\lambda'(\varphi), \varphi\rangle := &M(\upsilon_p(\varphi))\int_\Omega |\nabla\varphi|^{p(z)}dz + M(\upsilon_q(\varphi))\int_\Omega |\nabla\varphi|^{q(z)}dz \\ &+ \theta\int_\Omega \left(|\varphi|^{p(z)} + |\varphi|^{q(z)}\right)dz - \int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz - \lambda\int_\Omega |\varphi|^{\beta(z)}dz = 0. \end{aligned} \tag{3}$$

Following the consolidated approach designed in Brown-Wu [33], for each $\varphi \in W_0^{1,p(z)}(\Omega)$ we introduce the fibering map $\psi_\varphi : t \to \mathcal{J}_\lambda(t\varphi)$ $(t > 0)$ defined by

$$\begin{aligned} \psi_\varphi(t) = &\mathcal{M}\left(\int_\Omega t^{p(z)}\frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right) + \mathcal{M}\left(\int_\Omega t^{q(z)}\frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right) - \lambda\int_\Omega \frac{t^{\beta(z)}}{\beta(z)}|\varphi|^{\beta(z)}dz \\ &+ \theta\int_\Omega \left(t^{p(z)}\frac{|\varphi|^{p(z)}}{p(z)} + t^{q(z)}\frac{|\varphi|^{q(z)}}{q(z)}\right)dz - \int_\Omega t^{1-\alpha(z)}\frac{f(z)|\varphi|^{1-\alpha(z)}}{1-\alpha(z)}dz. \end{aligned}$$

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The corresponding first derivative is given as

$$\begin{aligned}\psi'_{\varphi}(t) &= M\left(\int_{\Omega} t^{p(z)}\frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_{\Omega} t^{p(z)-1}|\nabla\varphi|^{p(z)}dz\\ &+ M\left(\int_{\Omega} t^{q(z)}\frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_{\Omega} t^{q(z)-1}|\nabla\varphi|^{q(z)}dz - \lambda\int_{\Omega} t^{\beta(z)-1}|\varphi|^{\beta(z)}dz\\ &+ \theta\int_{\Omega}\left(t^{p(z)-1}|\varphi|^{p(z)} + t^{q(z)-1}|\varphi|^{q(z)}\right)dz - \int_{\Omega} t^{-\alpha(z)}f(z)|\varphi|^{1-\alpha(z)}dz. \end{aligned} \tag{4}$$

Furthermore, for the second derivative, we have the formula

$$\begin{aligned}\psi''_{\varphi}(t) &= M'\left(\int_{\Omega} t^{p(z)}\frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\left(\int_{\Omega} t^{p(z)-1}|\nabla\varphi|^{p(z)}dz\right)^2\\ &+ M\left(\int_{\Omega} t^{p(z)}\frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_{\Omega}(p(z)-1)t^{p(z)-2}|\nabla\varphi|^{p(z)}dz\\ &+ M'\left(\int_{\Omega} t^{q(z)}\frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\left(\int_{\Omega} t^{q(z)-1}|\nabla\varphi|^{q(z)}dz\right)^2\\ &+ M\left(\int_{\Omega} t^{q(z)}\frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_{\Omega}(q(z)-1)t^{q(z)-2}|\nabla\varphi|^{q(z)}dz\\ &+ \theta\int_{\Omega}\left((p(z)-1)t^{p(z)-2}|\varphi|^{p(z)} + (q(z)-1)t^{q(z)-2}|\varphi|^{q(z)}\right)dz\\ &+ \int_{\Omega}\alpha(z)t^{-\alpha(z)-1}f(z)|\varphi|^{1-\alpha(z)}dz - \lambda\int_{\Omega}(\beta(z)-1)t^{\beta(z)-2}|\varphi|^{\beta(z)}dz. \end{aligned} \tag{5}$$

We can now underline the strong link between the energy functional and the fibering map. Precisely, if φ is a local minimizer of $\mathcal{J}_{\lambda}(\cdot)$, then the corresponding fibering map $\psi_{\varphi}(\cdot)$ admits a local minimum at $t = 1$. Moreover, using the derivative in formulas (3) and (4), we easily deduce that

$$\varphi \in \mathcal{A} \text{ if and only if } \psi'_{\varphi}(1) = 0.$$

In turn, one has $t\varphi \in \mathcal{A}$ if and only if $\psi'_{\varphi}(t) = 0$. Equivalently, we can say that each element in $\mathcal{A}$ corresponds to a stationary point for $\psi_{\varphi}(t)$. From Equation (5), we get

$$\begin{aligned}\psi''_{\varphi}(1) &= M'\left(\int_{\Omega}\frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\left(\int_{\Omega}|\nabla\varphi|^{p(z)}dz\right)^2 + M\left(\int_{\Omega}\frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_{\Omega}(p(z)-1)|\nabla\varphi|^{p(z)}dz\\ &+ M'\left(\int_{\Omega}\frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\left(\int_{\Omega}|\nabla\varphi|^{q(z)}dz\right)^2 + M\left(\int_{\Omega}\frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_{\Omega}(q(z)-1)|\nabla\varphi|^{q(z)}dz\\ &+ \theta\int_{\Omega}\left((p(z)-1)|\varphi|^{p(z)} + (q(z)-1)|\varphi|^{q(z)}\right)dz + \int_{\Omega}\alpha(z)f(z)|\varphi|^{1-\alpha(z)}dz\\ &- \lambda\int_{\Omega}(\beta(z)-1)|\varphi|^{\beta(z)}dz. \end{aligned}$$

Based on this, we can separate our Nehari manifold $\mathcal{A}$ into three disjoint parts, then minimizing the energy functional $\mathcal{J}_{\lambda}(\varphi)$ on two of them. Precisely, we set

$$\begin{aligned}\mathcal{A}^+ &= \{\varphi \in \mathcal{A} : \psi''_{\varphi}(1) > 0\},\\ \mathcal{A}^- &= \{\varphi \in \mathcal{A} : \psi''_{\varphi}(1) < 0\},\\ \mathcal{A}^0 &= \{\varphi \in \mathcal{A} : \psi''_{\varphi}(1) = 0\}.\end{aligned}$$

Hence, $\mathcal{A}^+$, $\mathcal{A}^-$, and $\mathcal{A}^0$ refer to local minima, local maxima, and points of inflection, respectively. In the next result, we prove the first main properties for the functional $\mathcal{J}_{\lambda}(\varphi)$ on the corresponding Nehari manifold $\mathcal{A}$.

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Lemma 1. *Assume that hypotheses* $(A_\alpha), (A_f), (A_m), (A_M)$ *hold, then the functional* $\mathcal{J}_\lambda$, *associated to problem* (P_0), *is coercive and bounded below on* $\mathcal{A}$.

