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Vitali Theorems for Varying Measures

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Abstract: The classical Vitali theorem states that, under suitable assumptions, the limit of a sequence of integrals is equal to the integral of the limit functions. Here, we consider a Vitali-type theorem of the following form $\int f_n\, dm_n \to \int f\, dm$ for a sequence of pair $(f_n, m_n)_n$ and we study its asymptotic properties. The results are presented for scalar, vector and multivalued sequences of m_n-integrable functions f_n. The convergences obtained, in the vector and multivalued settings, are in the weak or in the strong sense for Pettis and McShane integrability. A list of known results on this topic is cited and new results are obtained when the ambient space Ω is not compact.

Keywords: setwise convergence; Weak and Vague convergence of measures; Vitali theorem; uniform integrability; equi-integrability; Pettis integral; McShane integral

MSC: 28B20; 28C15; 46G10; 49J05; 54C60

1. Introduction

The Vitali convergence theorem [1], which owes his name to the Italian mathematician Vitali, is a generalization of the Dominated convergence theorem developed by Lebesgue. It is a characterization of the convergence of a sequence $(f_n)_n$ in $L^p(m)$ in terms of uniform integrability and convergence in the m-measure of $(f_n)_n$. Our goal is to show an analog of Vitali's classic theorem for varying measures, namely when $(f_n)_n$ and $(m_n)_n$ are simultaneously convergent in some sense. In particular, we want to identify sufficient conditions for the following chart to hold, for the different types of convergence for varying measures. Obviously, we will have different versions depending on the assumptions we use on the sequence of measurable functions $(f_n)_n$ and on the varying measures $(m_n)_n$.

$$\begin{array}{ccccc} m_n & \& & f_n & \longmapsto & \displaystyle\int_E f_n\, dm_n = \nu_n(E) \\ \downarrow & & \downarrow & & \downarrow \\ m & \& & f & \longmapsto & \displaystyle\int_E f dm = \nu(E) \end{array} \qquad \text{for every } E \in \mathcal{A}. \tag{1}$$

Additionally, the convergence of $\nu_n(E)$ to $\nu(E)$ may be considered in a strong or weak sense, when the functions involved are vector-(multi)valued. Fundamental tools for obtaining results of this type are the "absolute" or "uniform" continuity of integrals. There is a wide literature on the convergence of measures, since it has applications, for example, in probability and statistics, stochastic processes, control and game theories, symmetric diffusion processes, transportation problems, signal and image processing, neural networks, symmetric operators, continuous dependence for measuring differential inclusions and measuring differential equations [2–14].

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In Section 2, some results, related to (1) for the convergence theorems of the type of Fatou, Monotone, Lebesgue or Vitali are highlighted, emphasizing both the hypotheses on the space Ω, on the sequence of functions $(f_n)_n$ and on the sequence of measures $(m_n)_n$. There is a large literature on this subject, the first paper on which was published by Serfozo [15]. This problem was recently addressed by various authors in [16–20]. Some of these results are given in Section 2 and are obtained for sequences of scalar or vector functions when the varying measures are finite and countably additive or probabilities. In [21,22], some of these results have been extended to the multivalued case with weakened assumptions, while in [23] the varying measures considered are only subadditive. Above all, we will highlight the results of these last three quoted papers because they will be the starting point for the new part that is contained in Section 3. Here, (1) is considered to have a stronger definition of integrability and results are given, either existing (the compact case) or new (the noncompact case), when the approximating sequence $(f_n)_n$ is integrable in the sense of McShane.

2. The State of the Art

Let $(\Omega, \mathcal{A})$ be a measurable space with $\mathcal{A}$ be a σ-algebra, $\mathcal{F}(\Omega)$ be the space of measurable functions and let $\mathcal{M}(\Omega)$ be the vector space of finite real-valued measures on $(\Omega, \mathcal{A})$. Let $\mathcal{M}_+(\Omega)$ be the cone of non-negative elements of $\mathcal{M}(\Omega)$. Let $|m|$ be the total variation of a measure m and $m^{\pm}$ be its positive and negative parts, respectively. The symbol $m \ll \nu$ denotes the absolute continuity of m with respect to ν.

Let X be a Banach space with dual X^* and B_{X^*} be the unit ball of X^*. We denote with $cwk(X)$ the family of all nonempty, convex, weakly compact subsets of X. For every $C, D \in cwk(X)$, let $s(x^*, C) = \sup\{\langle x^*, x\rangle : x \in C\}$, for each $x^* \in X^*$ be the *support function of* C and $d_H(C, D) := \sup_{||x^*|| \leq 1} |s(x^*, C) - s(x^*, D)|$ is the Hausdorff metric on the hyperspace $cwk(X)$ in which we consider the Minkowski addition ($C + D := \{c + d : c \in C, d \in D\}$) and the standard multiplication by non-negative scalars, see [24] for other properties.

When working with varying measures $(m_n)_n \subset \mathcal{M}(\Omega)$ different types of convergence can be considered. Concerning the setwise and the convergence in total variation, no more conditions are imposed on the measurable space $(\Omega, \mathcal{A})$, while for the weaker convergences a topology is necessary, Ω is supposed to be a locally compact Hausdorff space and $\mathcal{A}$ is its Borel σ-algebra (in this case we use the symbol $\mathcal{B}$). In these latter cases, let $C(\Omega)$, $C_0(\Omega)$, $C_c(\Omega)$ and $C_b(\Omega)$ be the families of all continuous functions, and their subfamilies that vanish at infinity have compact support and are bounded.

Definition 1. *Let m and m_n be in $\mathcal{M}(\Omega)$, we say that a sequence $(m_n)_n$*

(1a) *Converges vaguely to m ($m_n \xrightarrow{v} m$) if $\int_\Omega h dm_n \to \int_\Omega h dm$, for all $h \in C_0(\Omega)$;*
(1b) *Converges weakly to m ($m_n \xrightarrow{w} m$) if $\int_\Omega h dm_n \to \int_\Omega h dm$, for all $h \in C_b(\Omega)$;*
(1c) *Converges setwisely to m ($m_n \xrightarrow{s} m$) if $\lim_n m_n(A) = m(A)$ for all $A \in \mathcal{A}$ or, equivalently, if $\int_\Omega h dm_n \to \int_\Omega h dm$, for all bounded $h \in \mathcal{F}(\Omega)$, since simple functions are dense in the space of bounded measurable functions;*
(1d) *Converges in total variation to m ($m_n \xrightarrow{tv} m$) if $|m - m_n|(\Omega) \to 0$. Then $(m_n)_n$ is convergent to m uniformly on $\mathcal{A}$, ([25], Section 2);*
(1e) *Is bounded if $\sup_n |m_n|(\Omega) < +\infty$;*
(1f) *Is uniformly absolutely continuous with respect to m ($m_n \ll_{ac} m$), if for every $\varepsilon > 0$ there exists $\rho > 0$ such that $\sup_n m_n(E) < \varepsilon$ when $E \in \mathcal{A}$ and $m(E) < \rho$.*

Interesting comparisons among all these definitions are given in [25–27]. In general, the setwise convergence is stronger than both the vague and the weak convergences; weak convergence is stronger than vague convergence. Moreover if $(m_n)_n$ is simultaneously vaguely convergent and uniformly absolutely continuous with respect to m, then it converges weakly to m, while if $(m_n)_n$ is vaguely convergent to $m \geq m_n$ for every $n \in \mathbb{N}$ then it converges setwisely, (see [22], Remark 2.2 and Proposition 2.3).

