Note on Commuting Quasi-nilpotent Unbounded Operators

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Abstract

Are explored the spectral properties for an unbounded operator U for which there exists an injective quasi-nilpotent unbounded operator N such that $UN = NU$. Several important properties in spectral theory are considered.

1 Introduction and preliminary results

As it is well-study in [4]-[13], one of the most interesting properties in the local spectral theory concern the stability of certain spectra of a bounded linear operator U under commuting quasi-nilpotent operators. In this paper we consider a similar problem for unbounded operators. In this work we shall consider the version of this property for an $(U, D(U))$ closed linear operator in a dense subspace of a Hilbert space $\mathcal{H}$. We extend some of the results established in the bounded case to an unbounded linear operator. First we begin with some preliminary notations and remarks.

Let $(U, D(U))$ be a (possibly unbounded) closed linear operator in $\mathcal{H}$. Clearly we define $D(U^2) := \{x \in D(U) : \quad Ux \in D(U)\}$ and, in general, for $n \geq 2$ we put $D(U^n) := \{x \in D(U^{n-1}) : \quad U^{n-1}x \in D(U)\}$ and $U^n(x) = U(U^{n-1}x)$. It is worth mentioning that nothing guarantees, in general, that $D(U^k)$ does not reduce to the null subspace $\{0\}$, for some $k \in \mathbb{N}$. For this reason powers of an unbounded operator could be of little use in many occasions. Throughout this paper if D is linear subspace of $\mathcal{H}$ a function

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$f : \Omega \to D$ is analytic if $f : \Omega \to \mathcal{H}$ is analytic and $f^n(x) \in D$ for every $x \in \Omega$, and $n \in \mathbb{N}$.

Let $(U, D(U))$ be a closed linear operator in $\mathcal{H}$. As usual, the spectrum of $(U, D(U))$ is defined as the set

$$\sigma(U) := \{\mu \in \mathbb{C} : \mu I - U \text{ is not a bijection of } D(U) \text{ onto } \mathcal{H}\}.$$

The set $\rho(U) = \mathbb{C} \setminus \sigma(U)$ is called the resolvent set of $(U, D(U))$, while the map $R(\mu, U) : \rho(U) \ni \mu \mapsto (\mu I - U)^{-1}$ is called the resolvent of $(U, D(U))$.

Following [3] and [14] let $(U, D(U))$ be a closed operator in $\mathcal{H}$.

Definition 1.1

- A point $\mu \in \mathbb{C}$ is said to be in the *local resolvent set* of $x \in \mathcal{H}$, denoted by $\rho_U(x)$, if there exist an open neighborhood $\mathcal{U}$ of μ in $\mathbb{C}$ and an analytic function $f : \mathcal{U} \to D(T)$ which satisfies

$$(\mu I - U) f(\mu) = x \quad \text{for all } \mu \in \mathcal{U}. \tag{1.1}$$

- The *local spectrum* $\sigma_U(x)$ of U at $x \in \mathcal{H}$ is the set defined by $\sigma_U(x) := \mathbb{C} \setminus \rho_U(x)$ and obviously $\sigma_U(x) \subseteq \sigma(U)$, and $\sigma_U(x)$ is a closed subset of $\mathbb{C}$.

Definition 1.2 Let $(U, D(U))\,, D := D(U)$, be a closed linear operator in $\mathcal{H}$ such that $U^n(D) \subseteq D$. The *hyperrange* of U is the subspace

$$U^\infty(D) := \bigcap_{n \in \mathbf{N}} U^n(D).$$

Also in this case two classical quantities associated with an operator U on a vector space D there corresponds:

$$\{0\} = \ker U^0 \subseteq \ker U \subseteq \ker U^2 \cdots .$$

and

$$D = U^0(D) \supseteq U(D) \supseteq U^2(D) \cdots .$$

The *ascent* of U is the smallest positive integer $p = p(U)$ such that $\ker U^p = \ker U^{p+1}$. If such p does not exist we let $p = +\infty$. The *descent* of U is defined to be the smallest integer $q = q(U)$ such that $U^{q+1}(D) = U^q(D)$. If such q does not exist we let $q = +\infty$.

Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. We denote by $\mathcal{L}(\mathcal{D})$ the set of all closable linear operators from $\mathcal{D}$ to $\mathcal{D}$ and $\mathcal{L}^{\dagger}(\mathcal{D})$ be the space consisting of all its elements which leave, together with their adjoints, the domain $\mathcal{D}$ invariant. Then $\mathcal{L}(\mathcal{D})$ is a algebra with respect to the usual operations and $\mathcal{L}^{\dagger}(\mathcal{D})$ is a subalgebra of $\mathcal{L}(\mathcal{D})$ with identity $\mathbb{1}$. (For the definitions and in general for the details can be found [1]). Let $\mathfrak{M}$ be an $O-*$ algebra on $\mathcal{D}$.

Let $\alpha(U) :=$ dim ker U and $\beta(U) := \operatorname{codim} U(X)$. Also in this case the class of all *upper semi-Fredholm operators* is defined by

$$\Phi_+(X) := \{U \in \mathcal{L}(\mathcal{D}) : \alpha(U) < \infty \text{ and } U(X) \text{ is closed}\},$$

while the class all *lower semi-Fredholm operators* is defined by

$$\Phi_-(X) := \{U \in \mathcal{L}(\mathcal{D}) : \beta(U) < \infty\}.$$

If $U \in \Phi_+(X) \cup \Phi_-(X)$ the *index* of U, also in this case, is defined by $\operatorname{ind} T = \alpha(T) - \beta(U)$. It is well known that if $\beta(U) < \infty$ then $U(X)$ is closed. An operator $U \in \mathcal{L}(\mathcal{D})$ is said to be *unbounded below* if is injective and has closed range. The *approximate point spectrum*,also in this case, is defined by

$$\sigma_{\mathrm{ap}}(U) := \{\mu \in \mathbb{C} : \mu I - U \text{ is not unbounded below}\},$$

while the *surjectivity spectrum* is defined as

$$\sigma_{\mathrm{s}}(U) := \{\mu \in \mathbb{C} : \mu I - U \text{ is not onto}\}.$$

If U^* denotes the *dual* of U it is well known that $\sigma_{\mathrm{ap}}(U) = \sigma_{\mathrm{s}}(U^*)$ and $\sigma_{\mathrm{s}}(U) = \sigma_{\mathrm{ap}}(U^*)$. Let $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ the class of all *Fredholm operators* . An operator $U \in \mathcal{L}(\mathcal{D})$ is said to be a *Weyl operator* if $U \in \Phi(X)$ and $\operatorname{ind} U = 0$, $U \in \mathcal{L}(\mathcal{D})$ is said to be *upper semi-Weyl* if $U \in \Phi_+(X)$ and $\operatorname{ind} U \leq 0$, $U \in \mathcal{L}(\mathcal{D})$ is said to be *lower semi-Weyl* if $U \in \Phi_-(X)$ and $\operatorname{ind} U \geq 0$. Denote by $\sigma_{\mathrm{w}}(U)$, $\sigma_{\mathrm{uw}}(U)$ and $\sigma_{\mathrm{lw}}(U)$ the Weyl spectrum, the upper semi-Weyl spectrum and the lower semi-Weyl spectrum , respectively. Clearly,

$$\sigma_{\mathrm{uw}}(U) \subseteq \sigma_{\mathrm{ap}}(U) \quad \text{and} \quad \sigma_{\mathrm{lw}}(U) \subseteq \sigma_{\mathrm{s}}(U)$$

holds for every $U \in \mathcal{L}(\mathcal{D})$. There is a duality:

$$\sigma_{\mathrm{uw}}(U) = \sigma_{\mathrm{lw}}(U^*) \quad \text{and} \quad \sigma_{\mathrm{lw}}(U) = \sigma_{\mathrm{uw}}(U^*),$$

2 Injective quasi-nilpotent operators

An operator $N \in \mathcal{L}(\mathcal{D})$ is said to be *quasi-nilpotent* if $\sigma(N) = \{0\}$. A quasi-nilpotent operator on an infinite-dimensional Banach space cannot be onto, since $\sigma_s(N) \neq \emptyset$.