Proof. We first show that the functional $\mathcal{J}_\lambda(\varphi)$ is bounded below for $\varphi \in \mathcal{A}$. So, using $\varphi \in \mathcal{A}$ and the pairing $\langle \mathcal{J}'_\lambda(\varphi), \varphi\rangle$ (recall equation (3)), we deduce that

$$M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)}dz + M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)}dz$$
$$+\theta\int_\Omega \left(|\varphi|^{p(z)} + |\varphi|^{q(z)}\right)dz - \int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz = \lambda\int_\Omega |\varphi|^{\beta(z)}dz. \tag{6}$$

Now, to give detailed calculations for the boundedness part, we should distinguish the case when $\|\varphi\| \geq 1$ and the case when $\|\varphi\| < 1$. But, as estimates of both cases are similar, here we obtain the claim only for $\|\varphi\| > 1$. This also serves to conclude the coercivity. So, for $\|\varphi\| \geq 1$, Remark 1, inequality (6), (A_m), and Poincaré inequality lead to the following estimate from below for the energy functional

$$\begin{aligned}
\mathcal{J}_\lambda(\varphi) &= \mathcal{M}\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right) + \mathcal{M}\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right) - \lambda\int_\Omega \frac{|\varphi|^{\beta(z)}}{\beta(z)}dz\\
&\quad + \theta\int_\Omega \left(\frac{|\varphi|^{p(z)}}{p(z)} + \frac{|\varphi|^{q(z)}}{q(z)}\right)dz - \int_\Omega \frac{f(z)|\varphi|^{1-\alpha(z)}}{1-\alpha(z)}dz\\
&\geq \frac{1-m_0}{p_+}M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)}dz\\
&\quad + \frac{1-m_0}{q_+}M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_\Omega |\nabla\varphi|^{q(z)}dz\\
&\quad + \theta\int_\Omega \left(\frac{|\varphi|^{p(z)}}{p_+} + \frac{|\varphi|^{q(z)}}{q_+}\right)dz - \frac{\lambda}{\beta^-}\int_\Omega |\varphi|^{\beta(z)}dz\\
&\quad - \frac{1}{1-\alpha_-}\int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz \quad \text{(by Remark 1)}\\
&\geq \left(\frac{1-m_0}{p_+} - \frac{1}{\beta_-}\right)M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)}dz\\
&\quad - \left(\frac{1}{1-\alpha_-} + \frac{1}{\beta_-}\right)\int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz\\
&\geq \left(\frac{1-m_0}{p_+} - \frac{1}{\beta_-}\right)\frac{\varpi_0}{(a_2-1)p_+^{a_2-1}}\left(\int_\Omega |\nabla\varphi|^{p(z)}dz\right)^{a_2}\\
&\quad - \left(\frac{1}{1-\alpha_-} + \frac{1}{\beta_-}\right)\int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz\\
&\geq \left(\frac{1-m_0}{p_+} - \frac{1}{\beta_-}\right)\frac{\varpi_0}{(a_2-1)p_+^{a_2-1}}\|\varphi\|^{a_2p_-} - \left(\frac{1}{1-\alpha_-} + \frac{1}{\beta_-}\right)\int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz.
\end{aligned}$$

Since $a_2p_- > 2-\alpha_-$, passing to the limit as $\|\varphi\| \to +\infty$, we obtain that $\mathcal{J}_\lambda(\varphi) \to +\infty$ too. We conclude that $\mathcal{J}_\lambda(\varphi)$ is coercive and bounded below in $\mathcal{A}$. □

Remark 2. In the setting of Lemma 1, for $\varphi \in W_0^{1,p(z)}(\Omega)$ with $\|\varphi\| \geq 1$ and $t \geq 1$, we get the inequalities

$$\begin{aligned}
\frac{\varpi_0}{(a_2-1)(p_+)^{a_2-1}}t^{a_2p_- -1}\|\varphi\|^{a_2p_-} &\leq M\left(\int_\Omega t^{p(z)}\frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega t^{p(z)-1}|\nabla\varphi|^{p(z)}dz\\
&\leq \frac{\varpi_1}{(a_1-1)(p_-)^{a_1-1}}t^{a_1p_+-1}\|\varphi\|^{a_1p_+} \quad \text{(by } (A_m)\text{)}
\end{aligned} \tag{7}$$

and

$$\theta\int_\Omega \left(t^{p(z)-1}|\varphi|^{p(z)} + t^{q(z)-1}|\varphi|^{q(z)}\right)dz \leq Ct^{a_1p_+-1}\|\varphi\|^{a_1p_+} \tag{8}$$

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for some $C > 0$. Furthermore, the inequality (7) remains true if we consider q in place of p.

In the next result, we focus on the fibering map $\psi_\varphi(t)$ and show that it admits two critical points. To compact the calculations, we can use the following notation:

$$Z_\varphi = \int_\Omega |\nabla\varphi|^{p(z)}dz, \quad \overline{Z}_\varphi = \int_\Omega |\nabla\varphi|^{q(z)}dz,$$
$$Y_\varphi = \theta\int_\Omega |\varphi|^{p(z)}dz, \quad \overline{Y}_\varphi = \theta\int_\Omega |\varphi|^{q(z)}dz,$$
$$Q_\varphi = \int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz, \quad K_\varphi = \int_\Omega |\varphi|^{\beta(z)}dz.$$

We remark that the quantity Q_φ is finite by (A_α) and (A_f).

Lemma 2. *Assume that hypotheses $(A_\alpha), (A_f), (A_m), (A_M)$ hold, then there exists some constant $\tilde{\lambda}_0 > 0$ such that for all $\lambda \in (0, \tilde{\lambda}_0)$ the fibering map ψ_φ admits at least two critical points. Furthermore, we have*:

i. *If $\varphi \in \mathcal{A}^+$, then there exist $1 = t_1 < t_2$ such that $t_2\varphi \in \mathcal{A}^-$, t_1 and t_2 are a local minimum and a local maximum point, respectively*;

ii. *If $\varphi \in \mathcal{A}^-$, then there exist $t_1 < t_2 = 1$ such that $t_1\varphi \in \mathcal{A}^+$, t_1 and t_2 are a local minimum and a local maximum point, respectively*.

Proof. We distinguish the case when $\|\varphi\| \geq 1$ and the case when $\|\varphi\| < 1$. To avoid repetition, we develop the proof only in the first case. So, we first assume $\|\varphi\| \geq 1$ and $t \geq 1$.

