FRACTIONAL-ORDER GENERALIZATION OF TRANSPORT EQUATIONS IN FRACTAL POROUS MEDIA

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The anomalous flux of a viscous fluid across a porous media with power-law scaling of the geometrical features of the pores is dealt with in the paper. It has been shown that, assuming a linear force-flux relation for the motion in the porous solid, then a generalized version of the Darcy equation has been obtained with the aid of Riemann-Liouville fractional derivative. The order of the derivative is related to the scaling property of the considered media yielding an appropriate physical picture of the use fractional-order Darcy equation recently used in scientific literature.
Keywords: Anomalous diffusion; Fractional derivative; Hausdorff dimension; Transport equations.

1. Introduction

Mass diffusion across porous medua is usually dealt with by means of linear force-flux relations that relating the mass transfer rate to the gradient of pressure. However modern applications in coupled multi-field problems encountered in recent problems of biological mechanics require, however, the introduction of more complete operators to account for scale as well as of non-uniform geometrical features of the considered domain. In this regard the use of fractional-order differential calculus proved to be one of the most powerful

method to handle physical complexity (1). In the paper it will be assumed that the flow occurs in a self-similar porous volume with prescribed fractal dimension d and it will be shown that in presence of flow of a viscous

fluid in the Poiseuille laminar regime, the flux of the initially contained fluid is related to the pressure gradient fluctuations by means of a Riemann-Liouville fractional-order derivative. The differentiation-order is related to the fractal dimension of the porous volume yielding a direct connections among fractals and fractional differential operators.

2. Materials and methods

Let us assume that the flux of a Newtonian fluid occurs in a fractal-like three-dimensional porous media with anomalous dimension of the cross sectional area $[A_F] = L^d$ where $1 \leq d \leq 2$ is the Hausdorff dimension of the fractal set (2). The overall volume of the porous solid is assumed as a subset of the 3d Euclidean volume V_E that is $VF \subseteq VE$ with dimension $[V_F] = L^{1+d}$. If, instead, the usual Euclidean measure is used, then the cross-section area measure AF = 0 and the volume measure VF = 0. Let us assume that a coordinate system {O;x1;x2;x3} is attached to the Euclidean volume surrounding VF and let us define l and h, respectively, the Euclidean length of the solid and the Euclidean measure of a

characteristic dimension of the generator. Under these circumstances the cross-sectional area and the volume measure of the medium reads,
respectively, $A_F \propto h^d$ and $V_F \propto h^d l$ with an appropriate proportionality coefficient that depends on the shape of the cross-sectional area. Let us assume that the velocity field of the moving fluid across the material pores is v^T (x1;x2;x3) = [0;0;v(X; t)] where x^T = [x1;x2;x3] is the coordinate vector. The gravity is neglected for the considered motion and the driving force of the moving fluid is the pressure gradient Δp(t) that is maintained in range with values of the Reynolds number Re = vδ/ν where δ is the hydraulic diameter and ν is the kinematic viscosity (νμ/ρ), lower than the value corresponding to the transition regime for the assumed fluid motion. We assume that the fluid motion occurs in a self-similar fractal set with cross-sectional area $A_f^{(k)}$ given as $A_f^{(k)} = \xi_F \left(\frac{h}{a^{f_k}}\right)^2 b^{g_k}$ where ξ_F is a shape factor that depends on the shape of the self-similar object, b^{g_k} is the number of pre-fractal at resolution k, a^{f_k} is the scale resolution factor of the k^{th} pre-fractal that is the division factor of the k^{th} lenght to yield an object self similar to the parent set and a, b, f_k and g_k depend on the fractal construction rule; for instance, the Sierpinski Triangle yields a = 3, b = 2, f_k = k and g_k = 3k.
As we assume a self-similar fractal set, without any non-linear transformations among different scales, the geometric dependent exponents fk and gk may be defined as appropriate linear functions of the resolution level k , as f_k=rk ; g_k=sk where $r, s \in \mathbb{N} - \{0\}$ are the resolution factors. The lack of the geometric invariance of the Euclidean metric yields that the outgoing flux across the different channels of the porous medium is not uniform at any resolution scale. This consideration yield that the overall outgoing flux is provided by the algebraic contribution of the outgoing fluxes at x_3=l from the channels with cross-sections represented by the material pores as:

$$q(t) = \sum_{k=0}^{\infty} q_k(t) = \sum_{k=0}^{\infty} A_F^{(k)} \bar{v}_k(t) \tag{1}$$

where $\bar{v}_k(l,t)$ is the mean velocity field at cross-section x_3 = l
Under the assumption of small Reynolds number and the purely viscous linear fluid flow we assume a linear relation among the mean velocity field and the pressure gradient as from the Hagen-Poiseuille law:

$$\bar{v}_k = \frac{\alpha_S \Delta p R^2}{\mu l} = \frac{\alpha_s \alpha_r \Delta p}{\mu l}\left(\frac{h}{a^{rk}}\right)^2 \tag{2}$$

