CENTRAL POLYNOMIALS OF GRADED ALGEBRAS: CAPTURING THEIR EXPONENTIAL GROWTH

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Abstract. Let G be a finite abelian group and let A be an associative G-graded algebra over a field of characteristic zero. A central G-polynomial is a polynomial of the free associative G-graded algebra that takes central values for all graded substitutions of homogeneous elements of A. We prove the existence and the integrability of two limits called the central G-exponent and the proper central G-exponent that give a quantitative measure of the growth of the central G-polynomials and the proper central G-polynomials, respectively. Moreover, we compare them with the G-exponent of the algebra.

1. Introduction

Let G be a finite abelian group and let A be a G-graded algebra over a field of characteristic zero. If we denote by $F\langle X, G\rangle$ the free associative G-graded algebra, freely generated over F by the set X of variables, then a G-polynomial $f \in F\langle X, G\rangle$ is a central G-polynomial of A if f takes values in $Z(A)$, the center of A. In case f vanishes on A, then it is called a G-polynomial identity of A, otherwise f is a proper central G-polynomial.

Central polynomials were first studied after a famous conjecture of Kaplansky asserting that the algebra $M_n(F)$ of $n \times n$ matrices over F has proper central polynomials (see [21]). Later on, such a conjecture was proved independently by Formanek and Razmyslov in [13] and [26]. Nowadays very few is known about T-spaces of central polynomials. For instance, in [7] the T-space of central polynomials of the Grassmann algebra in characteristic different from 2 was computed, while a similar result for the same algebra was achieved in [8] for central polynomials with involution. On the other hand, there exist algebras with a non-trivial center having no proper central polynomials (see [19, Lemma 1]).

By using an idea of Regev (see [27]), here we are interested in a quantitative approach in order to get information about how many central polynomials a given G-graded algebra has. To this end, we consider for any $n \geq 1$ the space P_n^G of multilinear graded polynomials of degree n and we attach to it three numerical sequences: $c_n^G(A)$, the dimension of P_n^G modulo the G-polynomial identities of A; $c_n^{G,z}(A)$, the dimension of P_n^G modulo the central G-polynomials of A; $\delta_n^G(A)$, the dimension of the space of multilinear central G-polynomials of degree n modulo the graded identities of A. They are called the G-graded, central G-graded and proper central G-graded codimension sequence, respectively, and for all $n \geq 1$,

$$c_n^G(A) = c_n^{G,z}(A) + \delta_n^G(A).$$

It follows that one can gather informations about $c_n^G(A)$ by knowing how $c_n^{G,z}(A)$ behaves and viceversa (see for instance [6]).

The asymptotic behavior of the G-graded codimension sequence was extensively studied in the past years. In fact, in [15] it was proved that if A satisfies a non-trivial ordinary polynomial identity (PI-algebra), then $c_n^G(A)$ is exponentially bounded, moreover in [2], [3] and [14] the authors proved the existence and the integrability of the graded exponent, $\exp^G(A)$, also for non-abelian groups. Such a result was generalized in [22] in the setting of H-module algebras, where H is a semisimple finite dimensional Hopf algebra.

In this paper we want to prove an analogous result for the central and the proper central G-polynomials. Because of the previous relation among the codimension sequences, it is clear that $c_n^{G,z}(A)$ and $\delta_n^G(A)$ are exponentially bounded, provided that A is a PI-algebra. Furthermore, what can we say about the central

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G-exponent, $\exp^{G,z}(A)$, and the proper central G-exponent, $\exp^{G,\delta}(A)$? We will prove that such exponents exist and they are non-negative integers by showing an explicit way to compute them. Moreover, we compare the central G-exponent with the G-exponent by proving that either $\exp^{G,z}(A) = \exp^G(A)$ or $\exp^{G,z}(A) = 0$. Similar results were recently achieved for ordinary algebras in [19] and [20] and for algebras with involution in [25].

2. The basic setting

Let F be a field of characteristic zero, G be a finite abelian group and A be a G-graded algebra over F, i.e. $A = \bigoplus_{g\in G} A_g$, where the A_g's are vector subspaces of A such that $A_gA_h \subseteq A_{gh}$ for all $g, h \in G$. We refer to such subspaces as the homogeneous component of A and we say that the element a has homogeneous degree g if $a \in A_g$ for some $g \in G$. In this case we write $|a|_G = g$ or simply $|a| = g$ if no ambiguity arises.

Let $F\langle X\rangle$ be the free associative algebra on a countable set X of variables $x_1, x_2, \ldots$. One can define on such an algebra a G-grading in a natural way: write $X = \bigcup_{g\in G} X_g$, where $X_g = \{x_{1,g}, x_{2,g}, \ldots\}$ are disjoint sets and the elements of X_g have homogeneous degree g. If we denote by $\mathcal{F}_g$ the subspace of $F\langle X\rangle$ spanned by all monomials in the variables of X having homogeneous degree g, then $F\langle X\rangle = \bigoplus_{g\in G} \mathcal{F}_g$ is a G-graded algebra called the free associative G-graded algebra of countable rank over F. We shall denote it by $F\langle X, G\rangle$.

From now on, let $G = \{g_1, \ldots, g_s\}$. A G-graded polynomial, or simply a G-polynomial,

$$f = f(x_{1,g_1}, \ldots, x_{t_1,g_1}, \ldots, x_{1,g_s}, \ldots, x_{t_s,g_s})$$

of $F\langle X, G\rangle$ is a G-graded identity (or simply graded identity) of A, and we write $f \equiv 0$, if

$$f(a_{1,g_1}, \ldots, a_{t_1,g_1}, \ldots, a_{1,g_s}, \ldots, a_{t_s,g_s}) = 0$$

for all $a_{i,g_i} \in A_{g_i}$, $t_i \geq 0$, for all $1 \leq i \leq s$.

Let $\mathrm{Id}^G(A) = \{f \in F\langle X, G\rangle \mid f \equiv 0 \text{ on } A\}$ be the ideal of graded identities of A. It is easily seen that $\mathrm{Id}^G(A)$ is a T_G-ideal, i.e. it is an ideal invariant under all graded endomorphisms of $F\langle X, G\rangle$.

Notice that if for some $i \geq 1$ we set $x_i = x_{i,g_1} + \cdots + x_{i,g_s}$, then $F\langle X\rangle$ is naturally embedded into $F\langle X, G\rangle$ so that we can look at the (ordinary) identities of A as a special kind of graded identities.

Since $\mathrm{char} F = 0$, then $\mathrm{Id}^G(A)$ is determined by the multilinear G-polynomials it contains. Thus, for all $n \geq 1$, one can define

$$P_n^G = \mathrm{span}_F\{x_{\sigma(1),g_{i_1}} \cdots x_{\sigma(n),g_{i_n}} \mid \sigma \in S_n, g_{i_1}, \ldots, g_{i_n} \in G\}$$

as the space of multilinear G-polynomials in the graded variables $x_{1,g_{i_1}}, \ldots, x_{n,g_{i_n}}$, $g_{i_j} \in G$. Here S_n stands for the symmetric group on the integers $\{1, \ldots, n\}$. The T_G-ideal $\mathrm{Id}^G(A)$ is determined by the sequence of subspaces $P_n^G \cap \mathrm{Id}^G(A)$, $n \geq 1$, and we can construct the quotient space

$$P_n^G(A) = \frac{P_n^G}{P_n^G \cap \mathrm{Id}^G(A)}.$$

The non-negative integer

$$c_n^G(A) = \dim_F P_n^G(A),\ n \geq 1,$$

is called the nth codimension of A and the asymptotic behavior of the corresponding sequence is in some sense a quantitative measure of how many identities are satisfied by A. In [15] it was proved that if A satisfies a non-trivial ordinary polynomial identity, then such a sequence is exponentially bounded. Moreover, in [2, 3, 14] the authors captured this exponential growth by proving the existence and the integrability of the limit

$$\exp^G(A) = \lim_{n\to+\infty} \sqrt[n]{c_n^G(A)},$$

called the G-exponent of A. We highlight that such a result was achieved for any finite group.

The study of multilinear G-polynomials can be reduced to the one of smaller spaces in the following way. Take $n \geq 1$ and write $n = n_1 + \cdots + n_s$, where $n_1, \ldots, n_s \geq 0$. Then define $P_{n_1,\ldots,n_s}$ as the subspace of P_n^G of multilinear G-polynomials in which the first n_1 variables have homogeneous degree g_1, the next n_2

variables have homogeneous degree g_2, and so on. It is clear that P_n^G is the direct sum of such subspaces with $n_1+\cdots+n_s=n$ as well as $P_n^G(A)$ that inherits this decomposition. Thus, one defines

$$P_{n_1,\dots,n_s}(A)=\frac{P_{n_1,\dots,n_s}}{P_{n_1,\dots,n_s}\cap \mathrm{Id}^G(A)}$$

and sets

$$c_{n_1,\dots,n_s}(A)=\dim_F P_{n_1,\dots,n_s}(A).$$

Notice that, given $n_1,\dots,n_s$, there are $\binom{n}{n_1,\dots,n_s}$ subspaces isomorphic to $P_{n_1,\dots,n_s}$. Therefore, for all $n\geq 1$,

$$c_n^G(A)=\sum_{n_1+\cdots+n_s=n}\binom{n}{n_1,\dots,n_s}c_{n_1,\dots,n_s}(A).$$

Since there is a one-to-one correspondence between T_G-ideals and G-varieties of algebras, often it is convenient to translate all the objects we have introduced into the language of G-varieties. Thus if $\mathcal{V}=\mathrm{var}^G(A)$ is the G-variety generated by the G-graded algebra A, then we write $\mathrm{Id}^G(\mathcal{V})=\mathrm{Id}^G(A)$, $c_n^G(\mathcal{V})=c_n^G(A)$ and the growth of $\mathcal{V}$ is the growth of the codimension sequence $c_n^G(\mathcal{V})$.

3. Grassmann envelope and $G\times\mathbb{Z}_2$-graded algebras

In this section we introduce a useful tool that one can use in order to reduce the problem of computing the T_G-ideal of graded identities of any G-graded algebra to that of the so-called Grassmann envelope of a suitable finite dimensional $G\times\mathbb{Z}_2$-graded algebra, where $\mathbb{Z}_2$ is the cyclic group of order 2 in additive notation.

