A PROOF OF THE GHOUSSOUB-PREISS THEOREM BY THE $\varepsilon-$PERTURBATION OF BREZIS-NIRENBERG

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Abstract. In this note, a proof of the Ghoussoub-Preiss theorem is presented by using the $\varepsilon-$perturbation as introduced by Brezis-Nirenberg. Thus, besides the deformation lemma, other advanced tools such as the Radon measures space, sub-differential, or the theory of non-differentiable functions, are avoided. Our new argument is a lemma of local type which is used in combination with other main ingredients like the Ekeland variational principle and the pseudo-gradient lemma, for which a new proof is proposed as a consequence of the Michael selection theorem.

1. Introduction

In the Calculus of Variations and in Nonlinear Analysis one of the most important results of the second half of the twentieth century is surely the Ambrosetti-Rabinowitz theorem [2, Theorem 2.1]. The limiting case on "zero altitude" situation, has been subsequently wisely tackled by Pucci-Serrin [25, Theorem 1] and finally beautifully solved by Ghoussoub-Preiss [13, Theorem (1)]. The primary proof of the Ambrosetti-Rabinowitz theorem is based, among the others, on a deformation lemma. Subsequently, Aubin-Ekeland [4, Theorem 5.5.5] gave a proof based on the Ekeland variational principle where advanced tools as the Radon measures space and the notion of sub-differential are applied. Then, Ghoussoub-Preiss presented a proof based on the Ekeland variational principle using an $\varepsilon-$perturbation which allows to take in addition the limiting case on "zero altitude" and to give further information on the localization of the critical point. The $\varepsilon-$perturbation introduced by Ghoussoub-Preiss is a non-regular function, so it is handled by sophisticated arguments as those cited above and previously exploited also by Aubin-Ekeland. Finally, Brezis-Nirenberg in [8] introduced a new different type of $\varepsilon-$perturbation which is more regular than the one used by Ghoussoub-Preiss, presenting so a proof of the mountain pass theorem, including the Ambrosetti-Rabinowitz theorem, which is based again on the Ekeland variational principle, but where the above complex tools are avoided. However, the profound result of Brezis-Nirenberg does not include fully the one given by Ghoussoub-Preiss where, in the limiting case, the obtained critical point is well localized on the boundary of a suitable ball.

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The aim of this paper is to give a proof of the Ghoussoub-Preiss theorem by using, in particular, the $\varepsilon-$perturbation introduced by Brezis-Nirenberg, so that some technical tools as those recalled before can be avoided. Precisely, here, a lemma which ensures a point of "Palais-Smale type" belonging to a small ball is established by using just the $\varepsilon-$perturbation given by Brezis-Nirenberg (see Section 3). Then, a proof of the Ghoussoub-Preiss theorem based on the Ekeland variational principle and such a local lemma, beside of the pseudo-gradient lemma, is presented (see Section 5).

This paper is organized as follows. In Section 2, the Ambrosetti-Rabinowitz theorem is recalled and a proof obtained by following the ideas of Brezis-Nirenberg is reported. In such a case the $\varepsilon-$perturbation is not needed. The unique main tools are the Ekeland variational principle and the pseudo-gradient lemma which, here, is obtained as a consequence of the Michael selection theorem. In Section 3, there is our main result, that is a local lemma which is a fundamental tool for a new proof of the mountain pass theorem including the limiting cases, while in Section 4, the classical seminal results of Pucci-Serrin are recalled, presenting a proof which is a consequence of the local lemma. Finally, in Section 5, a proof of the Ghoussoub-Preiss theorem based again on the local lemma is established.

For a complete overview on the mountain pass theorem regarding its other formulations or extensions (as linking theorem, or Cerami condition) or types of proof (as through qualitative deformation lemma) we refer to the exhaustive books of Willem [28] and Ekeland [11], while for further references and the historical development we cite the book of Jabry [16]. We also refer to seminal books of Ambrosetti-Malchiodi [1], Aubin-Ekeland [4], Ghoussoub [15], Mawhin-Willem [18], Motreanu-Motreanu-Papageorgiou [20], Motreanu-Radulescu [21], Peral [22], Perera-Schechter [23], Rabinowitz [26], for further studies and completions.

2. The Ambrosetti-Rabinowitz theorem

Let E be a real Banach space and $f : E \to \mathbb{R}$ a function of class C^1. As usual, for $c \in \mathbb{R}$, a sequence $\{x_n\} \subseteq E$ such that

1. $f(x_n) \to c$,
2. $f'(x_n) \to 0$,

is called a Palais-Smale sequence at level c (briefly $(PS)_c-$sequence) for f. If any $(PS)_c-$sequence for f admits a subsequence strongly converging in E, we say that f satisfies the *Palais-Smale condition at level* c, shortly $(PS)_c-$condition. Moreover, we say that f satisfies the *Palais-Smale condition*, shortly $(PS)-$condition if it satisfies the Palais-Smale condition at level c for all $c \in \mathbb{R}$.

In this section, we report the classical mountain pass theorem, as essentially given by Ambrosetti-Rabinowitz (see [2] and Theorem 2.3 below), giving a proof directly deduced by the one established by Brezis-Nirenberg in [8]. The proof is based on the

Ekeland variational principle, by exploiting a lemma on the pseudo-gradient, that we recall here.

Proposition 2.1. *Let N be a metric space, E a real Banach space and $T : N \to E^*$ a continuous function. Then, for each $\varepsilon > 0$ there is a continuous function $v_\varepsilon : N \to E$ such that for each $\xi \in N$ one has*

$$\|v_\varepsilon(\xi)\| \leq 1; \qquad \|T(\xi)\|_{E^*} \leq \langle T(\xi), v_\varepsilon(\xi)\rangle + \varepsilon.$$

A direct proof of Proposition 2.1 can be found in Rabinowitz [26, Theorem A.2]. Here, we prove it as a consequence of the following version of the Michael selection theorem (see for instance Aubin-Cellina [3, proof of Theorem 1 pag. 82]).

Theorem 2.1. (Michael Theorem). *Let N be a metric space, E a real Banach space and $G : N \to 2^E$ a lower semicontinuous multifunction with nonempty and convex values. Then, the multifunction $\overline{G} : N \to 2^E$ defined by $\overline{G}(t) = \overline{G(t)}$ for all $t \in N$ admits a continuous selection.*

Proof of Proposition 2.1. Fix $\varepsilon > 0$ and let $G_\varepsilon : N \to 2^E$ be the multifunction defined by

$$G_\varepsilon(t) = \{v \in E : \|v\| \leq 1; \|T(t)\|_{E^*} < \langle T(t), v\rangle + \varepsilon\}$$

for all $t \in N$. Fix $t_0 \in N$. From $\|T(t_0)\|_{E^*} = \sup_{\|v\|\leq 1}\langle T(t_0), v\rangle$ one has that $G_\varepsilon(t_0)$ is nonempty and a simple computation shows that it is a convex set. Now, we claim that G_ε is lower semicontinuous. Let $A \subseteq E$ be an open set and fix $\bar{t} \in G^-(A) = \{t \in N : G_\varepsilon(t) \cap A \neq \emptyset\}$. From $G_\varepsilon(\bar{t}) \cap A \neq \emptyset$ there is $v \in A$ such that $\|T(\bar{t})\|_{E^*} < \langle T(\bar{t}), v\rangle + \varepsilon$. Therefore, there is a neighborhood U of $\bar{t}$ such that $\|T(t)\|_{E^*} < \langle T(t), v\rangle + \varepsilon$ for all $t \in U$, that is $U \subseteq G^-_\varepsilon(A)$. Hence, $G^-_\varepsilon(A)$ is an open set and our claim is proved. Now, from the Michael selection theorem there exists a continuous function $v_\varepsilon : N \to E$ such that

$$v_\varepsilon(t) \in \overline{G_\varepsilon(t)} \subseteq \{v \in E : \|v\| \leq 1; \|T(t)\|_{E^*} \leq \langle T(t), v\rangle + \varepsilon\}$$

for all $t \in N$ and the conclusion is achieved.

□

Now, we recall the Ambrosetti-Rabinowitz theorem, giving a proof which is basically due to Brezis-Nirenberg.

Theorem 2.2. *Let E be a real Banach space and $f \in C^1(E)$. Fix $x_0 \in E$ and $R > 0$ and assume that there are $x, y \in E$, with $\|x - x_0\| < R < \|y - x_0\|$ such that*

$$\inf_{\|z-x_0\|=R} f(z) > \max\{f(x), f(y)\}. \tag{G_1}$$

Put

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)),$$

where $\Gamma = \Big\{\gamma \in C([0,1],E) : \gamma(0)=x, \gamma(1)=y\Big\}$.

Then, f admits a $(PS)_c$−sequence.

Proof. Put $a = \max\{f(x),\ f(y)\}$. Condition (G_1) implies that

$$c > a. \tag{G_1'}$$

Indeed, for each $\gamma \in \Gamma$ there is $t_\gamma \in]0,1[$ such that $\|\gamma(t_\gamma) - x_0\| = R$, so that one has $f(\gamma(t_\gamma)) \geq \inf_{\|z-x_0\|=R} f(z)$. It follows that $\max_{t\in[0,1]} f(\gamma(t)) \geq \inf_{\|z-x_0\|=R} f(z)$ for each $\gamma \in \Gamma$. Hence, from (G_1) one has $c \geq \inf_{\|z-x_0\|=R} f(z) > \max\{f(x), f(y)\} = a$, so (G_1') is proved. Moreover, in particular, one has

$$\max_{t\in[0,1]} f(\gamma(t)) > a \tag{2.1}$$

for all $\gamma \in \Gamma$.