where α_s is an appropriate coefficient depending upon the shape of the channel cross-section and R is the scale-dependent hydraulic radius of the fluid channel. The hydraulic radius may be always expressed as proportional to the characteristic dimensions (base, height) of the cross-section by a proper radius factor α_r as in latter eq. (2) $R = \frac{h\sqrt{\alpha_r}}{a^{rk}}$
In the following we will restrict our attention to the volume of fluid that leaves the porous solid at the time instant t, namely q_E(t). Neglecting inertia forces, the outgoing time in which the fluid columns at scale k reaches the right side at x_3 = l may be evaluated as $t_k = \left|\frac{l}{\bar{v}_k}\right| = \frac{\mu l^2}{\alpha_s \alpha_r h^2 \Delta p} a^{2rk}$ after straightforward manipulations, the effective volume of fluid
abandoning the solid reads

$$q(t) = \sum_{k=0}^{\infty} q_k(t)\langle t_k - t\rangle^0 = \frac{\alpha_s \alpha_r \mathrm{h}^4 \Delta p \xi_F}{\mu l} \sum_{k=0}^{\infty} \left(\frac{b^s}{a^{4r}}\right)^k \langle t_k - t\rangle^0 \tag{3}$$

where $\langle\cdot\rangle^0$ is a zero-order singularity function. The expression of the outgoing flux may be continualized for every time t in the control volume as

$$q_E(t) = f(x) = \begin{cases} 0, \; 0 < t \leq t_0 \\ \propto \left(\frac{\Delta p}{l}\right)^{1-\beta} t^{-\beta}, \; t \geq t_0 \end{cases} \tag{4}$$

that is depicted in fig. 1 for the Sierpinsky triangle porous solid. The exponent of the power law is related to the fractal dimension of the set as

$$\beta = \frac{2-d}{2} \tag{5}$$

3. Results

The experimental data showed in previous section allow for a fractional order generalization of the transport equation. Indeed, assuming a continualized counterpart for f_k=rε and g_k = sε where r and s are real order resolution factors, the outgoing flux reads:

$$q_E(x_3, t) \propto \left(\frac{\partial p}{\partial x_3}\right)^{1-\beta} \tag{6}$$

that, upon a proper linearization of the pressure gradient term as a perturbation of the pressure field $p(x_3, t) = p_0(x_3, t) + \bar{p}(x_3, t)$ where $\bar{p}$ is the pressure perturbation, it reads

$$q_E(x_3, t) \propto \left(\frac{\partial p_0}{\partial x_3}\right)^{-\beta} \left(D_0^{\beta} + \frac{\partial \bar{p}}{\partial x_3}\right)(\mathrm{t}) \tag{7}$$

that represents a linear type force-flux relation across a porous fractal media in terms of Riemann-Liouville fractional derivative (3;4).

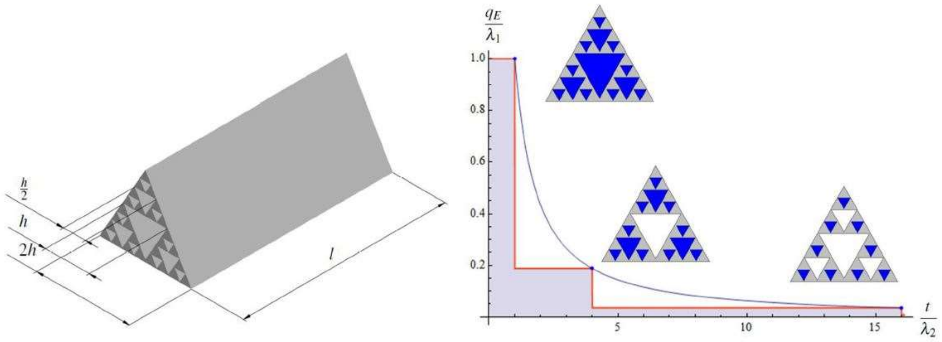

Fig. 1. Left: Sierpinsky triangle-like fractal porous solid. Right: non-dimensional outgoing flux across the fractal porous media

4. Conclusions

In the present study it has been shown that the mechanics involved in the outgoing flux of a viscous fluid in a laminar regime across a fractal porous media is capable to describe the state curdling described in previous papers 5. Fractional-order operators have been obtained under three main assumptions: i) the existence of a continualized scale parameter e to describe the self-similar fractal pores of the media; ii) a proper linearization of the non-linear force-flux relation around an initial value of the pressure gradient and iii) the local form of the force-flux relation obtained as the length of the control volume tends to zero. As far as these three assumptions have been introduced in the model, a fractional-order generalization of laminar transport equation is obtained in terms of the Riemann-Liouville fractional-order operators with differentiation order related to the fractal dimension of the porous set.

References

1. Podlubny I, Fractional Differential Equations,Accademic Press,1993 .
2. Le M´ehaute A, Crepy G, Introduction to transfer and motion in fractal media: The geometry of kinetics, Solid State Ionics,9-10:17–30, 1983.
3. Nigmatullin RR, Le M´ehaute A, Is there geometrical/physical meaning of the fractional integral with complex exponent?, J Non-Cryst Solids,351:2888-2899, 2005.
4. Samko SG, Kilbas AA, Marichev OI, Fractional Integrals and Derivatives: Theory and Applications, Taylor & Francis,1987.
5. Alaimo G, Zingales M, Fractional order generalization of laminar transport equations, Comm Nlin Sci Num Sim,in press.