We put

$$\mathbf{N}_i(X) := \begin{array}{ll} \{T \in \mathcal{L}^\dagger(\mathcal{D}) & : \text{there exists an injective quasi-nilpotent operator} \\ N \in \mathcal{L}^\dagger(\mathcal{D}) & \text{such that } UN = NU\}. \end{array}$$

Note that a nilpotent operator $N \neq 0$ cannot be injective, since if ker $N = \{0\}$ and $N^\nu = 0$, then $X = \ker N^\nu = \{0\}$. We start with the following result that has a central role in this paper.

We consider now some easy examples of unbounded operator $(U, D(U))$ (see [4])

In this example we give a a family of linear operators $U_{v_k} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ with commutes with an injective quasi-nilpotent operator. Let N_2 be the matrix

$$N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix},$$

where, for $i, j = 1, 2, \dots,$

$$q_{ij} = \begin{cases} 0 & \text{if } i < j + 1 \\ \frac{1}{j} & \text{if } i = j + 1 \\ 0 & \text{if } i > j + 1 \end{cases}$$

Clearly,

$$N_2(x_1, x_2, x_3, \dots) = (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots) \quad \text{for all } (x_1, x_2, x_3, \dots) \in \ell^2(\mathbb{N}),$$

then N_2 is injective and quasi-nilpotent. If N_2 is a weighted shift with non zero weights which tend to zero, then N_2 is a one-to-one quasi-nilpotent operator

Put $\mathbf{e}^k := (0,\dots,1,0,\dots)$, with $\mathbf{e}^k_i = \delta_{ik}$, let $U_{\mathbf{e}^k}$ be the following operetor:

$$U_{\mathbf{e}^k} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{k} & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\binom{k+1}{2}} & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\binom{k+2}{3}} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix},$$

where the generic element b_{ij} is given by:

$$b_{ij} = \begin{cases} 0 & \text{if } i < j+k-1 \\ \frac{1}{\binom{i-1}{i-k}} & \text{if } i = j+k-1 \\ 0 & \text{if } i > j+k-1 \end{cases}$$

Then, by arbitrarily choosing $k \geq 2$ and $\mu_k \in \mathbb{R}$, we obtain a family of matrices $U_{\mathbf{v}_k} := \mu_k U_{\mathbf{e}^k}$ that commute with N_2: $N_2\, U_{\mathbf{v}_k} = U_{\mathbf{v}_k}\, N_2$. Moreover it is easy to verify that $\forall k \geq 2$, $\mu_k \in \mathbb{R}$, $U_{\mathbf{v}_k}$ is a bounded linear operator; clearly $N_2 = U_{\mathbf{e}^2}$.

More generally, if we define $N_n := U_{\mathbf{e}^n}$, then, $\forall n \geq 2$, N_n is quasi-nilpotent, injective, with the property that N_n commute with $U_{\mathbf{v}_k}$, $\forall k \geq 2$, $\mu_k \in \mathbb{R} :\ N_n\, U_{\mathbf{v}_k} = U_{\mathbf{v}_k}\, N_n$.

We finally observe that the linear span $D := \langle U_{\mathbf{e}^1}, \dots, U_{\mathbf{e}^k}, \dots \rangle$ is an integral domain. If $B \in D$, then B is a matrix of the following type:

$$B = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \mu_2 & \mu_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \mu_3 & \frac{\mu_2}{2} & \mu_1 & 0 & 0 & 0 & 0 & \dots \\ \mu_4 & \frac{\mu_3}{3} & \frac{\mu_2}{3} & \mu_1 & 0 & 0 & 0 & \dots \\ \mu_5 & \frac{\mu_4}{4} & \frac{\mu_3}{6} & \frac{\mu_2}{4} & \mu_1 & 0 & 0 & \dots \\ \mu_6 & \frac{\mu_5}{5} & \frac{\mu_4}{10} & \frac{\mu_3}{10} & \frac{\mu_2}{5} & \mu_1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix},$$