By hypothesis (A_m), we know that $0 < a_1p_+ < \beta_-$; hence, using the estimates (7) and (8) (see Remark 2) we obtain

$$\begin{aligned}
&\frac{\varpi_0}{(a_2-1)(p_+)^{a_2-1}}t^{a_2q_- -1}\|\varphi\|^{a_2q_-} - t^{-\alpha_-}Q_\varphi - \lambda t^{\beta_+-1}K_\varphi \\
&\quad\leq \frac{\varpi_0 t^{a_2p_- -1}}{(a_2-1)(p_+)^{a_2-1}}Z_\varphi^{a_2} + \frac{\varpi_0 t^{a_2q_- -1}}{(a_2-1)(q_+)^{a_2-1}}\overline{Z}_\varphi^{a_2} + t^{p_- -1}Y_\varphi + t^{q_- -1}\overline{Y}_\varphi - t^{-\alpha_-}Q_\varphi - \lambda t^{\beta_+-1}K_\varphi \\
&\quad\leq \psi'_\varphi(t) \\
&\quad\leq \frac{\varpi_1 t^{a_1p_+ -1}}{(a_1-1)(p_-)^{a_1-1}}Z_\varphi^{a_1} + \frac{\varpi_1 t^{a_1q_+ -1}}{(a_1-1)(q_-)^{a_1-1}}\overline{Z}_\varphi^{a_1} + t^{p_+-1}Y_\varphi + t^{q_+-1}\overline{Y}_\varphi - t^{-\alpha_+}Q_\varphi - \lambda t^{\beta_- -1}K_\varphi \\
&\quad\leq \left\{\frac{2\varpi_1}{(a_1-1)(q_-)^{a_1-1}} + C\right\}t^{a_1p_+-1}\|\varphi\|^{a_1p_+} - t^{-\alpha_+}Q_\varphi - \lambda t^{\beta_- -1}K_\varphi.
\end{aligned} \tag{9}$$

On the other hand, assuming $\|\varphi\| \geq 1$ and $t \leq 1$, we get

$$\begin{aligned}
&\frac{\varpi_0}{(a_2-1)(p_+)^{a_2-1}}t^{a_2p_+ -1}\|\varphi\|^{a_2q_-} - t^{-\alpha_+}Q_\varphi - \lambda t^{\beta_- -1}K_\varphi \\
&\quad\leq \frac{\varpi_0 t^{a_2p_+ -1}}{(a_2-1)(p_+)^{a_2-1}}Z_\varphi^{a_2} + \frac{\varpi_0 t^{a_2q_+ -1}}{(a_2-1)(q_+)^{a_2-1}}\overline{Z}_\varphi^{a_2} + t^{p_+-1}Y_\varphi + t^{q_+-1}\overline{Y}_\varphi - t^{-\alpha_+}Q_\varphi - \lambda t^{\beta_- -1}K_\varphi \\
&\quad\leq \psi'_\varphi(t) \\
&\quad\leq \frac{\varpi_1 t^{a_1p_- -1}}{(a_1-1)(p_-)^{a_1-1}}Z_\varphi^{a_1} + \frac{\varpi_1 t^{a_1q_- -1}}{(a_1-1)(q_-)^{a_1-1}}\overline{Z}_\varphi^{a_1} + t^{p_- -1}Y_\varphi + t^{q_- -1}\overline{Y}_\varphi - t^{-\alpha_-}Q_\varphi - \lambda t^{\beta_+-1}K_\varphi \\
&\quad\leq \left\{\frac{2\varpi_1}{(a_1-1)(q_-)^{a_1-1}} + C\right\}t^{q_- -1}\|\varphi\|^{a_1p_+} - t^{-\alpha_-}Q_\varphi - \lambda t^{\beta_- -1}K_\varphi.
\end{aligned} \tag{10}$$

If we set $g(t) = t^\epsilon\|\varphi\|^{\tilde{\epsilon}} - t^{-\iota}Q_\varphi - \lambda t^\varsigma K_\varphi$ for $\varsigma > \epsilon > 0 > -\iota$, then we note that $g(t) = 0$ when

$$t^{-\iota-\epsilon}Q_\varphi + \lambda t^{\varsigma-\epsilon}K_\varphi = \|\varphi\|^{\tilde{\epsilon}}. \tag{11}$$

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Now, we denote

$$\tilde{g}(t) := t^{-\iota-\epsilon}Q_\varphi + \lambda t^{\varsigma-\epsilon}K_\varphi,$$

which has its derivative given by

$$\begin{aligned}\tilde{g}'(t) &= (-\iota-\epsilon)t^{-\iota-\epsilon-1}Q_\varphi + (\varsigma-\epsilon)\lambda t^{\varsigma-\epsilon-1}K_\varphi \\ &= t^{-\iota-\epsilon-1}[(-\iota-\epsilon)Q_\varphi + (\varsigma-\epsilon)\lambda t^{\varsigma+\iota}K_\varphi].\end{aligned}$$

Corresponding to the value

$$t^* = \left(\frac{(\iota+\epsilon)Q_\varphi}{\lambda(\varsigma-\epsilon)K_\varphi}\right)^{\frac{1}{\varsigma+\iota}},$$

since $\varsigma > \epsilon > 0 > -\iota$, we know that

$$\begin{aligned}\tilde{g}'(t) &< 0 \quad \text{for } t \in (0, t^*), \\ \tilde{g}'(t) &> 0 \quad \text{for } t \in (t^*, +\infty).\end{aligned}$$

Hence, from Equation (11), we deduce that $\tilde{g}(t^*) = \|\varphi\|^{\tilde{\epsilon}}$, namely we have

$$\|\varphi\|^{\tilde{\epsilon}} = (\varsigma+\iota)\left(\frac{Q_\varphi^{\varsigma-\epsilon}K_\varphi^{\epsilon+\iota}}{(\varsigma-\epsilon)^{\varsigma-\epsilon}(\epsilon+\iota)^{\epsilon+\iota}}\right)^{\frac{1}{\varsigma+\iota}}\lambda^{\frac{\iota+\epsilon}{\varsigma+\iota}}.$$

Consequently, Equation (11) has two solutions, say t_1 and t_2 with $t_1 < t_2$, for all $\lambda \in (0, \tilde{\lambda}_0)$, where the parameter value $\tilde{\lambda}_0$ is given by

$$\tilde{\lambda}_0 = \left(\frac{\|\varphi\|^{\tilde{\epsilon}}}{\varsigma+\iota}\right)^{\frac{\varsigma+\iota}{\epsilon+\iota}}\left(\frac{Q_\varphi^{\varsigma-\epsilon}K_\varphi^{\epsilon+\iota}}{(\varsigma-\epsilon)^{\varsigma-\epsilon}(\epsilon+\iota)^{\epsilon+\iota}}\right)^{\frac{-1}{\epsilon+\iota}}.$$

