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Decidability in integral conuclear residuated lattices

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Residuated lattices form a wide umbrella class of algebras which allows us to treat from a uniform perspective various superficially quite divergent classes of algebras with implication-like or division-like operations, such as Heyting algebras and lattice-ordered groups. In addition to the familiar constructions of universal algebra (homomorphic images, subalgebras, and so on), the structure theory of residuated lattices prominently features two further constructions, namely so-called nuclear and conuclear images. We will focus on the latter.

A *conucleus* is an interior operator on a residuated lattice whose image is a submonoid. A $\wedge$-*conucleus* moreover preserves binary meets, while a $\vee$-*conucleus* preserves binary joins. We call a conucleus *negative* if its image is a subset of the negative cone.

Crucially, the image of the conucleus $\Box$ on a residuated lattice $\mathbf{A}$ can also be equipped with the structure of a residuated lattice $\mathbf{A}_\Box$, albeit one which is not a subalgebra of the original. Such residuated lattices are called the *conuclear images* of $\mathbf{A}$, and algebras of the form $\langle \mathbf{A}, \Box \rangle$ will be called *conuclear expansions* of $\mathbf{A}$. The conuclear image construction allows us to represent residuated lattices of a general kind (Heyting algebras, cancellative commutative residuated lattices) as sitting inside residuated lattices of a more special kind (Boolean algebras, Abelian lattice-ordered groups) and leverage our deeper understanding of the latter class.

Besides their instrumental role in the theory of residuated lattices, conuclei also have a logical importance of their own, being generalizations of S4 modal box operators on Boolean algebras. Studying *conuclear residuated lattices*, i.e. residuated lattices equipped with a conucleus, thus amounts to studying substructural variants of the classical modal logic S4.

In this talk, we will consider the problem of deciding the validity of quasi-equations, or equivalently universal sentences, in varieties of integral conuclear residuated lattices. We sketch the tools needed to prove decidability results for varieties of conuclear integral residuated lattices, focusing on conuclear expansions of Abelian ℓ-groups (Abelian lattice-ordered groups), Abelian ℓ-group cones (the variety of negative cones of Abelian ℓ-groups), and MV-algebras.

Our main result is the following batch of decidability theorems. We prove them by algebraic means, i.e. by identifying small enough generating classes for these varieties. For example, the first item is a direct consequence of the fact that the variety of Abelian ℓ-group cones equipped with a conucleus is generated by the conuclear expansions of subalgebras of the negative cones of finite lexicographic powers of the Abelian ℓ-group $\mathbb{Z}$.

Theorem 1. *The following varieties have a decidable universal theory:*

(i) Abelian ℓ-group cones with a conucleus (or a $\wedge$-conucleus, or a $\vee$-conucleus),

(ii) Abelian ℓ-groups with a negative conucleus (or a neg. $\wedge$-conucleus, or a neg. $\vee$-conucleus),

(iii) MV-algebras with a conucleus (or a $\wedge$-conucleus, or a $\vee$-conucleus).

In logical terms, these result state that particular versions of S4 Łukasiewicz modal logic and S4 Abelian logic have a decidable deducibility problem.

The conuclear images of the above classes have been described by Montagna & Tsinakis [7] in their seminal study of conuclei on ℓ-groups as follows. (We abbreviate integral commutative residuated lattices by ICRL. *Fully distributive* means that $\cdot$ and $\vee$ distribute over $\wedge$.)

Theorem 2 (Montagna & Tsinakis [7]).

(i) The conuclear images of Abelian ℓ-group cones = cancellative ICRL*s.*

(ii) The $\wedge$-conuclear images of Abelian ℓ-group cones = fully distributive cancellative ICRL*s.*

(iii) The $\vee$-conuclear images of Abelian ℓ-group cones = semilinear cancellative ICRL*s.*

Combining the above results of Montagna & Tsinakis with our decidability results now immediately yields the following corollary.

Theorem 3. *The following varieties have a decidable universal theory:*

(i) cancellative ICRL*s,*

(ii) fully distributive cancellative ICRL*s,*

(iii) semilinear cancellative ICRL*s* [3].

This settles positively the long-standing problem of whether the (quasi-)equational theory of integral cancellative commutative residuated lattices is decidable [6, Problem 26], which had remained open since the first systematic study of cancellative residuated lattices [1] and which had not seen substantial progress since Horčík [3] settled the semilinear case. The big open problem remains the decidability of the (quasi-)equational theory of conuclear Abelian ℓ-groups.