Let E denote the infinite dimensional Grassmann algebra generated by the elements $1,e_1,e_2,\dots$ subject to the relations $e_ie_j=-e_je_i$, for all $i\neq j$. Let $E=E_0\oplus E_1$ be its natural $\mathbb{Z}_2$-grading, where

$$E_0=\mathrm{span}_F\{e_{i_1}\cdots e_{i_{2k}}\mid 1\leq i_1<\cdots<i_{2k},\ k\geq 0\}$$

and

$$E_1=\mathrm{span}_F\{e_{i_1}\cdots e_{i_{2k+1}}\mid 1\leq i_1<\cdots<i_{2k+1},\ k\geq 0\}.$$

Moreover, let $A=\bigoplus_{(g,i)\in G\times\mathbb{Z}_2}A_{(g,i)}$ be a $G\times\mathbb{Z}_2$-graded algebra. Then A has an induced $\mathbb{Z}_2$-grading, $A=A_0\oplus A_1$, where $A_0=\bigoplus_{g\in G}A_{(g,0)}$ and $A_1=\bigoplus_{g\in G}A_{(g,1)}$, and an induced G-grading $A=\bigoplus_{g\in G}A_g$ where, for all $g\in G$, $A_g=A_{(g,0)}\oplus A_{(g,1)}$.

Then, the Grassmann envelope of A is defined as

$$E(A)=(A_0\otimes E_0)\oplus(A_1\otimes E_1).$$

On one hand, it has a natural $G\times\mathbb{Z}_2$-grading induced by the one of A, i.e. $E(A)=\bigoplus_{(g,i)\in G\times\mathbb{Z}_2}E(A)_{(g,i)}$ where $E(A)_{(g,i)}=A_{(g,i)}\otimes E_i$. On the other, it has an induced G-grading by setting $E(A)_g=(A_{(g,0)}\otimes E_0)\oplus(A_{(g,1)}\otimes E_1)$.

In case of ordinary polynomial identities, a celebrated theorem of Kemer states that an arbitrary algebra satisfying a non-trivial polynomial identity over a field of characteristic zero has the same identities as the Grassmann envelope $E(A)$ of a finite dimensional $\mathbb{Z}_2$-graded algebra A (see [23]). This result was independently extended in the setting of graded algebras in [1] and [29] by proving the following theorem.

Theorem 1. *Let R be a G-graded algebra satisfying a non-trivial ordinary polynomial identity. Then there exists a finite dimensional $G\times\mathbb{Z}_2$-graded algebra A such that $Id^G(R)=Id^G(E(A))$.*

It is worth mentioning that in [1] the result was proved also for non-abelian groups.

By the Wedderburn–Malcev decomposition, we write

$$A=A'+J$$

where A' is a maximal semisimple subalgebra of A, which we may assume to be $G\times\mathbb{Z}_2$-graded by [28], and $J=J(A)$ its Jacobson radical, which is a graded ideal of A. Also we can write $A'=A_1\oplus\cdots\oplus A_k$, where $A_1,\dots,A_k$ are $G\times\mathbb{Z}_2$-graded simple algebras (or simply $G\times\mathbb{Z}_2$-simple algebras). The description of such algebras is given in the following theorem proved by Bahturin, Sehgal and Zaicev in [5].

Theorem 2. *Let A be a finite dimensional $G \times \mathbb{Z}_2$-simple algebra. Then there exist a subgroup H of $G \times \mathbb{Z}_2$, a 2-cocycle $\alpha : H \times H \to F^*$, where the action of H on F is trivial, an integer m and an m-tuple $(g_1 = 1, g_2, \dots, g_m) \in (G \times \mathbb{Z}_2)^m$ such that A is isomorphic as $G \times \mathbb{Z}_2$-graded algebras to*

$$R = F^\alpha H \otimes M_m(F),$$

where $R_g = span_F\{b_h \otimes e_{ij} \mid g = g_i^{-1} h g_j\}$. Here $b_h \in F^\alpha H$ is a representative of $h \in H$.

In order to simplify the notation we shall use the elements of the group instead of their representatives. According to the previous result, it turns out that $Z(A) \cong Z(F^\alpha H) \otimes I_m$, where I_m is the $m \times m$ identity matrix.

As a way to prove Theorem 1, the authors defined a map, denoted by $\tilde{}$, relating the $G \times \mathbb{Z}_2$-identities of a $G \times \mathbb{Z}_2$-graded algebra A to the ones of $E(A)$.

First they introduced the following notation. In the free $G \times \mathbb{Z}_2$-graded algebra $F\langle X, G \times \mathbb{Z}_2\rangle$, write $x_{i,(g,0)} = y_{i,g}$ and $x_{i,(g,1)} = z_{i,g}$, for all $g \in G$ and $i \geq 1$. Now let $f \in F\langle X, G \times \mathbb{Z}_2\rangle$ be a multilinear polynomial in the variables $z_1, \dots, z_m$, $y_1, \dots, y_t$ and write f in the form

$$f = \sum_{\substack{\sigma \in S_m \\ W = (w_0, w_1, \dots, w_m)}} \alpha_{\sigma,W} w_0 z_{\sigma(1)} w_1 \cdots w_{m-1} z_{\sigma(m)} w_m$$

where $z_1, \dots, z_m$ are homogeneous variables of degree $(g, 1)$, $g \in G$, $w_0, w_1, \dots, w_m$ are (eventually empty) monomials in variables of homogeneous degree $(g, 0)$, $g \in G$, and $\alpha_{\sigma,W} \in F$. Then define

$$\tilde{f} = \sum_{\substack{\sigma \in S_m \\ W = (w_0, w_1, \dots, w_m)}} (\operatorname{sgn} \sigma) \alpha_{\sigma,W} w_0 z_{\sigma(1)} w_1 \cdots w_{m-1} z_{\sigma(m)} w_m.$$

According to [18], the map $\tilde{}$ is such that

1. $\tilde{\tilde{f}} = f$;
2. f is a $G \times \mathbb{Z}_2$-identity of $E(A)$ if and only if $\tilde{f}$ is a $G \times \mathbb{Z}_2$-identity of A;
3. for any subset Z of variables $\{z_1, \dots, z_m\}$, f is alternating on Z if and only if $\tilde{f}$ is symmetric on Z.

4. On central G-polynomials

In this section we shall introduce the main object of the paper. Let R be a G-graded algebra, then a G-polynomial $f(x_{1,g_{i_1}}, \dots, x_{n,g_{i_n}}) \in F\langle X, G\rangle$ is a central G-polynomial (or simply a central polynomial) of R if $f(a_1, \dots, a_n) \in Z(R)$ for all homogeneous elements $a_1 \in R_{g_{i_1}}, \dots, a_n \in R_{g_{i_n}}$. If f takes only the zero value, then it is clear that f is a graded identity of R, otherwise we say that f is a proper central G-polynomial of R.

Let $\mathrm{Id}^{G,z}(R) = \{f \in F\langle X, G\rangle \mid f \text{ is a central } G\text{-polynomial of } R\}$. Then $\mathrm{Id}^{G,z}(R)$ is a T_G-space of $F\langle X, G\rangle$, i.e. a vector space invariant under all G-graded endomorphisms of the free G-graded algebra.

We set

$$P_n^{G,z}(R) = \frac{P_n^G}{P_n^G \cap \mathrm{Id}^{G,z}(R)},$$

and

$$\Delta_n^G(R) = \frac{P_n^G \cap \mathrm{Id}^{G,z}(R)}{P_n^G \cap \mathrm{Id}^G(R)}.$$

Notice that $\Delta_n^G(R)$ corresponds to the space of multilinear proper central G-polynomials of R in n fixed homogeneous variables.

It can be easily checked that if R_1 and R_2 are two G-graded algebras such that $\mathrm{Id}^G(R_1) = \mathrm{Id}^G(R_2)$, then $\mathrm{Id}^{G,z}(R_1) = \mathrm{Id}^{G,z}(R_2)$ and so $\Delta_n^G(R_1) = \Delta_n^G(R_2)$ for all $n \geq 1$.

Moreover, define $c_n^{G,z}(R) = \dim_F P_n^{G,z}(R)$ and $\delta_n^G(R) = \dim_F \Delta_n^G(R)$, $n \geq 1$, as the sequences of central G-codimensions and proper central G-codimensions, respectively. Then

$$c_n^G(R) = c_n^{G,z}(R) + \delta_n^G(R), \tag{1}$$

for all $n \geq 1$. If R is a PI-algebra, then $c_n^G(R)$ is exponentially bounded and so are $c_n^{G,z}(R)$ and $\delta_n^G(R)$ and our aim is to capture their exponential growth. In particular, we are asking whether the two limits

$$\exp^{G,z}(R) = \lim_{n\to+\infty} \sqrt[n]{c_n^{G,z}(R)} \quad \text{and} \quad \exp^{G,\delta}(R) = \lim_{n\to+\infty} \sqrt[n]{\delta_n^G(R)}$$

exist and, in case of affirmative answer, compute them.

To this end, as we already did for $P_n^G(R)$, one can split $\Delta_n^G(R)$ into the direct sum of subspaces

$$\Delta_{n_1,\dots,n_s}(R) = \frac{P_{n_1,\dots,n_s} \cap \mathrm{Id}^{G,z}(R)}{P_{n_1,\dots,n_s} \cap \mathrm{Id}^G(R)},$$

where $n_1 + \cdots + n_s = n$, and set $\delta_{n_1,\dots,n_s}(R) = \dim_F \Delta_{n_1,\dots,n_s}$ so that

$$\delta_n^G(R) = \sum_{n_1+\cdots+n_s=n} \binom{n}{n_1,\dots,n_s} \delta_{n_1,\dots,n_s}(R).$$

In order to guess the proper central G-exponent $\exp^{G,\delta}(R)$, we now introduce the following notation.

Let A be a finite dimensional $G \times \mathbb{Z}_2$-graded algebra such that $\mathrm{Id}^G(R) = \mathrm{Id}^G(E(A))$ and write $A = A_1 \oplus \cdots \oplus A_m + J$, where $A_1, \dots, A_m$ are $G \times \mathbb{Z}_2$-simple algebras and J is the Jacobson radical of A. We say that A is reduced if $A_{i_1} J A_{i_2} J \cdots J A_{i_m} \neq 0$, where $i_1, \dots, i_m \in \{1, \dots, m\}$ are all distinct.

Definition 1. *A semisimple $G \times \mathbb{Z}_2$-graded subalgebra $B = A_{i_1} \oplus \cdots \oplus A_{i_k}$, where $i_1, \dots, i_k \in \{1, \dots, m\}$ are all distinct, is called centrally admissible for $E(A)$ is there exists a proper central G-polynomial*

$$f(x_{1,g_{j_1}}, \dots, x_{r,g_{j_r}})$$

of $E(A)$ such that $r \geq k$ and $f(a_1, \dots, a_r) \neq 0$, for some homogeneous elements $a_1 \in E(A_{i_1})_{g_{j_1}}, \dots, a_k \in E(A_{i_k})_{g_{j_k}}$, $a_{k+1} \in E(A)_{g_{j_{k+1}}}, \dots, a_r \in E(A)_{g_{j_r}}$.