where $\mu_i \in \mathbb{R}$, $i = 1, 2, \dots$, and the generic element b_{ij} is

$$b_{ij} = \begin{cases} 0 & \text{if } \; i < j \\ \frac{\mu_{i-j+1}}{\binom{i-1}{j-1}} & \text{if } \; i \geq j \end{cases}$$

Let $\mathcal{M} := \{B \in D : \sup_{i,j} |b_{ij}| < \infty\}$. Then for every $B \in \mathcal{M}$ there exists $c_B \in \mathbb{R}$, such that $|b_{ij}| \leq c_B$, $\forall i, j$. Let l^∞ be the Banach space of bounded sequences, $x \in l^\infty$ and let $c_x := \sup_i |x_i|$. Let us consider now $y = Bx$. Then

$$y_k = \sum_{j=1}^{k} \frac{\mu_j x_{k+1-j}}{\binom{k-1}{j-1}},$$

so

$$\forall k \geq 1, \; |y_k| \leq |c_A c_x| \sum_{j=1}^{k} \frac{1}{\binom{k-1}{j-1}}.$$

Since

$$\lim_{k \to \infty} \sum_{j=1}^{k} \frac{1}{\binom{k-1}{j-1}} =: S < +\infty,$$

then $y = Bx \in l^\infty$, hence $A : l^\infty \to l^\infty$, and so A is bounded.

But in general in this example : $l^2 \to l^2$ is a unbounded operator with commutes with an injective quasi-nilpotent operator N_2.

3 Results

Following [4] but studing unbounded operators we have

Theorem 3.1 *If $U \in \mathbf{N}_i(X)$, then $\alpha(U) < \infty$ if and only if U is injective. If the dual N^* of a quasi-nilpotent operator N is injective then $\beta(U) < \infty$ if and only if U is onto.*

Proof. Reasoning in a similar way to the case of bounded operators [4] we define if $\alpha(U) < \infty$ we put $Y := \ker U$. Cleraly, Y is invariant under N, also in this case and the restriction $(\mu I - N)|Y$ is injective for all $\mu \neq 0$. By assumption Y is finite-dimensional, then $(\mu I - Q)|Y$ is surjective for all $\mu \neq 0$. Thus $\sigma(N|Y) \subseteq \{0\}$. Moreover $N|Y$ is injective and hence $N|Y$ is surjective. Consequently, $\sigma(N|Y) = \emptyset$, from which we conclude, also in this case, that

$Y = \{0\}$. Thus U is injective. The converse it is easily demonstrate.

Let U be a lower semi-Fredholm operator. Thus U^*, the dual of U, is an upper semi-Fredholm operator, therefore $\alpha(U^*) = \dim \ker U < \infty$. Moreover Q^* is injective and quasi-nilpotent and $U^*N^* = N^*U^*$ thus T^* is injective. Moreover the range of U^* is closed, since $U(X)$ is closed, so U^* is unbounded below. By duality then U is onto. The converse is easily demonstrate. □

Theorem 3.2 *Let $U \in \mathbf{N}_i(X)$. Then*

$$\sigma_{\mathrm{uf}}(U) = \sigma_{\mathrm{ap}}(U). \tag{3.1}$$

Proof. Reasoning in a similar way to the case of bounded operators [4] we define if $\mu \notin \sigma_{\mathrm{uf}}(U)$ then $\mu I - U \in \Phi_+(X)$, so $\alpha(\mu I - U) < \infty$. Since N commutes also with $\mu I - U$ it then follows, by Theorem 3.1, that $\mu I - U$ is injective. Since $(\mu I - U)(X)$ is closed then $\mu \notin \sigma_{\mathrm{ap}}(U)$. Hence $\sigma_{\mathrm{ap}}(U) \subseteq \sigma_{\mathrm{uf}}(U)$. The opposite inclusion is always true for every operator, since every unbounded below operator is upper semi-Fredholm. Therefore, $\sigma_{\mathrm{uf}}(U) = \sigma_{\mathrm{ap}}(U)$. □