As a side result, we observe that in some cases, the class of all conuclear expansions of K-algebras, for a universal class K of integral residuated lattices, inherits the FEP (the Finite Embeddability Property) from K. This yields further decidability results.

Theorem 4. *Consider a universal class* K *of (bounded) integral residuated lattices with locally finite monoid reducts, e.g. an n-potent variety of (bounded)* ICRL*s. If* K *has the FEP, then so do the classes of conuclear expansions and of conuclear images of* K*-algebras.*

This is useful in unifying a number of existing results. For example, Boolean algebras have the FEP due to being locally finite, so by the above theorem the varieties of Heyting algebras and S4 modal Boolean algebras inherit the FEP [5, 4]. Applying the result a second time yields that the variety of conuclear Heyting algebras has the FEP [2]. Similarly, the variety of conuclear Gödel algebras has the FEP thanks to the local finiteness of Gödel algebras.

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Layer decomposition for balanced residuated chains

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In [6], the decomposition method called *layer-algebra decomposition* was introduced for even and odd involutive FL_e-chains. The central idea is to partition the universe X into the equivalence classes of the relation

$$x \sim y \quad \Longleftrightarrow \quad \tau(x) = \tau(y),$$

where the *local-unit map* $\tau \colon X \to X$ is defined by $\tau(x) = x \to x$. This map sends X onto the set of positive idempotent elements of X, and $\tau(x)$ is called the *local unit* of x. For each positive idempotent element u, put

$$L_u := \{\, x \in X : \tau(x) = u \,\}.$$

For the class considered in [6], each L_u is closed under the operations of the constant-free reduct of the original algebra. Thus, the algebra decomposes into its *layer algebras* $\{\mathbf{L}_u\}$. Since $\tau(x) = x$ if and only if x is positive and idempotent, every $\sim$-class contains a unique positive idempotent element; hence the layers are canonically indexed by these elements.

Moreover, the layers form a direct system: whenever $u \leq v$, the connecting homomorphism $L_u \to L_v$ is given by $x \mapsto xv$. The original algebra can then be reconstructed from this system by means of Płonka sums [10], together with the directed lexicographic order of [6].

A natural next step is to identify further classes of residuated structures, beyond even and odd involutive FL_e-chains, that admit analogous layer decompositions. This program has led to decomposition and reconstruction results for:

- finite commutative idempotent involutive residuated lattices [9];
- finite involutive po-semilattices [8];
- locally integral involutive po-semigroups [5];
- locally integral involutive semirings [4];
- balanced residuated partially ordered monoids [2];
- balanced residuated partially ordered semigroups [3];
- balanced residuated posets [7];
- locally integral involutive residuated structures [11];
- Bochvar algebras [1].

In this talk, we present recent results on direct-system representations of balanced residuated chains. We identify the broadest class of balanced residuated chains admitting such a representation. In the finite case, the representation also yields a Clifford-style ordinal-sum decomposition. We also discuss related consequences, including enumeration results and cone representations.

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Interpolation and Amalgamation in bi-intuitionistic logics

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In the context of non-classical logics, the study of the interpolation property is often done through algebraic methods. In logical settings, the amalgamation property turns out to be equivalent to *deductive interpolation* (DIP) for a large class of systems (see e.g. [**Metcalfe2014**]). This has been applied for instance to systems of modal logic and intuitionistic logic, where the *Craig interpolation property* (CIP) has also been a subject of extensive study [**Maksimova1992**, **Maksimova1977**].

In this paper we study interpolation properties for bi-intuitionistic logic BiIPC. The algebraic semantics of this system, so-called *bi-Heyting algebras* have been the subject of extensive study. In this context, a particular extension of BiInt has been of interest, namely the *logic of co-trees*, i.e.:

$$\mathsf{BiGD} := \mathsf{BiInt} \oplus (p \to q) \vee (q \to p),$$

as studied for example in [**Bezhanishvili2024**, **Martins2025**], which serves as a bi-intuitionistic counterpart of the well-known Gödel-Dummett logic, with the corresponding algebras being deemed *bi-Gödel algebras*.

Definition 1. A logic L has the:

1. *Deductive interpolation property (DIP)*, if for any sets of variables $\overline{p}, \overline{q}, \overline{r}$ and formulas $\phi(\overline{p}, \overline{q}), \psi(\overline{p}, \overline{r})$ with $\phi(\overline{p}, \overline{q}) \vdash_L \psi(\overline{p}, \overline{r})$, there exists $\chi(\overline{p})$ such that $\phi(\overline{p}, \overline{q}) \vdash_L \chi(\overline{p})$ and $\chi(\overline{p}) \vdash_L \psi(\overline{p}, \overline{r})$.

