RELAXATION FOR A CLASS OF CONTROL SYSTEMS WITH UNILATERAL CONSTRAINTS

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ABSTRACT. We consider a nonlinear control system involving a maximal monotone map and with a priori feedback. We assume that the control constraint multifunction $U(t,x)$ is nonconvex valued and only lsc in the $x \in \mathbb{R}^N$ variable. Using the Q-regularization (in the sense of Cesari) of $U(t,\cdot)$, we introduce a relaxed system. We show that this relaxation process is admissible.

1. INTRODUCTION

We consider the following nonlinear control system

$$(1) \qquad \begin{cases} -x'(t) \in A(x(t)) + f(t,x(t))u(t) & \text{for a.a. } t \in T = [0,b], \\ x(0) = x_0, \quad u(t) \in U(t,x(t)) & \text{for a.a. } t \in T. \end{cases}$$

In this system $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map. We do not assume that $D(A) = \mathbb{R}^N$ (recall that $D(A) = \{x \in \mathbb{R}^N : A(x) \neq \emptyset\}$, the domain of $A(\cdot)$). This way we incorporate in our framework systems with unilateral constraints (differential variational inequalities). The control constraint multifunction $U : T \times \mathbb{R}^N \to 2^{\mathbb{R}^m} \setminus \{\emptyset\}$ is state-dependent (feedback control system). Moreover, $U(\cdot,\cdot)$ has nonconvex values and for a.a. $t \in T$, $U(t,\cdot)$ is lower semicontinuous (lsc for short).

Our goal is to study the relaxation properties of this system. More precisely, we want to introduce a bigger control system (known as the "relaxed system") in which (1) is embedded and the following two properties hold:

(a) Every original state is also a relaxed state.
(b) The set of original states is dense in $C(T,\mathbb{R}^N)$ in the set of relaxed states, which is closed.

To achieve this, we need to convexify (1). Usually this convexification is done by replacing $U(t,x)$ with $\overline{\text{conv}}\, U(t,x)$. This approach leads to an admissible relaxed system provided that $U(t,\cdot)$ is Hausdorff continuous (h-continuous for short) (see De Blasi-Pianigiani [5] and Papageorgiou [13]). Since in our case $U(t,\cdot)$ is only lsc, the aforementioned classical approach to relaxation fails. Instead, motivated by the pioneering work of Cesari [4] (see also the recent work of Liu-Liu [12]), we replace $U(t,\cdot)$ by its Q-regularization, which is defined as follows. For every $\delta > 0$, we set

$$U_\delta(t,x) = \overline{\text{conv}}\, U(t, x + \delta B_1) \quad \text{for all } (t,x) \in T \times \mathbb{R}^N,$$

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where $B_1 = \{y \in \mathbb{R}^N : |y| < 1\}$. Then the "Q-regularization" of the multifunction $U(t, \cdot)$ is defined by

$$V(t, x) = \bigcap_{\delta > 0} U_\delta(t, x) \quad \text{for all } (t, x) \in T \times \mathbb{R}^N$$

(see Cesari [4], p. 292). Then the "relaxed" version of (1) is the following control system

$$\begin{cases} -x'(t) \in A(x(t)) + f(t, x(t))u(t) & \text{for a.a. } t \in T = [0, b], \\ x(0) = x_0, \quad u(t) \in V(t, x(t)) & \text{for a.a. } t \in T. \end{cases} \tag{2}$$

We will show that this relaxation is admissible in the sense that it satisfies the two fundamental requirements (a) and (b) mentioned above.

Suppose $A(x) = \partial\varphi(x)$ with $\varphi \in \Gamma_0(\mathbb{R}^N)$ = the convex and lower semicontinuous $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$-valued functions which are not identically $+\infty$ (here $\partial\varphi(\cdot)$ denotes the subdifferential in the sense of convex analysis of $\varphi(\cdot)$). Then problem (1) is a control problem for a variational inequality with a priori feedback (since the control constraint set is state-dependent). Variational inequalities arise in a variety of applied problems. For example, if $\varphi = i_C$ = the indicator function of a nonempty, closed, convex cone, that is

$$i_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C, \end{cases}$$

then $\partial i_C(x) = N_C(x)$ = normal cone to C at x. Then such systems arise in mathematical economics in resource allocation problems (see Henry [10]).

The above case of a subdifferential operator $A(\cdot)$ extends also "gradient systems". In the present formulation we do not require the potential function to be differentiable. This leads to a gradient inclusion as (1).

In general the presence of the maximal monotone map $A(\cdot)$ which need not be everywhere defined, introduces constraints which the state of the system needs to satisfy. To the best of our knowledges the relaxation properties of such systems have not been examined before. Only the case $A \equiv 0$ has been considered (see, for example, Buttazzo [3], Section 5.3).

2. Mathematical Background

Let Y, Z be Banach spaces and $K : Y \to Z$. We say that $K(\cdot)$ is "compact", if it is continuous and maps bounded sets in Y to relatively compact sets in Z. We say that $K(\cdot)$ is "completely continuous", if $y_n \xrightarrow{w} y$ in Y, implies that $K(y_n) \to K(y)$ in Z. In general these two concepts are distinct. However, if Y is reflexive, then complete continuity implies compactness. Moreover, if in addition $K(\cdot)$ is linear, then the two notions coincide (see Gasiński-Papageorgiou [8]).

We will use the following basic result from the topological fixed point theory, known as the "Leray-Schauder Alternative Theorem" (see Granas-Dugundji [9], Theorem 5.4, p. 124).

Theorem 1. *If Y is a Banach space, $K : Y \to Y$ is compact and*

$$S = \{y \in Y : y = \lambda K(y),\ 0 < \lambda < 1\},$$

then one of the following statements is true:

(a) *S is unbounded.*

(b) $K(\cdot)$ *has a fixed point* (*that is, there exists* $\widehat{y} \in Y$ *such that* $\widehat{y} = K(\widehat{y})$).

Suppose that Y, Z are Hausdorff topological spaces and $G : Y \to 2^Z \setminus \{\emptyset\}$ a multifunction (a set valued function). We say that $G(\cdot)$ is "lower semicontinuous" (lsc for short) resp. "upper semicontinuous" (usc for short), if for every $U \subseteq Z$ open, the set

$$G^-(U) = \{y \in Y : G(y) \cap U \neq \emptyset\},$$
$$\text{resp.} \quad G^+(U) = \{y \in Y : G(y) \subseteq U\},$$

is open in Y. If Z is a metric space with metric $d_Z(\cdot,\cdot)$, then $G(\cdot)$ is lsc if and only if for every $z \in Z$, $y \to d_Z(z, G(y)) = \inf[d_Z(z,v) : v \in G(y)]$ is an upper semicontinuous $\mathbb{R}_+$-valued function (see Hu-Papageorgiou [11], Proposition 2.26, p. 45). If $\operatorname{Gr} G = \{(y,z) \in Y \times Z : z \in G(y)\}$ (the graph of $G(\cdot)$) is closed and $G(\cdot)$ is locally compact (that is, for every $y \in Y$, there exists an open set U with $y \in U$ and such that $\overline{G(U)} = \overline{\bigcup_{y' \in U} G(y')}$ is compact in Z), then $G(\cdot)$ is usc (see Hu-Papageorgiou [11], Proposition 2.23, p. 43). If $G(\cdot)$ is both lsc and usc, then we say that $G(\cdot)$ is "continuous" (or "Vietoris continuous").