If $(m_n)_n$ and m are measures in $\mathcal{M}(\Omega)$ and if the sequences $(m_n^{\pm})_n$ are setwise convergent to $m^{\pm}$, respectively, then $(m_n)_n$ is setwise convergent to m and the reverse implication generally fails, see for example ([21], Remark 2.2).

Finally, according to ([28], Corollary 8.1.8 and Remark 8.1.11), if Ω is an arbitrary completely regular space and let $m, (m_n)_n \subset \mathcal{M}(\Omega)$ with m Radon and $\lim_n m_n(\Omega) = m(\Omega)$, the convergence (1a) is equivalent to the following (Portmanteau result):

(1g) for any closed set $F \subset \Omega$, $\limsup_n m_n(F) \leq m(F)$.

Regarding $\ll_{ac}$, for those who are interested we quote [29] and the references therein.

Definition 2. *Let $(m_n)_n \subset \mathcal{M}_+(\Omega)$ and $(f_n)_n : \Omega \to \mathbb{R}$ be a sequence of measurable functions. We say that $(f_n)_n$*

- *Has uniformly absolutely continuous (m_n)-integrals on Ω* (**u.a.c.**), *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $A \in \mathcal{A}$ with $m_n(A) < \delta$ then $\int_A |f_n| \, dm_n < \varepsilon$, for every $n \in \mathbb{N}$.*
- *Is uniformly (m_n)-integrable on Ω* (**u.i.**), *if $\lim_{\alpha \to +\infty} \sup_n \int_{\{|f_n| > \alpha\}} |f_n| \, dm_n = 0$.*

It is obvious that if a sequence $(f_n)_n$ of measurable functions is uniformly bounded, then it is uniformly $(m_n)_n$-integrable for an arbitrary sequence $(m_n)_n$ such that $\sup_n m_n(\Omega) < +\infty$. Moreover, for a bounded sequence of measures $(m_n)_n$, the sequence $(f_n)_n$ is uniformly (m_n)-integrable on Ω if and only if it has uniformly absolutely continuous (m_n)-integrals and $\sup_n \int_\Omega |f_n| \, dm_n < +\infty$ ([21], Proposition 2.6). Finally, in [15], the uniform absolute continuity is given in a partly different form, but Serfozo's one and (**u.a.c.**) are equivalent. Many results in the literature are related to (**u.i.**); in this note we put in evidence sufficient conditions that use the weakest condition (**u.a.c.**).

All the results of Sections 2.1 and 2.2 resulting from [21,22] are given for $m, (m_n)_n \subset \mathcal{M}_+(\Omega)$ unless otherwise specified. The quoted results from other papers are given in general from sequences of probability measures or equibounded sequences of measures.

2.1. The Nontopological Case

In this subsection sufficient conditions are given for the problem (1) when the sequence of varying measures converges setwisely in an arbitrary space. The setwise convergence is a high power tool since it permits strong results to be obtained. Let $(\Omega, \mathcal{A})$ be a measurable space, with $\mathcal{A}$ an arbitrary σ-algebra. For existing results of type (1) in the literature, we would like to cite some results of Fatou, Monotone, Dominated or Vitali types:

- In [15], we have Theorems 2.4, 2.7 and 2.8. In the first theorem, a liminf setwise-type convergence is considered to obtain a Dominated convergence theorem under suitable hypotheses on a sequence $(g_n)_n$ that dominates $(|f_n|)_n$, while the other two are necessary and sufficient conditions to obtain a Vitali-type theorem and a Lebesgue theorem under tightness and **u.i.** conditions. All the results are given for scalar functions.
- In [16], in Theorem 2.2, the authors give a Fatou- and a Lebesgue-type theorem under an inequality of the $\liminf_n m_n$ on each Borelian set; in particular for the Dominated convergence theorem the sequence (m_n) is equibounded by a measure ν. In this case, the authors weaken the setwise convergence, but need a topology on the space and the Borel σ-algebra. Both results are given scalar functions.
- In [20], Theorem 4.2, Corollaries 5.3 and 6.2 give Fatou-, Lebesgue- and Monotone-type results for the setwise convergence of a sequence of probability measures, respectively. All the results are given for scalar functions.

In [21], first the authors consider the case of scalar integrands and obtain the following result for finite, non-negative measures using the uniform absolute continuity of the integrals and the setwise converges for the measures.

Theorem 1 ([21], Theorem 2.11)**.** *Let $f, f_n \in \mathcal{F}(\Omega)$. Suppose that*

(1i) $f_n(t) \to f(t)$, *in m-measure;*

(1ii) $f, (f_n)_n$ *satisfy* **(u.a.c.)***;*

(1iii) $m_n \xrightarrow{s} m$.

Then $f \in L^1(m)$ and for all $E \in \mathcal{A}$,

$$\lim_{n\to\infty} \int_E f_n dm_n = \int_E f dm.$$

An analogous result follows for signed measures when the convergence of $(f_n)_n$ is in $|m|$-measure and **(u.a.c.)** is considered with respect to $m_n^{\pm}$, ([21], Corollary 2.13).

Moreover in [20], Corollary 5.3, a similar result is obtained with different hypotheses, which implies the equiboundedness of the sequence $\left(\int_\Omega |f_n| dm_n \right)$ which is not required in Theorem 1.

A simple example of application of the previous theorem can be the following:

Example 1. Let $\Omega = (0,1]$, $\mathcal{B}$ the Borel σ-algebra, λ the Lebesgue measure and $m_n(E) = (1 + \frac{1}{n})\lambda(E)$, or $m_n(E) = \int_E (1 + \sin(nx))dx$ for every $n \in \mathbb{N}$. $(m_n)_n \subset \mathcal{M}^+(\Omega)$ and $m_n \xrightarrow{s} \lambda$. For every $n \in \mathbb{N}$ let $A_n = [2^{-(n+1)}, 2^{-n}]$. We divide each interval A_n into n pairwise disjoint equi-measurable intervals, $B_n^{(1)}, B_n^{(2)}, \ldots, B_n^{(n)}$ of λ-measure $n^{-1} \cdot 2^{-(n+1)}$. For every $j \in \{1, 2, \ldots, n\}$ let $g_n^{(j)}(x) = x^{-1} \cdot 1_{B_n^{(j)}}(x)$. Let $(f_n)_{n\in\mathbb{N}}$ the sequence of functions defined as follows:

$$\left(g_1^{(1)}, g_2^{(1)}, g_2^{(2)}, g_3^{(1)}, g_3^{(2)}, g_3^{(3)}, g_4^{(1)}, g_4^{(2)}, g_4^{(3)}, g_4^{(4)}, \ldots, g_n^{(1)}, g_n^{(2)}, \ldots, g_n^{(n)}, g_{n+1}^{(1)}, \ldots \right)$$

obtained by the construction $f_n \to 0$ pointwise and then in λ-measure. Moreover, the sequence $(f_n)_n$ is **(u.i.)** with respect to λ and so it is **(u.i.)** with respect to $(m_n)_n$. By [21], Proposition 2.6, the sequence $(f_n)_n$ is **(u.a.c.)**. Then, the pair (m_n, f_n) satisfies Theorem 1, but the sequence (f_n) is not dominated by an integrable function since, if such a function exists, then it will dominate $x^{-1} \notin L^1((0,1])$.