2. *Craig interpolation property (CIP)*, if for any sets of variables $\overline{p}, \overline{q}, \overline{r}$ and formulas $\phi(\overline{p}, \overline{q}), \psi(\overline{p}, \overline{r})$ with $\vdash_L \phi(\overline{p}, \overline{q}) \to \psi(\overline{p}, \overline{r})$, there exists $\chi(\overline{p})$ such that $\vdash_L \phi(\overline{p}, \overline{q}) \to \chi(\overline{p})$ and $\vdash_L \chi(\overline{p}) \to \psi(\overline{p}, \overline{r})$.

Definition 2. Let $\mathcal{K}$ be a category of structures with a partial order, and maps which are order-preserving. A tuple (A, B_0, B_1, f_0, f_1) of structures, where $f_i : A \to B_i$ is a monomorphism is called an *amalgamation triple*. We say that a given $C \in \mathcal{K}$ is an *amalgam* if there are two monomorphisms $g_0 : B_0 \to C$ and $g_1 : B_1 \to C$ such that $g_0 f_0 = g_1 f_1$.

We say that a given amalgam is a *superamalgam* if it satisfies the following: whenever $b_0 \in B_0$ and $b_1 \in B_1$ and $g_0(b_0) \leq g_1(b_1)$, there is some $a \in A$ such that $b_0 \leq f_0(a)$ and $f_1(a) \leq b_1$ (and similarly, with B_0 and B_1 swapped).

As a basic observation, following Maksimova's techniques, we have:

Theorem 3. *For L a bi-superintuitionistic logic, we have:*

1. *L has the deductive interpolation property if and only if $\mathsf{Var}(L)$ has the amalgamation property;*

2. *L has the Craig interpolation property if and only if $\mathsf{Var}(L)$ has the superamalgamation property.*

As a first step towards an analysis of these properties, we provide a Maksimova-style characterization of all extensions of biGD with the (Craig) interpolation properties.

We consider the following logics (see Figure 1):

1. $\mathsf{CPC} = \mathsf{Log}(C_1)$ is classical logic;

2. $\mathsf{BiLC} = \mathsf{Log}(\{C_n : n \in \omega\})$ and $\mathsf{BiLC}_2 = \mathsf{Log}(C_2)$;

3. $\mathsf{BiF}_2 = \mathsf{Log}(F_2)$ and $\mathsf{BiF} = \mathsf{Log}(\{F_n : n \in \omega\})$;

4. $\mathsf{BiLFC} = \mathsf{Log}(\{\mathfrak{C}_n : n \in \omega\})$.

Theorem 4. *Let $L \supseteq \mathsf{BiGD}$ be a bi-si logic. Then:*

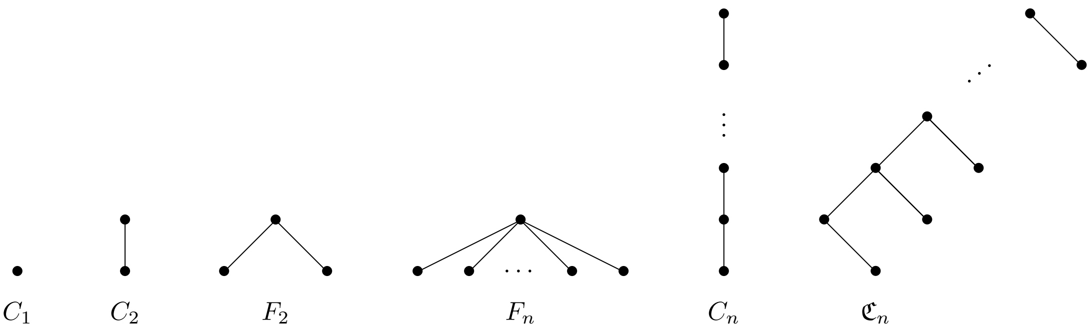

Figure 1: Characteristic posets

1. *L has the CIP if and only if $L \in \{\mathsf{BiGD}, \mathsf{BiLC}, \mathsf{BiLC}_2, \mathsf{CPC}\}$.*
2. *L has the DIP and lacks the CIP if and only if $L \in \{\mathsf{BiF}_2, \mathsf{BiF}\}$.*