Remark 1. *If B is centrally admissible for $E(A)$ of maximal dimension, then $\widehat{B} = B + J$ is reduced.*

Proof. Without loss of generality, we may assume that $B = A_1 \oplus \cdots \oplus A_k$. Moreover, let f be a multilinear proper central G-polynomial as in Definition 1. Since $E(A_i)E(A_j) = 0$ for all $i \neq j$ and f is not a graded identity of $E(A)$, then

$$E(A_{i_1})E(J)E(A_{i_2})E(J) \cdots E(J)E(A_{i_k}) \neq 0$$

for some permutation $(i_1, \dots, i_k)$ of $(1, \dots, k)$. Thus $A_{i_1} J A_{i_2} J \cdots J A_{i_k} \neq 0$, that is, $\widehat{B} = B + J$ is reduced. $\square$

The next two sections are devoted to the computation of an upper and lower bound for the proper central G-codimension sequence. In particular, we claim that $\exp^{G,\delta}(E(A)) = d$, where d is the maximal dimension of a centrally admissible subalgebra for $E(A)$. The last section studies the central G-exponent compared with the G-graded (ordinary) exponent. There we will prove that either such exponents are equal or the central G-exponent is zero.

5. The upper bound for $\delta_n^G(R)$

In this section we shall determine an upper bound for the proper central G-codimensions of a G-graded PI-algebra.

Let $\mathcal{S}$ be the free supercommutative algebra over F of countable rank (see [4]). Recall that $\mathcal{S}$ is defined by its universal property: we let $T_1 = \{\xi_{i,j}\}$, $T_2 = \{\eta_{i,j}\}$ be countable sets, then $\mathcal{S} = F\langle T_1, T_2\rangle$ is the algebra with 1 generated by T_1, T_2 subject to the condition that the elements of T_1 are central and the elements of T_2 anticommute among themselves. The algebra $\mathcal{S}$ has a natural $\mathbb{Z}_2$-grading $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$ by requiring that the variables $\xi_{i,j}$ are of homogeneous degree zero and the variables $\eta_{i,j}$ are of homogeneous degree one. Notice that the Grassmann algebra E can be viewed as a $\mathbb{Z}_2$-graded subalgebra of $\mathcal{S}$ if one identifies the generating elements e_k with the elements $\eta_{i,j}$. Hence $\mathcal{S} \cong E \otimes F[\xi_{i,j}]$, $\mathcal{S}_0 \cong E_0 \otimes F[\xi_{i,j}]$ and $\mathcal{S}_1 \cong E_1 \otimes F[\xi_{i,j}]$.

We recall our general setting: $A = A_1 \oplus \cdots A_m + J$ is a finite dimensional $G \times \mathbb{Z}_2$-graded algebra over an algebraically closed field F, $u \geq 0$ is the smallest integer such that $J^{u+1} = 0$ and $G = \{g_1, \dots, g_s\}$ is an abelian group. One can define the superenvelope of A to be

$$\mathcal{S}(A) = \mathcal{S}_0 \otimes \mathcal{A}_0 \oplus \mathcal{S}_1 \otimes \mathcal{A}_1 \cong E(A) \otimes F[\xi_{i,j}],$$

where $\mathcal{A}_0 = \bigoplus_{g\in G} A_{(g,0)}$ and $\mathcal{A}_1 = \bigoplus_{g\in G} A_{(g,1)}$.

Clearly $\mathcal{S}(A)$ has an induced G-grading where $\mathcal{S}(A)_g \cong E(A)_g \otimes F[\xi_{i,j}]$, $g \in G$, and $\mathrm{Id}^G(\mathcal{S}(A)) = \mathrm{Id}^G(E(A))$. Fix a basis $\mathcal{B}$ of A of homogeneous elements respect to the $G \times \mathbb{Z}_2$-grading which is union of a basis of J and a basis for each of the A_i. Let $\mathcal{B} = \{a_1, \dots, a_r, b_1, \dots, b_t\}$ where $\{a_1, \dots, a_r\}$ is a basis of A_0 and $\{b_1, \dots, b_t\}$ is a basis of A_1. For a fixed $n \geq 1$, we choose nr variables $\xi_{i,j} \in T_1$, $i = 1, \dots, n$, $j = 1, \dots, r$ and nt variables $\eta_{i,j} \in T_2$, $i = 1, \dots, n$, $j = 1, \dots, t$. For $i = 1, \dots, n$, $g \in G$, we define the generic elements:

$$U_{i,g} = \sum \xi_{i,r_j} \otimes a_{r_j} + \sum \eta_{i,t_j} \otimes b_{t_j} \in \mathcal{S}(A),$$

where the first sum runs over all r_j such that a_{r_j} is of homogeneous degree $(g, 0)$ and the second one runs over all t_j such that b_{t_j} is of homogeneous degree $(g, 1)$. We denote by $\mathcal{H}$ the G-graded subalgebra generated by the generic elements $U_{i,g}$, $i = 1, \dots, n, g \in G$.

Proposition 1. *The algebra $\mathcal{H}$ is isomorphic to the relatively free G-graded algebra for $E(A)$ in ns graded generators.*

Proof. Let ψ be the graded homomorphism of the free G-graded algebra $F\langle x_{1,g_1}, \dots, x_{n,g_1}, \dots, x_{1,g_s}, \dots, x_{n,g_s}\rangle$ to the algebra $\mathcal{H}$ mapping $x_{i,g_i} \mapsto U_{i,g_i}$. We shall prove that $Ker\psi = \mathrm{Id}^G(E(A))$. Since $\mathrm{Id}^G(E(A)) = \mathrm{Id}^G(S(A)) \subseteq \mathrm{Id}^G(\mathcal{H})$ we have that $\mathrm{Id}^G(E(A)) \subseteq Ker\psi$.

Now let $f = f(x_{1,g_1}, \dots, x_{n,g_1}, \dots, x_{1,g_s}, \dots, x_{n,g_s}) \in Ker\psi$, i.e., $f(U_{1,g_1}, \dots, U_{n,g_1}, \dots, U_{1,g_s}, \dots, U_{n,g_s}) = 0$ and consider arbitrary homogeneous elements of $E(A)$

$$c_{i,g_l} = \sum \alpha_{i,r_j} \otimes a_{r_j} + \sum \beta_{i,t_j} \otimes b_{t_j},$$

$i = 1, \dots, n$, $l = 1, \dots s$, where the first sum runs over all r_j such that a_{r_j} is of homogeneous degree $(g_l, 0)$, the second one runs over all t_j such that b_{t_j} is of homogeneous degree $(g_l, 1)$, $\alpha_{i,j} \in E_0$ and $\beta_{i,j} \in E_1$. Consider then the homorphism $\rho : A \otimes F[\xi_{i,j}, \eta_{l,k}] \to A \otimes E$ which is the identity on A and maps $\xi_{i,j} \mapsto \alpha_{i,j}$, $\eta_{l,k} \mapsto \beta_{l,k}$. This maps U_{i,g_l} to c_{i,g_l} and $f(c_{1,g_1}, \dots, c_{n,g_1}, \dots, c_{1,g_s}, \dots, c_{n,g_s}) = \rho(f(U_{1,g_1}, \dots, U_{n,g_1}, \dots, U_{1,g_s}, \dots, U_{n,g_s}))$. This says that if f vanishes when computed on the generic elements U_{i,g_l}, it is a graded identity for $E(A)$. □

As a consequence we have that a G-polynomial f is a G-identity of $E(A)$ if and only if it vanishes on the generic elements $U_{i,g}, g \in G, i \geq 1$. Now let ψ the homomorphism defined above and define

$$\mathcal{Z}_n = \psi(P_n^G \cap \mathrm{Id}^{G,z}(E(A))) = \Big\{ f(U_{1,g_{i_1}}, \dots, U_{n,g_{i_n}}) \mid f(x_{1,g_{i_1}}, \dots, x_{n,g_{i_n}}) \in P_n^G \cap \mathrm{Id}^{G,z}(E(A)) \Big\}.$$

It is clear that, since $\mathrm{Id}^G(E(A)) = \ker\psi$, then

$$\delta_n^G(E(A)) = \dim_F \ \mathcal{Z}_n$$

and, for fixed non negative integers $n_1, \dots, n_s$ such that $n = n_1 + \dots + n_s$

$$\delta_{n_1,\dots,n_s}(E(A)) = \dim_F \ \mathcal{Z}_{n_1,\dots,n_s} \tag{2}$$

where $\mathcal{Z}_{n_1,\dots,n_s} = \psi(P_{n_1,\dots,n_s} \cap \mathrm{Id}^{G,z}(E(A))) \subseteq \mathcal{Z}_n$.

Next we shall prove the following result.

Lemma 1. *There exist constants $C > 0$ and v such that*

$$\delta_n^G(E(A)) \leq Cn^v d^n,$$

where d is the maximal dimension of a centrally admissible algebra for $E(A)$.

Proof. We start by computing an upper bound of $\dim_F \ \mathcal{Z}_{n_1,\dots,n_s}$.