Suppose that X is a metric space with metric $d_X(\cdot,\cdot)$. On the family of nonempty, bounded, closed sets in X (denoted by $P_{bf}(X)$), we can define a metric $h(\cdot,\cdot)$, known as the "Hausdorff metric", by

$$h(A,B) = \max\{\sup[d_X(a,B) : a \in A], \sup[d_X(b,A) : b \in B]\},$$

for all $A, B \subseteq P_{bf}(X)$. If X is a complete metric space, then so is $(P_{bf}(X), h)$.

If Y is a Hausdorff topological space and $G : Y \to P_{bf}(X)$, we say that $G(\cdot)$ is "Hausdorff continuous" ("h-continuous" for short), if it is continuous from Y into the metric space $(P_{bf}(X), h)$. In general continuity and h-continuity of multifunctions are distinct notions. However, for multifunctions with compact values the two notions coincide (see Hu-Papageorgiou [11], Corollary 2.69, p. 62).

Now let (Ω, Σ, μ) be a finite measure space and X a separable Banach space. We introduce the following families of X:

$$P_{f(c)}(X) = \{A \subseteq X : \text{nonempty, closed (and convex)}\},$$
$$P_{(w)k(c)}(X) = \{A \subseteq X : \text{nonempty, (w-) compact (and convex)}\}.$$

A multifunction $F : \Omega \to 2^X \setminus \{\emptyset\}$ is said to be measurable, if for all $U \subseteq X$ open, we have $F^-(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma$. This is equivalent to saying that for all $x \in X$, the $\mathbb{R}_+$-valued function $\omega \to d(x, F(\omega)) = \inf[\|x - y\|_X : y \in F(\omega)]$ is Σ-measurable (see Hu-Papageorgiou [11], Proposition 1.4, p. 142). If $F(\cdot)$ is measurable, then $F(\cdot)$ is "graph measurable", that is,

$$\operatorname{Gr} F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \otimes B(X),$$

with $B(X)$ being the Borel σ-field of X. If Σ is μ-complete and $F(\cdot)$ has nonempty closed values, then measurability and graph measurability are equivalent notions (see Hu-Papageorgiou [11], Theorem 2.4, p. 156).

Given $1 \leq p \leq +\infty$ and a multifunction $F : \Omega \to 2^X \setminus \{\emptyset\}$, we introduce the set

$$S_F^p = \{f \in L^p(\Omega, X) : f(\omega) \in F(\omega) \ \mu\text{-a.e. on } \Omega\}.$$

This set may be empty. If $F(\cdot)$ is graph measurable, then $S_F^p \neq \emptyset$ if and only if $\omega \to \inf[\|v\|_X : v \in F(\omega)]$ belongs in $L^p(\Omega)$. The set S_F^p is "decomposable", that is, if $(A, f, g) \in \Sigma \times S_F^p \times S_F^p$ we have $\chi_A f + \chi_{A^c} g \in S_F^p$.

Here for a given set $E \subseteq \Omega$, by $\chi_E(\cdot)$ we denote the characteristic function of E, defined by

$$\chi_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{if } \omega \in E^c = \Omega \setminus E. \end{cases}$$

Since $\chi_{E^c} = 1 - \chi_E$, we see that the notion of decomposability formally looks like that of convexity, only now the coefficients in the "linear" combination are functions. Decomposability turns out to be a good substitute for convexity and several results valid for convex sets have their counterparts for decomposable sets (see Fryszkowski [7]). One such result is the next theorem due to Bressan-Colombo [1] and Fryszkowski [6], which extends the celebrated Michael selection theorem (see, for example, Hu-Papageorgiou [11], Theorem 4.6, p. 92).

Theorem 2. *If Y is a separable metric space and $G : Y \to P_f(L^1(\Omega, X))$ is a lsc multifunction with decomposable values, then there exists a continuous map $g : Y \to L^1(\Omega, X)$ such that $g(y) \in G(y)$ for all $y \in Y$.*

If X is a Banach space and $C \subseteq X$ is nonempty, then we define

$$|C| = \sup[\|c\|_X : c \in C].$$

Let X^* denote the topological dual of the Banach space X and let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair (X^*, X). Consider a map $A : X \to 2^{X^*}$ with $D(A) = \{x \in X : A(x) \neq \emptyset\}$ (the domain of A) nonempty. We say that $A(\cdot)$ is "monotone" if

$$\langle x^* - y^*, x - y \rangle \geq 0 \quad \text{for all } (x, x^*), (y, y^*) \in \operatorname{Gr} A.$$

We say that $A(\cdot)$ is "strictly monotone", if the above inequality is strict when $x \neq y$. Finally we say that $A(\cdot)$ is "maximal monotone", if it is monotone and

$$\langle x^* - y^*, x - y \rangle \geq 0 \quad \text{for all } (x, x^*) \in \operatorname{Gr} A \Rightarrow (y, y^*) \in \operatorname{Gr} A.$$

This is equivalent to saying that $\operatorname{Gr} A$ is maximal with respect to inclusion among the graphs of all monotone maps from X to 2^{X^*}. Zorn's lemma implies that every monotone map admits a maximal monotone extension. It is easy to see that for a maximal monotone map $A(\cdot)$, $\operatorname{Gr} A$ is sequentially closed in $X \times X^*_{w^*}$ and in $X_w \times X^*$. Here by $X^*_{w^*}$ (resp. X_w) we denote the space X^* equipped with the w^*-topology (resp. X equipped with the w-topology).

3. Hypotheses

In this section we introduce the hypotheses on the data of the problem. These hypotheses will be valid throughout the rest of the paper.

H(A): $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map such that $0 \in A(0)$.

Remark 1. Since we do not assume that $D(A) = \mathbb{R}^N$, we incorporate in our framework systems with unilateral constraints (differential variational inequalities).

H(f): $f : T \times \mathbb{R}^N \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^N)$ is a function such that:

(i) for all $x \in \mathbb{R}^N$, $t \to f(t, x)$ is measurable;

(ii) for every $c > 0$, there exists a function $k_c \in L^1(T)$ such that

$$\|f(t, x) - f(t, y)\|_{\mathcal{L}} \leq k_c(t)|x - y| \quad \text{for a.a. } t \in T, \text{ all } |x|, |y| \leq c;$$

(iii) $\|f(t, x)\|_{\mathcal{L}} \leq a(t)[1 + |x|]$ for a.a. $t \in T$, all $x \in \mathbb{R}^N$, with $a \in L^1(T)$.