For the multivalued case we recall that any map $\Gamma : \Omega \to cwk(X)$ is said to be *scalarly measurable* if for each $x^* \in X^*$ is measurable the scalar function $t \to s(x^*, \Gamma(t))$; given a measure m, Γ is said to be *scalarly integrable with respect to m* if for each $x^* \in X^*$ is integrable the scalar function $t \to s(x^*, \Gamma(t))$ and Γ is said to be *Pettis integrable* in $cwk(X)$ with respect to m if Γ is scalarly m-integrable and for every $E \in \mathcal{A}$, $M_\Gamma(E) \in cwk(X)$ exists so that

$$s(x^*, M_\Gamma(E)) = \int_E s(x^*, \Gamma)\, dm \text{ for each } x^* \in X^*$$

and in such a case $\int_E \Gamma\, dm := M_\Gamma(E)$. We denote by $\mathcal{P}(m, cwk(X))$ the class of all scalarly m-integrable and $cwk(X)$-valued multifunctions which are Pettis m-integrable. Similarly, we write $\mathcal{P}(m, X)$ for vector-valued functions. For results concerning the Pettis integrability of vector-(multi)valued functions, see for example [24,30–37].

The **(u.a.c.)** takes the following form for sequences of multifunctions: $(\Gamma_n)_n$ has *uniformly absolute continuous scalar (m_n)-integrals* on Ω **(u.a.c.s.)**, if,

- For every $\varepsilon > 0$, a positive δ exists so that for every $n \in \mathbb{N}$ and $E \in \mathcal{A}$ it is:

$$\sup_{\|x^*\| \le 1} \int_E |s(x^*, \Gamma_n)|\, dm_n < \varepsilon \quad \text{when } m_n(E) < \delta. \tag{2}$$

A first result of the multivalued case is the following:

Theorem 2 ([21], Theorem 3.2)**.** *Let $\Gamma : \Omega \to cwk(X)$ be scalarly m-integrable, and for every $n \in \mathbb{N}$, let $\Gamma_n : \Omega \to cwk(X)$ be in $\mathcal{P}(m_n, cwk(X))$. Then $\Gamma \in \mathcal{P}(m, cwk(X))$ if*

(2i) $(\Gamma_n)_n$ *satisfies* (**u.a.c.s.**);

(2ii) $m_n \xrightarrow{s} m$;

(2iii) $\lim_{n\to\infty} \int_E s(x^*, \Gamma_n)\, dm_n = \int_E s(x^*, \Gamma)\, dm$, *for all $E \in \mathcal{A}$ and for all $x^* \in X^*$.*

To prove the Pettis m-integrability of Γ, it is sufficient to show that it is determined by a weakly countably generated subspace of X ([35], Theorem 2.5). As a consequence of Theorem 2, we have

Theorem 3 ([21], Theorem 3.3)**.** *Let $\Gamma : \Omega \to cwk(X)$ be scalarly measurable and for every $n \in \mathbb{N}$ let $\Gamma_n : \Omega \to cwk(X)$ be in $\mathcal{P}(m_n, cwk(X))$. If*

(3i) *For each $x^* \in X^*$, $s(x^*, \Gamma_n)$ converges in m-measure to $s(x^*, \Gamma)$;*

(3ii) Γ *and* $(\Gamma_n)_n$ *satisfy* (**u.a.c.s.**);

(3iii) $m_n \xrightarrow{s} m$,

then $\Gamma \in \mathcal{P}(m, cwk(X))$ and for all $x^* \in X^*$ and for all $E \in \mathcal{A}$

$$\lim_{n\to\infty} s\left(x^*, \int_E \Gamma_n\, dm_n\right) = s\left(x^*, \int_E \Gamma\, dm\right).$$

Moreover, if the convergence in m-measure is strengthened with a scalar equiconvergence in measure with respect to the sequence $(m_n)_n$ and the setwise convergence is substituted with the convergence in variation, a stronger result is obtained.

Theorem 4 ([21], Theorem 3.5)**.** *Let $\Gamma : \Omega \to cwk(X)$ be scalarly measurable and for every $n \in \mathbb{N}$ let $\Gamma_n : \Omega \to cwk(X)$ be in $\mathcal{P}(m_n, cwk(X))$. If*

(4i) *For all $\rho > 0$, $\lim_{n\to\infty} \sup_{\|x^*\|\le 1} m_n\{\omega \in \Omega \colon |s(x^*, \Gamma_n(\omega)) - s(x^*, \Gamma(\omega))| > \rho\} = 0$;*

(4ii) Γ *and* $(\Gamma_n)_n$ *satisfy* (**u.a.c.s.**);

(4iii) *Γ has uniformly absolutely continuous scalar m integral;*

(4iv) $m_n \xrightarrow{tv} m$,

then $\Gamma \in \mathcal{P}(m, cwk(X))$ and uniformly in $E \in \mathcal{A}$

$$\lim_{n\to\infty} d_H\Big(\int_E \Gamma_n\, dm_n, \int_E \Gamma\, dm\Big) = 0.$$

These results can be applied directly to the vector-valued case. Indeed, in the vector case $\Gamma(\omega) = \{f(\omega)\}$ for every $\omega \in \Omega$. The corresponding results are [21], Theorem 3.7 and [21], Theorem 3.9. Therefore we can deduce the vector case directly from multivalued case. Here, we report, as an example, the vector-valued formulation of Theorem 3 for which in [21], Theorem 3.7, a direct proof is also given that makes use of a Grothendieck characterization of weakly compact sets.

Corollary 1 ([21], Theorem 3.7)**.** *Let $f_n : \Omega \to X$ be in $\mathcal{P}(m_n, X)$. If*

(1j) *$x^* f_n(t) \to x^* f(t)$, in m-measure for each $x^* \in X^*$;*

(1jj) $f, (f_n)_n$ *satisfy* (**u.a.c.**);

(1jjj) $m_n \xrightarrow{s} m$,

then $f \in \mathcal{P}(m, X)$ and, weakly in X, we have

$$\lim_{n\to\infty} \int_\Omega f_n\, dm_n = \int_\Omega f dm.$$

2.2. The Topological Case

Sometimes in applications it is difficult, at least technically, to prove that the sequence of measures converges for every measurable set. Therefore, other types of convergence could be considered, based on the structure of the topological space Ω. Following [28], we assume that Ω is only an arbitrary locally compact Hausdorff space and $\mathcal{B}$ is its Borel σ-algebra. All the measures we will consider on $(\Omega, \mathcal{B})$ are finite. Moreover a measure m is Radon if it is inner regular in the sense of approximation by compact sets. For existing results of type (1) in the literature, in this setting, we quote, for example,