Take a nonzero element $f = f(U_{1,g_{i_1}}, \dots, U_{n,g_{i_n}}) = \psi(f(x_{1,g_{i_1}}, \dots, x_{n,g_{i_n}}))$ in $\mathcal{Z}_{n_1,\dots,n_s}$ (this says that $f(x_{1,g_{i_1}}, \dots, x_{n,g_{i_n}})$ is a proper central G-polynomial) and by replacing each $U_{i,g}$, write f as a linear combination of elements of the type

$$\Gamma \otimes c, \ \Gamma = \gamma_{1,j_1}\gamma_{2,j_2}\cdots\gamma_{n,j_n} \tag{3}$$

where $c \in \mathcal{B}$ is an element of the above basis of A and γ_{k,j_k} is equal either to ξ_{k,j_k} or to η_{k,j_k}. Thus we shall estimate $\dim_F \ \mathcal{Z}_{n_1,\dots,n_s}$ through an estimate of the number of possible monomials Γ which can appear as coefficients of the elements in $\mathcal{Z}_{n_1,\dots,n_s}$ in the chosen basis. We shall call them *nonzero monomials.* Since each variable $\xi_{i,j}$ or $\eta_{i,l}$ is attached to a basis element of some homogeneous degree, we shall say that $\xi_{i,j}$ or $\eta_{i,l}$ is a radical variable or a semisimple variable of some homogeneous degree. Take a nonzero monomial $\Gamma = \Gamma_r\Gamma_s$ of type (3) where Γ_r contains i radical variables and Γ_s contains $n-i$ semisimple variables. Write $i = i_1 + \cdots + i_s$ where $i_1 \leq n_1$ is the number of radical variables of homogeneous degree g_1, $i_2 \leq n_2$ the number of radical variables of homogeneous degree g_2 and so on. Now this element is nonzero only if $i \leq u$. The number of possible monomials Γ_r containing $i = i_1 + \cdots + i_s$ radical variables with $i_j \leq n_j$, $j = 1, \ldots, s$ is at most $\binom{n_1}{i_1}\cdots\binom{n_s}{i_s}(\dim J)^i \leq c_1 n^a$, for some constants c_1 and a. Now each such monomial involves $n-i$ semisimple variables with $n'_j = n_j - i_j$ variables of homogeneous degree g_j, $j = 1, \ldots, s$ which come from some distinct simple components $A_{l_1}, \ldots, A_{l_k}$. This means that $C = A_{l_1} \oplus \cdots \oplus A_{l_k}$ is a centrally admissible subalgebra for $E(A)$.

Now, fix a centrally admissible subalgebra C and let $d^C = \dim C$, $d_i^C = |\mathcal{B} \cap C_{(g_i,0)}|$ and $l_i^C = |\mathcal{B} \cap C_{(g_i,1)}|$. How many possible monomials with the semisimple variables coming from C can we write? If there are $i = i_1 + \ldots + i_s$ radical variables, such number is at most

$$\binom{n_1}{i_1}\cdots\binom{n_s}{i_s}(\dim J)^i (d_1^C + l_1^C)^{n'_1}\cdots(d_s^C + l_s^C)^{n'_s}.$$

Hence such centrally admissible subalgebra C may contribute with at most

$$\sum_{i_1+\ldots+i_s \leq u} \binom{n_1}{i_1}\cdots\binom{n_s}{i_s}(\dim J)^{i_1+\cdots+i_s}(d_1^C + l_1^C)^{n'_1}\cdots(d_s^C + l_s^C)^{n'_s} \leq cn^v(d_1^C + l_1^C)^{n_1}\cdots(d_s^C + l_s^C)^{n_s}$$

monomials, for some constants c, v. Thus if $C_1, \ldots, C_w$ are all the centrally admissible subalgebras we get that

$$\delta_{n_1,\ldots,n_s}(E(A)) \leq (\dim A)\ cn^v \sum_{i=1}^{w}(d_1^{C_i} + l_1^{C_i})^{n_1}\cdots(d_s^{C_i} + l_s^{C_i})^{n_s}.$$

Hence:

$$\delta_n^G(E(A)) = \sum_{n_1+\cdots+n_s=n}\binom{n}{n_1,\ldots,n_s}\delta_{n_1,\ldots,n_s}(E(A)) \leq$$

$$c'n^v\sum_{i=1}^{w}\sum_{n_1+\cdots+n_s=n}\binom{n}{n_1,\ldots,n_s}(d_1^{C_i} + l_1^{C_i})^{n_1}\cdots(d_s^{C_i} + l_s^{C_i})^{n_s} = c'n^v\sum_{i=1}^{w}(d^{C_i})^n \leq c''n^v d^n,$$

where d is the maximal dimension of a centrally admissible algebra for $E(A)$. □

6. The lower bound for $\delta_n^G(R)$

In this section a lower bound for the proper central G-codimension sequence will be found. As consequence, we shall compute the proper central G-exponent.

Recall that $G = \{g_1, \ldots, g_s\}$ and R is a G-graded algebra. We shall use for a fixed $n = n_1 + \cdots + n_s \geq 1$, the representation theory of the group $S_{n_1} \times \cdots \times S_{n_s}$, where S_{n_i} is the symmetric group of order n_i, $1 \leq i \leq s$. In fact, the spaces $P_{n_1,\ldots,n_s}(R)$ and $\Delta_{n_1,\ldots,n_s}(R)$ become $S_{n_1} \times \cdots \times S_{n_s}$-modules via the permutation action of the variables of the same homogeneous degree, i.e., S_{n_1} permutes the variables of homogeneous degree g_1, S_{n_2} those of homogeneous degree g_2, and so on (see [12, 18]).

In what follows, if $\lambda(1) \vdash n_1, \ldots, \lambda(s) \vdash n_s$ are partitions, we write $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(s)) \vdash (n_1, \ldots, n_s)$ and we say that $\langle\lambda\rangle$ is a multipartition of $n = n_1 + \cdots + n_s$. Moreover, if for every $1 \leq i \leq s$, $T_{\lambda(i)}$ is a Young tableau of shape $\lambda(i)$ and $e_{T_{\lambda(i)}}$ is the corresponding minimal essential idempotent of FS_{n_i}, then we denote by $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, \ldots, T_{\lambda(s)})$ the multitableau of shape $\langle\lambda\rangle$ and by $e_{T_{\langle\lambda\rangle}} = e_{T_{\lambda(1)}}\cdots e_{T_{\lambda(s)}}$ the corresponding idempotent of $F(S_{n_1} \times \cdots \times S_{n_s})$. Furthermore, given integers $d, l, t \geq 0$, we define the hook-shaped partition

$$h(d, l, t) = (\underbrace{t + l, \ldots, t + l}_{d}, \underbrace{l, \ldots, l}_{t}).$$

We start by recalling some results on finite dimensional $G \times \mathbb{Z}_2$-simple algebras.

Lemma 2 ([14], Lemma 9)**.** *Let B be a finite dimensional $G \times \mathbb{Z}_2$-simple algebra over an algebraically closed field F. Let $\dim_F B_{(g_i,0)} = p_i$ and $\dim_F B_{(g_i,1)} = q_i$, $1 \le i \le s$. Then, for any positive integer t there exist a multipartition $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s))$ with*

$$h(p_i, q_i, 2t - dim_F B) \le \lambda(i) \le h(p_i, q_i, 2t),$$

$1 \le i \le s$, and a multitableau $T_{\langle\lambda\rangle}$ such that $e_{T_{\langle\lambda\rangle}} f \notin Id^G(E(B))$, for a suitable multilinear G-polynomial f with $\deg f = 2t \dim_F B$.

Lemma 3 ([14], Lemma 10)**.** *Let B be a finite dimensional $G \times \mathbb{Z}_2$-simple algebra over an algebraically closed field F, then for any non-zero homogeneous elements $b_1, b_2 \in B$ there exist homogeneous elements $c_1, c_2 \in B$ such that $c_1 b_1 c_2 = b_2$.*

From now until the end of the section, let $A = A_1 \oplus \cdots \oplus A_p + J$ be a finite dimensional $G \times \mathbb{Z}_2$-graded algebra, where $A_1, \dots, A_p$ are $G \times \mathbb{Z}_2$-simple algebras and J is the Jacobson radical of A. Moreover, let $B = B_1 \oplus \cdots \oplus B_k \subseteq A$ be centrally admissible for $E(A)$ of maximal dimension, $\dim_F B = d$, where $B_1, \dots, B_k$ are $G \times \mathbb{Z}_2$-simple algebras, and let

$$f = f(x_{1,g_{i_1}}, \dots, x_{r,g_{i_r}})$$

be a multilinear central proper G-polynomial of $E(A)$ with $r \ge k$, such that $f(a_1, \dots, a_r) \ne 0$ for some homogeneous elements $a_1 \in E(B_1)_{g_{i_1}}, \dots, a_k \in E(B_k)_{g_{i_k}}, a_{k+1} \in E(A)_{g_{i_{k+1}}}, \dots, a_r \in E(A)_{g_{i_r}}$. Notice that, since B is centrally admissible of maximal dimension, then $f \notin \mathrm{Id}^G(E(\widehat{B}))$, where $\widehat{B} = B + J$. Thus we assume, as we may, that $a_1, \dots, a_r$ are homogeneous elements of $E(\widehat{B})$.

Moreover, we say that a multilinear G-polynomial p has homogeneous degree g if any evaluation of p belongs to the homogeneous component g of the algebra. Notice that such a definition is not ambiguous since p is multilinear and G is abelian.

Lemma 4. *Let $f_1, \dots, f_k$ be multilinear G-polynomials on distinct sets of homogeneous variables (different from those ones in f) such that $f_i \notin Id^G(E(B_i))$, $i = 1, \dots, k$. Then the multilinear polynomial*

$$f' = f(u_1 f_1 v_1, \dots, u_k f_k v_k, x_{k+1,g_{i_{k+1}}}, \dots, x_{r,g_{i_r}})$$

where $u_1, v_1, \dots, u_k, v_k$ are new homogeneous variables such that $|u_j|_G |f_j|_G |v_j|_G = g_{i_j}$, $j = 1, \dots, k$, is a proper central G-polynomial of $E(\widehat{B})$.

Proof. Notice that since f is a central G-polynomial of $E(A)$ then it follows that f' is also a central G-polynomial. Let $a_1 = b_1 \otimes w_1, \dots, a_r = b_r \otimes w_r$, $w_i \in E$ be the homogeneous elements in the $G \times \mathbb{Z}_2$-grading such that

$$f(b_1 \otimes w_1, \dots, b_r \otimes w_r) \ne 0,$$

$b_i \in B_i, i = 1, \dots, k$. Now, for $t = 1, \dots, r$, if $b_t \in \widehat{B}_{(g_{i_t},0)}$ (resp. $b_t \in \widehat{B}_{(g_{i_t},1)}$), we set $x_{t,g_{i_t}} = y_{g_{i_t}}$ (resp. $x_{t,g_{i_t}} = z_{g_{i_t}}$), a variable of homogeneous degree $(g_{i_t}, 0)$ (resp. $(g_{i_t}, 1)$). In this way we can regard f as a $G \times \mathbb{Z}_2$-polynomial and, by the property of $\tilde{\ }$, we can write

$$f(b_1 \otimes w_1, \dots, b_r \otimes w_r) = \widetilde{f}(b_1, \dots, b_r) \otimes w_1 \cdots w_r \ne 0. \tag{4}$$