Let $\Psi : \Gamma \to \mathbb{R}$ be the function defined by $\Psi(\gamma) = \max_{t\in[0,1]} f(\gamma(t))$ for all $\gamma \in \Gamma$. We observe that Γ is closed in $C([0,1],E)$ endowed with the usual metric (that is, $d(\gamma_1,\gamma_2) = \|\gamma_1 - \gamma_2\|_\infty = \max_{t\in[0,1]} \|\gamma_1(t) - \gamma_2(t)\|$) and Ψ is lower semi-continuous being the upper envelope of continuous functions. Moreover, Ψ is bounded from below since $\inf_\Gamma \Psi = c > a$.

Our aim is to prove that for every $\varepsilon > 0$ there are $p_\varepsilon \in \Gamma$ and $t_\varepsilon \in [0,1]$ such that

$$c \leq f(p_\varepsilon(t_\varepsilon)) \leq c + \varepsilon \tag{PS1}$$

and

$$\|f'(p_\varepsilon(t_\varepsilon))\|_{E^*} \leq \varepsilon. \tag{PS2}$$

To this end, fix $\varepsilon > 0$ and $\gamma_\varepsilon \in \Gamma$ such that $\Psi(\gamma_\varepsilon) \leq \inf_\Gamma \Psi + \dfrac{\varepsilon}{2}$. Therefore, from the Ekeland Variational Principle (see, for instance, Mawhin-Willem [18, Theorem 4.1 page 75]) there is $p_\varepsilon \in \Gamma$ such that

$$\Psi(p_\varepsilon) \leq \Psi(\gamma_\varepsilon) \tag{E1}$$

and

$$\Psi(p_\varepsilon) \leq \Psi(q) + \frac{\varepsilon}{2}\|p_\varepsilon - q\|_\infty, \quad \forall q \in \Gamma. \tag{E2}$$

Moreover, taking in mind that $f \in C^1(E)$, from Proposition 2.1 there is a continuous function $v_\varepsilon : [0,1] \to E$ such that for each $t \in [0,1]$ one has

$$\|v_\varepsilon(t)\| \leq 1; \qquad \|f'(p_\varepsilon(t))\|_{E^*} \leq \langle f'(p_\varepsilon(t)), v_\varepsilon(t)\rangle + \frac{\varepsilon}{2}.$$

Now, put

$$M(p_\varepsilon) = \{t \in [0,1] : f(p_\varepsilon(t)) = \max_{[0,1]} f(p_\varepsilon)\}.$$

Observe that $M(p_\varepsilon)$ is a nonempty closed set. Moreover, from (2.1) one has $\Psi(p_\varepsilon) > a$ for which $M(p_\varepsilon) \cap \{0,1\} = \emptyset$.

Put

$$\alpha(t) = \frac{\inf_{z\in K^*} |t-z|}{\inf_{z\in K^*} |t-z| + \inf_{l\in M(p_\varepsilon)} |t-l|}, \quad \forall\, t \in [0,1],$$

where $K^* = \{0,1\}$. Clearly, α is locally Lipschitz and $0 \le \alpha(t) \le 1$ as well as $\alpha(0) = \alpha(1) = 0$, $\alpha(t) = 1$ for all $t \in M(p_\varepsilon)$. So, setting $w_\varepsilon(t) = \alpha(t) v_\varepsilon(t)$ and $q_h(t) = p_\varepsilon(t) - h w_\varepsilon(t)$ for all $t \in [0,1]$ and $h > 0$, one has

$$\|f'(p_\varepsilon(t))\|_{E^*} \le \langle f'(p_\varepsilon(t)), w_\varepsilon(t)\rangle + \frac{\varepsilon}{2} \tag{2.2}$$

for all $t \in M(p_\varepsilon)$, since in this case $v_\varepsilon(t) = w_\varepsilon(t)$. Moreover, one has $d(q_h, p_\varepsilon) = \|p_\varepsilon - q_h\|_\infty = h \max_{t\in[0,1]} \|w_\varepsilon(t)\| \le h$, for which $q_h \to p_\varepsilon$ when $h \to 0^+$.
Given $M(q_h) = \{t \in [0,1] : f(q_h(t)) = \max_{[0,1]} f(q_h)\}$, let $h_n \to 0^+$ and $t_{h_n} \in M(q_{h_n})$. Therefore, there is $\tilde{t} \in [0,1]$ (and a subsequence relabelled again t_{h_n}) such that $t_{h_n} \to \tilde{t}$. So, it follows that $\Psi(p_\varepsilon) \le \liminf_{n\to+\infty} \Psi(q_{h_n}) = \liminf_{n\to+\infty} [f(q_{h_n}(t_{h_n})] = f(p_\varepsilon(\tilde{t}))$, namely $\tilde{t} \in M(p_\varepsilon)$. Therefore, from (2.2) one has

$$\|f'(p_\varepsilon(\tilde{t}))\|_{E^*} \le \langle f'(p_\varepsilon(\tilde{t})), w_\varepsilon(\tilde{t})\rangle + \frac{\varepsilon}{2} \tag{2.3}$$

Moreover, from (E2) one has

$$\Psi(q_{h_n}) - \Psi(p_\varepsilon) + \frac{\varepsilon}{2}\|q_{h_n} - p_\varepsilon\|_\infty \ge 0,$$

$$\max_{t\in[0,1]} f(p_\varepsilon(t) - h_n w_\varepsilon(t)) - \max_{t\in[0,1]} f(p_\varepsilon(t)) + \frac{\varepsilon}{2} h_n \|w_\varepsilon\|_\infty \ge 0,$$

$$f(p_\varepsilon(t_{h_n}) - h_n w_\varepsilon(t_{h_n})) - \max_{t\in[0,1]} f(p_\varepsilon(t)) + \frac{\varepsilon}{2} h_n \ge 0,$$

$$f(p_\varepsilon(t_{h_n})) - h_n \langle f'(p_\varepsilon(t_{h_n}), w_\varepsilon(t_{h_n})\rangle + o(h_n) - \max_{t\in[0,1]} f(p_\varepsilon(t)) + \frac{\varepsilon}{2} h_n \ge 0.$$

Since $\max_{t\in[0,1]} f(p_\varepsilon(t)) \ge f(p_\varepsilon(t_{h_n}))$, it follows

$$f(p_\varepsilon(t_{h_n})) - h_n \langle f'(p_\varepsilon(t_{h_n}), w_\varepsilon(t_{h_n})\rangle + o(h_n) - f(p_\varepsilon(t_{h_n})) + \frac{\varepsilon}{2} h_n \ge 0,$$

$$-h_n \langle f'(p_\varepsilon(t_{h_n}), w_\varepsilon(t_{h_n})\rangle + o(h_n) + \frac{\varepsilon}{2} h_n \ge 0,$$

$$h_n \langle f'(p_\varepsilon(t_{h_n}), w_\varepsilon(t_{h_n})\rangle \le \frac{\varepsilon}{2} h_n + o(h_n),$$

$$\langle f'(p_\varepsilon(t_{h_n}), w_\varepsilon(t_{h_n})\rangle \le \frac{\varepsilon}{2} + \frac{o(h_n)}{h_n}.$$

Thus, if we put $t_\varepsilon = \bar{t}$, one has

$$\langle f'(p_\varepsilon(t_\varepsilon), w_\varepsilon(t_\varepsilon)\rangle > \le \frac{\varepsilon}{2}. \tag{2.4}$$

Hence, from (2.3) and (2.4) one has

$$\|f'(p_\varepsilon(t_\varepsilon))\|_{E^*} \le \varepsilon,$$

and (PS2) is verified.
Moreover, from (E1) we obtain $c \leq \Psi(p_\varepsilon) \leq \Psi(\gamma_\varepsilon) \leq c + \dfrac{\varepsilon}{2}$, that is $c \leq f(p_\varepsilon(t_\varepsilon)) \leq c + \dfrac{\varepsilon}{2}$ and also (PS1) is verified.

At this point, by picking $\varepsilon = \dfrac{1}{n}$, $n \in \mathbb{N}^+$, and $x_n = p_{\frac{1}{n}}(t_{\frac{1}{n}})$, the sequence $\{x_n\}$ ensures the conclusion.

$\square$

Remark 2.1. As seen in the proof, the conclusion of Theorem 2.2 holds under the more general condition (G_1'). Hence, actually one has "*If $c > a$, then f admits a $(PS)_c-$sequence.*"

We now give a definition that we will use in the following.