Remark 2. Hypotheses H(f)(i), (ii) imply that $(t,x) \to f(t,x)$ is $\mathcal{L}_T \otimes B(\mathbb{R}^N)$-measurable with $\mathcal{L}_T$ being the Lebesgue σ-field of T and $B(\mathbb{R}^N)$ is the Borel σ-field of $\mathbb{R}^N$ (see Hu-Papageorgiou [11], Proposition 1.6, p. 142).

H(U): $U : T \times \mathbb{R}^N \to P_k(\mathbb{R}^m)$ is a multifunction such that:
- (i) $(t,x) \to U(t,x)$ is an $\mathcal{L}_T \otimes B(\mathbb{R}^N)$-measurable multifunction;
- (ii) for a.a. $t \in T$, $x \to U(t,x)$ is lsc;
- (iii) $|U(t,x)| \le M$ for a.a. $t \in T$, all $x \in \mathbb{R}^N$, some $M > 0$.

Remark 3. We know that if for $x \in \mathbb{R}^N$, $t \to U(t,x)$ is measurable and for a.a. $t \in T$, $x \to U(t,x)$ is lsc, then $(t,x) \to U(t,x)$ need not be $\mathcal{L}_T \otimes B(\mathbb{R}^N)$-measurable (see Hu-Papageorgiou [11], Example 7.2, p. 227). Hypothesis H(U)(i) implies that if $x : T \to \mathbb{R}^N$ is measurable, then $t \to U(t,x(t))$ is an $\mathcal{L}_T$-measurable multifunction (superpositional measurability). For example, if $\theta_i : T \times \mathbb{R}^N \to \mathbb{R}$, $\eta_i : T \times \mathbb{R}^N \to \mathbb{R}$, $i = 1, \dots, m$, are measurable functions such that for all $t \in T$

$$\theta_i(t,\cdot) \text{ is lower semicontinuous,}$$
$$\eta_i(t,\cdot) \text{ is upper semicontinuous,}$$

with θ_i bounded below, η_i bounded above for all $i = 1, \dots, m$, then the multifunction $U(t,x) = \{u = (u_i)_{i=1}^m \in \mathbb{R}^m : \theta_i(t,x) \le u_i \le \eta_i(t,x) \text{ for all } i = 1, \dots, m\}$ satisfies hypotheses H(U).

H_0: $x_0 \in \overline{D(A)}$.

We introduce the following two subsets of $C(T,\mathbb{R}^N) \times L^1(T,\mathbb{R}^m)$:

$$\begin{aligned} P_c = &\text{ the set of admissible state-control pairs for system (2)} \\ = &\{(x,u) \in C(T,\mathbb{R}^N) \times L^1(T,\mathbb{R}^m) : (x,u) \text{ solves (2)}\}, \\ P = &\text{ the set of admissible state-control pairs for system (1)} \\ = &\{(x,u) \in C(T,\mathbb{R}^N) \times L^1(T,\mathbb{R}^m) : (x,u) \text{ solves (1)}\}. \end{aligned}$$

Also we define

$$\mathcal{S} = \mathrm{proj}_{C(T,\mathbb{R}^N)} P \quad \text{and} \quad \mathcal{S}_c = \mathrm{proj}_{C(T,\mathbb{R}^N)} P_c.$$

These are the sets of admissible states and of admissible relaxed states respectively.

4. Relaxation

First we show that the original system (1) has a nonempty set of admissible state-control pairs.

Proposition 1. *If hypotheses H(A), H(f), H(U), H_0 hold, then $P \neq \emptyset$.*

Proof. We set

$$F(t,x) = f(t,x)U(t,x) = \bigcup_{u \in U(t,x)} f(t,x)u.$$

Evidently $F(t,x) \in P_k(\mathbb{R}^N)$ for all $(t,x) \in T \times \mathbb{R}^N$ and

$$|F(t,x)| \le \widehat{a}(t)[1 + |x|] \quad \text{for a.a. } t \in T, \text{ all } x \in \mathbb{R}^N, \text{ with } \widehat{a} \in L^1(T) \tag{3}$$

(see hypotheses H(f)(iii), H(U)(iii)).

Hypothesis H(U)(i) and Theorem 7.24, p. 236 of Hu-Papageorgiou [11] imply that we can find a sequence $u_n : T \times \mathbb{R}^N \to \mathbb{R}^m$, $n \in \mathbb{N}$, of $\mathcal{L}_T \otimes B(\mathbb{R}^N)$-measurable functions such that

$$U(t,x) = \overline{\{u_n(t,x)\}}_{n\geq 1} \quad \text{for all } (t,x) \in T \times \mathbb{R}^N.$$

Since $f(t,x) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^N)$ we have

$$F(t,x) = \overline{\{f(t,x)u_n(t,x)\}}_{n\geq 1} \quad \text{for all } (t,x) \in T \times \mathbb{R}^N,$$

$$\Rightarrow \quad (t,x) \to F(t,x) \text{ is measurable (see Hu-Papageorgiou [11], Theorem 2.4, p. 156).}$$

Also, suppose that $x_n \to x$ in $\mathbb{R}^N$ and let $v \in F(t,x)$. Then

$$v = f(t,x)u \text{ with } u \in U(t,x).$$

Since $U(t,\cdot)$ is lsc (see hypothesis H(U)(ii)), we can find $u_n \in U(t,x_n)$, $n \in \mathbb{N}$, such that $u_n \to u$ in $\mathbb{R}^m$ (see Hu-Papageorgiou [11], Proposition 2.6, p. 37). Then

$$v_n = f(t,x_n)u_n \to f(t,x)u = v \text{ in } \mathbb{R}^N$$

(see hypothesis H(f)(ii)). Since $v_n \in F(t,x_n)$ for all $n \in \mathbb{N}$, as above we conclude that for a.a. $t \in T$, $F(t,\cdot)$ is lsc.

We consider the following multivalued Cauchy problem

$$-x'(t) \in A(x(t)) + F(t,x(t)) \quad \text{for a.a. } t \in T,\ x(0) = x_0. \tag{4}$$

Consider the multifunction $N_F : C(T,\mathbb{R}^{\mathbb{N}}) \to 2^{L^1(T,\mathbb{R}^N)}$ defined by $N_F(x) = S^1_{F(\cdot,x(\cdot))}$ for all $x \in C(T,\mathbb{R}^N)$. The measurability of $F(\cdot,\cdot)$ implies that $t \to F(t,x(t))$ is $\mathcal{L}_T$-measurable and so by hypotheses H(f)(iii), H(U)(iii), $N_F(u) \in P_{wk}(L^1(T,\mathbb{R}^N))$.