- The paper [15] (Theorems 3.3 and 3.5) in locally compact second countable and Hausdorff spaces, when the sequence of scalar functions $(f_n)_n$ converges continuously to f, for vague and weak convergences, respectively. Under a domination condition with a suitable sequence $(g_n)_n$ in the first result while, in the other, the uniform (m_n)-integrability of the sequence $(f_n)_n$, with $f_n \geq 0$ for every $n \in \mathbb{N}$, together with a condition for $m_n(\{f_n > t\})$, a Lebesgue's result is given. In [15], Lemma 3.2, a Fatou's result is obtained.
- The paper [16] (Section 3), where a Monotone convergence result is obtained for locally compact separable metric spaces requiring weak convergence of the sequence of measures and that the space X is a Banach lattice.
- The papers [19,20], where Fatou's, Monotone and Dominated convergence results are obtained in metric spaces, for sequences of scalar lower semi (equi)continuous functions $(f_n^-)_n, f^+$ satisfying an asymptotic uniform (m_n)-integrability, when the sequence of measures converges weakly.

In [22], a Vitali result in the scalar case is obtained:

Theorem 5. ([22], Theorem 3.4) *Let m be a Radon measure and let $f, f_n \in \mathcal{F}(\Omega)$. Suppose that*

(5i) $f_n(t) \to f(t)$, *m-a.e. with $f \in C(\Omega)$;*
(5ii) $(f_n)_n$ *and f satisfy* (**u.a.c.**)*;*
(5iii) $m_n \xrightarrow{v} m$ *and* $m_n \ll_{ac} m$.

Then, $f \in L^1(m)$ and for every $E \in \mathcal{B}$

$$\lim_{n \to \infty} \int_E f_n \, dm_n = \int_E f dm. \tag{3}$$

The result of this theorem is still valid if we replace convergence m almost everywhere with convergence in the m measure. The formula (3) relies on the Urisohn's and a Portmanteau Lemma and it was first proven for arbitrary compact sets and finally for Borelian sets, since m is a Radon measure. This allows us to deduce the setwise convergence of $(m_n)_n$, considering $f_n = f = 1$ for every $n \in \mathbb{N}$ ([22], Corollary 3.5).

Using the scalar case and the support functions, an analogous result is obtained for the multivalued case.

Theorem 6 ([22], Theorem 4.2)**.** *Let m be a Radon measure and Γ be a scalarly continuous multifunction and $\Gamma_n \in \mathcal{P}(m_n, cwk(X))$ for every $n \in \mathbb{N}$. If*

(6i) $(\Gamma_n)_n$ *and Γ satisfy* (**u.a.c.s.**)*;*
(6ii) $s(x^*, \Gamma_n) \to s(x^*, \Gamma)$ *m-a.e. for each $x^* \in X^*$;*
(6iii) $m_n \xrightarrow{v} m$ *and* $m_n \ll_{ac} m$,

then $\Gamma \in \mathcal{P}(m, cwk(X))$ and, for every $x^* \in X^*$ and $E \in \mathcal{B}$, it is

$$\lim_{n \to \infty} s\Big(x^*, \int_E \Gamma_n \, dm_n\Big) = s\Big(x^*, \int_E \Gamma \, dm\Big).$$

We observe in the previous theorem that the null set in (6ii) may depend on $x^* \in X^*$.

If we assume by hypothesis

$$\lim_{n\to\infty}\int_A s(x^*,\Gamma_n)\,dm_n = \int_A s(x^*,\Gamma)\,dm.$$

We can remove the assumption on Ω and the convergence of the sequence $(m_n)_n$ and we can obtain the next result in a general measure space without any topology.

Theorem 7 ([22], Proposition 4.4)**.** *Let Γ be a scalarly m-integrable multifunction and $\Gamma_n \in \mathcal{P}(cwk(X), m_n)$ for every $n \in \mathbb{N}$. Suppose that*

(7i) *$(\Gamma_n)_n$ satisfies* (**u.a.c.s.**)*;*
(7ii) *$m_n \ll_{ac} m$;*
(7iii) *for every $E \in \mathcal{A}$ and $x^* \in X^*$ it is*

$$\lim_{n\to\infty}\int_E s(x^*,\Gamma_n)\,dm_n = \int_E s(x^*,\Gamma)\,dm.$$

Then $\Gamma \in \mathcal{P}(m, cwk(X))$.

Recently, another paper was published on this subject [38], for the weak convergence of integrals, referred to the vector values case, when Ω is a metric space, X is a complete paranormed vector spaces and the sequence of probability measures converges weakly. Therefore, in this case, the target space X is more general, but the sequence of measures is equibounded.

2.3. The Nonadditive Case

Additionally the study of nonadditive measures has played an important role because of its applications in probability, statistics and in all applied sciences where uncertainty must be considered. Therefore, in this case, we quote a Monotone convergence result of type (1). Let Ω be a locally compact Hausdorff space, $\mathcal{P}(\Omega)$ the family of all subsets of Ω and $\mathcal{A}$ be a σ-algebra of subsets of Ω. We denote with the symbol $B(\Omega)$ the family of bounded, real valued functions. We begin by recalling the scalar nonadditive measures: let $m : \mathcal{A} \to \mathbb{R}^+_0$ be a *a submeasure* (in the sense of Drewnowski [39]), namely with $m(\emptyset) = 0$, *monotone* (if $m(A) \le m(B)$, for every $A, B \in \mathcal{A}$, with $A \subseteq B$) and *subadditive* (if $m(A \cup B) \le m(A) + m(B)$, for every disjoint sets $A, B \in \mathcal{A}$) . Let $\overline{m} : \mathcal{P}(\Omega) \to [0, +\infty]$ be the variation of m defined by $\overline{m}(E) = \sup\{\sum_{i=1}^{n} |m(C_i)|, \{C_i\}_{i=1}^n \subset \mathcal{A},\ C_i \subseteq E,\ C_i \cap C_j = \emptyset\}$, for every $E \in \mathcal{P}(\Omega)$, m is said to be *of finite variation* (on $\mathcal{A}$) if $\overline{m}(\Omega) < +\infty$. For the properties of $\overline{m}$, we refer for example to [40–42].

Definition 3. *A sequence of submeasures $(m_n)_n$ setwise converges to a submeasure m if, for every $E \in \mathcal{A}$, $\lim_{n\to\infty} \overline{m_n - m}(E) = 0$.*

Since $|m_n(E) - m(E)| \le \overline{m_n - m}(E)$ for every $E \in \mathcal{A}$, the convergence given in Definition 3 is the (1c); the converse does not hold in general. Nevertheless, from [43], Remark 1, in the countable additive case the two definitions coincide, so we use the same notation.

We denote by the symbol $ck(\mathbb{R}^+_0)$ the family of all nonempty convex compact subsets of $\mathbb{R}^+_0$. We consider on $ck(\mathbb{R}^+_0)$ the weak interval order: $[a, b] \preceq [c, d]$ if and only if $a \le c$ and $b \le d$ and a multiplication $[a, b] \cdot [c, d] = [ac, bd]$. For what concerns the (lattice) weak order $\preceq$ and for its meaning and uses, we refer to [44].