Immediately it is worth noting that from $g(t) \to -\infty$ as $t \to 0^+$ and $g(t) \to -\infty$ as $t \to +\infty$, one can deduce that $g'(t_1) > 0$ and $g'(t_2) < 0$, i.e., the function $g(t)$ is initially increasing and eventually decreasing for $\varsigma > \epsilon > 0 > -\iota$. Hence, combining the inequalities (9) and (10), one can show that the first derivative of the fibering map, $\psi'_\varphi(t)$, has the graph between the graphs of the functions $\psi^1_\varphi(t)$ and $\psi^2_\varphi(t)$ given by

$$\psi^1_\varphi(t) = \frac{\varpi_0}{(a_2-1)(p_+)^{a_2-1}}t^{a_2q_--1}\|\varphi\|^{a_2q_-} - t^{-\alpha_-}Q_\varphi - \lambda t^{\beta_+-1}K_\varphi$$

and

$$\psi^2_\varphi(t) = \left\{\frac{2\varpi_1}{(a_1-1)(q_-)^{a_1-1}} + C\right\}t^{a_1p_+-1}\|\varphi\|^{a_1p_+} - t^{-\alpha_+}Q_\varphi - \lambda t^{\beta_--1}K_\varphi,$$

or $\psi'_\varphi(t)$ has its graph between the graphs of the functions $\psi^3_\varphi(t)$ and $\psi^4_\varphi(t)$ given by

$$\psi^3_\varphi(t) = \frac{\varpi_0}{(a_2-1)(p_+)^{a_2-1}}t^{a_2p_+-1}\|\varphi\|^{a_2q_-} - t^{-\alpha_+}Q_\varphi - \lambda t^{\beta_--1}K_\varphi$$

and

$$\psi^4_\varphi(t) = \left\{\frac{2\varpi_1}{(a_1-1)(q_-)^{a_1-1}} + C\right\}t^{q_--1}\|\varphi\|^{a_1p_+} - t^{-\alpha_-}Q_\varphi - \lambda t^{\beta_--1}K_\varphi.$$

Keeping in mind the above discussion about the behavior of $g(t) = t^\epsilon\|\varphi\|^{\tilde{\epsilon}} - t^{-\iota}Q_\varphi - \lambda t^\varsigma K_\varphi$ (for $\varsigma > \epsilon > 0 > -\iota$), then one can conclude that the fibering map $\psi_\varphi(t)$ admits at least two critical points, precisely a local minimum at t_1 and a local maximum at t_2 with $0 < t_1 < t_2 < +\infty$ (recall that $\psi'_\varphi(t) \to -\infty$ as $t \to 0^+$ and $\psi'_\varphi(t) \to -\infty$ as $t \to +\infty$). Furthermore, for $\varphi \in \mathcal{A}^+$ we retrieve $1 = t_1 < t_2$ and $t_2\varphi \in \mathcal{A}^-$. The similar reasoning leads to the same conclusion in the case when $\|\varphi\| < 1$, but we omit the details. This establishes the proof of claim (i). Claim (ii) follows the above steps too; this time, for $\varphi \in \mathcal{A}^-$, we get $t_1 < t_2 = 1$ and $t_1\varphi \in \mathcal{A}^+$. □

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In the next result, we can briefly conclude that certain local minimizers, on the Nehari manifold, of the energy functional $\mathcal{J}_\lambda(\varphi)$ associated to problem (P_0) are exactly critical points of $\mathcal{J}_\lambda(\varphi)$.

Lemma 3. *If φ_0 is a local minimizer of the functional $\mathcal{J}_\lambda$ on $\mathcal{A}$ and $\varphi_0 \notin \mathcal{A}^0$, then φ_0 is a critical point of $\mathcal{J}_\lambda$.*

Proof. Starting from a local minimizer of $\mathcal{J}_\lambda(\varphi)$ on $\mathcal{A}$, say $\varphi_0 \in \mathcal{A}$, one can follow the Lagrange multiplier method to find a real number $l_0 \in \mathbb{R}$ satisfying

$$\mathcal{J}'_\lambda(\varphi_0) = l_0 \Psi'_\lambda(\varphi_0),$$

where the functional $\Psi_\lambda(\varphi)$ is exactly the first derivative of the fibering map associate to the energy functional $\mathcal{J}_\lambda(\varphi)$, i.e.,

$$\begin{aligned}\Psi_\lambda(\varphi_0) = \psi'_{\varphi_0}(1) = M\left(\int_\Omega \frac{|\nabla\varphi_0|^{p(z)}}{p(z)}dz\right)\int_\Omega |\nabla\varphi_0|^{p(z)}dz \\ + M\left(\int_\Omega \frac{|\nabla\varphi_0|^{q(z)}}{q(z)}dz\right)\int_\Omega |\nabla\varphi_0|^{q(z)}dz - \lambda\int_\Omega |\varphi_0|^{\beta(z)}dz \\ + \theta\int_\Omega \left(|\varphi_0|^{p(z)} + |\varphi_0|^{q(z)}\right)dz - \int_\Omega f(z)|\varphi_0|^{1-\alpha(z)}dz.\end{aligned}$$

We note that

$$\langle \mathcal{J}'_\lambda(\varphi_0), \nu\rangle = l_0\langle \Psi'_\lambda(\varphi_0), \nu\rangle \text{ for } \nu \in W_0^{1,p(z)}(\Omega), \tag{12}$$

and, keeping in mind that $\varphi_0 \in \mathcal{A}$, we conclude the following

$$\langle \mathcal{J}'_\lambda(\varphi_0), \varphi_0\rangle = \Psi_\lambda(\varphi_0) = 0. \tag{13}$$

We easily deduce that

$$\langle \Psi'_\lambda(\varphi_0), \varphi_0\rangle = \psi''_{\varphi_0}(1) + \Psi_\lambda(\varphi_0) = \psi''_{\varphi_0}(1).$$

It gives us

$$\langle \Psi'_\lambda(\varphi_0), \varphi_0\rangle \neq 0 \quad \text{for } \varphi_0 \notin \mathcal{A}^0.$$

The arguments in Equations (12) and (13) lead to the conclusion that the null value of the multiplier $l_0 = 0$. This shows that φ_0 is a critical point of $\mathcal{J}_\lambda(\varphi)$. □

In the next result, we show that the Nehari submanifold corresponding to points of inflection is empty, for suitable values of the parameter λ.

Lemma 4. *Assume that hypotheses $(A_\alpha), (A_f), (A_m), (A_M)$ hold, then there exists some constant $\lambda_0^* > 0$ such that for all $\lambda \in (0, \lambda_0^*)$ the Nehari submanifold $\mathcal{A}^0$ is empty.*

Proof. Consider $\varphi \in \mathcal{A}^0$, so that $\psi''_\varphi(1) = 0$. Namely, we have

$$\begin{aligned}M'\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\left(\int_\Omega |\nabla\varphi|^{p(z)}dz\right)^2 \\ + M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega (p(z)-1)|\nabla\varphi|^{p(z)}dz \\ + M'\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\left(\int_\Omega |\nabla\varphi|^{q(z)}dz\right)^2 \\ + M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_\Omega (q(z)-1)|\nabla\varphi|^{q(z)}dz \\ + \theta\int_\Omega \left((p(z)-1)|\varphi|^{p(z)} + (q(z)-1)|\varphi|^{q(z)}\right)dz \\ + \int_\Omega \alpha(z)f(z)|\varphi|^{1-\alpha(z)}dz - \lambda\int_\Omega (\beta(z)-1)|\varphi|^{\beta(z)}dz = 0.\end{aligned} \tag{14}$$