Definition 2.1. *A nonempty set $D \subseteq E$ is said to separate two points x and y if $x, y \notin D$ and for each $\gamma \in \Gamma$ there is $t_\gamma \in]0,1[$ such that $\gamma(t_\gamma) \in D$.*

Remark 2.2. A simple computation shows that condition (G_1') is equivalent to the existence of a set V [respectively: a closed set D] which separates x and y such that

$$\inf_V f > a \qquad [\text{respectively}: \quad \inf_D f > a].$$

Indeed, if $c > a$, we can choose

$$V = \{z \in E : \exists \gamma_z \in \Gamma, \exists t_z \in]0,1[\text{ with } z = \gamma_z(t_z) \text{ and } \max_{t \in [0,1]} f(\gamma_z(t)) = f(\gamma_z(t_z))\}$$

and $D = \overline{V}$.
Hence, one also has "*If there is a set V which separates x and y such that* $\inf_V f > a$*, then f admits a $(PS)_c-$sequence*".

A consequence of Theorem 2.2 is the following result.

Theorem 2.3. (Ambrosetti-Rabinowitz Theorem). *Let E be a real Banach space and $f \in C^1(E)$. Fix $x_0 \in E$ and $R > 0$ and assume that there are $x, y \in E$, with $\|x - x_0\| < R < \|y - x_0\|$ such that*

$$\inf_{\|z - x_0\| = R} f(z) > \max\{f(x), f(y)\}. \qquad (G_1)$$

Put

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)),$$

where $\Gamma = \left\{\gamma \in C([0,1], E) : \gamma(0) = x, \gamma(1) = y\right\}$ *and assume also that*

(*i*) *f satisfies $(PS)_c-$condition.*

Then, there is $\bar{x} \in E$, which is different from x and y, such that

$$f(\bar{x}) = c$$

and

$$f'(\bar{x}) = 0.$$

Proof. From Theorem 2.2 and assumption (i), there is a $(PS)_c-$sequence $\{x_n\}$ such that $x_n \to \bar{x}$. Hence, taking into account that $f \in C^1$, one has $f(\bar{x}) = c$ and $f'(\bar{x}) = 0$. Moreover, since $c > a = \max\{f(x),\ f(y)\}$, as seen in the proof of Theorem 2.2, one has $f(\bar{x}) = c > a = \max\{f(x),\ f(y)\}$ and the conclusion is achieved. □

Remark 2.3. The statement of Theorem 2.3 is essentially the same as given in [2], even if here the mountain pass geometry is expressed in a form which appears more general and, as already in the literature, the Palais-Smale condition is considered at level c. We recall that the original proof is based on a deformation lemma that involves the Palais-Smale condition in the same spirit of Clark [10]. Here, the proof is based on the preliminary construction of a $(PS)_c-$sequence given in Theorem 2.2 where the Palais-Smale condition is not required. There are other results in the literature where the same conclusion of Theorem 2.2 is obtained by a deformation lemma, without assuming the Palais-Smale condition. Generally, the approach based on deformation lemmas makes use of more sophisticated techniques. For instance, a qualitative deformation lemma as proved by Willem represents a nice prototype of this kind of approach and we refer just to the exhaustive book of Willem [28] for more details on this argument.

3. A local lemma

In this section we establish our main result, that is, a lemma which will be used for the proof of the mountain pass theorem in the limiting cases and for a more precise conclusion in order to have a localization of the $(PS)_c-$ sequence. It takes full advantage of the $\varepsilon-$perturbation as introduced by Brezis-Nirenberg. We also say that the basic idea is deduced by that of Pucci-Serrin, that is, roughly speaking, to keep the function above the critical value in a certain region of the space. The perturbation of Brezis-Nirenberg is therefore fundamental for our purposes.
Precisely, we have the following result.

Lemma 3.1. *Let E be a real Banach space, $f \in C^1(E)$ and $c \in \mathbb{R}$. Fix $\varepsilon \in]0, 1[$ and assume that there exists $z \in E$ such that*

(H_1) $f(z) < c + \dfrac{\varepsilon^2}{48}$;

(H_2) $f(w) \geq c$ *for every* $w \in B(z, \varepsilon)$.

Then, there is $x_\varepsilon \in E$ such that

(*i*) $|f(x_\varepsilon) - c| \leq \varepsilon^2$
(*ii*) $\|f'(x_\varepsilon)\|_{E^*} \leq \varepsilon$
(*iii*) $d(x_\varepsilon, z) \leq \varepsilon$.

Proof. Since f is continuous, from assumption (H_1) there is $\sigma > 0$ such that

$$f(w) < c + \frac{\varepsilon^2}{48}, \tag{3.1}$$

for all $w \in B(z, \sigma)$. Moreover, it is not restrictive assume that $\sigma < \frac{\varepsilon}{4}$.
Now, fix $\bar{w} \in B(z, \sigma)$, with $\bar{w} \neq z$, and put

$$\bar{\gamma}(t) = t\bar{w} + (1-t)z$$

for all $t \in [0, 1]$. We have $\|\gamma(t) - z\| = t\|\bar{w} - z\| < \sigma$ for all $t \in [0, 1]$, namely

$$\bar{\gamma}([0, 1]) \subseteq B(z, \sigma),$$

so that, in view of (3.1),

$$f(\bar{\gamma}(t)) < c + \frac{\varepsilon^2}{48}. \tag{3.2}$$

for all $t \in [0, 1]$.

Denote $\bar{x} = z$, $\bar{y} = \bar{w}$ and put

$$\bar{\Gamma} = \bar{\Gamma}(\bar{x}, \bar{y}) = \Big\{\gamma \in C([0, 1], E) : \gamma(0) = \bar{x}, \gamma(1) = \bar{y}\Big\},$$

so that, it is clear that $\bar{\gamma} \in \bar{\Gamma}$.

Consider the Lipschitz continuous function $d : [0, 1] \to [0, 1/2]$ defined by

$$d(t) = \min\{t,\ 1-t\} \quad \forall t \in [0, 1]$$

and put

$$F_\varepsilon(p, t) = f(p(t)) + \frac{\varepsilon^2}{12} d(t) \tag{3.3}$$

for all $p \in \bar{\Gamma}$ and for all $t \in [0, 1]$. It is clear that the function $\Psi_\varepsilon : \bar{\Gamma} \to \mathbb{R}$ defined by

$$\Psi_\varepsilon(p) = \max_{t \in [0,1]} F_\varepsilon(p, t)$$

is lower semi-continuous and bounded from below, being $\Psi_\varepsilon(\gamma) \geq \max\{f(\bar{x}),\ f(\bar{y})\}$ for every $\gamma \in \bar{\Gamma}$.

Moreover, put

$$\bar{a} = \max\{f(\bar{x}),\ f(\bar{y})\}, \qquad \bar{c} = \inf_{\gamma \in \bar{\Gamma}} \max_{t \in [0,1]} f(\gamma(t))$$

and

$$c_\varepsilon = \inf_{p \in \bar{\Gamma}} \max_{t \in [0,1]} F_\varepsilon(p, t).$$

We now point out some claims.

Claim 1. For every $p \in \bar{\Gamma}$ such that $\|p - \bar{\gamma}\|_\infty \leq \frac{\varepsilon}{4}$ one has that

$$p(t) \in B(z, \varepsilon/2).$$

Claim 2. For every $p \in \bar{\Gamma}$ such that $\|p - \bar{\gamma}\|_\infty \leq \frac{\varepsilon}{4}$ one has that

$$f(p(t)) \geq c$$

for all $t \in [0,1]$.

Claim 3. The following condition holds

$$c \leq \bar{a} \leq \bar{c} < c + \frac{\varepsilon^2}{48}. \tag{3.4}$$

Claim 4. The following condition holds

$$\bar{c} \leq c_\varepsilon \leq \bar{c} + \frac{\varepsilon^2}{24}. \tag{3.5}$$

Claim 5. The following condition holds

$$c \leq c_\varepsilon < c + \frac{\varepsilon^2}{16}. \tag{3.6}$$

Claim 6. For every $p \in \bar{\Gamma}$ such that $f(p(1/2)) \geq c$ one has that

$$\Psi_\varepsilon(p) > \bar{a}. \tag{3.7}$$

Let us verify Claim 1. Fix $p \in \bar{\Gamma}$ such that $\|p - \bar{\gamma}\|_\infty \leq \dfrac{\varepsilon}{4}$ and observe that, for all $t \in [0,1]$

$$\|p(t) - z\| \leq \|p(t) - \bar{\gamma}(t)\| + \|\bar{\gamma}(t) - z\| \leq \|p - \bar{\gamma}\|_\infty + \|\bar{\gamma}(t) - z\| < \frac{\varepsilon}{4} + \sigma < \frac{\varepsilon}{2},$$

and the conclusion is achieved.

Let us verify Claim 2. It follows directly from Claim 1 and assumption (H_2).

Let us verify Claim 3. Owing to assumption (H_2), in particular, $f(z) \geq c$ and for every $\gamma \in \bar{\Gamma}$ one has

$$c \leq f(z) = f(\bar{x}) \leq \bar{a} \leq \max_{t \in [0,1]} f(\gamma(t)),$$

namely $c \leq \bar{a} \leq \bar{c}$. On the other hand, taking in mind (3.2), it is clear that

$$\bar{c} \leq \max_{t \in [0,1]} f(\bar{\gamma}(t)) < c + \frac{\varepsilon^2}{48}$$

and the claim is proved.