Definition 4. *Let $M : \mathcal{A} \to ck(\mathbb{R}^+_0)$. M is an* interval valued multisubmeasure *if*

(4a) $M(\emptyset) = \{0\}$;
(4b) $M(A) \preceq M(B)$ *for every $A, B \in \mathcal{A}$ with $A \subseteq B$ (monotonicity);*
(4c) $M(A \cup B) \preceq M(A) + M(B)$ *for every disjoint sets $A, B \in \mathcal{A}$ (subadditivity).*

In literature the monotone multimeasures that satisfy $M(\emptyset) = \{0\}$ are also called set valued fuzzy measures. For the results on this subject, see for example [40,42,45].

Remark 1. *Given two submeasures $\mu_1, \mu_2 : \mathcal{A} \to \mathbb{R}_0^+$ with $\mu_1(E) \leq \mu_2(E)$ for every $E \in \mathcal{A}$ let $M : \mathcal{A} \to ck(\mathbb{R}_0^+)$ be defined by $M(E) = [\mu_1(E), \mu_2(E)]$. According to [42], Remark 3.6, M is a multisubmeasure with respect to the weak interval order $\preceq$ if and only if μ_1, μ_2 are submeasures. Moreover M is monotone or finitely additive if and only if the set functions μ_1 and μ_2 are the same (see [41], Proposition 2.5, Remark 3.3). Moreover, following [23], Definition 2.2, $M_n = [\mu_{n,1}, \mu_{n,2}]$ setwise converges to $M = [\mu_1, \mu_2]$ if and only if $\mu_{n,i}$ setwise converges to μ_i for $i = 1, 2$.*

In this framework we consider the Riemann–Lebesgue integrability studied in [43,46]. If P and P' are two partitions of Ω, then P' is *finer than* P ($P' \geq P$), if every set of P' is included in some set of P. All the partitions we consider in this subsection are countable.

Definition 5. *Let $f : \Omega \to \mathbb{R}_0^+$ be a function and $\mu : \mathcal{A} \to \mathbb{R}_0^+$ be a set function. f is Riemann–Lebesgue (RL) μ-integrable (on Ω) if $b \in \mathbb{R}_0^+$ exists such that for every $\varepsilon > 0$ a partition P_ε of Ω exists so that for every partition $P = \{A_n\}_{n\in\mathbb{N}}$ of Ω with $P \geq P_\varepsilon$, f is bounded on every A_n, with $\mu(A_n) > 0$ and for all $t_n \in A_n$, $n \in \mathbb{N}$, $\sum_{n=0}^{+\infty} f(t_n)\mu(A_n)$ is convergent and $|\sum_{n=0}^{+\infty} f(t_n)\mu(A_n) - b| < \varepsilon$. b is called the Riemann-Lebesgue μ-integral of f on Ω and is denoted by $(RL)\int_\Omega f dm$.*

Analogously we can define the Riemann–Lebesgue integrability for the interval multifunctions $G : \Omega \to ck(\mathbb{R}_0^+)$, see for example ([12], Definition 6).

Remark 2. *Given an interval multifunction $G = [g_1, g_2]$, with $g_1, g_2 : \Omega \to \mathbb{R}_0^+$ and $g_1(t) \leq g_2(t)$ for all $t \in \Omega$, for every tagged partition $\mathscr{P} = \{(A_n, t_n),\ n \in \mathbb{N}\}$ of Ω, we have that*

$$\{\sum_{n=1}^{\infty} x_n,\ x_n \in [g_1(t_n)\mu_1(A_n), g_2(t_n)\mu_2(A_n)],\ n \in \mathbb{N}\} =$$
$$= \sum_{n=1}^{\infty} [g_1(t_n)\mu_1(A_n), g_2(t_n)\mu_2(A_n)] = \sum_{n=1}^{\infty} G(t_n) \cdot M(A_n) \in ck(\mathbb{R}_0^+)$$

Moreover, according to [12], Proposition 2, G it is RL integrable with respect to M on Ω if and only if g_i are RL integrable with respect to μ_i, $i = 1, 2$ and

$$(RL)\int_\Omega G dM = \left[(RL)\int_\Omega g_1 d\mu_1,\ (RL)\int_\Omega g_2 d\mu_2\right].$$

Therefore, for every submeasure μ which is a selection of M and every $g \in B(\Omega)$ with $g_1(t) \leq g(t) \leq g_2(t)$ then $\int_\Omega g d\mu \in (RL)\int_\Omega G dM$.

Theorem 8 ([23], Theorem 4.2)**.** *Let $G_n = [g_{n,1}, g_{n,2}]$ be a sequence of bounded interval valued multifunctions and $M_n = [\mu_{n,1}, \mu_{n,2}]$ be multisubmeasures. Suppose that there exist an interval valued multisubmeasure $M := [\mu_1, \mu_2]$ with μ_2 of bounded variation and a bounded multifunction $G = [g_1, g_2]$ such that*

(8i) *$G_n \preceq G_{n+1}$ for every $n \in \mathbb{N}$ and $d_H(G_n, G) \to 0$ uniformly on Ω;*
(8ii) *$M_n \preceq M_{n+1} \preceq M$ for every $n \in \mathbb{N}$ and $(M_n)_n$ setwise converges to M,*

then

$$\lim_{n\to\infty} d_H\left((RL)\int_\Omega G_n\, dM_n, (RL)\int_\Omega G\, dM\right) = 0.$$

3. McShane Integrability

In Sections 2.1 and 2.2, the vector valued case was derived from the multivalued one when we consider Pettis integrability. In the case of the McShane integral it is possible to proceed in the same way. However, thanks to the Rådström embedding, for the McShane integral it is possible to deduce at once the multivalued case from the vector valued case. We recall that the Rådström embedding is a map $i : cwk(X) \to \ell_\infty(B_{X^*})$ that is additive, isometric and positively homogeneous, see for example [24,47]. Therefore, if $\Gamma_n' s$ are McShane integrable multifuctions, then the vector-valued functions $i \circ \Gamma_n$ are and viceversa. So the assumptions for multifunctions can be shifted to the corresponding $i \circ \Gamma_n' s$ allowing the convergence to be obtained from the vector case. In general, the Rådström embedding of a Pettis integrable multifunction could not be Pettis integrable. If we ask for a stronger notion of integrability, also under setwise convergence, some topological conditions are needed (see for example [48–51]).

3.1. The Compact Case

Let $(\Omega, \mathcal{B})$ be a compact measure space with a topology $\tau \subset \mathcal{B}$. A *McShane partition* of Ω is a family $\{(E_i, t_i)\}_{i \le p}$ such that $E_1, ..., E_p$ is a finite disjoint cover of Ω by elements of $\mathcal{B}$ and $t_i \in \Omega, i = 1, \ldots, p$. A *gauge* Δ on Ω is a function $\Delta : \Omega \to \tau$ such that for every $t \in \Omega$ it is $t \in \Delta(t)$. A McShane partition $\{(E_i, t_i)\}_{i \le p}$ is *subordinated* to a gauge Δ if $E_i \subset \Delta(t_i)$ for $i = 1, ..., p$.