Claim 1: $N_F(\cdot)$ is lsc.

It suffices to show that for all $y \in L^1(T,\mathbb{R}^N)$, the $\mathbb{R}_+$-valued function $x \to d(y, N_F(x))$ is upper semicontinuous on $C(T,\mathbb{R}^N)$ (see Section 2). To this end, let $\{x_n\}_{n\geq 1} \subseteq C(T,\mathbb{R}^N)$ be a sequence such that

$$x_n \to x \text{ in } C(T,\mathbb{R}^N) \text{ and } d(y, N_F(x_n)) \geq \lambda \quad \text{for some } \lambda > 0, \text{ all } n \in \mathbb{N}. \tag{5}$$

We have

$$\begin{aligned}
d(y,N_F(x_n)) &= \inf[\|y-h\|_1 : h \in N_F(x_n)] \\
&= \inf\left[\int_0^b |y(t)-h(t)|dt : h \in N_F(x_n)\right] \\
&= \int_0^b \inf[|y(t)-v| : v \in F(t,x_n(t))]dt \\
&\quad \text{(see Hu-Papageorgiou [11], Theorem 3.24, p. 183)} \\
&= \int_0^b d(y(t), F(t,x_n(t)))dt, \\
\Rightarrow \quad \limsup_{n\to+\infty} d(y,N_F(x_n)) &\leq \int_0^b \limsup_{n\to+\infty} d(y(t),F(t,x_n(t)))dt \\
&\quad \text{(by Fatou's lemma, see (5) and (3))} \\
&\leq \int_0^b d(y(t),F(t,x(t)))dt \\
&\quad \text{(see (5) and recall that for a.a. } t \in T,\ F(t,\cdot) \text{ is lsc)}
\end{aligned}$$

$$
\begin{aligned}
&= d(y, N_F(x)), \\
\Rightarrow \quad & \lambda \le d(y, N_F(x)), \\
\Rightarrow \quad & N_F(\cdot) \text{ is lsc.}
\end{aligned}
$$

This proves Claim 1.

Clearly $N_F(\cdot)$ has decomposable values. So, we can apply Theorem 2 (the Bressan-Colombo-Fryszkowski selection theorem) and produce a continuous map $\xi : C(T, \mathbb{R}^N) \to L^1(T, \mathbb{R}^N)$ such that

$$\xi(x) \in N_F(x) \quad \text{for all } x \in C(T, \mathbb{R}^N).$$

Given $h \in L^1(T, \mathbb{R}^N)$, we consider the following Cauchy problem

$$-x'(t) \in A(x(t)) + h(t) \quad \text{for a.a. } t \in T,\ x(0) = x_0. \tag{6}$$

According to Proposition 3.8, p. 82 of Brezis [2], problem (6) has a unique solution $x_h \in W^{1,1}((0,b), \mathbb{R}^N) = AC^1(T, \mathbb{R}^N)$. So, we can define the solution map $K : L^1(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$ by $K(h) = x_h$.

<u>Claim 2</u>: $K(\cdot)$ is completely continuous.

Suppose that $h_n \xrightarrow{w} h$ in $L^1(T, \mathbb{R}^N)$ and let $x_n = K(h_n)$, $x = K(h)$. By the Dunford-Pettis theorem (see Papageorgiou-Winkert [14], Theorem 4.1.18, p.289), we know that $\{h_n\}_{n\ge1} \subseteq L^1(T, \mathbb{R}^N)$ is uniformly integrable. So, invoking Theorem 2.3.2, p. 64, of Vrabie [15], we have that

$$\{x_n\}_{n\ge1} \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact.}$$

By passing to a suitable subsequence if necessary, we may assume that

$$x_n \to \widehat{x} \text{ in } C(T, \mathbb{R}^N). \tag{7}$$

Hypothesis H(A) implies that

$$
\begin{aligned}
& (x_n'(t) - x'(t), x_n(t) - x(t))_{\mathbb{R}^N} \le (h(t) - h_n(t), x_n(t) - x(t))_{\mathbb{R}^N} \text{ for a.a. } t \in T, \\
\Rightarrow\ & \frac{1}{2}\frac{d}{dt}|x_n(t) - x(t)|^2 \le (h(t) - h_n(t), x_n(t) - x(t))_{\mathbb{R}^N} \text{ for a.a. } t \in T, \\
\Rightarrow\ & |x_n(t) - x(t)|^2 \le 2\int_0^t (h(s) - h_n(s), x_n(s) - x(s))_{\mathbb{R}^N} ds, \\
\Rightarrow\ & x_n(t) \to x(t) \quad \text{for all } t \in T, \\
\Rightarrow\ & \widehat{x} = x \text{ (see (7)) and so } x_n \to x \text{ in } C(T, \mathbb{R}^N).
\end{aligned}
$$

Therefore for the original sequence we have $x_n \to x$ in $C(T, \mathbb{R}^N)$, and so $K(\cdot)$ is completely continuous. This proves Claim 2.

We consider the map $\varphi : C(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$ defined by

$$\varphi = K \circ \xi.$$

Evidently $\varphi(\cdot)$ is continuous (see Claim 2 and recall that $\xi(\cdot)$ is continuous). If $B \subseteq C(T, \mathbb{R}^N)$ is bounded, then on account of (3) we have that

$$\xi(B) \subseteq L^1(T, \mathbb{R}^N) \text{ is uniformly integrable.}$$

So, by the Dunford-Pettis theorem and Claim 2, we have that

$$(K \circ \xi)(B) \subseteq C(T, \mathbb{R}^N) \text{ is relatively compact.}$$

Therefore we have that

$$x \to \varphi(x) = (K \circ \xi)(x) \text{ is compact.} \tag{8}$$

Consider the set

$$S = \{x \in C(T, \mathbb{R}^N) : x = \lambda\varphi(x),\, 0 < \lambda < 1\}.$$

Claim 3: $S \subseteq C(T, \mathbb{R}^N)$ is bounded.