Let us verify Claim 4. For every $p \in \bar{\Gamma}$ observe that

$$\bar{c} \leq \max_{t \in [0,1]} f(p(t)) \leq \max_{t \in [0,1]} F_\varepsilon(p,t) \leq \max_{t \in [0,1]} f(p(t)) + \frac{\varepsilon^2}{12} \max_{t \in [0,1]} d(t) = \max_{t \in [0,1]} f(p(t)) + \frac{\varepsilon^2}{24},$$

namely (3.5) holds.

Let us verify Claim 5. It follows straightforwardly from Claim 3 and Claim 4. Indeed, from (3.4) and (3.5) one has

$$c \leq \bar{c} \leq c_\varepsilon \leq \bar{c} + \frac{\varepsilon^2}{24} < \left(c + \frac{\varepsilon^2}{48}\right) + \frac{\varepsilon^2}{24} = c + 3\frac{\varepsilon^2}{48},$$

namely (3.6).

Let us verify Claim 6. Fix $p \in \bar{\Gamma}$ such that $f(p(1/2)) \geq c$. Then, it is clear that

$$\Psi_\varepsilon(p) \geq F_\varepsilon(p, 1/2) = f(p(1/2)) + \frac{\varepsilon^2}{24} \geq c + \frac{\varepsilon^2}{24}.$$

On the other hand, exploiting Claim 3 and recalling that $f(z) \geq c$, one has

$$\Psi_\varepsilon(p) \geq c + \frac{\varepsilon^2}{24} > \left(\bar{a} - \frac{\varepsilon^2}{48}\right) + \frac{\varepsilon^2}{24} = \bar{a} + \frac{\varepsilon^2}{48} > \bar{a}.$$

Incidentally, we explicitly wish to point out that, in verifying all the previous claims, the only moment where assumption (H_2) is fully used is in the proof of Claim 2. On the contrary, for all the other claims, the only condition that we used is that $f(z) \geq c$ (see also next Remark 3.1).

Now, we can apply the Ekeland variational principle in correspondence of ε and $\bar{\gamma}$ previously introduced. Indeed, from (3.2) and Claim 3, one has

$$\max_{t\in[0,1]} F_\varepsilon(\bar{\gamma}, t) \leq \max_{t\in[0,1]} f(\bar{\gamma}(t)) + \frac{\varepsilon^2}{24} < c + \frac{\varepsilon^2}{48} + \frac{\varepsilon^2}{24} \leq \bar{a} + \frac{\varepsilon^2}{16} \leq c_\varepsilon + \frac{\varepsilon^2}{16},$$

that is,

$$\Psi_\varepsilon(\bar{\gamma}) < \inf_{\bar{\Gamma}} \Psi_\varepsilon + \frac{\varepsilon^2}{16}. \tag{3.8}$$

Therefore (see, for instance, [18, Theorem 4.1 and Remark 4.1]), there is $p_\varepsilon \in \bar{\Gamma}$ such that

$$\|p_\varepsilon - \bar{\gamma}\|_\infty \leq \frac{\varepsilon}{4}; \tag{E0}$$

$$\Psi_\varepsilon(p_\varepsilon) \leq \Psi_\varepsilon(\bar{\gamma}) \tag{E1'}$$

and

$$\Psi_\varepsilon(p_\varepsilon) \leq \Psi_\varepsilon(q) + \frac{\varepsilon}{4}\|p_\varepsilon - q\|_\infty, \quad \forall q \in \bar{\Gamma}. \tag{E2'}$$

Now, in view of (E0) we can recall Claim 2 and then Claim 6 in order to conclude that

$$\Psi_\varepsilon(p_\varepsilon) > \bar{a}. \tag{3.9}$$

At this point, we can argue in a similar way of the proof of Theorem 2.2. We give a sketch below to facilitate the reader.

Indeed, taking in mind that $f \in C^1(E)$ from Proposition 2.1 there is a continuous function $v_\varepsilon : [0,1] \to E$ such that for each $t \in [0,1]$ one has

$$\|v_\varepsilon(t)\| \leq 1; \qquad \|f'(p_\varepsilon(t))\|_{E^*} \leq \quad \langle f'(p_\varepsilon(t)), v_\varepsilon(t)\rangle + \frac{\varepsilon}{2}.$$

Put

$$M(p_\varepsilon) = \{t \in [0,1] : F_\varepsilon(p_\varepsilon, t) = \max_{s\in[0,1]} F_\varepsilon(p_\varepsilon, s)\}.$$

Observe that $M(p_\varepsilon)$ is a nonempty closed set. Moreover, from (3.9) one has $M(p_\varepsilon) \cap \{0,\ 1\} = \emptyset$.

Put

$$\alpha(t) = \frac{\inf_{z\in K^*} |t-z|}{\inf_{z\in K^*} |t-z| + \inf_{l\in M(p_\varepsilon)} |t-l|},$$

where $K^* = \{0,1\}$. Clearly, α is locally Lipschitz and $0 \leq \alpha(t) \leq 1$ as well as $\alpha(0) = \alpha(1) = 0$, $\alpha(t) = 1$ for all $t \in M(p_\varepsilon)$. So, setting $w_\varepsilon(t) = \alpha(t)v_\varepsilon(t)$ and $q_h(t) = p_\varepsilon(t) - hw_\varepsilon(t)$ for all $t \in [0,1]$ and $h > 0$, one has

$$\|f'(p_\varepsilon(t))\|_{E^*} \leq \quad \langle f'(p_\varepsilon(t)), w_\varepsilon(t)\rangle + \frac{\varepsilon}{2} \tag{3.10}$$

for all $t \in M(p_\varepsilon)$, since in this case $v_\varepsilon(t) = w_\varepsilon(t)$. Moreover, one has $d(q_h, p_\varepsilon) = \|p_\varepsilon - q_h\|_\infty = h\max_{t\in[0,1]} \|w_\varepsilon(t)\| \leq h$, for which $q_h \to p_\varepsilon$ when $h \to 0^+$.

Given $M(q_h) = \{t \in [0,1] : F_\varepsilon(q_h, t) = \max_{s\in[0,1]} F_\varepsilon(q_h, s)$, let $h_n \to 0^+$ be and $t_{h_n} \in M(q_{h_n})$. Therefore, there is $\tilde{t} \in [0,1]$ (and a subsequence relabelled again t_{h_n}) such that $t_{h_n} \to \tilde{t}$. So, it follows $\Psi_\varepsilon(p_\varepsilon) \leq \liminf_{n\to+\infty} \Psi_\varepsilon(q_{h_n}) = \liminf_{n\to+\infty} F_\varepsilon(q_{h_n}, t_{h_n}) = F_\varepsilon(p_\varepsilon, \tilde{t})$, namely $\tilde{t} \in M(p_\varepsilon)$. Therefore, from (3.10) one has

$$\|f'(p_\varepsilon(\tilde{t}))\|_{E^*} \leq \quad \langle f'(p_\varepsilon(\tilde{t})), w_\varepsilon(\tilde{t})\rangle + \frac{\varepsilon}{2}. \tag{3.11}$$

Moreover, from (E2') one has

$$\Psi_\varepsilon(q_{h_n}) - \Psi_\varepsilon(p_\varepsilon) + \frac{\varepsilon}{4}\|q_{h_n} - p_\varepsilon\|_\infty \geq 0,$$

$$\max_{t\in[0,1]} \left[f(p_\varepsilon(t) - h_n w_\varepsilon(t)) + \frac{\varepsilon^2}{12}d(t)\right] - \Psi_\varepsilon(p_\varepsilon) + \frac{\varepsilon}{4}h_n\|w_\varepsilon\|_\infty \geq 0,$$

$$f(p_\varepsilon(t_{h_n}) - h_n w_\varepsilon(t_{h_n})) + \frac{\varepsilon^2}{12}d(t_{h_n}) - \Psi_\varepsilon(p_\varepsilon) + \frac{\varepsilon}{4}h_n \geq 0,$$

$$f(p_\varepsilon(t_{h_n})) - h_n\langle f'(p_\varepsilon(t_{h_n}), w_\varepsilon(t_{h_n})\rangle + o(h_n) + \frac{\varepsilon^2}{12}d(t_{h_n}) - \Psi_\varepsilon(p_\varepsilon) + \frac{\varepsilon}{4}h_n \geq 0.$$

Since $\Psi_\varepsilon(p_\varepsilon) \geq f(p_\varepsilon(t_{h_n})) + \frac{\varepsilon^2}{12}d(t_{h_n})$, it follows

$$-h_n\langle f'(p_\varepsilon(t_{h_n}), w_\varepsilon(t_{h_n})\rangle + o(h_n) + \frac{\varepsilon}{4}h_n \geq 0,$$

$$h_n\langle f'(p_\varepsilon(t_{h_n}), w_\varepsilon(t_{h_n})\rangle \leq \frac{\varepsilon}{4}h_n + o(h_n),$$

$$\langle f'(p_\varepsilon(t_{h_n}), w_\varepsilon(t_{h_n})\rangle \leq \frac{\varepsilon}{4} + \frac{o(h_n)}{h_n}$$

Therefore, if we put $t_\varepsilon = \tilde{t}$, one has

$$\langle f'(p_\varepsilon(t_\varepsilon), w_\varepsilon(t_\varepsilon)\rangle \leq \frac{\varepsilon}{4}. \tag{3.12}$$