Definition 6. *$f : \Omega \to X$ is said to be m^{MS}-integrable on Ω with m^{MS}- integral $w \in X$ if for every $\varepsilon > 0$ a gauge Δ exists such that for each partition $\{(E_i, t_i)\}_{i \le p}$ subordinated to Δ, we have $\left\| \sum_{i=1}^{p} f(t_i) m(E_i) - w \right\| < \varepsilon$. In this case, we set $w := (MS) \int_\Omega f dm$.*

For this kind of integration see for example [48,52–54].

Definition 7. *A sequence $f_n : \Omega \to X$ of m_n^{MS}-integrable functions is (m_n)-equi-integrable on Ω, if for every $\varepsilon > 0$ there exists a gauge Δ such that*

$$\left\| \sum_{i=1}^{p} f_n(t_i) m_n(E_i) - (MS) \int_\Omega f_n dm_n \right\| < \varepsilon \tag{4}$$

for each partition $\{(E_i, t_i)\}_{i \le p}$ subordinated to Δ and every $n \in \mathbb{N}$.

We can observe that, thanks to [48], Theorem 1N, the inequality (4) holds for every $E \in \mathcal{B}$, as highlighted in [21], Theorem 3.11. Therefore, we have

Theorem 9 ([21], Theorem 3.11)**.** *Let $m,\ m_n \subset \mathcal{M}_+(\Omega)$, $n \in \mathbb{N}$, and let $f, f_n : \Omega \to X$. If*

(9i) *$(f_n)_n$ is (m_n)-equi-integrable on Ω;*
(9ii) *$f_n(t) \to f(t)$, for all $t \in \Omega$;*
(9iii) *$m_n \xrightarrow{s} m$,*

then f is m^{MS}-integrable and, for all $E \in \mathcal{B}$,

$$\lim_{n \to \infty} (MS) \int_E f_n \, dm_n = (MS) \int_E f dm. \tag{5}$$

Moreover if condition (9iii) is replaced with the convergence in total variation ($m_n \xrightarrow{tv} m$), then (5) holds uniformly in $E \in \mathcal{B}$.

The multivalued case [21], Theorem 3.13, follows using the Rådström embedding.

3.2. The Noncompact Case

We now want to extend the previous results for the compact case to a more general case, we limit ourselves to the vector case because the multivalued case follows similarly.

Let now $(\Omega, \tau, \mathcal{B})$ be a nonempty, locally compact Hausdorff space and let m be a quasi-Radon and outer regular measure. Let $\mathcal{M}^F(\Omega)$ be the subset of $\mathcal{M}_+(\Omega)$ consisting of the outer regular and quasi-Radon measures. A series $\sum_{i\in\mathbb{N}} x_n$ exists unconditionally for the norm topology of the Banach space X if and only if for every $\varepsilon > 0$ a finite set $J = J(\varepsilon) \subset \mathbb{N}$ exists such that for every finite set I with $J \subset I \subset \mathbb{N}$ we have

$$\Big\| \sum_{i\in\mathbb{N}} x_i - \sum_{i\in I} x_i \Big\| < \varepsilon. \tag{6}$$

We say that a sequence of series $\left(\sum_{i\in\mathbb{N}} x_i^n\right)_n$ is *uniformly unconditionally convergent* if for every $\eta > 0$ a finite set $J = J(\eta) \subset \mathbb{N}$ exists so that for every $n \in \mathbb{N}$, we have

$$\Big\| \sum_{i\in\mathbb{N}} x_i^n - \sum_{i\in I} x_i^n \Big\| < \eta \quad \text{for every } I \supset J,\ I \subset \mathbb{N},\ \#I < +\infty. \tag{7}$$

A *generalized McShane partition* of Ω is a family $\{(A_i, t_i)\}_{i\in\mathbb{N}}$ such that $(A_n)_n$ is a sequence of disjoint measurable sets of finite measure such that of $m(\Omega \setminus \cup_n A_n) = 0$ and $t_i \in \Omega$ for each $n \in \mathbb{N}$. A *gauge* Δ on Ω is a function $\Delta : \Omega \to \tau$ such that $t \in \Delta(t)$ for every $t \in \Omega$. We say that a generalized McShane partition $\{(A_i, t_i)\}_{i\in\mathbb{N}}$ is *subordinated* to a gauge Δ if $A_i \subset \Delta(t_i)$ for all $i \in \mathbb{N}$.

Definition 8 ([48], Definition 1A)**.** *A function $f : \Omega \to X$ is said to be m-generalized McShane (m^{gMS})-integrable on Ω with m^{gMS}-integral $w \in X$ if for every $\varepsilon > 0$ a gauge Δ exists such that for each generalized McShane partition $\{(E_i, t_i)\}_{i\in\mathbb{N}}$ subordinated to Δ, we have*

$$\limsup_{n\to\infty} \Big\| \sum_{i=1}^{n} f(t_i) m(E_i) - w \Big\| < \varepsilon. \tag{8}$$

We set $w := (gMS) \int_\Omega f dm$.

Remark 3. *By [48], Theorem 1N, we know that a function f is $m^{(gMS)}$-integrable if and only if $f\chi_E$ is $m^{(gMS)}$-integrable for every subset $E \in \mathcal{B}$ and $(gMS) \int_E f dm = (gMS) \int_\Omega f\chi_E dm$. In this case we can set $\Delta_E(t) = E \cap \Delta(t)$ for every $t \in \Omega$. Moreover, by [48], Remark of Theorem 1N, if E is such that $m(\Omega \setminus E) = 0$ then $(gMS) \int_E f dm = (gMS) \int_\Omega f dm$.*

Finally, according to [48], Corollary 2D, f is $m^{(gMS)}$- integrable if and only if for every $\varepsilon > 0$ there exists a gauge Δ such that: for each generalized McShane partition $\{(A_i, t_i)\}_{i\in\mathbb{N}}$ subordinated to Δ, $\sum_{i\in\mathbb{N}} f(t_i) m(A_i)$ exists unconditionally for the norm topology of X and it is

$$\Big\| \sum_{i\in\mathbb{N}} f(t_i) m(A_i) - w \Big\| < \varepsilon. \tag{9}$$

We begin with a convergence results involving the m^{gMS}-integrability of the limit function f. To obtain this we need the definition of equi-integrability for series, so we extend (4) in the following form:

Definition 9. *Let m_n, $n = 1, 2, \ldots$ be measures in $\mathcal{M}^F(\Omega)$. We say that a sequence of $m_n^{(gMS)}$-integrable functions $f_n : \Omega \to X$ is (m_n)-g-equi-integrable on Ω, if for every $\varepsilon > 0$ a gauge Δ exists so that for each generalized McShane partition $\{(A_i, t_i)\}_{i\in\mathbb{N}}$ of Ω subordinated to Δ and for every $E \in \mathcal{B}$ the sequence of series $(\sum_{i\in\mathbb{N}} f_n(t_i) m_n(A_i \cap E))_n$ is uniformly unconditionally convergent and*

$$\Big\| \sum_{i\in\mathbb{N}} f_n(t_i) m_n(A_i \cap E) - (gMS) \int_E f_n \, dm_n \Big\| < \varepsilon \quad \text{for every } n \in \mathbb{N}. \tag{10}$$