The proof of this theorem consists of several main parts, which we outline:

1. For a logic L of finite depth (above biIPC), we show that L has the CIP iff L is CPC or BiLC_2.
2. For locally tabular logics, we show that having the DIP implies being one of biLC, biLC_2, biF, biF_2 or CPC, relying on a characterization of local tabularity above biGD provided by [**Martins2025**]: we show that a logic which is not one of the above must contain all finite combs, and hence, be contained in biLFC.
3. We show that any logic $L \subseteq \mathsf{biLFC}$ which has superamalgamation must contain a 3-fork. Using a bi-intuitionistic analogue of Maksimova's amalgamation trick, we prove that this implies that all finite co-trees must be models for L, and hence $L = \mathsf{biGD}$.
4. Finally, using a construction of colimits of free Gödel algebras due to Carai [**caraifreeproducts**], we prove that biGD has the Craig interpolation property.

Hence, we obtain a full characterization of extensions of biGD with the CIP. Beyond biGD we make a few observations:

Theorem 5. *In the lattice* $\mathrm{Ext}(\mathsf{biIPC})$ *of super-bi-intuitionistic logics, there exists:*

1. *Infinitely many tabular logics with the DIP and without the CIP.*
2. *Infinitely many tabular logics without the DIP.*
3. *Infinitely many logics with the DIP such that their intuitionistic fragment lacks the DIP.*

A completeness theorem for topological doctrines

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As is well known, topological spaces provide a complete semantics for propositional S4 logic, where formulas are interpreted as subsets of a fixed space X and where the modality $\Diamond$ is interpreted as the closure operator. Following [Ghi90], we lift this semantics to *first-order* S4 logic in analogy with the classical set-based semantics of first-order Boolean logic. More precisely, in the mono-sorted case:

- Relational symbols of arity n are interpreted as subspaces of X^n, for a fixed space X.
- Functional symbols are interpreted as *continuous* maps.
- Boolean operations are interpreted as intersection, union and complementation.
- Existential quantification is interpreted by taking the direct image.
- Substitution of free variables is interpreted by taking the inverse image.

Compared to the case of first-order Boolean logic, the main subtlety is that substitution cannot be treated as a secondary construction on formulas: it is at the same level as the other logical connectives such as conjunction, disjunction, quantification and the modalities. The reason is that taking the inverse image by a continuous map does not commute with the modalities in topological spaces: we only have $\Diamond f^{-1}(A) \subseteq f^{-1}(\Diamond A)$ in general. Equality fails for instance when f is the diagonal $X \to X \times X$ of a non-discrete space X.

In order to make these ideas precise, we use the formalism of categorical logic. The base setup is developed in [GM25b]. In order to work with modal theories in an algebraic way, we define *modal categories*, which relate to first-order modal logic in the same way as Boolean categories relate to first-order Boolean logic. The main peculiarity is that in this setting, the bare categorical structure is not enough to specify a theory. Indeed, the notion of an "embedding" (subobject) differs from the notion of an "injection," as can be seen in the category of topological spaces, for instance. We use a proper factorization system $(\mathcal{E}, \mathcal{M})$ to encode this data: the "embeddings" are the morphisms in $\mathcal{M}$ while the "injections" are the monomorphisms.

A *lax morphism* of modal algebras is a map f preserving the Boolean operations and satisfying $\Diamond f(A) \leq f(\Diamond A)$.

Definition 1. A (Boolean) *modal category* is a left exact category E equipped with:

1. A stable proper factorization system $(\mathcal{E}, \mathcal{M})$.
2. For all $X \in \mathsf{E}$, a structure of modal algebra on the poset $\mathrm{Sub}_{\mathcal{M}}(X)$ of $\mathcal{M}$-subobjects of X, making $\mathrm{Sub}_{\mathcal{M}}$ a functor from E^{op} to the category of modal algebras and lax morphisms.

Moreover, we require that $\Diamond A = m^* \Diamond m A$ for each $m : X \hookrightarrow Y$ in $\mathcal{M}$ and each $A \in \mathrm{Sub}_{\mathcal{M}}(X)$ (we use m^* for inverse image along m and m itself for its left adjoint, namely direct image).

In [GM25a], we use this framework to state and answer to the following question:

What is the first-order modal theory of topological spaces?