Hence, from (3.11) and (3.12) one has

$$\|f'(p_\varepsilon(t_\varepsilon))\|_{E^*} \leq \Big(\frac{1}{2} + \frac{1}{4}\Big)\varepsilon. \tag{3.13}$$

Moreover, from (E1') and (3.8) we obtain $c_\varepsilon \leq \Psi_\varepsilon(p_\varepsilon) \leq \Psi_\varepsilon(\bar{\gamma}) \leq c_\varepsilon + \frac{\varepsilon^2}{16}$ and recalling (3.6) of Claim 5 one has $c \leq \Psi_\varepsilon(p_\varepsilon) \leq \Big(c + \frac{\varepsilon^2}{16}\Big) + \frac{\varepsilon^2}{16} \leq c + \frac{\varepsilon^2}{8}$, that is,

$$c \leq \Psi_\varepsilon(p_\varepsilon) \leq c + \frac{\varepsilon^2}{8}.$$

Hence,

$$c \leq \Psi_\varepsilon(p_\varepsilon) = F_\varepsilon(p_\varepsilon, t_\varepsilon) \leq f(p_\varepsilon(t_\varepsilon)) + \frac{\varepsilon^2}{24},$$

that leads to

$$c - \frac{\varepsilon^2}{24} \leq f(p_\varepsilon(t_\varepsilon)) \leq c + \frac{\varepsilon^2}{8}. \tag{3.14}$$

From (E0) and Claim 1 one has

$$d(p_\varepsilon(t_\varepsilon), z) \leq \frac{\varepsilon}{2}. \tag{3.15}$$

At this point, by picking $x_\varepsilon = p_\varepsilon(t_\varepsilon)$, from (3.13), (3.14), (3.15) one has

$$\|f'(x_\varepsilon)\|_{E^*} \leq \frac{3}{4}\varepsilon, \quad c - \frac{\varepsilon^2}{8} \leq f(x_\varepsilon) \leq c + \frac{\varepsilon^2}{8}, \quad d(x_\varepsilon, z) \leq \frac{\varepsilon}{2}$$

and the proof is complete. □

Remark 3.1. A careful analysis of the above proof points out that the crucial condition (3.9) is obtained exploiting Claim 6. Hence, as already observe inside its proof, Lemma 3.1 is true also without assuming (H_2), provided that one has

$$f(z) \geq c$$

and

$$f(p_\varepsilon(1/2)) \geq c$$

where p_ε is the path obtained by the Ekeland Variational Principle (see (E0)–(E2')).

Remark 3.2. We wish to stress that, in Lemma 3.1, c is a completely arbitrary number. Hence, when it is the case, we can choose $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$, with, as usual, $\Gamma = \{\gamma \in C^0([0,1], E) : \ \gamma(0) = x, \ \gamma(1) = y\}$, and use Lemma 3.1 as a new tool for obtaining a $(PS)_c-$sequence in the classical mountain pass theorem framework. Indeed, the arbitrariness of c allows to apply Lemma 3.1 to further more general situations, covering also settings of linking-type, as we will see in the next two sections (see Theorem 4.1 and Remark 5.7).

4. The Pucci-Serrin theorem

In this section, we recall the Pucci-Serrin theorem and its seminal corollary. We give a novel proof which is a consequence of the Lemma 3.1 presented in the previous section. Precisely we have the following result.

Theorem 4.1. *Let E be a real Banach space and $f \in C^1(E)$. Fix $x_0 \in E$ and $R_1, R_2 > 0$ and assume that there are $x, y \in E$, with $\|x - x_0\| < R_1 < R_2 < \|y - x_0\|$ such that*

$$\inf_{R_1<\|z-x_0\|<R_2} f(z) \geq \max\{f(x), f(y)\}. \tag{G_2}$$

Put

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)),$$

where $\Gamma = \Big\{\gamma \in C([0,1], E) : \gamma(0) = x, \gamma(1) = y\Big\}$.

Then, f admits a $(PS)_c-$sequence. In particular, if $c = a := \max\{f(x), \ f(y)\}$, then f admits a bounded $(PS)_c-$sequence $\{x_n\}$ such that $d(x_n, \partial B) \leq \dfrac{1}{n}$ for all $n \in \mathbb{N}$, where $\partial B = \{z \in E : \|z - x_0\| = (R_1 + R_2)/2\}$.

Proof. Put $R = (R_1 + R_2)/2$. Clearly, from (G_2) one has

$$\inf_{\|z-x_0\|=R} f(z) \geq \inf_{R_1<\|z-x_0\|<R_2} f(z) \geq \max\{f(x), f(y)\}.$$

If one has the strict inequality (or $c > \inf_{\|z-x_0\|=R} f(z)$ if $\inf_{\|z-x_0\|=R} f(z) = a = \max\{f(x), \ f(y)\}$) then ($G_1'$) holds (with $R = (R_1 + R_2)/2$) and Theorem 2.2 (see also Remark 2.1) ensures the conclusion. So, we can assume

$$c = \inf_{\|z-x_0\|=R} f(z) = \max\{f(x), f(y)\}.$$

Therefore, from $\inf_{\|z-x_0\|=R} f(z) = c$, for all $n \in \mathbb{N}$ such that $\dfrac{1}{n} < (R_2 - R_1)/2$ there is $z_n \in \partial B$ such that $f(z_n) < c + \dfrac{(1/n)^2}{48}$ and, taking (G_2) into account, $f(w) \geq a = c$

for all $w \in B(z_n, \frac{1}{n})$. Hence, Lemma 3.1, see also Remark 3.2, ensures that there is a sequence $\{x_n\}$ such that

$$c-(1/n)^2 \leq f(x_n) \leq c+(1/n)^2, \quad \|f'(x_n)\|_{E^*} \leq (1/n), \quad d(x_n, \partial B) \leq d(x_n, z_n) \leq (1/n)$$

for all $n > 2/(R_2 - R_1)$ and the conclusion is achieved.

□

As consequences of Theorem 4.1 we obtain the following two famous classical results of Pucci-Serrin.

Theorem 4.2. (Pucci-Serrin Theorem). *Let E be a real Banach space and $f \in C^1(E)$. Fix $x_0 \in E$ and $R_1, R_2 > 0$ and assume that there are $x, y \in E$, with $\|x - x_0\| < R_1 < R_2 < \|y - x_0\|$ such that*

$$\inf_{R_1 < \|z-x_0\| < R_2} f(z) \geq \max\{f(x), f(y)\}. \qquad (G_2)$$

Put

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)),$$

where $\Gamma = \Big\{\gamma \in C([0,1], E) : \gamma(0) = x, \gamma(1) = y\Big\}$ and assume also that

(*i*) *f satisfies $(PS)_c-$condition.*

Then, there is $\bar{x} \in E$, which is different from x and y, such that

$$f(\bar{x}) = c$$

and

$$f'(\bar{x}) = 0.$$

In particular, if $c = a$ then $\|\bar{x} - x_0\| = (R_1 + R_2)/2$.

Definition 4.1. *A point $x \in E$ is said to be a local minimum point of $f : E \to \mathbb{R}$ if there exists a neighbourhood V of x such that $f(z) \geq f(x)$ for every $z \in V$.*

Corollary 4.1. (Pucci-Serrin Corollary). *Let E be a real Banach space and $f \in C^1(E)$. Assume that f admits two different local minimum points $x, y \in E$. Put*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)),$$

where $\Gamma = \Big\{\gamma \in C([0,1], E) : \gamma(0) = x, \gamma(1) = y\Big\}$ and assume also that

(*i*) *f satisfies $(PS)_c-$condition.*

Then, f admits a third distinct critical point.

Remark 4.1. We explicitly observe that when, instead of (G_1), the so called zero altitude occurs, namely

$$\inf_{\|z-x_0\|=R} f(z) = \max\{f(x),\ f(y)\},$$

from the Pucci-Serrin theorem the existence of a critical point can be again obtained. However, for this end, the (PS)–condition must be assumed. Moreover, in such a case, the level of the obtained critical point may be higher or such a critical point may coincide with one of the end points x or y (see, for instance, de Figueiredo-Solimini [12]). In particular, such a result can be obtained by a characterization of the mountain pass geometry (see [5, Theorem 2.1] or [6, Theorem 2.1]), that we report below in a suitable form.

Proposition 4.1. *Let E be a real Banach space and $f \in C^1(E)$. Fix $x_0 \in E$ and $R > 0$ and assume that f fulfills (PS) and it is bounded from below or on $B(x_0, R)$ or on $E \setminus \bar{B}(x_0, R)$. Then, the following assertions are equivalent:*

(*a*) *there are $x, y \in E$, with $\|x - x_0\| < R < \|y - x_0\|$, such that*

$$\inf_{\|z-x_0\|=R} f(z) \geq \max\{f(x), f(y)\}; \qquad (G_3)$$

(*b*) *f admits at least one global minimum or on $B(x_0, R)$ or on $E\setminus\bar{B}(x_0, R)$ which is not strictly global on the whole E.*

Remark 4.2. Taking into account the above proposition, assuming (G_3) and the $(PS)-$condition, the existence of a critical point follows from Theorems 2.3 and 4.2, and Corollary 4.1. The Ghoussoub-Preiss theorem, that we will see in the next section, gives a more precise result. Indeed, under only the $(PS)_c-$condition, it ensures that, when the equality holds, the critical point is again at level c and it belongs to a set that separates x and y, namely it is distinct from x and y.