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We now distinguish the case when $\|\varphi\| \geq 1$ and the case when $\|\varphi\| < 1$. In the first situation, i.e., for $\varphi \in W_0^{1,p(z)}(\Omega)$ with $\|\varphi\| \geq 1$, combining assumptions (A_m) and (A_M) together with Equations (3), and (14), we derive the following estimate

$$
\begin{aligned}
0 &= M'\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\left(\int_\Omega |\nabla\varphi|^{p(z)}dz\right)^2 + M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega p(z)|\nabla\varphi|^{p(z)}dz \\
&\quad + M'\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\left(\int_\Omega |\nabla\varphi|^{q(z)}dz\right)^2 + M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_\Omega q(z)|\nabla\varphi|^{q(z)}dz \\
&\quad + \theta\int_\Omega \left(p(z)|\varphi|^{p(z)} + q(z)|\varphi|^{q(z)}\right)dz - \int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz \\
&\quad + \int_\Omega \alpha(z)f(z)|\varphi|^{1-\alpha(z)}dz - \lambda\int_\Omega \beta(z)|\varphi|^{\beta(z)}dz \\
&\leq \frac{m_0}{1-m_0}\frac{M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)}{\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz}\left(\int_\Omega |\nabla\varphi|^{p(z)}dz\right)^2 + p_+M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)}dz \\
&\quad + \frac{m_0}{1-m_0}\frac{M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)}{\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz}\left(\int_\Omega |\nabla\varphi|^{q(z)}dz\right)^2 + q_+M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)}dz \\
&\quad - \lambda\beta_-\int_\Omega |\varphi|^{\beta(z)}dz + \theta\int_\Omega \left(p_+|\varphi|^{p(z)} + q_+|\varphi|^{q(z)}\right)dz - (1-\alpha_+)\int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz \\
&\leq \frac{m_0p_+}{1-m_0}M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)}dz + p_+M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)}dz \\
&\quad + \frac{m_0q_+}{1-m_0}M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_\Omega |\nabla\varphi|^{q(z)}dz + q_+M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)}dz \\
&\quad - \lambda\beta_-\int_\Omega |\varphi|^{\beta(z)}dz + \theta\int_\Omega \left(p_+|\varphi|^{p(z)} + q_+|\varphi|^{q(z)}\right)dz - (1-\alpha_+)\int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz \\
&= \frac{p_+}{1-m_0}M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)}dz + \frac{q_+}{1-m_0}M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_\Omega |\nabla\varphi|^{q(z)}dz \\
&\quad - \lambda\beta_-\int_\Omega |\varphi|^{\beta(z)}dz + \theta\int_\Omega \left(p_+|\varphi|^{p(z)} + q_+|\varphi|^{q(z)}\right)dz - (1-\alpha_+)\int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz \\
&= \left(\frac{p_+}{1-m_0} - \beta_-\right)M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)}dz \\
&\quad + \left(\frac{q_+}{1-m_0} - \beta_-\right)M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_\Omega |\nabla\varphi|^{q(z)}dz \\
&\quad + (\theta-\beta_-)\int_\Omega \left(p_+|\varphi|^{p(z)} + q_+|\varphi|^{q(z)}\right)dz + (\beta_- - (1-\alpha_+))\int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz \\
&\leq -\left(\beta_- - \frac{p_+}{1-m_0}\right)\frac{\varpi_0}{p_+^{a_2-1}(a_2-1)}\left(\int_\Omega |\nabla\varphi|^{p(z)}dz\right)^{a_2} \\
&\quad - \left(\beta_- - \frac{q_+}{1-m_0}\right)\frac{\varpi_0}{q_+^{a_2-1}(a_2-1)}\left(\int_\Omega |\nabla\varphi|^{q(z)}dz\right)^{a_2} + (\beta_- - (1-\alpha_+))\int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz \\
&\leq -\left(\beta_- - \frac{p_+}{1-m_0}\right)\frac{\varpi_0}{p_+^{a_2-1}(a_2-1)}\|\varphi\|^{a_2q_-} + (\beta_- - (1-\alpha_+))c_0,
\end{aligned}
$$

where $c_0 = \int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz < +\infty$ by (3). The latter inequality implies the following bound above for the norm

$$
\|\varphi\|^{a_2q_-} \leq \frac{(\beta_- - (1-\alpha_+))c_0}{\Xi}, \tag{15}
$$

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where we set the constant

$$\Xi := \left(\beta_- - \frac{p_+}{1-m_0}\right)\frac{\varpi_0}{p_+^{a_2-1}(a_2-1)}.$$

On the other hand, we have the estimate

$$\begin{aligned}
0 &= M'\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\left(\int_\Omega |\nabla\varphi|^{p(z)}dz\right)^2 + M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega p(z)|\nabla\varphi|^{p(z)}dz \\
&\quad + M'\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\left(\int_\Omega |\nabla\varphi|^{q(z)}dz\right)^2 + M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_\Omega q(z)|\nabla\varphi|^{q(z)}dz \\
&\quad + \theta\int_\Omega \left(p(z)|\varphi|^{p(z)} + q(z)|\varphi|^{q(z)}\right)dz - \int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz \\
&\quad + \int_\Omega \alpha(z)f(z)|\varphi|^{1-\alpha(z)}dz - \lambda\int_\Omega \beta(z)|\varphi|^{\beta(z)}dz \\
&\geq \varpi_0\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)^{a_2-2}\left(\int_\Omega |\nabla\varphi|^{p(z)}dz\right)^2 + p_-M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)}dz \\
&\quad + \varpi_0\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)^{a_2-2}\left(\int_\Omega |\nabla\varphi|^{q(z)}dz\right)^2 + q_-M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_\Omega |\nabla\varphi|^{q(z)}dz \\
&\quad - (1-\alpha_-)\int_\Omega f(z)|\varphi|^{1-\alpha(z)}dz - \beta_+\lambda\int_\Omega |\varphi|^{\beta(z)}dz \\
&= \varpi_0\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)^{a_2-2}\left(\int_\Omega |\nabla\varphi|^{p(z)}dz\right)^2 \\
&\quad + (p_- - (1-\alpha_-))M\left(\int_\Omega \frac{|\nabla\varphi|^{p(z)}}{p(z)}dz\right)\int_\Omega |\nabla\varphi|^{p(z)}dz \\
&\quad + \varpi_0\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)^{a_2-2}\left(\int_\Omega |\nabla\varphi|^{q(z)}dz\right)^2 \\
&\quad + (q_- - (1-\alpha_-))M\left(\int_\Omega \frac{|\nabla\varphi|^{q(z)}}{q(z)}dz\right)\int_\Omega |\nabla\varphi|^{q(z)}dz \\
&\quad - (\beta_+ - (1-\alpha_-))\lambda\int_\Omega |\varphi|^{\beta(z)}dz \\
&\geq \frac{\varpi_0}{p_+^{a_2-2}}\left(\int_\Omega |\nabla\varphi|^{p(z)}dz\right)^{a_2} + \frac{\varpi_0}{(a_2-1)p_+^{a_2-1}}(p_- - (1-\alpha_-))\left(\int_\Omega |\nabla\varphi|^{p(z)}dz\right)^{a_2} \\
&\quad - (\beta_+ - (1-\alpha_-))\lambda\int_\Omega |\varphi|^{\beta(z)}dz \\
&\geq \frac{\varpi_0}{p_+^{a_2-2}}\|\varphi\|^{a_2q_-} - (\beta_+ - (1-\alpha_-))c_1\lambda\|\varphi\|^{\beta_+},
\end{aligned}$$