Theorem 3.3 *Let $U \in \mathbf{N}_i(X)$. Then*

$$\sigma_{\mathrm{ub}}(U) = \sigma_{\mathrm{uw}}(U) = \sigma_{\mathrm{ap}}(U) \quad \text{and} \quad \sigma_{\mathrm{b}}(U) = \sigma_{\mathrm{w}}(U) = \sigma(U).$$

Proof. The first equality is a consequence of 3.2, since

$$\sigma_{\mathrm{uf}}(U) \subseteq \sigma_{\mathrm{uw}}(U) \subseteq \sigma_{\mathrm{ub}}(U) \subseteq \sigma_{\mathrm{ap}}(U).$$

Moreover note first that $\sigma_{\mathrm{w}}(U) \subseteq \sigma_{\mathrm{b}}(U) \subseteq \sigma(U)$. If $\mu \notin \sigma_{\mathrm{w}}(U)$ then $\alpha(\mu I - U) = \beta(\mu I - U) < \infty$. By Theorem 3.1 we have $\alpha(\mu I - U) = \beta(\mu I - U) = 0$, so $\mu \notin \sigma(U)$ and hence $\sigma(U) \subseteq \sigma_{\mathrm{w}}(U)$. The converse is easily demonstrate . □

Theorem 3.4 *If $U \in \mathbf{N}_i(X)$ is such that $\sigma_{\mathrm{ap}}(U)$ has no hole, then*

$$\sigma_{\mathrm{uw}}(U) = \sigma_{\mathrm{ap}}(U) = \sigma_{\mathrm{w}}(U) = \sigma(U). \tag{3.2}$$

Proof. Reasoning in a similar way to the case of bounded operators [4] we define

$$\rho_{\mathrm{sf}}^{-}(U) := \{\mu \in \mathbb{C} : \mu I - U \in \Phi_{\pm}(X) \text{ and ind}\,(\mu I - U) < 0\},$$

$\rho_{\mathrm{w}}(U) := \mathbb{C} \setminus \sigma_{\mathrm{w}}(U)$ and $\rho_{\mathrm{uw}} := \mathbb{C} \setminus \sigma_{\mathrm{uw}}(U)$. By of course $\rho_{\mathrm{sf}}^{-}(U)$, $\rho_{\mathrm{w}}(U)$ and $\rho_{\mathrm{uw}}(U)$, also in this case, are open. It is easily seen that

$$\rho_{\mathrm{uw}}(U) = \rho_{\mathrm{sf}}^{-}(U) \cup \rho_{\mathrm{w}}(U). \tag{3.3}$$

Now, suppose that $\sigma_{\mathrm{ap}}(U)$ has no hole and set $\rho_{\mathrm{ap}}(U) := \mathbb{C} \setminus \sigma_{\mathrm{ap}}(U)$. By Theorem 3.3 we have $\rho_{\mathrm{uw}}(T) = \rho_{\mathrm{ap}}(U)$, so $\rho_{\mathrm{uw}}(U)$ is connected by assumption. From (3.3) we know that $\rho_{\mathrm{uw}}(U) = \rho_{\mathrm{sf}}^{-}(U) \cup \rho_{\mathrm{w}}(U)$ and both the two sets $\rho_{\mathrm{sf}}^{-}(U)$ and $\rho_{\mathrm{w}}(U)$ are open. Therefore $\rho_{\mathrm{sf}}^{-}(U) = \emptyset$, so $\rho_{\mathrm{uw}}(U) = \rho_{\mathrm{w}}(U)$ and hence $\sigma_{\mathrm{uw}}(U) = \sigma_{\mathrm{w}}(U)$.This implies, taking into account Theorem 3.3, that the equalities (3.2) hold. $\square$

4 Concluding Remark

In this work, only some results obtained for limited operators have been generalized to the case of non-limited operators. Clearly in many situations this is not possible or is under study. Naturally the reasoning is to determine a dense and stable domain with respect to the operators involved. The proofs and arguments valid for bounded operators are not obvious when switching to unbounded operators. But in the applications and in different situations it is precisely these operators that are involved as in the case of representation theory for insance.