Remark 4. *An immediate consequence of Definition 9 and of [48], Theorem 1N, is the (m_n)-g-equi-integrability of the sequence $(f_n)_n$ with respect to every $E \in \mathcal{B}$. In fact it is enough to* *take $\Delta_E(t) = \Delta(t) \cap E$ for every $t \in \Omega$. In particular if Ω is a compact Radon measure space this is exactly [21], Definition 3.10, since the McShane integrability is given in terms of finite partitions as in Definition 7, so the concept of (m_n)-equi-integrability on compact sets coincides with that of (m_n)-g-equi-integrability. If $m_n = m$ for every $n \in \mathbb{N}$, this is the classical condition of equi-integrability.*

On noncompact sets, the two definitions differ, being the definition of (m_n)-g-equi-integrability stronger than that of (m_n)-equi-integrability. In particular, if $\Omega = \cup_k \Omega_k$, Ω_k compact for every $k \in \mathbb{N}$, and $(f_n)_n$ is (m_n)-g-equi-integrable on Ω, then $(f_n)_n$ is (m_n)-equi-integrable on Ω_k for each $k \in \mathbb{N}$. For the converse implication, we can observe that it is false in general. Let $\Omega = (0,1]$ and $\Omega_k = [\frac{1}{k}, 1]$. Take $f_n(x) = x^{-1}\chi_{[\frac{1}{n},1]}$ and let $m_n = m$ be the Lebesgue measure. Then the collection $\{f_n : n \in N\}$ restricted to a separate Ω_k consists of a finite family, since $f_m = f_k$ on Ω_k for every $m \geq k$. Consequently it is equi-integrable on each Ω_k. However, it is not g-equi-integrable on $(0,1]$ because, otherwise, by [55], Theorem 4, $f(x) = x^{-1}$ will be integrable.

Theorem 10. *Let m and $(m_n)_n$ be measures in $\mathcal{M}^F(\Omega)$, let $f_n : \Omega \to X$ be $m_n^{(gMS)}$-integrable functions, $n \in \mathbb{N}$. Suppose that*

(10i) *$(f_n)_n$ is (m_n)-g-equi-integrable on Ω;*

(10ii) *$(f_n)_n$ converges pointwise to a m^{gMS}-integrable function f;*

(10iii) *$m_n \xrightarrow{s} m$.*

Then, for all $E \in \mathcal{B}$,

$$\lim_{n\to\infty} (gMS) \int_E f_n \, dm_n = (gMS) \int_E f dm. \tag{11}$$

Moreover, if we substitute condition (10iii) *with the convergence in total variation ($m_n \xrightarrow{tv} m$), then* (11) *holds uniformly in $E \in \mathcal{B}$.*

Proof. Let $\varepsilon > 0$ be fixed. Let $\Delta = \Delta(\varepsilon)$ be a gauge on Ω satisfying (10i), then for every generalized McShane partition $\{(A_i, t_i)\}_{i\in\mathbb{N}}$ subordinated to Δ and for every $E \in \mathcal{B}$ the sequence of series $(\sum_{i\in\mathbb{N}} f_n(t_i) m_n(A_i \cap E))_n$ is uniformly unconditionally convergent and

$$\left\| \sum_{i\in\mathbb{N}} f_n(t_i) m_n(A_i \cap E) - (gMS) \int_E f_n \, dm_n \right\| < \varepsilon. \tag{12}$$

Without loss of generality, we may assume that corresponding to ε the same gauge Δ works for the function f that is m^{gms}-integrable. Now, let $\{(A_i, t_i)\}_{i\in\mathbb{N}}$ be a fixed generalized McShane partition subordinated to Δ. We will now show that the sequence

$$\left(\sum_{i\in\mathbb{N}} f_n(t_i) m_n(A_i \cap E) \right)_n \tag{13}$$

is Cauchy. We fix $\sigma > 0$. Since the sequence of the series $(\sum_{i\in\mathbb{N}} f_n(t_i) m_n(A_i \cap E))_n$ is uniformly unconditionally convergent, let $J(\sigma)$ be a finite subset of $\mathbb{N}$ such that, for every finite set I with $J(\sigma) \subset I \subset \mathbb{N}$ and for each $n \in \mathbb{N}$,

$$\left\| \sum_{i\in\mathbb{N}} f_n(t_i) m_n(A_i \cap E) - \sum_{i\in I} f_n(t_i) m_n(A_i \cap E) \right\| < \sigma. \tag{14}$$

We fix $J \subset I \subset \mathbb{N}$ with I finite. The pointwise convergence of f_n to f and the setwise convergence of m_n to m implies that

$$\lim_{n\to\infty} \sum_{i\in I} f_n(t_i) m_n(A_i \cap E) = \sum_{i\in I} f(t_i) m(A_i \cap E). \tag{15}$$

Corresponding to $\sigma > 0$ choose $n_0 \in \mathbb{N}$ so that if $n, s > n_0$

$$\left\| \sum_{i\in I} f_n(t_i)m_n(A_i\cap E) - \sum_{i\in I} f_s(t_i)m_s(A_i\cap A) \right\| < \sigma. \tag{16}$$

Then, if $n, s > n_0$, by (14) and (16)

$$\begin{aligned} &\left\| \sum_{i\in\mathbb{N}} f_n(t_i)m_n(A_i\cap E) - \sum_{i\in\mathbb{N}} f_s(t_i)m_s(A_i\cap E) \right\| \leq \\ &\left\| \sum_{i\in\mathbb{N}} f_n(t_i)m_n(A_i\cap E) - \sum_{i\in I} f_n(t_i)m_n(A_i\cap E) \right\| + \\ &\left\| \sum_{i\in I} f_n(t_i)m_n(A_i\cap E) - \sum_{i\in I} f_s(t_i)m_s(A_i\cap E) \right\| + \\ &\left\| \sum_{i\in\mathbb{N}} f_s(t_i)m_s(A_i\cap E) - \sum_{i\in I} f_s(t_i)m_s(A_i\cap E) \right\| < 3\sigma. \end{aligned}$$

Then the sequence in (13) is Cauchy.