where c_1 means the proper embedding constant. This way, we get the following bound below for the norm

$$\|\varphi\|^{\beta_+ - a_2q_-} \geq \frac{\Theta}{(\beta_+ - (1-\alpha_-))c_1\lambda}, \tag{16}$$

where we involve the constant

$$\Theta := \frac{\varpi_0}{p_+^{a_2-2}}.$$

Manipulating the two bounds given by (15) and (16), we deduce that

$$\left(\frac{\Theta}{(\beta_+ - (1-\alpha_-))c_1\lambda}\right)^{\frac{1}{\beta_+ - a_2q_-}} \leq \left(\frac{(\beta_- - (1-\alpha_+))c_0}{\Xi}\right)^{\frac{1}{a_2q_-}}.$$

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This inequality permits us to retrieve suitable values of the parameter λ in the form

$$\lambda \geq \frac{\Theta}{(\beta_+ - (1-\alpha_-))c_1}\left(\frac{(\beta_- - (1-\alpha_+))c_0}{\Xi}\right)^{1-\frac{\beta_+}{a_2 q_-}} := \overline{\lambda}_1.$$

We can repeat the similar calculation process as above, starting this time from a function $\varphi \in \mathcal{A}_0$ with $\|\varphi\| < 1$. Hence, we obtain that the parameter λ has to satisfy the following inequality

$$\lambda \geq \frac{\Theta}{(\beta_+ - (1-\alpha_-))c_1}\left(\frac{(\beta_- - (1-\alpha_+))c_0}{\Xi}\right)^{1-\frac{\beta_-}{a_2 q_+}} := \overline{\lambda}_2.$$

Taking $\lambda_0^* = \min\{\overline{\lambda}_1, \overline{\lambda}_2\}$, we immediately conclude that

$$\mathcal{A}^0 = \emptyset \ \text{ for } \lambda \in (0, \lambda_0^*),$$

i.e., the Nehari submanifold corresponding to points of inflection is empty. □

4 | Main Result

In this section, we conclude the proof of our existence and multiplicity theorem, Theorem 1 will be proved in two steps by working separately on the submanifolds $\mathcal{A}^+$ and $\mathcal{A}^-$. Precisely, we first establish the existence of a local minimum for $\mathcal{J}_\lambda(\varphi)$ on $\mathcal{A}^+$. The proof is mainly based on Lemma 1, which gives us sufficient conditions for the coercivity and boundedness below of the functional $\mathcal{J}_\lambda(\varphi)$.

Remark 3. In what follows, we seek global minimizers of $\mathcal{J}_\lambda$ on the subsets $\mathcal{A}^+$ and $\mathcal{A}^-$. This is distinct from the fibering classification of these sets, which pertains to the behavior of the map $t \mapsto \mathcal{J}_\lambda(t\varphi)$ at $t = 1$.

Lemma 5. *Assume that hypotheses* $(A_\alpha), (A_f), (A_m), (A_M)$ *hold and* $\lambda^* = \min\{\tilde{\lambda}_0, \lambda_0^*\}$, *then for all* $\lambda \in (0, \lambda^*)$ *the functional* $\mathcal{J}_\lambda$, *associate to problem* (P_0), *admits a minimizer on* $\mathcal{A}^+$.

Proof. By hypotheses, invoking Lemma 1, we know that the energy functional $\mathcal{J}_\lambda(\varphi)$ is bounded below on the Nehari manifold $\mathcal{A}$, and hence on the submanifold $\mathcal{A}^+$. This means that we can construct a minimizing sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}^+$ satisfying

$$\lim_{n\to+\infty} \mathcal{J}_\lambda(\varphi_n) = \inf_{\varphi\in\mathcal{A}^+} \mathcal{J}_\lambda(\varphi).$$

Invoking Lemma 1, we also know that the energy functional is coercive on $\mathcal{A}$, which means that the above minimizing sequence is bounded in $W_0^{1,p(z)}(\Omega)$. It follows that one can find a function $\overline{\varphi} \in W_0^{1,p(z)}(\Omega)$ such that, up to a subsequence (yet denoted by $\{\varphi_n\}_{n\in\mathbb{N}}$) we have weak convergence in the Sobolev space $W_0^{1,p(z)}(\Omega)$ and strong convergence in suitable Lebesgue spaces $L^{\vartheta(z)}(\Omega)$, i.e.,

$$\begin{aligned} &\varphi_n \rightharpoonup \overline{\varphi} \ \text{ in } \ W_0^{1,p(z)}(\Omega), \\ &\varphi_n \to \overline{\varphi} \ \text{ in } \ L^{\vartheta(z)}(\Omega), \quad 1 \leq \vartheta(z) < p^*(z). \end{aligned}$$

We need to prove that the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ strongly converges to $\overline{\varphi}$ in $W_0^{1,p(z)}(\Omega)$ too. To this aim, we argue by contradiction. Hence, we get

$$\begin{aligned} &\int_\Omega |\overline{\varphi}|^{\vartheta(z)} dz = \lim_{n\to+\infty} \int_\Omega |\varphi_n|^{\vartheta(z)} dz \ \text{ for } 1 \leq \vartheta(z) < p^*(z), \\ &\int_\Omega |\nabla\overline{\varphi}|^{\vartheta(z)} dz < \liminf_{n\to+\infty} \int_\Omega |\nabla\varphi_n|^{\vartheta(z)} dz \ \text{ for } \theta \in \{p, q\}, \end{aligned}$$

and since $M : (0,\infty) \to (0,\infty)$ is a continuous increasing function (by (A_m)), we have

$$M\left(\int_\Omega |\nabla\overline{\varphi}|^{\vartheta(z)} dz\right) < \liminf_{n\to+\infty} M\left(\int_\Omega |\nabla\varphi_n|^{\vartheta(z)} dz\right) \ \text{ for } \theta \in \{p, q\}.$$