Let $f_i = f_i(x_1^i, \dots, x_{n_i}^i)$, $i = 1, \dots, k$. Since f_i is not a G-graded identity of $E(B_i)$, there exist homogeneous elements in the $G \times \mathbb{Z}_2$-grading $\bar{x}_1^i, \dots, \bar{x}_{n_i}^i \in B_i$, $e_1^i, \dots, e_{n_i}^i \in E$ such that $f_i(\bar{x}_1^i \otimes e_1^i, \dots, \bar{x}_{n_i}^i \otimes e_{n_i}^i) \ne 0$. As above, if we see f_i as a $G \times \mathbb{Z}_2$-polynomial we can write

$$f_i(\bar{x}_1^i \otimes e_1^i, \dots, \bar{x}_{n_i}^i \otimes e_{n_i}^i) = \widetilde{f}_i(\bar{x}_1^i, \dots, \bar{x}_{n_i}^i) \otimes e_1^i \cdots e_{n_i}^i \ne 0.$$

Let $\widetilde{f}_i(\bar{x}_1^i, \dots, \bar{x}_{n_i}^i) = b_i' \ne 0$, for some homogeneous (in the $G \times \mathbb{Z}_2$-grading) element $b_i' \in B_i$. By Lemma 3, for any $i = 1, \dots, k$, one can choose homogeneous elements $a_i, c_i \in B_i$ such that $a_i b_i' c_i = b_i$. Therefore the polynomial $u_i \widetilde{f}_i v_i$ takes the value b_i by evaluating $u_i, x_1^i, \dots, x_{n_i}^i, v_i$ in $a_i, \bar{x}_1^i, \dots, \bar{x}_{n_i}^i, c_i$, respectively. Let h_i, h_i' be elements of E of the same homogeneous degree (in the $\mathbb{Z}_2$-grading) as a_i, c_i, respectively. Then for $i = 1, \dots, k$, we get

$$\begin{aligned}&(a_i \otimes h_i) f_i(\bar{x}_1^i \otimes e_1^i, \dots, \bar{x}_{n_i}^i \otimes e_{n_i}^i)(c_i \otimes h_i') \\ &= a_i \widetilde{f}_i(\bar{x}_1^i, \dots, \bar{x}_{n_i}) c_i \otimes h_i e_1^i \cdots e_{n_i}^i h_i' = b_i \otimes h_i e_1^i \cdots e_{n_i}^i h_i'.\end{aligned}$$

Now if we extend the evaluations of the $u_if_iv_i$ to an evaluation φ of f' by specializing the remaining variables as in (4) we get

$$\varphi(f') = \widetilde{f}'(a_1, \bar{x}_1^1, \ldots, \bar{x}_{n_1}^1, c_1, \ldots, a_k, \bar{x}_1^k, \ldots, \bar{x}_{n_k}^k, c_k, b_{k+1}, \ldots, b_r) \otimes h_1e_1^1 \cdots e_{n_1}^1 h_1' \cdots h_k e_1^k \cdots e_{n_k}^k h_k' w_{k+1} \cdots w_r$$
$$= \widetilde{f}(b_1, \ldots, b_r) \otimes h_1e_1^1 \cdots e_{n_1}^1 h_1' \cdots h_k e_1^k \cdots e_{n_k}^k h_k' w_{k+1} \cdots w_r.$$

Since E is the infinite dimensional Grassmann algebra, we can choose homogeneous elements h_i, h_i', e_i^j, w_i in E such that

$$h_1e_1^1 \cdots e_{n_1}^1 h_1' \cdots h_k e_1^k \cdots e_{n_k}^k h_k' w_{k+1} \cdots w_r \neq 0.$$

Hence, the polynomial f' takes a non-zero value and the proof of the lemma is complete. $\square$

Now we describe a technique of gluing Young tableaux that one can find in [18, Chapter 6].

For every $1 \leq i \leq k$, let $\lambda^i \vdash n_i$ be a partition with the property

$$h(d_i, l_i, t_i - s_i) \leq \lambda^i \leq h(d_i, l_i, t_i), \tag{5}$$

and

$$t_i - s_i \geq \max\{t_{i+1} + d_{i+1}, t_{i+1} + l_{i+1}\} \tag{6}$$

where d_i, l_i, t_i, s_i, $1 \leq i \leq k$, are fixed integers.

As we know, we can associate a partition to the corresponding Young diagram. Then, by the above relations, we can glue the first row of λ^{i+1} to the (d_i+1)th row of λ^i, the second row of λ^{i+1} to the (d_i+2)th row of λ^i and so on. As a result we obtain a new partition $\lambda^i \star \lambda^{i+1}$ of the integer $n_i + n_{i+1}$. Along this lines by gluing together $\lambda^1, \ldots, \lambda^k$, we get the partition $\mu = \lambda^1 \star \cdots \star \lambda^k$ of $n = \sum_{i=1}^k n_i$ with the property

$$h(d, l, t_k - s_k) \leq \mu \leq h(d, l, t)$$

where $d = \sum_{i=1}^k d_i$, $l = \sum_{i=1}^k l_i$ and $t \geq \max\{t_1 + d_1 - d, t_1 + l_1 - l\}$.

In a similar way we can glue also Young tableau. Let $T_{\lambda^1}, \ldots, T_{\lambda^k}$ be young tableaux corresponding to the partition $\lambda^1, \ldots, \lambda^k$, respectively. Now we define new tableaux as follows. Set $T'_{\lambda^1} = T_{\lambda^1}$ and, for $2 \leq i \leq k$, let T'_{λ^i} be the Young tableau obtained from T_{λ^i} by adding the integer $\sum_{j=1}^{i-1} n_j$ to all entries of T_{λ^i}. Then we denote by

$$T_\mu = T'_{\lambda^1} \star \cdots \star T'_{\lambda^k}$$

the Young tableau obtained gluing together the tableaux $T'_{\lambda^1}, \ldots, T'_{\lambda^k}$ according to the previous procedure. It is clear that the tableau so obtained is filled up with the distinct integers $1, 2, \ldots, n$, where $n = \sum_{i=1}^k n_i$. Moreover, by [16, Lemma 14],

$$e_{T_\mu} = e_{T'_{\lambda^1}} \cdots e_{T'_{\lambda^k}} + b \tag{7}$$

where $b \in \operatorname{span}_F\{\sigma \in S_n \mid \sigma(N_i) \nsubseteq N_i \text{ for some } 1 \leq i \leq k\}$ and N_i denotes the set of integers in the tableau T'_{λ^i}.

Next we apply the gluing technique to multitableaux. For all $1 \leq j \leq k$ consider the multipartition $\langle \lambda^j \rangle = (\lambda^j(1), \ldots, \lambda^j(s)) \vdash (n^j(1), \ldots, n^j(s))$, and suppose that for all $1 \leq i \leq s$, the partitions $\lambda^1(i), \ldots, \lambda^k(i)$ satisfy conditions (5) and (6), i.e., they are glueable. Then, by the previous arguments, we obtain a new multipartition $\langle \mu \rangle = (\mu(1), \ldots, \mu(s))$, where $\mu(i) = \lambda^1(i) \star \cdots \star \lambda^k(i)$, $1 \leq i \leq s$.

Set $n_0 = 0$, $n^0(i) = 0$ and $n_i = \sum_{j=1}^k n^j(i)$, $1 \leq i \leq s$. Let $T_{\langle \lambda^j \rangle} = (T_{\lambda^j(1)}, \ldots, T_{\lambda^j(s)})$, $1 \leq j \leq k$, be the multitableau corresponding to the multipartition $\langle \lambda^j \rangle$. If we add the integer $n_0 + n_1 + \cdots + n_{i-1} + n^0(i) + \cdots + n^{j-1}(i)$ to all entries of $T_{\lambda^j(i)}$, $1 \leq i \leq s$, $1 \leq j \leq k$, we get a new multitableau which we denote by $T'_{\langle \lambda^j \rangle} = (T'_{\lambda^j(1)}, \ldots, T'_{\lambda^j(s)})$. Then we define

$$T_{\langle \mu \rangle} = T'_{\langle \lambda^1 \rangle} \star \cdots \star T'_{\langle \lambda^k \rangle} = \left(T'_{\lambda^1(1)} \star \cdots \star T'_{\lambda^k(1)}, \ldots, T'_{\lambda^1(s)} \star \cdots \star T'_{\lambda^k(s)} \right).$$

We denote by $N^j(i)$ the set of integers filled in $T'_{\lambda^j(i)}$, $1 \leq i \leq s$, $1 \leq j \leq k$. Moreover, let $N(i) = N^1(i) \cup \cdots \cup N^k(i)$, $1 \leq i \leq s$. As a consequence of (7), we get the following.

Lemma 5. *Let $\langle \lambda^j \rangle = (\lambda^j(1), \ldots, \lambda^j(s))$, $1 \le j \le k$, be multipartitions and suppose that for all $1 \le i \le s$, the partitions $\lambda^1(i), \ldots, \lambda^k(i)$ satisfy conditions* (5) *and* (6)*. Also, let $T_{\langle \lambda^j \rangle} = (T_{\lambda^j(1)}, \ldots, T_{\lambda^j(s)})$ be multitableau corresponding to $\langle \lambda^j \rangle$. If $T_{\langle \mu \rangle} = T'_{\langle \lambda^1 \rangle} \star \cdots \star T'_{\langle \lambda^k \rangle}$, then*

$$e_{T_{\langle \mu \rangle}} = e_{T'_{\lambda^1(1)}} \cdots e_{T'_{\lambda^k(1)}} \cdots e_{T'_{\lambda^1(s)}} \cdots e_{T'_{\lambda^k(s)}} + \gamma,$$

where $\gamma \in span_F\{\sigma \in S_{n_1} \times \cdots \times S_{n_s} \mid \sigma(N^j(i)) \nsubseteq N^j(i),\ for\ some\ 1 \le i \le s,\ 1 \le j \le k\}$.

In the next lemma we apply the gluing technique in order to construct a proper central G-polynomial that will be very useful. To this end, for all $1 \le i \le k$, let $d_i = \dim_F B_{(g_i,0)}$ and $l_i = \dim_F B_{(g_i,1)}$.

Lemma 6. *For any positive integer $t \ge 2k \dim_F \widehat{B}$ there exist an integer n, a multipartition $\langle \mu \rangle = (\mu(1), \ldots, \mu(s))$ of n with*

$$h(d_i, l_i, 2t - 4\dim \widehat{B}) \le \mu(i) \le h(d_i, l_i, 2t),$$

$1 \le i \le s$, and a multitableau $T_{\langle \mu \rangle}$ such that $e_{T_{\langle \mu \rangle}} f'$ is a proper central G-polynomial of $E(\widehat{B})$ for some multilinear G-polynomial f' with $\deg f' = n + k + r$.