We find two additional axioms, besides the obvious S4 axioms from the propositional case:

1. A "product independence axiom" which states that $\Diamond$ distributes over conjunctions of formulas sharing no common variable:

$$(\Diamond\varphi)[\overline{x}] \wedge (\Diamond\psi)[\overline{y}] \iff \Diamond(\varphi[\overline{x}] \wedge \psi[\overline{y}])$$

(here notations like $\varphi[\overline{x}], \psi[\overline{y}]$ obviously refer to inverse images along projections in internal logic).

2. A "loop contraction axiom," which expresses a continuity condition on certain composable loops of binary relations.

Theorem 1 ([GM25a])**.** *A small modal category embeds in a power of the modal category of topological spaces if and only if it satisfies S*4 *axioms, the product independence axiom and the loop contraction axiom.*

We give details about the loop contraction axiom. A *partial map* from X to Y is a morphism $f : D \to Y$ defined on an $\mathcal{M}$-subobject D of X. Its graph is an internal binary relation $G_f \subseteq X \times Y$ whose transpose we denote by $G_f^t \subseteq Y \times X$. A *loop* is a sequence of internal binary relations $R_1, R_2, \ldots, R_n$ such that the composite $R_1 \circ \cdots \circ R_n$ makes sense and such that the codomain of R_1 coincides with the domain of R_n. This object is called the *anchor* of the loop. We define the *acceptable loops* by induction:

1. The empty loop is acceptable (on any anchor).

2. Identities can be added anywhere in an acceptable loop to produce a new acceptable loop.

3. If w and w' are acceptable loops, then w, w' is acceptable.

4. If w is acceptable and if f is a partial map, then G_f, w, G_f^t is acceptable.

5. If $R_1, \ldots, R_n$ and $S_1, \ldots, S_n$ are acceptable, then $R_1 \times S_1, \ldots, R_n \times S_n$ is acceptable.

The loop contraction axiom states that for any acceptable loop $R_1, \ldots, R_n$ and for any subspace A of the anchor, it holds that

$$\Diamond R_1 \Diamond R_2 \Diamond \cdots \Diamond R_n A \leq \Diamond A.$$

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On actions and split extensions in varieties of hoops

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Abstract

BL-algebras were introduced by P. Hájek in [6] as the algebraic semantics of *Basic Logic*, the logic of continuous t-norms. They capture the common fragment of the three most relevant *many-valued* logics, namely *Łukasiewicz Logic*, *Gödel Logic*, and *Product Logic*. It is a classical result [10] that, up to isomorphism, every continuous t-norm behaves locally as one of the following three fundamental ones: the Łukasiewicz t-norm, defined by $x \cdot_L y = \max\{x+y-1, 0\}$, the Gödel t-norm, defined by $x \cdot_G y = \min\{x, y\}$, and the product t-norm, defined by $x \cdot_P y = xy$. Every continuous t-norm naturally induces a *residuation*, or implication,

$$x \to y = \sup\{z \in [0,1] \mid z \cdot x \leqslant y\}.$$

In [6], the author studied in detail the residuated structures associated with the three fundamental continuous t-norms, whose corresponding varieties of algebras may be described as follows: the variety MVAlg generated by $([0,1], \cdot_L, \to_L, \max, \min, 0, 1)$ defines the class of *MV-algebras* [2], which form the algebraic semantics of Łukasiewicz Logic [3]. Algebras in the variety GAlg generated by $([0,1], \cdot_G, \to_G, \max, \min, 0, 1)$ are called *Gödel algebras*, and they form the equivalent algebraic semantics for *Gödel Logic*. The variety PAlg of *product algebras* is generated by $([0,1], \cdot_P, \to_P, \max, \min, 0, 1)$ and its associated propositional calculus is *Product Logic* [5]. Finally, P. Hájek introduced the variety BLAlg of *BL-algebras*, whose propositional calculus is *Basic Logic*. It was then shown that BLAlg is the variety generated by all algebras $([0,1], \cdot, \to, \max, \min, 0, 1)$, where $\cdot$ is a continuous t-norm on $[0,1]$ (see [1, 4]).

From a categorical point of view, the variety BLAlg, as well as its subvarieties MVAlg, GAlg, and PAlg, is an ideally exact category, see [7]. Moreover, if $\mathbf{2}$ denotes the two-element Boolean algebra, then the semi-abelian categories $(\mathsf{BLAlg} \downarrow \mathbf{2})$, $(\mathsf{MVAlg} \downarrow \mathbf{2})$, $(\mathsf{GAlg} \downarrow \mathbf{2})$, $(\mathsf{PAlg} \downarrow \mathbf{2})$ are equivalent, respectively, to the semi-abelian varieties of basic hoops, Wajsberg hoops, Gödel hoops, and product hoops.