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Furthermore, we can invoke Lemma 2 to conclude there is $t_1 > 0$ such that $t_1\overline{\varphi} \in \mathcal{A}^+$, which leads to the following condition on the first derivative of the fibering map

$$\psi'_{\overline{\varphi}}(t_1) = 0.$$

It follows, by using the Lebesgue dominated convergence theorem, that

$$\begin{aligned}
\liminf_{n\to+\infty} \psi'_{\varphi_n}(t_1) &\geq \liminf_{n\to+\infty} M\left(\int_\Omega t_1^{p(z)} \frac{|\nabla\varphi_n|^{p(z)}}{p(z)} dz\right) \int_\Omega t_1^{p(z)-1} |\nabla\varphi_n|^{p(z)} dz \\
&\quad + \liminf_{n\to+\infty} M\left(\int_\Omega t_1^{q(z)} \frac{|\nabla\varphi_n|^{q(z)}}{q(z)} dz\right) \int_\Omega t_1^{q(z)-1} |\nabla\varphi_n|^{q(z)} dz \\
&\quad + \theta \lim_{n\to+\infty} \int_\Omega \left(t_1^{p(z)-1}|\varphi_n|^{p(z)} + t_1^{q(z)-1}|\varphi_n|^{q(z)}\right) dz \\
&\quad - \limsup_{n\to+\infty} \int_\Omega t_1^{-\alpha(z)} f(z) |\varphi_n|^{1-\alpha(z)} dz - \lambda \limsup_{n\to+\infty} \int_\Omega t_1^{\beta(z)-1} |\varphi_n|^{\beta(z)} dz \\
&> M\left(\int_\Omega t_1^{p(z)} \frac{|\nabla\overline{\varphi}|^{p(z)}}{p(z)} dz\right) \int_\Omega t_1^{p(z)-1} |\nabla\overline{\varphi}|^{p(z)} dz \\
&\quad + M\left(\int_\Omega t_1^{q(z)} \frac{|\nabla\overline{\varphi}|^{q(z)}}{q(z)} dz\right) \int_\Omega t_1^{q(z)-1} |\nabla\overline{\varphi}|^{q(z)} dz \\
&\quad + \theta \int_\Omega \left(t_1^{p(z)-1}|\overline{\varphi}|^{p(z)} + t_1^{q(z)-1}|\overline{\varphi}|^{q(z)}\right) dz \\
&\quad - \int_\Omega t_1^{-\alpha(z)} f(z) |\overline{\varphi}|^{1-\alpha(z)} dz - \lambda \int_\Omega t_1^{\beta(z)-1} |\overline{\varphi}|^{\beta(z)} dz \\
&= \psi'_{\overline{\varphi}}(t_1) = 0.
\end{aligned}$$

We easily deduce that $\psi'_{\varphi_n}(t_1) > 0$ choosing $n > n_0$ sufficiently large. On the other hand, working with the appropriate first derivative of the fibering map $\psi'_{\varphi_n}(t)$ for $\varphi_n \in \mathcal{A}^+$, evidently we know that $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}^+$. So, we can appeal to Lemma 2(i) to conclude that

$$\begin{aligned}
&\psi'_{\varphi_n}(t) < 0 \quad \text{for } t \in (0,1), \text{all } n \in \mathbb{N}, \\
&\psi'_{\varphi_n}(1) = 0 \quad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Then, we also conclude that $t_1 > 1$ with $\psi'_{\varphi_n}(t_1) > 0$.

Referring to the fibering map, we know that $\psi_{t_1\overline{\varphi}}(1) < \psi_{t_1\overline{\varphi}}(t) < 0$, where $t \in (0,1)$. Let us choose $t = 1/t_1$, so that we have

$$\mathcal{J}_\lambda(t_1\overline{\varphi}) = \psi_{t_1\overline{\varphi}}(1) < \psi_{t_1\overline{\varphi}}\left(\frac{1}{t_1}\right) = \mathcal{J}_\lambda(\overline{\varphi}).$$

Consequently, we deduce that

$$\mathcal{J}_\lambda(t_1\overline{\varphi}) < \mathcal{J}_\lambda(\overline{\varphi}) \leq \lim_{n\to+\infty} \mathcal{J}_\lambda(\varphi_n) = \inf_{\varphi\in\mathcal{A}^+} \mathcal{J}_\lambda(\varphi).$$

This leads to contradiction with the fact that $t_1\overline{\varphi} \in \mathcal{A}^+$, and so we conclude that $\varphi_n \to \overline{\varphi}$ in $W_0^{1,p(z)}(\Omega)$ and

$$\mathcal{J}_\lambda(\overline{\varphi}) = \lim_{n\to+\infty} \mathcal{J}_\lambda(\varphi_n) = \inf_{\varphi\in\mathcal{A}^+} \mathcal{J}_\lambda(\varphi).$$

It remains to observe that $\mathcal{A}^0 = \emptyset$; hence, we have $\overline{\varphi} \neq 0$. □

Next, we establish the existence of a local minimum for $\mathcal{J}_\lambda(\varphi)$ on $\mathcal{A}^-$.

Lemma 6. *Assume that hypotheses* $(A_\alpha), (A_f), (A_m), (A_M)$ *hold and* $\lambda^* = \min\{\tilde{\lambda}_0, \lambda_0^*\}$, *then for all* $\lambda \in (0, \lambda^*)$ *the functional* $\mathcal{J}_\lambda$, *associate to problem* (P_0), *admits a minimizer on* $\mathcal{A}^-$.

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Proof. To conclude the claim, we follow the similar steps as given in the proof of the previous Lemma 5. That is, invoking Lemma 1, we know that the energy functional $\mathcal{J}_\lambda(\varphi)$ is bounded below on the Nehari manifold $\mathcal{A}$, and hence on the submanifold $\mathcal{A}^-$. This means that we can construct a minimizing sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}^-$ satisfying

$$\lim_{n\to+\infty} \mathcal{J}_\lambda(\varphi_n) = \inf_{\varphi\in\mathcal{A}^-} \mathcal{J}_\lambda(\varphi).$$

On the other hand the coercivity of $\mathcal{J}_\lambda(\varphi)$ on $\mathcal{A}$ says that $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}^-$ is bounded in $W_0^{1,p(z)}(\Omega)$. This implies there exists $\widehat{\varphi} \in W_0^{1,p(z)}(\Omega)$ such that, up to a subsequence (yet denoted by $\{\varphi_n\}_{n\in\mathbb{N}}$) we have weak convergence in the Sobolev space $W_0^{1,p(z)}(\Omega)$ and strong convergence in suitable Lebesgue spaces $L^{\vartheta(z)}(\Omega)$, namely

$$\begin{aligned} &\varphi_n \rightharpoonup \widehat{\varphi} \ \text{ in } \ W_0^{1,p(z)}(\Omega),\\ &\varphi_n \to \widehat{\varphi} \ \text{ in } \ L^{\vartheta(z)}(\Omega), \quad 1 \le \vartheta(z) < p^*(z). \end{aligned}$$