Proof. Let $p_i^j = \dim_F (B_j)_{(g_i,0)}$ and $q_i^j = \dim_F (B_j)_{(g_i,1)}$, for all $1 \le i \le s$, $1 \le j \le k$. Then $d_i = \sum_{j=1}^k p_i^j$ and $l_i = \sum_{j=1}^k q_i^j$, $1 \le i \le s$. Moreover, set $u_j = \dim_F B_j$, for all $1 \le j \le k$.

By Lemma 2, for any integer $t_j \ge 1$, there exists a multipartition $\langle \lambda^j \rangle = (\lambda^j(1), \ldots, \lambda^j(s)) \vdash (n^j(1), \ldots, n^j(s))$, a multitableau $T_{\langle \lambda^j \rangle} = (T_{\lambda^j(1)}, \ldots T_{\lambda^j(s)})$ and a multilinear G-polynomial h_j such that

$$h(p_i^j, q_i^j, 2t_j - u_j) \le \lambda^j(i) \le h(p_i^j, q_i^j, 2t_j),$$

for all $1 \le i \le s$, and $e_{T_{\langle \lambda^j \rangle}} h_j \notin \mathrm{Id}^G(E(B_j))$.

Let $t = t_1 \ge 2k \dim_F \widehat{B}$ be an arbitrary integer and for $2 \le l \le k$, define

$$r_l = u_{l-1} + \max\{p_1^l, \ldots, p_s^l, q_1^l, \ldots, q_s^l\}.$$

Also set $r'_l = r_l$ if r_l is even and $r'_l = r_l + 1$ if r_l is odd. Moreover, for all $1 \le l \le k-1$, we define

$$2t_{l+1} = 2t_l - r'_{l+1}.$$

Hence for all $1 \le l \le k$, it follows that

$$2t_l - u_l = 2t_{l+1} + r'_{l+1} - u_l \ge 2t_{l+1} + r_{l+1} - u_l = 2t_{l+1} + \max\{p_1^{l+1}, \ldots, p_s^{l+1}, q_1^{l+1}, \ldots, q_s^{l+1}\}.$$

Thus, conditions (5) and (6) hold and, for each $1 \le i \le s$, we can glue the partitions $\lambda^1(i), \ldots, \lambda^k(i)$. In this way we get a multipartition $\langle \mu \rangle = (\mu(1), \ldots, \mu(s))$, such that, for $1 \le i \le s$,

$$h(d_i, l_i, 2t_k - u_k) \le \mu(i) \le h(d_i, l_i, v_i)$$

for every $v_i \ge \max\{2t_1 + p_i^1 - d_i, 2t_1 + q_i^1 - l_i\}$.

Now we compute

$$\begin{aligned} 2t_1 - 2t_k &= \sum_{j=1}^{k-1} (2t_j - 2t_{j+1}) = \sum_{j=1}^{k-1} r'_{j+1} \le k + \sum_{j=1}^{k-1} r_{j+1} \\ &= k + \sum_{j=1}^{k-1} (u_j + \max\{p_1^{j+1}, \ldots, p_s^{j+1}, q_1^{j+1}, \ldots, q_s^{j+1}\}) \le k + 2\dim_F \widehat{B} \le 3\dim_F \widehat{B}. \end{aligned}$$

Thus,

$$2t_k - u_k \ge 2t_1 - 3\dim_F \widehat{B} - u_k \ge 2t_1 - 4\dim_F \widehat{B}.$$

Hence, recalling that $t = t_1$, we get for all $1 \le i \le s$,

$$h(d_i, l_i, 2t - 4\dim_F \widehat{B}) \le \mu(i) \le h(d_i, l_i, 2t).$$

Moreover, by the gluing technique shown above, we also obtain the multitableau $T_{\langle \mu \rangle} = (T'_{\lambda^1(1)} \star \cdots \star T'_{\lambda^k(1)}, \ldots, T'_{\lambda^1(s)} \star \cdots \star T'_{\lambda^k(s)})$.

For all $1 \leq j \leq k$, denote by f_j the polynomial $e_{T_{\langle\lambda^j\rangle}} h_j$ written in the set of variables $\{x_{i,g_l} \mid i \in N^j(r),\ 1 \leq r \leq s\}$, where $N^j(r)$ is the set of integers filled in $T'_{\langle\lambda^j\rangle}$. Hence the polynomial constructed in Lemma 4

$$f' = f(w_1 f_1 w'_1, \ldots, w_k f_k w'_k, x_{n+1,g_{i_{k+1}}}, \ldots, x_{n+r,g_{i_r}}),$$

where $w_1, \ldots, w_k, w'_1, \ldots, w'_k$ are new homogeneous variables distinct from $x_{n+1,g_{i_{k+1}}}, \ldots, x_{n+r,g_{i_r}}$ and those ones of $f_1, \ldots, f_k$, is a proper central G-polynomial of $E(\widehat{B})$ and $\deg f' = n+k+r$, where $n = n_1 + \cdots + n_s$. We claim that also $\bar{f} = e_{T_{\langle\mu\rangle}} f'$ is a proper central G-polynomial of $E(\widehat{B})$.

Notice that by construction $\bar{f}$ is a central polynomial, thus we have only to prove that $\bar{f} \notin \mathrm{Id}^G(E(\widehat{B}))$. To this end, let ψ be a non-zero evaluation of f'. Then by Lemma 5

$$\psi(\bar{f}) = \psi(e_{T'_{\lambda^1(1)}} \cdots e_{T'_{\lambda^k(1)}} \cdots e_{T'_{\lambda^1(s)}} \cdots e_{T'_{\lambda^k(s)}} f') + \psi(\gamma f')$$

where $\gamma \in \mathrm{span}_F\{\sigma \in S_{n_1} \times \cdots \times S_{n_s} \mid \sigma(N^j(i)) \nsubseteq N^j(i),$ for some $1 \leq i \leq s,\ 1 \leq j \leq k\}$. It readily follows that since $E(B_i)E(B_j) = 0$ if $i \neq j$, then $\psi(\gamma f') = 0$. Moreover, since $e^2_{T'_{\lambda^j(i)}} = \alpha^j_i e_{T'_{\lambda^j(i)}}$, for some non-zero integer α^j_i, and $f_j = e_{T'_{\lambda^j(1)}} \cdots e_{T'_{\lambda^j(s)}} h_j$, we also get that $e_{T'_{\lambda^j(1)}} \cdots e_{T'_{\lambda^j(s)}} f_j = \alpha^j_1 \cdots \alpha^j_s f_j$. Hence

$$\psi(\bar{f}) = \psi(e_{T'_{\lambda^1(1)}} \cdots e_{T'_{\lambda^k(1)}} \cdots e_{T'_{\lambda^1(s)}} \cdots e_{T'_{\lambda^k(s)}} f') = \alpha^j_1 \cdots \alpha^j_s \alpha^j_1 \cdots \alpha^j_s \psi(f') \neq 0,$$

as claimed. □

As a consequence of the previous result we get the following lemma which gives us the required lower bound for the proper central G-codimension sequence of $E(\widehat{B})$.

Lemma 7. *There exist constants $C > 0$ and a such that for n large enough,*

$$\delta^G_n(E(\widehat{B})) \geq C n^a d^n,$$

where $d = \dim_F B$.

Proof. Let N be any integer such that $N > (4+s)km^2 + 4m + r$, where $m = \dim_F \widehat{B}$ and $r = \deg f$, and let us divide $N - k\sum_{i=1}^s d_i l_i - 2m - r$ by $2d$. Then we have that $N = 2td + k\sum_{i=1}^s d_i l_i + 2m + r + v$ with $t \geq 2km$ and $0 \leq v < 2d$.

By Lemma 6, there exist an integer $n = \sum_{i=1}^s n_i$, a multipartition $\langle\mu\rangle = (\mu(1), \ldots, \mu(s)) \vdash (n_1, \ldots, n_s)$ with the property

$$h(d_i, l_i, 2t - 4m) \leq \mu(i) \leq h(d_i, l_i, 2t),$$

$1 \leq i \leq s$, a mutitableau $T_{\langle\mu\rangle}$ and a multilinear proper central G-polynomial f' such that $e_{T_{\langle\mu\rangle}} f'$ is a proper central G-polynomial for $E(\widehat{B})$ and $n \leq c = \deg f' \leq n + 2m + r$ and $N > n$.

Let now $\bar{f}$ be the polynomial obtained by $e_{T_{\langle\mu\rangle}} f'$ multiplying the variable u_1 by $x_{c+1,1_G} \cdots x_{N,1_G}$, where $x_{c+1,1_G}, \ldots, x_{N,1_G}$ are new G-graded variables of homogeneous degree 1_G. Since $E(B_1)$ is a unitary G-graded algebra, it follows that $\bar{f}$ is a proper central G-polynomial of $E(\widehat{B})$. Now, by the branching theorem (see [18, Theorem 2.4.3]) there exist a multipartition $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(s)) \vdash (N_1, \ldots, N_s)$ of N with $N_i \geq n_i$, for all $1 \leq i \leq s$, i.e., $\lambda(i) \geq \mu(i)$, $1 \leq i \leq s$, and a multitableau $T_{\langle\lambda\rangle}$ such that $e_{T_{\langle\lambda\rangle}} \bar{f}$ is a proper central G-polynomial of $E(\widehat{B})$.

Notice that

$$\begin{aligned} N - \sum_{i=1}^s |h(d_i, l_i, 2t-4m)| &= 2dt + k\sum_{i=1}^s d_i l_i + 2m + r + v - \sum_{i=1}^s d_i l_i - 2dt + 4dm \\ &\leq (k-1)\sum_{i=1}^s d_i l_i + 2m + r + 2d + 4dm = K \end{aligned}$$

and K is a constant. Hence for all $1 \leq i \leq s$, it follows that $N_i - |h(d_i, l_i, 2t-4m)| \leq K$, since $N_i \geq n_i \geq |h(d_i, l_i, 2t-4m)|$. Thus by [18, Lemma 6.2.4],

$$d_{\lambda(i)} \geq N_i^{-2K} d_{h(d_i, l_i, 2t-4m)},$$

for all $1 \leq i \leq s$, where $d_{\lambda(i)}$ is the degree of the irreducible S_{N_i}-character associated to $\lambda(i)$, $1 \leq i \leq s$. By taking into account [18, Lemma 6.2.5], it follows that

$$\delta^G_{N_1,\dots,N_s}(E(\widehat{B})) \geq \prod_{i=1}^{s} d_{\lambda(i)} \geq \prod_{i=1}^{s} N_i^{-2K} d_{h(d_i,l_i,2t-4m)} \geq \alpha \prod_{i=1}^{s} N_i^{K_i}(d_i+l_i)^{d_il_i+(2t-4m)(d_i+l_i)}$$
$$\geq \beta N^u \prod_{i=1}^{s}(d_i+l_i)^{(2t-4m)(d_i+l_i)},$$

for some constant $\alpha, \beta, K_1, \dots, K_s, u$.