Let $x \in S$. Then $\frac{1}{\lambda}x = K(\xi(x))$. So, we have

$$-\frac{1}{\lambda}x'(t) \in A\left(\frac{1}{\lambda}x(t)\right) + \xi(x)(t) \quad \text{for a.a. } t \in T,\, x(0) = x_0.$$

We take inner product with $x(t)$ and use hypothesis H(A). We obtain

$$\begin{aligned}
&\frac{1}{\lambda}(x'(t), x(t))_{\mathbb{R}^N} \le |\xi(x)(t)||x(t)| \quad \text{for a.a. } t \in T,\\
\Rightarrow\quad &\frac{1}{2}\frac{d}{dt}|x(t)|^2 \le |\xi(x)(t)||x(t)| \quad \text{for a.a. } t \in T \text{ (since } 0 < \lambda < 1),\\
\Rightarrow\quad &\frac{1}{2}|x(t)|^2 \le \frac{1}{2}|x_0|^2 + \int_0^t |\xi(x)(s)||x(s)|ds \quad \text{for all } t \in T\ ,\\
\Rightarrow\quad &|x(t)| \le |x_0| + \int_0^t |\xi(x)(s)|ds \quad \text{for all } t \in T\ ,\\
&\qquad \text{(see Brezis, [2], p. 157, Lemma A.5)}\\
\Rightarrow\quad &|x(t)| \le |x_0| + \int_0^t \widehat{a}(s)[1 + |x(s)|]ds \quad \text{for all } t \in T \text{ (see (3))}\\
\Rightarrow\quad &|x(t)| \le c_1 \quad \text{for some } c_1 > 0, \text{ all } t \in T, \text{ all } x \in S \text{ (by the Grönwall's inequality).}
\end{aligned}$$

Therefore $S \subseteq C(T, \mathbb{R}^N)$ is bounded and this proves Claim 3.

On account of (8) and Claim 3, we see that we can apply Theorem 1 (the Leray-Schauder Alternative Theorem) and find $\widehat{x} \in C(T, \mathbb{R}^N)$ such that

$$\widehat{x} = \varphi(\widehat{x}) = K(\xi(\widehat{x})).$$

We have

$$-\widehat{x}'(t) \in A(\widehat{x}(t)) + \xi(\widehat{x})(t) \quad \text{for a.a. } t \in T,\, \widehat{x}(0) = x_0. \tag{9}$$

Note that $\xi(\widehat{x})(t) \in F(t, \widehat{x}(t))$ for a.a. $t \in T$. We consider

$$D(t) = \{u \in U(t, \widehat{x}(t)) : \xi(\widehat{x})(t) = f(t, \widehat{x}(t))u\}.$$

By setting $D(\cdot)$ to be equal to $\{0\}$ on a Lebesgue-null set, we see that $D(t) \neq \emptyset$ for all $t \in T$. Also it is graph measurable. So, by the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [11], Theorem 2.14, p. 158), we can find a measurable map $\widehat{u} : T \to \mathbb{R}^m$ such that

$$\begin{aligned}
&\widehat{u}(t) \in D(t) \quad \text{for a.a. } t \in T,\\
\Rightarrow\quad &\xi(\widehat{x})(t) = f(t, \widehat{x}(t))\widehat{u}(t) \quad \text{for a.a. } t \in T,\\
\Rightarrow\quad &(\widehat{x}, \widehat{u}) \in P \text{ and so } P \neq \emptyset.
\end{aligned}$$

□

Next we examine the control constraint set $V(t, x)$ for the relaxed system (2).

Proposition 2. *If hypotheses H(U) hold, then*

(*a*) $(t, x) \to V(t, x)$ *is measurable;*

(*b*) *for a.a.* $t \in T$, $V(t, \cdot)$ *is usc.*

Proof. (a) Recall that $B_1 = \{x \in \mathbb{R}^N : |x| < 1\}$ and let $\{z_k\}_{k\geq 1} \subseteq B_1$ be dense. We set

$$U^k_{\frac{1}{n}}(t,x) = U\left(t, x + \frac{1}{n} z_k\right) \quad \text{for all } k,n \in \mathbb{N}.$$

We have that for a.a. $t \in T$

$$U_{\frac{1}{n}}(t,x) = \overline{\text{conv}}\, \overline{\bigcup_{k\geq 1} U^k_{\frac{1}{n}}(t,x)} \tag{10}$$

(see Cesari [4], p. 292 and Liu-Liu [12], Lemma 5).

For every $k,n \in \mathbb{N}$, the multifunction $(t,x) \to U^k_{\frac{1}{n}}(t,x)$ is measurable (see hypothesis H(U)(i)). It follows that $(t,x) \to \overline{\bigcup_{k\geq 1} U^k_{\frac{1}{n}}(t,x)}$ is measurable (see Hu-Papageorgiou [11], Proposition 1.40, p. 151 and Proposition 1.8, p. 143). From this we infer that $(t,x) \to U_{\frac{1}{n}}(t,x) = \overline{\text{conv}}\, \overline{\bigcup_{k\geq 1} U^k_{\frac{1}{n}}(t,x)}$ is measurable (see Hu-Papageorgiou [11], Proposition 2.26, p. 163). Recall that

$$V(t,x) = \bigcap_{n\geq 1} U_{\frac{1}{n}}(t,x).$$

Invoking Proposition 1.43, p. 152, of Hu-Papageorgiou [11], we conclude that

$$(t,x) \to V(t,x) \text{ is measurable.}$$

(b) Evidently for a.a. $t \in T$, $V(t,\cdot)$ is locally compact (see hypothesis H(U)(iii)). So, it suffices to show that

$$\text{Gr}\, V(t,\cdot) \subseteq \mathbb{R}^N \times \mathbb{R}^m \text{ is closed (see Section 2).}$$

For this purpose, let $\{(x_n, v_n)\}_{n\geq 1} \subseteq \text{Gr}\, V(t,\cdot)$ and assume that

$$x_n \to x \text{ in } \mathbb{R}^{\mathbb{N}} \text{ and } v_n \to v \text{ in } \mathbb{R}^m \text{ as } n \to +\infty. \tag{11}$$

Given $\delta > 0$, we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} & x_n \in x + \delta B_1 \quad \text{for all } n > n_0, \\ \Rightarrow \quad & v_n \in U_\delta(t,x) \quad \text{for all } n > n_0, \\ \Rightarrow \quad & v \in U_\delta(t,x) \quad \text{(see (11)).} \end{aligned}$$

Since $\delta > 0$ is arbitrary, we conclude that

$$\begin{aligned} & v \in V(t,x) \quad \text{(that is, } (x.v) \in \text{Gr}\, V(t,\cdot)), \\ \Rightarrow \quad & x \to V(t,x) \text{ is usc.} \end{aligned}$$

□

Recall that

$$\begin{aligned} \mathcal{S} &= \text{ set of admissible states of (1) } (\mathcal{S} = \text{proj}_{C(T,\mathbb{R}^N)} P), \\ \mathcal{S}_c &= \text{ set of admissible states of (2) } (\mathcal{S}_c = \text{proj}_{C(T,\mathbb{R}^N)} P_c), \end{aligned}$$

In what follows by $L^1(T,\mathbb{R}^m)_w$ we denote the Lebesgue space $L^1(T,\mathbb{R}^m)$ equipped with the weak topology.