Additionally, the sequence $\left((gMS)\int_E f_n dm_n\right)_n$ is Cauchy. Therefore, corresponding to $\varepsilon > 0$, let $\{(A_i, t_i)\}_{i\in\mathbb{N}}$ be a generalized McShane partition subordinated to Δ related to the condition (10.i). Therefore, by (12) and taking into account that (13) is Cauchy, for n, s being suitably large we have

$$\begin{aligned} \left\| (gMS)\int_E f_n\,dm_n - (gMS)\int_E f_s\,dm_s \right\| &\leq \left\| (gMS)\int_E f_n\,dm_n - \sum_{i\in\mathbb{N}} f_n(t_i)m_n(A_i\cap E) \right\| + \\ &+ \left\| \sum_{i\in\mathbb{N}} f_n(t_i)m_n(A_i\cap E) - \sum_{i\in\mathbb{N}} f_s(t_i)m_s(A_i\cap E) \right\| + \\ &+ \left\| \sum_{i\in\mathbb{N}} f_s(t_i)m_s(A_i\cap E) - (gMS)\int_E f_s dm_s \right\| < 3\varepsilon \end{aligned}$$

which also shows that the sequence $\left((gMS)\int_E f_n\,dm_n\right)_n$ is Cauchy, therefore it converges to $x_E \in X$. Let n_1 be an integer such that, for every $n \geq n_1$

$$\left\| (gMS)\int_E f_n\,dm_n - x_E \right\| < \varepsilon.$$

We will show that x_E is the m^{gMS}-integral of f. Since $\sum_{i\in\mathbb{N}} f(t_i)m(A_i\cap E)$ is unconditionally convergent let $I \subset \mathbb{N}$ be a finite set such that

$$\left\| \sum_{i\in\mathbb{N}} f(t_i)m(A_i\cap E) - \sum_{i\in I} f(t_i)m(A_i\cap E) \right\| < \varepsilon.$$

Moreover, since $\sum_{i\in I} f_n(t_i)m(A_i\cap E)$ are uniformly unconditionally convergent without loss of generality, we may assume that for the same set I the inequality (14) holds for $\sigma = \varepsilon$. Therefore, if n is suitably large we have

$$\begin{aligned} \left\| \sum_{i\in\mathbb{N}} f(t_i)m(A_i\cap E) - x_E \right\| &\leq \left\| \sum_{i\in\mathbb{N}} f(t_i)m(A_i\cap E) - \sum_{i\in I} f(t_i)m(A_i\cap E) \right\| + \\ &+ \left\| \sum_{i\in I} f(t_i)m(A_i\cap E) - \sum_{i\in I} f_n(t_i)m(A_i\cap E) \right\| + \\ &+ \left\| \sum_{i\in I} f_n(t_i)m(A_i\cap E) - \sum_{i\in\mathbb{N}} f_n(t_i)m_n(A_i\cap E) \right\| + \\ &+ \left\| \sum_{i\in\mathbb{N}} f_n(t_i)m_n(A_i\cap E) - (gMS)\int_E f_n dm_n \right\| + \\ &+ \left\| (gMS)\int_E f_n\,dm_n - x_E \right\| < 5\varepsilon. \end{aligned}$$

Therefore it follows that

$$\lim_{n} (gMS) \int_{E} f_n \, dm_n = (gMS) \int_{E} f dm.$$

Finally, if $m_n \stackrel{tv}{\to} m$, n_0 does not depend on E. Then, formula (15) holds uniformly on $\mathcal{B}$ and the convergence in formula (11) is uniform. □

Remark 5. *Theorem 10 is still valid if we replace the pointwise convergence of $(f_n)_n$ to f in condition (10ii) with the convergence in m-measure to f. In fact, by [48], Remark of Theorem 1N, we can pass to the convergence m-a.e., and from this, the theorem holds for every subsequence $(f_{n_k})_k$ of $(f_n)_n$. This implies that the result of this theorem is still valid for convergence in the m-measure because there would be a contradiction if, absurdly, a subsequence existed in which it is not valid.*

In this theorem, the m^{gMS}-integrability of the limit function f plays a fundamental role. To weaken this hypothesis and obtain the m^{gMS}-integrability as a thesis, we must introduce additional assumptions on Ω. Let Ω be a locally compact second countable Haudorff space (**lcsc**). Namely,

- There exists an increasing sequence of relatively compact spaces Ω_n such that $\Omega = \cup_n \Omega_n$ and $\overline{\Omega}_n \subseteq \Omega_{n+1}$ for every $n \in \mathbb{N}$.

From now on we suppose that $(m_n)_n$ is a sequence of measures in $\mathcal{M}^F(\Omega)$ and Ω such that has (**lcsc**). Thanks to Remark 4 we can apply Theorem 9 to each compact set $\overline{\Omega}_k$. Therefore, we have

Corollary 2. *Let m and $(m_n)_n$ be measures in $\mathcal{M}^F(\Omega)$, with m Radon and Ω that has* (**lcsc**)*. Let $f_n : \Omega \to X$ be $m_n^{(gMS)}$-integrable functions, $n \in \mathbb{N}$. Suppose that*

(2j) *The sequence $(f_n)_n$ is (m_n)-g-equi-integrable on Ω;*

(2jj) *The sequence $(f_n)_n$ converges pointwise to a Pettis m-integrable function f;*

(2jjj) $m_n \stackrel{s}{\to} m$.

Then, f is m^{gMS}-integrable on Ω and

$$\lim_{n\to\infty} (gMS) \int_{\Omega} f_n \, dm_n = (gMS) \int_{\Omega} f \, dm. \tag{17}$$

Proof. Let $\varepsilon > 0$ be fixed. Let $(\Omega_k)_k \subset \mathcal{B}$ be as in (**lcsc**). Since Ω_k is relatively compact, we can apply Theorem 9 ([21], Theorem 3.11) and we obtain that f is m^{MS}-integrable on Ω_k and there exists $n^* := n(k, \varepsilon) \in \mathbb{N}$ such that

$$\left\| (MS) \int_{\Omega_k} f_n \, dm_n - (MS) \int_{\Omega_k} f dm \right\| < \varepsilon$$

for every $n \geq n(k, \varepsilon)$. Then, by [48], Corollary 4.B, f is m^{gMS}-integrable on Ω, since it is Pettis m-integrable in Ω and m^{MS}-integrable in each Ω_k. We now apply Theorem 10. □

Remark 6. *Moreover, if we substitute condition* (2.iii) *with the convergence in total variation ($m_n \stackrel{tv}{\to} m$), then* (17) *holds uniformly in $E \in \mathcal{B}$. Finally, as in Remark 5, the result is still valid for convergence in the m-measure of $(f_n)_n$ to f.*

4. Discussion

This paper starts from the results obtained in [21,22] and quoted in Section 2. The research continue with an examination of convergence theorems for a sequence of McShane integrable functions with respect to a sequence of varying measures. To obtain new results on noncompact spaces, we refer to [48] for the theory of McShane integrability.

5. Conclusions

This paper describes sufficient conditions ensuring convergence in both a weak and strong sense of a sequence $(\int f_n \, dm_n)_n$ for scalar and vector or multivalued Pettis integrable functions when the sequence $(m_n)_n$ converges in some sense to a measure m. When convergence is setwise or in total variation, our ambient space is a general measure space, while in the case of vague convergence, the ambient space is a locally compact Hausdorff measure space. Both the case of countably additive measures and that of fuzzy measures has been exposed, together with a comparison with results known in the literature.

When we consider the McShane integrability we can pass from the vector case to the multivalued one. In this setting, we need a topology on the space; thus, $(\Omega, \mathcal{A}, m)$ is a Radon measure space with a topology τ and the Rådström embedding is the key to passing from a sequence of McShane integrable functions to a sequence of McShane integrable multifunctions. In this case, new results of Vitali type were also provided. For what concerns future research directions on this subject, we are studying the case of convergence results for fuzzy varying measures that converge weakly; this research is in progress.

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