The crucial step is now in establishing the strong convergence of ψ_n to $\widehat{\varphi}$ in $W_0^{1,p(z)}(\Omega)$. If this is not the case, then we have

$$\begin{aligned} \int_\Omega |\widehat{\varphi}|^{\vartheta(z)} dz &\le \lim_{n\to+\infty} \int_\Omega |\varphi_n|^{\vartheta(z)} dz \ \text{ for } 1 \le \vartheta(z) < p^*(z),\\ \int_\Omega |\nabla\widehat{\varphi}|^{\vartheta(z)} dz &< \liminf_{n\to+\infty} \int_\Omega |\nabla\varphi_n|^{\vartheta(z)} dz \ \text{ for } \theta \in \{p,q\},\\ M\left(\int_\Omega |\nabla\widehat{\varphi}|^{\vartheta(z)} dz\right) &< \liminf_{n\to+\infty} M\left(\int_\Omega |\nabla\varphi_n|^{\vartheta(z)} dz\right) \ \text{ for } \theta \in \{p,q\}. \end{aligned}$$

So, if we appeal to Lemma 2, we can conclude the existence of a value t_2 such that $t_2\widehat{\varphi} \in \mathcal{A}^-$, and this leads to

$$\psi'_{\widehat{\varphi}}(t_2) = 0.$$

It follows that

$$\begin{aligned} \lim_{n\to+\infty} \psi'_{\varphi_n}(t_2) &= \lim_{n\to+\infty} M\left(\int_\Omega t_2^{p(z)} \frac{|\nabla\varphi_n|^{p(z)}}{p(z)} dz\right) \int_\Omega t_2^{p(z)-1} |\nabla\varphi_n|^{p(z)} dz\\ &\quad + \lim_{n\to+\infty} M\left(\int_\Omega t_2^{q(z)} \frac{|\nabla\varphi_n|^{q(z)}}{q(z)} dz\right) \int_\Omega t_2^{q(z)-1} |\nabla\varphi_n|^{q(z)} dz\\ &\quad + \theta \lim_{n\to\infty} \int_\Omega \left(t_2^{p(z)-1}|\varphi_n|^{p(z)} + t_2^{q(z)-1}|\varphi_n|^{q(z)}\right) dz\\ &\quad - \lim_{n\to+\infty} \int_\Omega t_2^{-\alpha(z)} f(z) |\varphi_n|^{1-\alpha(z)} dz - \lambda \lim_{n\to+\infty} \int_\Omega t_2^{\beta(z)-1} |\varphi_n|^{\beta(z)} dz\\ &> M\left(\int_\Omega t_2^{p(z)} \frac{|\nabla\widehat{\varphi}|^{p(z)}}{p(z)} dz\right) \int_\Omega t_2^{p(z)-1} |\nabla\varphi_2|^{p(z)} dz\\ &\quad + M\left(\int_\Omega t_2^{q(z)} \frac{|\nabla\widehat{\varphi}|^{q(z)}}{q(z)} dz\right) \int_\Omega t_2^{q(z)-1} |\nabla\widehat{\varphi}|^{q(z)} dz\\ &\quad + \theta \int_\Omega \left(t_2^{p(z)-1}|\varphi_2|^{p(z)} + t_2^{q(z)-1}|\varphi_2|^{q(z)}\right) dz\\ &\quad - \int_\Omega t_2^{-\alpha(z)} f(z) |\widehat{\varphi}|^{1-\alpha(z)} dz - \lambda \int_\Omega t_2^{\beta(z)-1} |\widehat{\varphi}|^{\beta(z)} dz\\ &= \psi'_{\widehat{\varphi}}(t_2) = 0. \end{aligned}$$

This immediately gives us that $\psi'_{\varphi_n}(t_2) > 0$, choosing $n > n_0$ sufficiently large.

Corresponding to $\varphi_n \in \mathcal{A}^-$, the first derivative of the fibering map $\psi'_{\varphi_n}(t)$ is such that (recall Lemma 2(ii))

$$\begin{aligned} &\psi'_{\varphi_n}(t) < 0 \quad \text{for } t > 1, \text{all } n \in \mathbb{N},\\ &\psi'_{\varphi_n}(1) = 0 \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

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Since $\psi'_{\varphi_n}(t_2) > 0$, we deduce that $t_2 < 1$, and so we have

$$\mathcal{J}_\lambda(t_2\widehat{\varphi}) < \liminf_{n\to+\infty} \mathcal{J}_\lambda(t_2\varphi_n) \le \lim_{n\to+\infty} \mathcal{J}_\lambda(\varphi_n) = \inf_{\varphi\in\mathcal{A}^-} \mathcal{J}_\lambda(\varphi).$$

We reached a contradiction with $t_2\widehat{\varphi} \in \mathcal{A}^-$. Hence, the strong convergence in $W_0^{1,p(z)}(\Omega)$ occurs. The emptiness of submanifold $\mathcal{A}^0$ ensures that $\widehat{\varphi} \neq 0$. We conclude that the functional $\mathcal{J}_\lambda(\varphi)$, associate to problem (P_0), admits a minimizer on $\mathcal{A}^-$, for all $\lambda \in (0, \lambda^*)$. □

Now we can give the proof of Theorem 1 (i.e., the existence of two non-negative weak solutions to problem (P_0)).

Proof of Theorem 1.1. We know from Lemma 5 that one can find a function $\overline{\varphi} \in \mathcal{A}^+$ such that for all $\lambda \in (0, \lambda^*)$ we get

$$\mathcal{J}_\lambda(\overline{\varphi}) = \inf_{\varphi\in\mathcal{A}^+} \mathcal{J}_\lambda(\varphi).$$

Similarly, by Lemma 6 we can find a function $\widehat{\varphi} \in \mathcal{A}^-$ such that for all $\lambda \in (0, \lambda^*)$ we have

$$\mathcal{J}_\lambda(\widehat{\varphi}) = \inf_{\varphi\in\mathcal{A}^-} \mathcal{J}_\lambda(\varphi).$$

The expression of the energy functional at the beginning of Section 3 shows that

$$\mathcal{J}_\lambda(\varphi) = \mathcal{J}_\lambda(|\varphi|).$$

This obviously gives us that $|\overline{\varphi}| \in \mathcal{A}^+$, $|\widehat{\varphi}| \in \mathcal{A}^-$, and $|\overline{\varphi}|, |\widehat{\varphi}| \notin \mathcal{A}^0$. We know that such local minimizers are exactly critical points of $\mathcal{J}_\lambda$ on $\mathcal{A}$ (recall Lemma 3) and so they are two weak solutions to problem (P_0) on the Sobolev space $W_0^{1,p(z)}(\Omega)$ with $|\overline{\varphi}| \neq |\widehat{\varphi}|$. Consequently, it is not restrictive to assume that $\overline{\varphi}, \widehat{\varphi} \in W_0^{1,p(z)}(\Omega)$ are two distinct nontrivial non-negative weak solutions of problem (P_0).

Author Contributions

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Conflicts of Interest

The authors declare no conflicts of interest.

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