5. The Ghoussoub-Preiss theorem

In this section, we recall the Ghoussoub-Preiss theorem giving a novel proof based on the $\varepsilon-$perturbation as introduced by Brezis-Nirenberg. Precisely, such a proof is a combination of the ideas contained in the same Ghoussoub-Preiss work with those of Brezis-Nirenberg, which have been previously exploited in Lemma 3.1, which here is our main tool.

First, we examine the version of the Ghoussoub-Preiss theorem, Theorem 5.1, where the mountain pass geometry is given by (G_3) below, that is, roughly speaking, the original assumption of Ambrosetti-Rabinowitz where also the equality is considered. Next, we give the more general version of the Ghoussoub-Preiss theorem, Theorem 5.3, which includes the previous case and provides information in addition on $(PS)_c-$sequence in terms of distance.

Theorem 5.1. (Ghoussoub-Preiss Theorem). *Let E be a real Banach space and $f \in C^1(E)$. Fix $x_0 \in E$ and $R > 0$ and assume that there are $x, y \in E$, with $\|x - x_0\| < R < \|y - x_0\|$ such that*

$$\inf_{\|z-x_0\|=R} f(z) \geq \max\{f(x), f(y)\}. \tag{G_3}$$

Put

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)),$$

where $\Gamma = \Big\{\gamma \in C([0,1], E) : \gamma(0) = x, \gamma(1) = y\Big\}$.

Then, f admits a $(PS)_c$−sequence. In particular, if $c = a := \max\{f(x),\ f(y)\}$, then f admits a bounded $(PS)_c$−sequence $\{x_n\}$ such that $d(x_n, \partial B) \leq \dfrac{1}{n}$ for all $n \in \mathbb{N}$, where $\partial B = \{z \in E : \|z - x_0\| = R\}$.

Proof. Condition (G_3) implies that

$$\begin{cases} \textit{either} \quad c > a, \\ \textit{or } c = a \textit{ and for all } \ \gamma \in \Gamma \textit{ there is } t_\gamma \in]0,1[\textit{ such that } f(\gamma(t_\gamma)) \geq a. \end{cases} \tag{G_3'}$$

Indeed, for each $\gamma \in \Gamma$ there is $t_\gamma \in]0,1[$ such that $\|\gamma(t_\gamma) - x_0\| = R$, so that one has $f(\gamma(t_\gamma)) \geq \inf_{\|z-x_0\|=R} f(z)$. It follows that $\max_{t\in[0,1]} f(\gamma(t)) \geq \inf_{\|z-x_0\|=R} f(z)$ for each $\gamma \in \Gamma$. Hence, from (G_3) one has $c \geq \inf_{\|z-x_0\|=R} f(z) \geq \max\{f(x), f(y)\} = a$. So, our claim is proved.

We can assume

$$c = \inf_{\|z-x_0\|=R} f(z) = a \tag{5.1}$$

since otherwise, that is $c > a$, the conclusion follows from the proof of Theorem 2.2. Fixed $\varepsilon > 0$, we start from an idea of Ghoussoub-Preiss in order to get close to ∂B. Precisely, from $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$ there is $\bar{\gamma} \in \Gamma$ such that

$$c \leq \max_{t\in[0,1]} f(\bar{\gamma}(t)) < c + \frac{\varepsilon^2}{48}. \tag{5.2}$$

Put

$$T_0 = \sup\{t \in [0,1] : \bar{\gamma}(t) \in B(x_0, R), d(\bar{\gamma}(t), \partial B) \geq \varepsilon\}$$

and

$$T_1 = \inf\{t \in [T_0, 1] : \bar{\gamma}(t) \in E \setminus \bar{B}(x_0, R), d(\bar{\gamma}(t), \partial B) \geq \varepsilon\}.$$

Clearly, one has

$$d(\bar{\gamma}(t), \partial B) \leq \varepsilon \quad \forall t \in [T_0, T_1]. \tag{5.3}$$

Now, put $z_1 = \bar{\gamma}(T_0)$ and consider $B(z_1, \varepsilon)$ which is included in $B(x_0, R)$. We can have either $f(z_1) \geq c$ or $f(z_1) < c$. In the first case, if $\bar{w}_1 \in B(z_1, \sigma)$, with $\sigma < \frac{\varepsilon}{4}$, we argue as in Lemma 3.1 and we obtain a path p^1_ε, joining z_1 with $\bar{w}_1$

and such that $p^1_\varepsilon([0,1]) \subset B(z_1,\varepsilon) \subset B(x_0,R)$, which satisfies (E0), (E1'), (E2'). Setting $w_1 = p^1_\varepsilon(1/2)$ we can have either $f(w_1) \geq c$ or $f(w_1) < c$. Moreover, put $z_2 = \bar\gamma(T_1)$ and consider $B(z_2,\varepsilon)$ which is included in $E \setminus \bar B(x_0,R)$. We can have either $f(z_2) \geq c$ or $f(z_2) < c$. As above, in the first case, if $\bar w_2 \in B(z_2,\sigma)$, with $\sigma < \frac{\varepsilon}{4}$, we argue as in Lemma 3.1 and we obtain a path p^2_ε, joining z_2 with $\bar w_2$ and such that $p^2_\varepsilon([0,1]) \subset B(z_2,\varepsilon) \subset E \setminus \bar B(x_0,R)$ which satisfies (E0), (E1'), (E2'). Setting $w_2 = p^2_\varepsilon(1/2)$ we can have either $f(w_2) \geq c$ or $f(w_2) < c$.

At this point, put $m_1 = \min\{f(z_1),\ f(w_1)\}$, $m_2 = \min\{f(z_2),\ f(w_2)\}$, the two cases occurs

(A) $\max\{m_1,\ m_2\} \geq c$;
(B) $\max\{m_1,\ m_2\} < c$.

In the case (A) we apply Lemma 3.1 (see also Remark 3.1) for which (taking also (5.3) into account) there is $x_\varepsilon \in E$ such that

$$|f(x_\varepsilon) - c| \leq \varepsilon^2; \qquad \|f'(x_\varepsilon)\|_{E^*} \leq \varepsilon; \qquad d(x_\varepsilon, \partial B) \leq 2\varepsilon. \tag{5.4}$$

Indeed, in particular, we have $d(z_1,\partial B) = \varepsilon$ from (5.3) and $d(x_\varepsilon, z_1) \leq \varepsilon$ from Lemma 3.1.

In the case (B) observe that if $f(z_1) \geq c$ then $f(w_1) < c$ and, analogously, if $f(z_2) \geq c$ then $f(w_2) < c$. Denote by $\tilde x$ the point z_1 if $f(z_1) < c$ or the point w_1 otherwise. Similarly $\tilde y$ indicates z_2 or w_2 accordingly if $f(z_2) < c$ or not. Put

$$\tilde\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = \tilde x, \gamma(1) = \tilde y\},$$

let $\tilde p^i_\varepsilon$ be the restriction of p^i_ε joining z_i with w_i, where $i = 1,\ 2$, let $\hat\gamma$ be the restriction of $\bar\gamma$ joining z_1 with z_2 and consider the path $\tilde\gamma$ as follows

$$\tilde\gamma = \begin{cases} \hat\gamma & \text{if } f(z_1) < c,\ f(z_2) < c \\ (-\tilde p^1_\varepsilon) \cup \hat\gamma & \text{if } f(z_1) \geq c,\ f(z_2) < c \\ \hat\gamma \cup (-\tilde p^2_\varepsilon) & \text{if } f(z_1) < c,\ f(z_2) \geq c \\ (-\tilde p^1_\varepsilon) \cup \hat\gamma \cup (-\tilde p^2_\varepsilon) & \text{if } f(z_1) \geq c,\ f(z_2) \geq c. \end{cases}$$

Clearly it is always possible to parametrize $\tilde\gamma$ in such a way that $\tilde\gamma \in C^0([0,1],X)$; for example in the latest case, that is if $f(z_1) \geq c$ and $f(z_2) \geq c$, $\tilde\gamma$ joins $\tilde x = w_1$ to $\tilde y = w_2$, passing through z_1 and z_2, and we consider

$$\tilde\gamma(t) = \begin{cases} p^1_\varepsilon(1/2 - 2t) & \text{if } t \in [0,1/4[\\ \bar\gamma((T_1 - T_0)(2t - 1/2) + T_0) & \text{if } t \in [1/4,3/4[\\ p^2_\varepsilon(2t - 3/2) & \text{if } t \in [3/4,1]. \end{cases}$$

Similarly, we argue in the other previous three cases. So, in all cases one can verify that

$$\tilde\gamma \in \tilde\Gamma$$

and

$$f(\tilde\gamma(t)) < c + \frac{\varepsilon^2}{8} \qquad \forall t \in [0,1]. \tag{5.5}$$

Indeed, for $i = 1, 2$, put

$$c^i_\varepsilon = \inf_{p\in\bar\Gamma_i} \max_{s\in[0,1]} F_\varepsilon(p, s),$$

where $\bar\Gamma_i = \{\gamma \in C^0([0, 1], E) : \ \gamma(0) = z_i, \ \gamma(1) = \bar w_i\}$, while F_ε is defined in (3.3), and observe that (5.5) follows from (5.2) and (E1'), taking also into account (3.6) and (3.8), being

$$f(p^i_\varepsilon(t)) \le \Psi_\varepsilon(p^i_\varepsilon) \le c^i_\varepsilon + \frac{\varepsilon^2}{16} < c + \frac{\varepsilon^2}{16} + \frac{\varepsilon^2}{16}.$$