Let $r_i = d_i + l_i$, $1 \leq i \leq s$. Then, since $N_i \geq (2t-4m)r_i$, we get

$$\frac{N!}{N_1! \dots N_s!} \geq \frac{\left(\sum_{i=1}^{s}(2t-4m)r_i\right)!}{\prod_{i=1}^{s}((2t-4m)r_i)!}.$$

Thus

$$\delta^G_N(E(\widehat{B})) \geq \binom{N}{N_1,\dots,N_s}\delta^G_{N_1,\dots,N_s}(E(\widehat{B})) \geq N^u \frac{\left(\sum_{i=1}^{s}(2t-4m)r_i\right)!}{\prod_{i=1}^{s}((2t-4m)r_i)!}\prod_{i=1}^{s} r_i^{(2t-4m)r_i}.$$

Recalling Stirling formula $n! \simeq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, we get

$$\delta^G_N(E(\widehat{B})) \geq C' N^a \frac{\left(\sum_{i=1}^{s}(2t-4m)r_i\right)^{\sum_{i=1}^{s}(2t-4m)r_i}}{\prod_{i=1}^{s}((2t-4m)r_i)^{(2t-4m)r_i}}\prod_{i=1}^{s} r_i^{(2t-4m)r_i} = C' N^a \left(\sum_{i=1}^{s} r_i\right)^{(2t-4m)\sum_{i=1}^{s} r_i} = CN^a d^N$$

for some constants C', C and a. The proof is now complete. □

Theorem 3. *Let $E(A)$ be the Grassmann envelope of a finite dimensional $G \times \mathbb{Z}_2$-graded algebra A over a field F of characteristic zero. If there exists a centrally admissible $G \times \mathbb{Z}_2$-subalgebra for $E(A)$, then for n large enough, there exist constants $C_1, > 0$ C_2, a_1, a_2 such that*

$$C_1 n^{a_1} d^n \leq \delta^G_n(E(A)) \leq C_2 n^{a_2} d^n,$$

where d is the maximal dimension of a centrally admissible $G \times \mathbb{Z}_2$-subalgebra for $E(A)$. Thus the proper central G-exponent $\exp^{G,\delta}(E(A))$ exists and is a non-negative integer.

Proof. Since the codimensions do not change by extending the ground field, we may assume that F is algebraically closed. The proof follows from Lemmas 1 and 7, by noticing that $\delta^G_n(E(A)) \geq \delta^G_n(E(\widehat{B}))$, where $\widehat{B} = B + J$ and B is a centrally admissible $G \times \mathbb{Z}_2$-subalgebra for $E(A)$ of maximal dimension. □

In case $E(A)$ has proper central polynomials but A has no centrally admissible $G \times \mathbb{Z}_2$-subalgebras, then the following proposition holds.

Proposition 2. *If for $E(A)$ there are no centrally admissible subalgebras, then for n large enough, $\delta^G_n(E(A)) = 0$.*

Proof. If A is a nilpotent algebra, we have nothing to prove. So let us assume that $A = \bar{A} + J$, where $\bar{A}$ is a maximal semisimple $G \times \mathbb{Z}_2$-graded subalgebra of A, and A is not nilpotent. If $E(A)$ has no proper central polynomials then the result easily follows. Otherwise, let $h(x_{1,g_{i_1}}, \dots, x_{n,g_{i_n}})$ be a multilinear proper central G-polynomial of $E(A)$. Then there exist G-graded homogeneous elements $a_i \in E(A)$, $1 \leq i \leq n$, such that $h(a_1, \dots, a_n) \neq 0$. Since A has no centrally admissible $G \times \mathbb{Z}_2$-subalgebras, then $a_i \in E(J)$, for all $1 \leq i \leq n$, and $J^n \neq 0$. Thus it follows that $\delta^G_N(E(A)) = 0$ a soon as $J^N = 0$. □

Corollary 1. *If R is a G-graded algebra over a field of characteristic zero, then the proper central G-exponent $\exp^{G,\delta}(R)$ exists and is a non-negative integer. Moreover, $\exp^{G,\delta}(R) \leq \exp^G(R)$.*

7. The central exponent

In this section we shall study the central graded exponent $\exp^{G,z}(R)$, where R is any G-graded algebra, and we will compare it to the ordinary G-graded exponent.

To that end, we need some preliminary results about G-graded minimal varieties and their relation with central G-polynomials. Minimal varieties were completely described in [17], [11] and [10] in the setting of ordinary polynomial identities, identities with involution and identities with graded involution, respectively. Moreover, in [9] the authors studied minimal varieties for algebras graded by the cyclic group $\mathbb{Z}_p$. In what follows we present some basic results strictly related to our purpose.

We start by recalling the definition of minimal G-graded algebra.

Definition 2. *Let F be an algebraically closed field. A finite dimensional G-graded algebra A is called minimal if either A is G-simple or $A = A_1 \oplus \cdots \oplus A_m + J$, where*

1. *$A_1, \ldots, A_m$ are G-simple algebras and $m \geq 2$;*
2. *there exist homogeneous elements $w_{12}, w_{23}, \ldots, w_{m-1,m} \in J$ and minimal homogeneous idempotents $e_1 \in A_1, \ldots, e_m \in A_m$ such that*
$$e_1 w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i+1}, \quad 1 \leq i \leq m$$
and
$$w_{12} w_{23} \cdots w_{m-1,m} \neq 0;$$
3. *$w_{12}, w_{23}, \ldots w_{m-1,m}$ generate J as two-sided ideal of A.*

We highlight that in some papers, minimal algebras are called triangular algebras since their properties recall in some sense the ones of upper triangular matrix algebras.

Strictly connected to minimal algebras, there are the minimal varieties of exponential growth whose definition in the graded case is the following.

Definition 3. *Let $\mathcal{V}$ be a variety of G-graded algebras. Then $\mathcal{V}$ is said to be minimal of G-exponent d if $\exp^G(\mathcal{V}) = d$ and for any proper subvariety $\mathcal{U}$, $\exp^G(\mathcal{U}) < d$.*

The next theorem relates minimal G-varieties and Grassmann envelope of minimal algebras. To this end, notice that Definition 2 has to be properly translated in case of algebras graded by the group $G \times \mathbb{Z}_2$.

Theorem 4. *Let $\mathcal{V}$ be a minimal G-variety of G-exponent $d \geq 2$. Then there exists a minimal $G \times \mathbb{Z}_2$-graded algebra A such that $\mathcal{V} = var^G(E(A))$.*

Proof. By following the lines of [18, Lemma 8.1.5] with the necessary adaptations to the graded case, one can prove that there exists a minimal $G \times \mathbb{Z}_2$-graded algebra A such that $E(A) \in \mathcal{V}$ and $\exp^G(\mathcal{V}) = \exp^G(E(A)) = d$ (see also [25, Lemma 9]). Thus, by the minimality of $\mathcal{V}$, it readily follows that $\mathcal{V} = \mathrm{var}^G(E(A))$ as claimed. $\square$

Grassmann envelopes of minimal $G \times \mathbb{Z}_2$-graded algebras, which are not simple, have no proper central G-polynomial as proved in the following lemma.

Lemma 8. *Let $A = A_1 \oplus \cdots \oplus A_m + J$ be a minimal $G \times \mathbb{Z}_2$-graded algebra. If $m \geq 2$, then $E(A)$ has no proper central G-polynomials.*

Proof. Since A is minimal, we notice that $Z(A) \cap J = 0$. Moreover, if we set $Z(A)_0 = \bigoplus_{g \in G} Z(A)_{(g,0)}$, then one can easily check that $Z(E(A)) = Z(A)_0 \otimes E_0$, thus also
$$Z(E(A)) \cap E(J) = 0.$$
Let $f = f(x_{1,g_{i_1}}, \ldots, x_{n,g_{i_n}})$ be a multilinear central G-polynomial of $E(A)$. In order to reach our goal, we have to prove that $f \in \mathrm{Id}^G(E(A))$. To this end, let assume by contradiction that f is not a G-identity of $E(A)$.

Since $E(A_i)E(A_j) = 0$, for all $i \neq j$, and $Z(E(A)) \cap E(J) = 0$, then in particular we can assume that f is a proper central G-polynomial of $E(A_i)$ for some i. Moreover, notice that since $G \times \mathbb{Z}_2$ is abelian and f is multilinear, any evaluation of f on elements of $E(A)$ is $G\times\mathbb{Z}_2$-homogeneous. Hence there exist $G\times\mathbb{Z}_2$-homogeneous elements $a_1,\dots,a_n\in E(A_i)$ such that

$$f(a_1,\dots,a_n)=a\otimes h\in Z(A_i)_{(g,0)}\otimes E_0\subseteq Z(E(A)),$$

for some $g\in G$, $h\in E_0$, $a\neq 0$.

According to Theorem 2, if $A_i\cong F^\alpha H\otimes M_k(F)$, for some $k\geq 1$, $H\leq G\times\mathbb{Z}_2$ and α 2-cocycle, then we write $a=\beta g_0\otimes I_k$, where $\beta\in F^*$, $g_0=(g,0)\in H$ and I_k is the $k\times k$ identity matrix. Let suppose that l is the order of g_0, then

$$a^l=\beta^l\prod_{i=1}^{l-1}\alpha(g_0^i,g_0)\Big(1_{G\times\mathbb{Z}_2}\otimes I_k\Big)$$

is a non-zero central element of A_i. Thus in particular, $\Big(1_{G\times\mathbb{Z}_2}\otimes I_k\Big)\otimes h\in Z(E(A))$.

Denoted by $1_{A_i}=1_{G\times\mathbb{Z}_2}\otimes I_k$ the unit element of A_i, we have that if $1\leq i\leq m-1$

$$(1_{A_i}\otimes h)(w_{i,i+1}\otimes h')=w_{i,i+1}\otimes hh'\neq 0$$

for some $h'\in E$, whereas

$$(w_{i,i+1}\otimes h')(1_{A_i}\otimes h)=0.$$

Similarly,

$$(1_{A_m}\otimes h)(w_{m-1,m}\otimes h')=0$$

and

$$(w_{m-1,m}\otimes h')(1_{A_m}\otimes h)=w_{m-1,m}\otimes h'h\neq 0.$$

Hence we get a contradiction since we are assuming that $1_{A_i}\otimes h\in Z(E(A))$.