Proposition 3. *If hypotheses H(A), H(f), H(U), H_0 hold, then*

(a) *there exists $c_2 > 0$ such that $\|x\|_{C(T,\mathbb{R}^N)} \leq c_2$ for all $x \in \mathcal{S}_c$;*

(b) $P_c \subseteq C(T,\mathbb{R}^N) \times L^1(T,\mathbb{R}^m)_w$ *is compact.*

Proof. (a) Let $x \in \mathcal{S}_c$. Then we can find $v \in S^1_{V(\cdot,x(\cdot))}$ such that

$$-x'(t) \in A(x(t)) + f(t,x(t))v(t) \quad \text{for a.a. } t \in T,\ x(0) = x_0.$$

Taking inner product with $x(t) \in \mathbb{R}^N$ and using hypothesis H(A), we obtain

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 + (f(t,x(t))v(t),x(t))_{\mathbb{R}^N} \leq 0 \quad \text{for a.a. } t \in T,$$

$$\Rightarrow \quad \frac{1}{2}|x(t)|^2 \leq \frac{1}{2}|x_0|^2 + \int_0^t \|f(s,x(s))\|_{\mathcal{L}} M|x(s)|ds \quad \text{for all } t \in T$$
$$\text{(see hypothesis H(U)(iii))},$$

$$\Rightarrow \quad |x(t)| \leq |x_0| + \int_0^t M\|f(s,x(s))\|_{\mathcal{L}} ds \quad \text{for all } t \in T$$
$$\text{(see Brezis [2], Lemma A.5, p. 157)},$$

$$\Rightarrow \quad |x(t)| \leq |x_0| + \int_0^t Ma(s)[1+|x(s)|]ds \quad \text{for all } t \in T \text{ (see hypothesis H(f)(iii))},$$

$$\Rightarrow \quad |x(t)| \leq c_2 \quad \text{for some } c_2 > 0, \text{ all } t \in T, \text{ all } x \in \mathcal{S}_c \text{ (by the Gronwall's inequality).}$$

(b) Let $\{(x_n,u_n)\}_{n\geq 1} \subseteq P_c$. Hypothesis H(U)(iii) implies that $\{u_n\}_{n\geq 1} \subseteq L^1(T,\mathbb{R}^N)$ is uniformly integrable.

Then by Dunford-Pettis and Eberlein-Šmulian theorems, we may assume that

$$u_n \xrightarrow{w} u \text{ in } L^1(T,\mathbb{R}^m). \tag{12}$$

Also let

$$B = \{h \in L^1(T,\mathbb{R}^N) : |h(t)| \leq a(t)[1+c_2]M\}$$

(see hypotheses H(f)(iii), H(U)(iii)) and (a)). Then by the Dunford-Pettis theorem $B \subseteq L^1(T,\mathbb{R}^N)$ is w-compact. If $K : L^1(T,\mathbb{R}^N) \to C(T,\mathbb{R}^N)$ is the solution map from the proof of Proposition 1, then on account of Claim 2 in that proof, we have

$$K(B) \subseteq C(T,\mathbb{R}^N) \text{ is compact.}$$

Note that

$$\{x_n\}_{n\geq 1} \subseteq K(B).$$

So, we may assume that

$$x_n \to x \text{ in } C(T,\mathbb{R}^N). \tag{13}$$

Let $\eta_n(t) = f(t,x_n(t))u_n(t)$, $\eta(t) = f(t,x(t))u(t)$. Evidently

$$\{\eta_n,\eta\}_{n\geq 1} \subseteq L^1(T,\mathbb{R}^N).$$

In what follows, by $((\cdot,\cdot))$ we denote the duality brackets for the pair $(L^1(T,\mathbb{R}^N), L^\infty(T,\mathbb{R}^N))$. For $h \in L^\infty(T,\mathbb{R}^N)$ we have

$$((\eta_n,h)) = \int_0^b (\eta_n(t),h(t))_{\mathbb{R}^N} dt$$
$$= \int_0^b (u_n(t), f(t,x_n(t))^* h(t))_{\mathbb{R}^N} dt \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

We have

$$
\begin{aligned}
&|f(t,x_n(t))^* h(t) - f(t,x(t))^* h(t)| \\
&\leq \|f(t,x_n(t))^* - f(t,x(t))^*\|_{\mathcal{L}} |h(t)| \\
&= \|f(t,x_n(t)) - f(t,x(t))\|_{\mathcal{L}} |h(t)| \\
&\leq k_{c_2}(t)\|h\|_\infty \|x_n - x\|_{C(T,\mathbb{R}^N)} \to 0 \quad \text{for a.a.} t \in T \text{ (see (13))} \\
\Rightarrow \quad & f(\cdot, x_n(\cdot))^* h(\cdot) \to f(\cdot, x(\cdot))^* h(\cdot) \text{ in } L^1(T,\mathbb{R}^m) \\
& \text{(by the Lebesgue dominated convergence theorem).}
\end{aligned} \tag{15}
$$

From Theorem 3.4.12, p. 220, of Papageorgiou-Winkert [14], we know that on bounded subsets of $L^\infty(T,\mathbb{R}^m)$ the w^*-topology is metrizable. So, from hypothesis H(U)(iii) and (12), we have

$$
u_n \xrightarrow{w^*} u \text{ in } L^\infty(T,\mathbb{R}^m). \tag{16}
$$

From (15) and (16) it follows that

$$
\begin{aligned}
&((\eta_n, h)) \\
&= \int_0^b (f(t,x_n(t))u_n(t), h(t))_{\mathbb{R}^N} dt \to \int_0^b (f(t,x(t))u(t), h(t))_{\mathbb{R}^N} dt = ((\eta, h)),
\end{aligned}
$$

$$
\Rightarrow \quad \eta_n \xrightarrow{w} \eta \text{ in } L^1(T,\mathbb{R}^N). \tag{17}
$$

Let $\widetilde{x} \in W^{1,1}((0,b),\mathbb{R}^N) = AC^1(T,\mathbb{R}^N)$ be the unique solution of the following Cauchy problem

$$
-x'(t) \in A(x(t)) + \eta(t) \quad \text{for a.a. } t \in T, \; x(0) = x_0.
$$

As before (see the proof of Proposition 1), using hypothesis H(A) we have

$$
\begin{aligned}
&\frac{1}{2}|x_n(t) - \widetilde{x}(t)|^2 \leq \int_0^t (\eta(s) - \eta_n(s), x_n(s) - \widetilde{x}(s))_{\mathbb{R}^N} ds \quad \text{for all } n \in \mathbb{N}, \\
\Rightarrow \quad & x_n(t) \to \widetilde{x}(t) \quad \text{for all } t \in T \text{ (see (17))}, \\
\Rightarrow \quad & \widetilde{x} = x \text{ (see (13))}.
\end{aligned}
$$