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References

[1] J.-P. Antoine, A.Inoue, C.Trapani, *Partial *-algebras and their Operator Realizations*, Kluwer, Dordrecht, 2002

[2] P. Aiena, *Fredholm and Local Spectral Theory, With Applications to Multipliers*, (2004) Springer.

[3] P. Aiena, C. Trapani, S. Triolo *SVEP and local spectral radius formula for unbounded operators* Filomat, 2014, 28(2), pp. 263–273

[4] P. Aiena, F. Burderi, S. Triolo *On commuting quasi-nilpotent operators that are injective* Mathematical Proceedings of the Royal Irish Academy, 122A (2), 101–116 (2022)

[5] P. Aiena, M. L. Colasante, M. González. *Operators which have a closed quasi-nilpotent part.* Proc. Amer. Math. Soc. **130** (2002), 2701-2710.

[6] P. Aiena, T. L. Miller, M. M. Neumann, *On a localized single-valued extension property.* Math. Proc. Royal Irish Ac. **104A** (1), (2004), 17–34.

[7] P. Aiena, and O. Monsalve, *The single valued extension property and the generalized Kato decomposition property,* Acta Sci. Math. (Szeged) **67** (2001), 791–807.

[8] P. Aiena, O. Monsalve *Operators which do not have the single valued extension property.* J. Math. Anal. Appl. 250 (2000), no. 2, 435-448

[9] P. Aiena, and M. M. Neumann, *On the stability of the localized single-valued extension property under commuting perturbations,*, (2013), to appear in Proc. Amer. Math. Soc.

[10] I. Colojoară, C. Foiaş *Theory of generalized spectral operators.* Gordon and Breach, New York, 1968.

[11] N. Dunford *Spectral theory I. Resolution of the identity.* Pacific J. Math. **2** (1952), 559-614.

[12] N. Dunford *Spectral operators.* Pacific J. Math. **4** (1954), 321-54.

[13] N. Dunford, J. T. Schwartz *Linear operators, Part I.* (1967), Wiley, New York.

[14] J. K. Finch *The single valued extension property on a Banach space.* Pacific J. Math. **58** (1975), 61-9.

[15] E. K. Berberian *Some condition on an operator implying normality.* Proc. Amer. Math. Soc. **26** (1970), 277-281.

[16] N. B. Andersen and M. De Jeu *Local spectral radius formulas for a class of unbounded operators on Banach spaces* Journal of Operator Theory 69, Issue 2, 2013, Pages 435-461.

[17] D. Cichon, I.B. Jung and J. Stochel *Normality via local spectral radii* J. Operator Theory **61: 2** (2009), 253-278.

[18] K. B. Laursen, M. M. Neumann *An introduction to local spectral theory.* Clarendon Press, Oxford, 2000.

[19] M. Mbekhta *Sur la théorie spectrale locale et limite des nilpotents.* Proc. Amer. Math. Soc. **110** (1990), 621-31.

[20] M. A. Naimark, *On the square of a closed symmetric operator* (Russian) Dokl. Akad. Nauk SSSR **26** (1940) 207–208.

[21] P. R. Chernoff, *A semibounded closed symmetric operator whose square has trivial domain*, Proc. Amer. Math. Soc., 89/2 (1983) 289–290.

[22] S. Triolo, WQ*-Algebras of measurable operators, *Indian J. Pure Appl. Math.* 43(6): , December 2012, 601-617.

[23] P. Vrbová *On local spectral properties of operators in Banach spaces.* Czechoslovak Math. J. **23**(98) (1973a), 483-92.