Thus $f\in \mathrm{Id}^G(E(A_i))$ for all $1\leq i\leq m$ and so, according to what remarked above, $f\in\mathrm{Id}^G(E(A))$. □

We are now in a position to prove the existence of the central G-exponent of a G-graded algebra provided that its graded exponent is greater or equal than 2.

Theorem 5. *Let R be a G-graded algebra over a field of characteristic zero such that* $\exp^G(R)\geq 2$. *Then either $c_n^{G,z}(R)=0$ for all $n\geq 0$, or*

$$C_1n^{t_1}\exp^G(R)^n\leq c_n^{G,z}(R)\leq C_2n^{t_2}\exp^G(R)^n$$

for some constants $C_1>0$, C_2, t_1, t_2.

Proof. If R is commutative, then it is clear that $c_n^{G,z}(R)=0$ for all $n\geq 0$, so let us suppose that R is not commutative.

As we already did before, we may assume that F is an algebraically closed field and $R=E(A)$, where A is a finite dimensional $G\times\mathbb{Z}_2$-graded algebra.

By [14, Theorem 2], there exist constants $C_1>0$, C_2, t_1, t_2 such that

$$C_1n^{t_1}d^n\leq c_n^G(E(A))\leq C_2n^{t_2}d^n, \tag{8}$$

where $d=\exp^G(E(A))$. Hence by (1), we get that also $c_n^{G,z}(E(A))\leq C_2n^{t_2}d^n$. Thus we have to prove now the lower bound.

Let $\mathcal{V}=\mathrm{var}^G(E(A))$. Since any T_G-ideal is finitely generated, it readily follows that $\mathcal{V}$ contains a subvariety $\mathcal{U}$ which is minimal of G-exponent $d=\exp^G(\mathcal{U})=\exp^G(\mathcal{V})$. Hence by Theorem 4, there exists a minimal $G\times\mathbb{Z}_2$-graded algebra B such that $\mathcal{U}=\mathrm{var}^G(E(B))$.

Suppose first that B is not $G\times\mathbb{Z}_2$-simple. Then by Lemma 8, $E(B)$ has no proper central G-polynomials and therefore

$$\mathrm{Id}^{G,z}(E(A))\subseteq\mathrm{Id}^{G,z}(E(B))=\mathrm{Id}^G(E(B)).$$

Thus $c_n^{G,z}(E(A))\geq c_n^G(E(B))$ and, since $c_n^G(E(B))$ has a lower bound as in (8), we are done in this case.

Now suppose that B is $G\times\mathbb{Z}_2$-simple. If $E(B)$ is commutative, then $\mathrm{Id}^{G,z}(E(B))=\mathrm{Id}^G(E(B))$ and, as in the previous case, we get the desired result. Thus let us suppose that $E(B)$ is not commutative. Let $N=c_n^G(E(B))$ and let $f_1,\dots,f_N\in P_n^G$ be multilinear G-polynomials linearly independent modulo $\mathrm{Id}^G(E(B))$. We shall prove that $f_1x_{n+1},\dots,f_Nx_{n+1}$, where x_{n+1} is an extra variable of any homogeneous degree, are linearly independent modulo $\mathrm{Id}^{G,z}(E(B))$.

Suppose by contradiction that there exist not all zero scalars $\alpha_1, \dots, \alpha_N$ such that $\alpha_1 f_1 x_{n+1} + \cdots + \alpha_N f_N x_{n+1} \in \mathrm{Id}^{G,z}(E(B))$ and set $f = \alpha_1 f_1 + \cdots + \alpha_N f_N$. It is clear that $f \notin \mathrm{Id}^G(E(B))$.

If f is a central G-polynomial of $E(B)$, then it must be a proper central polynomial and there exists a non-zero evaluation φ such that $\varphi(f) \in Z(E(B)) = Z(B)_0 \otimes E_0$. Recall that $Z(B)_0 = \bigoplus_{g\in G} Z(B)_{(g,0)}$. In particular, if $B \cong F^\alpha H \otimes M_k(F)$ for some $k \geq 1$, $H \leq G \times \mathbb{Z}_2$ and α 2-cocycle, then we can assume $\varphi(f) = (a \otimes I_k) \otimes l_0$, where $a \in Z(F^\alpha H)$, I_k is the $k \times k$ identity matrix and $l_0 \in E_0$.

If $k > 1$, then we set $b = 1_{G\times\mathbb{Z}_2} \otimes e_{12}$ and we can extend φ to an evaluation φ' of fx_{n+1} such that $\varphi'(fx_{n+1}) = (a \otimes e_{12}) \otimes l_0 l \neq 0$, for some suitable $l \in E$. Since such an element is not central, we get a contradiction.

Suppose now $k = 1$. Since $E(B)$ is not commutative, then either B is not commutative or there exists an element $(g, 1) \in G \times \mathbb{Z}_2$ such that $B_{(g,1)} \neq 0$.

Suppose first that B is not commutative, write $a = \sum_{h\in H} \beta_h h$ and let $h_0 \in H$ be such that $\beta_{h_0} \neq 0$. Then there exists $h_1 \notin Z(F^\alpha H)$ and we set $b = h_0^{-1} h_1$. Thus

$$ab = \sum_{h\in H} \beta_h \alpha(h, h_0^{-1}h_1) h h_0^{-1} h_1 = \beta_{h_0} \alpha(h_0, h_0^{-1}h_1) h_1 + \sum_{h\neq h_0} \beta_h \alpha(h, h_0^{-1}h_1) h h_0^{-1} h_1,$$

where $\beta_{h_0}\alpha(h_0, h_0^{-1}h_1) \neq 0$. We get that ab is a non-zero non central element of $F^\alpha H$, since if $ab \in Z(F^\alpha H)$, then each summand should be a central element, thus in particular $h_1 \in Z(F^\alpha H)$, a contradiction. We now evaluate x_{n+1} in $b \otimes l$, for some suitable $l \in E$, so that φ can be extended to an evaluation φ' of fx_{n+1} such that

$$\varphi'(fx_{n+1}) = ab \otimes l_0 l \neq 0$$

that is a non central element of $E(B)$.

Now suppose that there exists $h' = (g, 1) \in G \times \mathbb{Z}_2$ such that $B_{(g,1)} \neq 0$. Then

$$ah' = \beta_{h_0}\alpha(h_0, h')h_0 h' + \sum_{h\neq h_0} \beta_h \alpha(h, h') h h' \neq 0.$$

Therefore, by evaluating x_{n+1} in $h' \otimes l_1$ where $l_1 \in E_1$, we get

$$\varphi'(fx_{n+1}) = ah' \otimes l_0 l_1 \neq 0$$

that is a non central element of $E(B)$ since $l_0 l_1 \in E_1$.

Finally, suppose that f is not a central G-polynomial of $E(B)$, then fx_{n+1} has a non central evaluation by specializing x_{n+1} with $1_B \otimes l$, for a suitable $l \in E_0$.

Thus in both cases we have a contradiction and $\alpha_1 = \cdots = \alpha_N = 0$.

We have proved that $c_n^G(E(B)) \leq c_{n+1}^{G,z}(E(B))$ and we are done. □

We now study G-graded algebras with G-exponent less or equal than 1. If R is such an algebra, then by [24, Lemma 2.1], we may assume that R is finite dimensional.

Lemma 9. *Let R be a finite dimensional G-graded algebra over an algebraically closed field such that $\exp^G(R) = 1$. If $c_n^{G,z}(R) = 0$ for some $n \geq 2$, then $R = C \oplus N$, where C is a commutative G-graded algebra and N is a nilpotent G-graded algebra.*

Proof. Write $R = A_1 \oplus \cdots \oplus A_m + J$, where $A_1, \dots, A_m$ are G-simple algebras and J is the Jacobson radical. Since $\exp^G(R) = 1$, then $A_i \cong F$ for all $1 \leq i \leq m$.

Denote by e the unit element of A_1 and set $J_0 = \{j \in J \mid ej = je = 0\}$ and $J_1 = \{j \in J \mid ej = je = j\}$. It is clear that J_0 and J_1 are G-graded ideals of R.

Moreover, remark that $J = J_0 \oplus J_1$. In fact, for all $j \in J$, we can write $j = j - ej + ej$ and since $c_n^{G,z}(R) = 0$, for some $n \geq 2$, in particular $x_1 \cdots x_n$ is a central G-polynomial of R, where $x_1, \dots, x_n$ are homogeneous variables of any degree. Therefore, $j - ej \in J_0$ and $ej \in J_1$.

Notice that $eA_i = A_i e = 0$ for all $2 \leq i \leq m$, hence $A_i J_1 = J_1 A_i = 0$ and

$$R = (A_1 + J_1) \oplus (A_2 \oplus \cdots \oplus A_m + J_0)$$

as G-graded algebras. Remark that for all $j \in J_1$, we can write $j = e^{n-1} j$, therefore j is a central element of R and $A_1 + J_1$ is a commutative G-graded algebra.

Applying the same arguments to $A_2 \oplus \cdots \oplus A_m + J_0$, we finally get that $R = C_1 \oplus \cdots \oplus C_m \oplus N$, where $C_1, \ldots, C_m$ are commutative and $N \subseteq J_0$ is nilpotent, as claimed. $\square$

Putting together the previous results we finally get the following theorem.

Theorem 6. *Let R be any G-graded algebra over a field of characteristic zero. Then its central G-exponent $\exp^{G,z}(R)$ exists. Moreover either $\exp^{G,z}(R) = \exp^{G}(R)$ or $\exp^{G,z}(R) = 0$.*

Proof. If $\exp^G(R) \geq 2$, then by Theorem 5 either $\exp^{G,z}(R) = \exp^G(R)$ or $\exp^{G,z}(R) = 0$.

Now suppose that $\exp^G(R) = 1$, then R can be assumed finite dimensional. Furthermore, since the codimension sequence does not change by extending the base field, we may assume F to be algebraically closed. Notice that in this case $c_n^{G,z}(R)$ is polynomially bounded and $\exp^{G,z}(R) \leq 1$.

If $c_n^{G,z}(R) \neq 0$ for all $n \geq 1$, then it readily follows that $\exp^{G,z}(R) = 1$. Otherwise there exists an integer n such that $c_n^{G,z}(R) = 0$, so Lemma 9 applies and $R = C \oplus N$, where C is commutative and N is nilpotent. Thus $\exp^{G,z}(R) = 0$.

Finally, if $\exp^G(R) = 0$, then R is nilpotent, $\exp^{G,z}(R) = 0$ and we are done. $\square$

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