Now, if $\Psi : \tilde\Gamma \to \mathbb{R}$ is defined by $\Psi(\gamma) = \max_{t\in[0,1]} f(\gamma(t))$ for all $\gamma \in \tilde\Gamma$, denoting $\tilde c = \inf_{\gamma\in\tilde\Gamma} \max_{t\in[0,1]} f(\gamma(t)) = \inf_{\tilde\Gamma} \Psi$ and $\tilde a = \max\{f(\tilde x), f(\tilde y)\}$, one has $\tilde c \ge c$ (since every path of $\tilde\Gamma$ passes through the boundary ∂B and (5.1) holds) and $c > \tilde a$, because we are in the case (B). So, we have

$$\tilde c > \tilde a. \tag{5.6}$$

In addition, from (5.5) (taking into account again that $\tilde c \ge c$) one has that $\tilde\gamma \in \tilde\Gamma$ is such that

$$\Psi(\tilde\gamma) < c + \frac{\varepsilon^2}{8} \le \inf_{\tilde\Gamma} \Psi + \frac{\varepsilon^2}{8}.$$

Therefore, from the Ekeland Variational Principle there is $\tilde p_\varepsilon \in \tilde\Gamma$ such that

$$\|\tilde p_\varepsilon - \tilde\gamma\|_\infty \le \frac{\varepsilon}{\sqrt 8}; \tag{E0*}$$

$$\Psi(\tilde p_\varepsilon) \le \Psi(\tilde\gamma) \tag{E1*}$$

and

$$\Psi(\tilde p_\varepsilon) \le \Psi(q) + \frac{\varepsilon}{\sqrt 8}\|\tilde p_\varepsilon - q\|_\infty, \quad \forall q \in \tilde\Gamma. \tag{E2*}$$

Arguing as in the proof of Theorem 2.2, exploiting (5.6), from (E1*) and (E2*) we get a $t_\varepsilon \in [0, 1]$ such that $f(p_\varepsilon(t_\varepsilon)) = \Psi(p_\varepsilon)$ and

$$\|f'(\tilde p_\varepsilon(t_\varepsilon))\|_{E^*} \le (\frac{1}{\sqrt 8} + \frac{1}{2})\varepsilon,$$

in addition to $\tilde c \le \Psi(\tilde p_\varepsilon) \le \Psi(\tilde\gamma) \le \tilde c + \frac{\varepsilon^2}{8}$, that is, $\tilde c \le f(\tilde p_\varepsilon(t_\varepsilon)) \le \tilde c + \frac{\varepsilon^2}{8}$.

Finally, from (E0*) one has $d(\tilde p_\varepsilon(t_\varepsilon), \partial B) \le d(\tilde p_\varepsilon(t_\varepsilon), \tilde\gamma(t_\varepsilon)) + d(\tilde\gamma(t_\varepsilon), \partial B) \le \left(\frac{1}{\sqrt 8} + 2\right)\varepsilon$.

Hence, setting $x_\varepsilon = \tilde p_\varepsilon(t_\varepsilon)$, taking into account that $c \le \tilde c \le f(x_\varepsilon) \le \tilde c + \frac{\varepsilon^2}{8}$ and that from (5.5) it follows that $\tilde c \le c + \frac{\varepsilon^2}{8}$, one has $c \le f(x_\varepsilon) \le c + \frac{\varepsilon^2}{4}$. For which, we have

$$|f(x_\varepsilon) - c| \leq \frac{\varepsilon^2}{4}; \qquad \|f'(x_\varepsilon)\|_{E^*} \leq \varepsilon; \qquad d(x_\varepsilon, \partial B) \leq 3\varepsilon. \tag{5.6}$$

In conclusion, by putting $\varepsilon = \dfrac{1}{3n}$ from (5.4) and (5.6), the sequence $\{x_n\}$ is a bounded $(PS)-$sequence such that $d(x_n, \partial B) \leq \dfrac{1}{n}$ for all $n \in \mathbb{N}$ and the theorem is proved.

□

A consequence of Theorem 5.1 is the following result.

Theorem 5.2. (Ghoussoub-Preiss Theorem). *Let E be a real Banach space and $f \in C^1(E)$. Fix $x_0 \in E$ and $R > 0$ and assume that there are $x, y \in E$, with $\|x - x_0\| < R < \|y - x_0\|$ such that*

$$\inf_{\|z - x_0\| = R} f(z) \geq \max\{f(x), f(y)\}. \tag{G_3}$$

Put

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)),$$

where $\Gamma = \Big\{\gamma \in C([0,1], E) : \gamma(0) = x, \gamma(1) = y\Big\}$ and assume also that

(i) f satisfies $(PS)_c-$condition.

Then, there is $\bar{x} \in E$, which is different from x and y, such that

$$f(\bar{x}) = c$$

and

$$f'(\bar{x}) = 0.$$

In particular, if $c = a := \max\{f(x),\ f(y)\}$ then $\|\bar{x} - x_0\| = R$.

Proof. It follows from Theorem 5.1. In particular, when $c = a$, the critical point $\bar{x} \in \partial B$, for which it is distinct from x and y.

□

Remark 5.1. Brezis-Nirenberg in [8] (see also Ghoussoub [14]) proved that the conclusion of Theorem 5.1 holds under the more general condition (G_3'). We observe that condition (G_3') is equivalent to the existence of a nonempty set $D \subseteq E$ which separates x and y such that $\inf_D f \geq a$ (see also Remark 2.2). So, summarizing, the assumption

(j) *there is a nonempty set $D \subseteq E$ which separates x and y such that*

$$\inf_D f \geq a$$

ensures, owing to [8, Theorem 1], the existence of $(PS)_c-$sequence for f. However, in this case, we do not obtain a useful localization of the $(PS)_c-$sequence and, in particular, when $(PS)_c-$condition holds, the critical point may coincides with one of the end points x or y. We recall that Brezis-Nirenberg proved such a result owing to their $\varepsilon-$perturbation that we have used in the proof of the Lemma 3.1 (see Brezis-Nirenberg [8, Theorem 1] for details).

For the sake of completeness we also give the more general version of the Ghoussoub-Preiss theorem.

Theorem 5.3. (Ghoussoub-Preiss Theorem). *Let E be a real Banach space and $f \in C^1(E)$. Fix $x, y \in E$, with $x \neq y$ and put*

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)),$$

where $\Gamma = \Big\{\gamma \in C([0,1], E) : \gamma(0) = x, \gamma(1) = y\Big\}$.
Assume that

(k) *there is a nonempty closed set $T \subseteq E$ which separates x and y such that*

$$\inf_T f = c.$$

Then, f admits a $(PS)_c-$sequence $\{x_n\}$ such that $d(x_n, T) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$.

Proof. First observe that $E \setminus T$ is not path-connected. Otherwise, since $x,\ y \in E \setminus T$ there exits $\gamma \in \Gamma$ such that $\gamma([0,1]) \subseteq E \setminus T$ in contradiction with (k). Hence, denote by X the path-connected component of x in $E \setminus T$, $V = \partial X$ and $Y = E \setminus \overline{X}$. We claim that

$$V \subseteq T. \tag{5.7}$$

Indeed, by contradiction, if $v \in V \cap (E \setminus T)$ we can denote by C_v the path-connected component of v in $E \setminus T$. Clearly, since $v \notin X$, one has

$$C_v \cap X = \emptyset. \tag{5.8}$$

On the other hand, C_v is an open set and, from $v \in V$, it follows that $C_v \cap X \neq \emptyset$, against (5.8).

From (5.7) and (k) it is clear that $\inf_V f \geq \inf_T f = c$. Moreover, simple connectedness arguments show that for every $\gamma \in \Gamma$ there exists $t_\gamma \in]0,1[$ such that $\gamma(t_\gamma) \in V$, namely V separates x and y (see Remark 2.2). Hence, for every $\gamma \in \Gamma$ one has

$$\max_{t\in[0,1]} f(\gamma(t)) \geq f(t_\gamma) \geq \inf_V f,$$

namely $c \geq \inf_V f$ and, in conclusion

$$c = \inf_{z\in V} f(z). \tag{5.1'}$$

Now, we observe that condition (G_3') is satisfied.

At this point, repeating the same computations detailed in the proof of Theorem 5.1 by starting immediately after the formula (5.1), replacing ∂B, $B(x_0,R)$ and $E \setminus \bar{B}(x_0,R)$ by V, X and Y respectively the existence of a $(PS)_c-$sequence $\{x_n\}$ such that $d(x_n,T) \le d(x_n,V) \le \dfrac{1}{n}$ for all $n \in \mathbb{N}$, is proved.