Therefore, finally we have

$$
\begin{aligned}
&(x_n, u_n) \to (x, u) \text{ in } C(T,\mathbb{R}^N) \times L^1(T,\mathbb{R}^m)_w, \\
\Rightarrow \quad & P_c \subseteq C(T,\mathbb{R}^N) \times L^1(T,\mathbb{R}^m)_w \text{ is compact (by the Eberlein-Šmulian theorem).}
\end{aligned}
$$

□

Proposition 4. *If hypotheses H(A), H(f), H(U), H_0 hold and $(\widehat{x}, \widehat{u}) \in P_c$, then we can find $e_n \in S^1_{\frac{1}{n}B_1}$ and $u_n \in S^1_{U(\cdot, \widehat{x}(\cdot) + e_n(\cdot))}$ such that $u_n \xrightarrow{w} \widehat{u}$ in $L^1(T,\mathbb{R}^m)$.*

Proof. We have

$$
\widehat{u} \in S^1_{V(\cdot, \widehat{x}(\cdot))} \subseteq S^1_{\overline{\text{conv}}\, U_{\frac{1}{n}}(\cdot, \widehat{x}(\cdot))}.
$$

Invoking Proposition 3.30, p. 185, of Hu-Papageorgiou [11], we can find $\widehat{u}_n \in S^1_{U(\cdot, \widehat{x}(\cdot) + \frac{1}{n}B)}$, $n \in \mathbb{N}$, such that

$$
\widehat{u}_n \xrightarrow{w} \widehat{u} \text{ in } L^1(T,\mathbb{R}^m). \tag{18}
$$

Consider the multifunction $H_n : T \to 2^{\mathbb{R}^N}$ defined by

$$H_n(t) = \left\{ e \in B_1 : \widehat{u}_n(t) \in U\left(t, \widehat{x}(t) + \frac{1}{n}e\right)\right\}$$
$$= \left\{ e \in B_1 : d\left(\widehat{u}_n(t), U\left(t, \widehat{x}(t) + \frac{1}{n}e\right)\right) = \vartheta_n(t,e) = 0\right\}$$

(recall that $B_1 = \{x \in \mathbb{R}^N : |x| < 1\}$). Setting $H_n(t) = \{0\}$ on a Lebesgue-null set, we will have that $H_n(t) \neq \emptyset$ for all $t \in T$. Evidently $\vartheta_n(t,e)$ is jointly measurable (see hypothesis H(U)(i)). Hence

$$\operatorname{Gr} H_n \in \mathcal{L}_T \otimes B(B_1).$$

Using the Yankov-von Neumann-Aumann selection theorem, we produce a measurable function $\widehat{e}_n : T \to \mathbb{R}^N$ such that

$$\widehat{e}_n(t) \in H_n(t) \quad \text{for a.a. } t \in T, \text{ all } n \in \mathbb{N}.$$

Set $e_n(t) = \frac{1}{n}\widehat{e}_n(t)$. Then $e_n \in S^1_{\frac{1}{n}B_1}$, $\widehat{u}_n \in S^1_{U(\cdot,\widehat{x}(\cdot)+e_n(\cdot))}$, $n \in \mathbb{N}$, and $\widehat{u}_n \xrightarrow{w} \widehat{u}$ in $L^1(T,\mathbb{R}^m)$ (see (18)). ☐

Now we are ready for the relaxation theorem which asserts that any state of the relaxed system (2) can be approximated in the $C(T,\mathbb{R}^N)$-norm with any degree of accuracy, by a state of the original system. Such a result has practical implications since it says that we can economize in the use of control functions.

Theorem 3. *If hypotheses H(A), H(f), H(U), H_0 hold, then* $\mathcal{S}_c = \overline{\mathcal{S}}^{C(T,\mathbb{R}^N)}$.

Proof. Let $(\widehat{x}, \widehat{u}) \in P_c$. According to Proposition 4, we can find sequences

$$\{u_n\}_{n\geq 1} \subseteq L^1(T,\mathbb{R}^m) \text{ and } \{e_n\}_{n\geq 1} \subseteq L^1(T,\mathbb{R}^N)$$

such that

$$(19) \qquad |e_n(t)| < \frac{1}{n} \text{ for a.a. } t \in T,\ u_n \in S^1_{U(\cdot,\widehat{x}(\cdot)+e_n(\cdot))},\ n \in \mathbb{N},\ u_n \xrightarrow{w} \widehat{u} \text{ in } L^1(T,\mathbb{R}^m).$$

Consider the function $\xi_n : T \times \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$\xi_n(t,x,v) = (f(t,\widehat{x}(t)+e_n(t))u_n(t) - f(t,x)v, \widehat{x}(t) + e_n(t) - x)_{\mathbb{R}^N}$$

for all $(t,x,v) \in T \times \mathbb{R}^N \times \mathbb{R}^m$. Evidently ξ_n is a Carathéodory function (that is, for all $(x,v) \in \mathbb{R}^N \times \mathbb{R}^m$, $t \to \xi_n(t,x,v)$ is measurable and for a.a. $t \in T$, $(x,v) \to \xi_n(t,x,v)$ is continuous). It follows that $(t,x,v) \to \xi_n(t,x,v)$ is $\mathcal{L}_T \otimes B(\mathbb{R}^N) \otimes B(\mathbb{R}^m)$-measurable (see Hu-Papageorgiou [11], Proposition 1.6, p. 142). Let

$$\widehat{G}_n(t,x) = U(t,x) \cap \left\{ v \in \mathbb{R}^m : \xi_n(t,x,v) < \frac{1}{n}\right\}.$$

Then $\widehat{G}_n(\cdot,\cdot)$ is graph measurable, while from hypothesis H(U)(ii) and Proposition 2.47, p. 53 of Hu-Papageorgiou [11], we have that for a.a. $t \in T$, $x \to \widehat{G}_n(t,x)$ is lsc. Therefore if $G_n(t,x) = \overline{\widehat{G}_n(t,x)}$ for $(t,x) \in T \times \mathbb{R}^N$, we have that $G_n(\cdot,\cdot)$ is graph measurable and for a.a. $t \in T$, $x \to G_n(t,x)$ is lsc.