□

Theorem 5.4. (Ghoussoub-Preiss Theorem). *Let E be a real Banach space and $f \in C^1(E)$. Fix $x,y \in E$, with $x \neq y$ and put*

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)),$$

where $\Gamma = \Big\{\gamma \in C([0,1],E) : \gamma(0) = x, \gamma(1) = y\Big\}$.
Assume that

(k) *there is a nonempty closed set $T \subseteq E$ which separates x and y such that*

$$\inf_T f = c$$

and assume also that

(i) *f satisfies $(PS)_c-$condition.*

Then, there is $\bar{x} \in T$ such that

$$f(\bar{x}) = c$$

and

$$f'(\bar{x}) = 0.$$

In particular, since $\bar{x} \in T$, it is different from x and y.

Proof. It follows from Theorem 5.3.

□

Remark 5.2. The original assumption of Ghoussoub-Preiss theorem is the following:
(k') *there is a closed set $F \subseteq E$ for which the set $H = F \cap \{x \in E : f(x) \ge c\}$ is such that x and y belong two disjoint connected components of $E \setminus H$.*

Indeed, assuming (k') and setting $D = H$, the set D separates x and y and one has $\inf_D \ge c$. So, for each $\gamma \in \Gamma$, fixed $t_D \in [0,1]$ such that $\gamma(t_D) \in D$, taking into account that $\max_{t\in[0,1]} f(\gamma(t)) \ge f(\gamma(t_D)) \ge \inf_D f$, it follows $\inf_D f = c$, that is (k) is verified.

Remark 5.3. From Theorems 5.3 and 5.4 we can replace condition (k) with

(j') *there is a nonempty closed set $D \subseteq E$ which separates x and y such that*

$$\inf_D f \ge a,$$

and the case $c = \inf_D f = a$ is exactly that examined in Theorem 5.3. So, for instance, the statement of Theorem 5.4 becomes as follows: *Assuming* (j') *and* (i)*, then there is* $\bar{x}$ *different by* x *and* y *such that* $f(\bar{x}) = c$ *and* $f'(\bar{x}) = 0$*. In particular, if* $c = a$ *then* $\bar{x} \in D$*.*

Remark 5.4. As already highlighted by Ghoussoub-Preiss, if T is bounded, Theorem 5.3 ensures the existence of a bounded $(PS)_c$−sequence. Therefore, in this case, the existence of a critical point follows by requiring only the $(WPS)_c$−condition (that is, when any bounded $(PS)_c$−sequence admits a subsequence strongly converging in E). In this direction, we recall [7, Theorem 3.3], where a type of $(WPS)_c$−condition is requested in order to obtain a critical point of mountain pass type.

Remark 5.5. We observe that the proof proposed in this section, compared with the profound original one provided by Ghoussoub-Preiss (see [13, Theorem (1)]), avoids some advanced arguments such as Radon measures space, the sub-differential or the theory of non-differentiable functions (see also the paper by Pucci-Radulescu [24]). In fact, roughly speaking, the ε−deformation given by Ghossoub-Preiss involves a non-regular function depending on the paths γ while here we use Lemma 3.1 where the ε−deformation is as given by Brezis-Nirenberg and it depends on the function $t \to d_{\mathbb{R}}(t, \{0, 1\})$ which is constant with respect to γ.

Remark 5.6. We recall that the conclusion of the Ghoussoub-Preiss theorem, as well as those of the others mountain pass theorems examined in this paper, can be expressed in a bit more general form. In fact, we have that *assuming* (G_3)*, then there is a* $(C)_c$*−sequence for* f, that is, a sequence $\{x_n\}$ such that $f(x_n) \to c$ and $(1 + \|x_n\|)f'(x_n) \to 0$. So, when the Cerami–condition (that is, if any $(C)_c$−sequence admits a subsequence strongly converging in E) is satisfied, then the existence of a critical point is achieved. It is enough to argue, step by step, as in the exhaustive book of Ekeland [11]. Precisely, taken the geodesic distance δ as metric on E (see [11, page 138]), one repeats as previously done in this paper with δ instead of d, by following the line traced in [11, Theorem 6 page 140]. Finally, we also observe that the existence of a bounded $(C)_c$−sequence for f is equivalent to the existence of a bounded $(PS)_c$−sequence for f.

Remark 5.7. Taking Remark 3.2 into account, Theorem 4.1 (and hence the Pucci-Serrin theorem) can be given in the more general setting involving the family $\mathcal{A} = \{\gamma \in C(J, E) : \gamma_{|J^*} = p^*{}_{|J^*}\}$, where J is a compact metric space, $J^* \subseteq J$ is a nonempty closed set and $p^* : J \to E$ is a continuous function. Of course, in this case $c = \inf_{\gamma \in \mathcal{A}} \max_{t \in J} f(\gamma(t))$. To do the same for the Ghoussoub-Preiss theorem, instead, an appropriate study must be undertaken in order to get closer to the closed set disjoint by $p^*(J^*)$ and which intersects every compact $p(J)$. In this direction, we refer to the profound result of Ghoussoub [14, Theorem 1] where the notion of a homotopy-stable family is exploited in order to get closer to the previous closed set (see [14, page 145]). We also recall that starting from the seminal papers by Chang

[9] and Szulkin [27], a version of the Ghoussoub theorem ([14, Theorem 1]) has been given in [17] in the powerful framework developed by Motreanu-Panagiotopoulos (see [19]), that is when f is given by a sum of a locally Lipschitz continuous function and a convex proper lower semicontinuous function. We refer to the fruitful books of Motreanu-Radulescu [21] and Motreanu-Motreanu-Papageorgiou [20] for more details on the theory of non-differentiable functions.

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References

[1] A. Ambrosetti and A. Malchiodi, *Nonlinear analysis and semilinear elliptic problems*, Cambridge University Press, Cambridge, 2007.

[2] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[3] J.-P. Aubin and A. Cellina, *Differential inclusions*, Springer-Verlag, Berlin, 1984.

[4] J.-P. Aubin and I. Ekeland, *Applied nonlinear analysis*, Wiley-Interscience, New York, 1984.

[5] G. Bonanno, *Relations between the mountain pass theorem and local minima*, Adv. Nonlinear Anal. **1** (2012), 205–220.

[6] G. Bonanno, *A characterization of the mountain pass geometry for functionals bounded from below*, Differential and Integral Equations **25** (2012), 1135–1142.

[7] G. Bonanno and P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differential Equations **244** (2008), 3031–3059.

[8] H. Brezis and L. Nirenberg, *Remarks on finding critical points*, Comm. Pure Appl. Math. **44** (1991), 939–963.

[9] K.-C. Chang, *Variational methods for non-differentiable functionals and their applications to partial differential equations,* J. Math. Anal. Appl. **80** (1981), 102–129.

[10] D. C. Clark, *A variant of the Lusternik-Schnirelman theory,* Indiana Univ. Math. J. **22** (1972), 65–74.

[11] I. Ekeland, *Convexity methods in hamiltonian mechanics*, Springer-Verlag, Heidelberg, 1990.

[12] D. G. de Figueiredo and S. Solimini, *A Variational approach to superlinear elliptic problems*, Comm. in Partial Differential Equations **9** (1984), 699–717.

[13] N. Ghoussoub and D. Preiss, *A general mountain pass principle for locating and classifying critical points*, Ann. I.H.P. Analyse Nonlinéaire **6** (1989), 321–330.

[14] N. Ghoussoub, *A min-max principle with a relaxed boundary condition*, Proc. Amer. Math. Soc. **117** (1993), 439–447.

[15] N. Ghoussoub, *Duality and perturbation methods in critical point theory*, Cambridge University Press, Cambridge, 1993.

[16] Y. Jabri, *The mountain pass theorem: variants, generalizations and some applications*, Cambridge University Press, Cambridge, 2003.

[17] R. Livrea and S.A. Marano, *Existence and classification of critical points for nondifferentiable functions*, Advances in Differential Equations **9** (2004), 961–978.

[18] J. Mawhin and M. Willem, *Critical point theory and hamiltonian systems*, Springer-Verlag, New York, 1989.
[19] D. Motreanu and P.D. Panagiotopoulos, *Minimax theorems and qualitative properties of the solutions of hemivariational inequalities*, Nonconvex Optim. Appl. 29, Kluwer, Dordrecht, 1998.
[20] D. Motreanu, V.V. Motreanu and N. Papageorgiou, *Topological and variational methods with applications to nonlinear boundary value problems*, Springer, New York, 2014.
[21] D. Motreanu and V. Radulescu, *Variational and non-variational methods in nonlinear analysis and boundary value problems*, Kluwer Academic Publishers, Dordrecht, 2003.
[22] I. Peral, *Multiplicity of solutions for the p-Laplacian*, International Center for Theoretical Physics Lecture Notes, Trieste, 1997.
[23] K. Perera and M. Schechter, *Topics in Critical Point Theory*, Cambridge University Press, Cambridge, 2013.
[24] P. Pucci and V. Radulescu, *The impact of the mountain pass theory in nonlinear analysis: a mathematical survay*, Boll. Unione Mat. Ital. **3** (2010), 543–582.
[25] P. Pucci and J. Serrin, *A mountain pass theorem*, J. Differential Equations **60** (1985), 142–149.
[26] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. Ser. Math., vol. 65, Amer. Math. Soc., Providence, 1986.
[27] A. Szulkin, *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems*, Ann. Inst. Henri Poincaré, **3** (1986), 77–109.
[28] M. Willem, *Minimax theorems*, Birkhäuser, Boston, 1996.

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