Let $C_0 = \overline{\mathcal{S}}^{C(T,\mathbb{R}^N)} \in P_k(C(T,\mathbb{R}^N))$ (see Proposition 3) and consider the multifunction $\widehat{H}_n : C_0 \to P_f(L^1(T,\mathbb{R}^m))$ defined by

$$\widehat{H}_n(x) = S^1_{G_n(\cdot,x(\cdot))} \quad \text{for all } x \in C_0.$$

As in the proof of Proposition 1 (see Claim 1), we can show that $\widehat{H}_n(\cdot)$ is lsc and of course has decomposable values. So, we can apply Theorem 2 and produce a continuous map $\widehat{h}_n : C_0 \to L^1(T, \mathbb{R}^m)$ such that

$$\widehat{h}_n(x) \in \widehat{H}_n(x) \quad \text{for all } x \in C_0.$$

Let $D = \{h \in L^1(T, \mathbb{R}^N) : |h(t)| \leq Ma(t)[1 + c_2] = a_0(t) \text{ for a.a. } t \in T\}$ (here $M > 0$ is as in hypothesis H(U)(iii) and $c_2 > 0$ as in Proposition 3). Recall that the solution map $K : L^1(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$ is completely continuous (see the proof of Proposition 1). The set $D \subseteq L^1(T, \mathbb{R}^N)$ is w-compact (by the Dunford-Pettis theorem). Therefore $K(D) \subseteq C(T, \mathbb{R}^N)$ is compact. We define

$$H_n(x) = \begin{cases} \{\widehat{h}_n(x)\} & \text{if } x \in C_0, \\ S^1_{U(\cdot, x(\cdot))} & \text{if } x \in K(D) \setminus C_0. \end{cases} \tag{20}$$

Clearly $H_n(\cdot)$ is lsc and has decomposable values. So, a new application of Theorem 2, gives a continuous map $h_n : K(D) \to L^1(T, \mathbb{R}^m)$ such that

$$h_n(x) \in H_n(x) \quad \text{for all } x \in K(D).$$

We consider the following Cauchy problem

$$-x'(t) \in A(x(t)) + f(t, x(t))h_n(x)(t) \quad \text{for a.a. } t \in T,\ x(0) = x_0.$$

This problem has a solution $\widehat{x}_n \in AC^1(T, \mathbb{R}^N)$ (see the proof of Proposition 1). We have

$$h_n(\widehat{x}_n) = \widehat{h}_n(\widehat{x}_n) \quad \text{for all } n \in \mathbb{N} \text{ (see (20))}.$$

We set $\widehat{u}_n = \widehat{h}_n(\widehat{x}_n) \in S^1_{G_n(\cdot, \widehat{x}_n(\cdot))} \subseteq S^1_{U(\cdot, \widehat{x}_n(\cdot))}$, $n \in \mathbb{N}$.

We have

$$\begin{aligned} &-\widehat{x}_n'(t) \in A(\widehat{x}_n(t)) + f(t, \widehat{x}_n(t))\widehat{u}_n(t) \quad \text{for a.a. } t \in T,\ \widehat{x}_n(0) = x_0,\ n \in \mathbb{N}, \\ &-\widehat{x}'(t) \in A(\widehat{x}(t)) + f(t, \widehat{x}(t))\widehat{u}(t) \quad \text{for a.a. } t \in T,\ \widehat{x}(0) = x_0. \end{aligned}$$

Recall that $K(D) \subseteq C(T, \mathbb{R}^N)$ is compact. Therefore

$$\{\widehat{x}_n\}_{n \geq 1} \subseteq C(T, \mathbb{R}^N) \quad \text{is relatively compact.}$$

So, we may assume that

$$\widehat{x}_n \to \widetilde{x} \text{ in } C(T, \mathbb{R}^N). \tag{21}$$

For every $n \in \mathbb{N}$ and every $t \in T$, we have

$$\begin{aligned} \frac{1}{2}|\widehat{x}(t) - \widehat{x}_n(t)|^2 \leq &\int_0^t (f(s, \widehat{x})\widehat{u} - f(s, \widehat{x} + e_n)u_n, \widehat{x}_n - \widehat{x})_{\mathbb{R}^N} ds \\ &+ \int_0^t (f(s, \widehat{x} + e_n)u_n - f(s, \widehat{x}_n)\widehat{u}_n, \widehat{x}_n - \widehat{x})_{\mathbb{R}^N} ds \\ &\text{(see hypothesis H(A)).} \end{aligned} \tag{22}$$

Note that

$$\begin{aligned} &\int_0^t (f(s, \widehat{x} + e_n)u_n - f(s, \widehat{x}_n)\widehat{u}_n, \widehat{x}_n - \widehat{x})_{\mathbb{R}^N} ds \\ &= \int_0^t (f(s, \widehat{x} + e_n)u_n - f(s, \widehat{x}_n)\widehat{u}_n, \widehat{x}_n - (\widehat{x} + e_n))_{\mathbb{R}^N}\, ds \end{aligned}$$

$$+\int_0^t (f(s,\widehat{x}+e_n)u_n - f(s,\widehat{x}_n)\widehat{u}_n, e_n)_{\mathbb{R}^N} ds$$

$$\leq \frac{b}{n} + \int_0^t (f(s,\widehat{x}+e_n)u_n - f(s,\widehat{x}_n)\widehat{u}_n, e_n)_{\mathbb{R}^N} ds \to 0. \tag{23}$$

(see (19) and recall the definition of $G_n(t,x)$)

In addition we have

$$\int_0^t (f(s,\widehat{x})\widehat{u} - f(s,\widehat{x}+e_n)u_n, \widehat{x}_n - \widehat{x})_{\mathbb{R}^N} ds$$

$$= \int_0^t (f(s,\widehat{x})\widehat{u} - f(s,\widehat{x}+e_n)\widehat{u}, \widehat{x}_n - \widehat{x})_{\mathbb{R}^N} ds$$

$$+ \int_0^t (f(s,\widehat{x}+e_n)\widehat{u} - f(s,\widehat{x}+e_n)u_n, \widehat{x}_n - \widehat{x})_{\mathbb{R}^N} ds$$

$$\to 0 \text{ as } n \to +\infty \quad \text{(see hypothesis H(f) and (19))}. \tag{24}$$

We return to (22), pass to the limit as $n \to +\infty$ and use (21), (23), (24). We obtain

$$\frac{1}{2}|\widehat{x}(t) - \widetilde{x}(t)|^2 \leq 0 \quad \text{for all } t \in T,$$

$$\Rightarrow \quad \widetilde{x} = \widehat{x}. \tag{25}$$

Therefore we have

$$\widehat{x}_n \to \widehat{x} \text{ in } C(T,\mathbb{R}^N) \text{ (see (21), (25))}.$$

Since $\widehat{x}_n \in \mathcal{S}$ for all $n \in \mathbb{N}$ and $\mathcal{S}_c \subseteq C(T,\mathbb{R}^N)$ is compact (see Proposition 3), we conclude that

$$\mathcal{S}_c = \overline{\mathcal{S}}^{C(T,\mathbb{R}^N)}.$$

□

Remark 4. We can have

$$P_c = \overline{P}^{C(T,\mathbb{R}^N)\times L^1(T,\mathbb{R}^m)_w}$$

provided we strengthen the conditions on $U(t,\cdot)$ namely we need to assume that $U(t,\cdot)$ is h-continuous (equivalently $U(t,\cdot)$ is Vietoris continuous). The proof remains essentially the same with minor modifications.

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