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Keywords: Semi-abelian category, internal action, split extension, strong section, hoop, BL-algebra, MV-algebra, Gödel algebra, product algebra

A key notion that can be studied in the context of semi-abelian categories is that of *internal action*, which generalizes classical algebraic notions such as group or Lie algebra actions and provides a description of split extensions (and, therefore, of retractions) in algebraic terms, namely by means of semidirect products.

The aim of this talk is to investigate actions and split extensions in the variety of hoops and in its subvarieties of basic, Wajsberg, Gödel, and product hoops. In particular, we focus on split extensions with *strong section* in the sense of W. Rump [11], which we describe in terms of *strong external actions*, i.e., pairs of maps satisfying a set of identities closely related to the axioms satisfied by the hoop, see [8, 9].

We prove that, for any hoop X, there is a natural isomorphism of functors

$$\mathrm{EAct}_{\mathrm{ss}}(-,X) \cong \mathrm{SplExt}_{\mathrm{ss}}(-,X)$$

between strong external actions on X and isomorphism classes of split extensions with strong section with kernel X. We also observe that the notion of split extension with strong section trivializes in the context of MV-algebras, whereas in the variety of Gödel hoops, strong external actions coincide with those in the variety of basic hoops.

Eventually, we show a connection between the notion of strong external action in the variety of hoops and the semidirect product construction introduced by W. Rump in the category of L-algebras [12].

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Rewriting Systems on Arbitrary Monoids

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Classical rewriting theory studies string rewriting systems, i.e. rewriting systems whose reductions take place in free monoids [1]. In this work we introduce a more general notion of rewriting in which the ambient algebraic structure is an arbitrary monoid. This generalization preserves the defining feature of string rewriting: a rule $s \to t$ may be applied by replacing xsy with xty. The key observation is that this operation only requires the ability to multiply elements on the left and on the right: the additional syntactic information carried by a free monoid of words over a set of generators is not needed for the basic rewriting mechanism itself.

There is also a logical motivation behind this effort to decouple rewriting from free monoids. If rewriting systems are viewed as first-order structures in a language containing a monoid operation together with a binary relation, then the classical restriction to free monoids is an external semantic condition: the class of free monoids is not first-order axiomatizable. Thus, in any approach that aims to use first-order or model-theoretic methods to study rewriting presentations, the free setting cannot be treated internally. Rewriting over arbitrary monoids removes this restriction and gives a formulation in which the ambient structure is simply a monoid equipped with a compatible reduction relation.

A *monoidal rewriting system* (MRS) is a triple $(M, \cdot, R)$, where $(M, \cdot)$ is a monoid and $R \subseteq M \times M$ is a binary relation. We write $a \to_R b$ when there exist $x, y, s, t \in M$ such that

$$a = xsy, \qquad b = xty, \qquad (s,t) \in R.$$

If $(M, \cdot, R)$ is Noetherian and confluent, every $u \in M$ has a unique normal form $\overline{u}$ [2], and the set of these normal forms inherits, under $a \,\bar{\cdot}\, b := \overline{a \cdot b}$, the structure of a monoid which we denote by $I(M, \cdot, R)$ and whose elements we call the irreducible elements of M.

We first show that every monoid admits a canonical convergent presentation in this generalized sense, followed by a classification of all Noetherian confluent MRS presenting a fixed monoid.

1 The Canonical Presentation of a Monoid

Let **NCRS** be the category of Noetherian confluent MRS and rewriting-preserving monoid homomorphisms (Here a rewriting-preserving monoid homomorphism $f : (A, \cdot, R) \to (B, *, L)$ is a monoid homomorphism $f : (A, \cdot) \to (B, *)$ such that $(s,t) \in R$ implies $f(s) \to_L^* f(t)$.)

We promote **NCRS** to a strict 2-category $\mathbf{NCRS}_2$ by declaring that, for parallel 1-cells $f, g : A \to B$, there is a unique 2-cell $f \Rightarrow g$ if and only if $f(a) \leftrightarrow_L^* g(a)$ for every $a \in A$, and we regard **Mon** as a locally discrete 2-category.

Given a monoid $(M, \cdot)$, let $M^+ = M \setminus \{1\}$ and define

$$G(M) := (F(M^+), \oplus, R_M), \qquad R_M = \{(\nu_M(a) \oplus \nu_M(b), \nu_M(ab)) : a, b \in M\}.$$

where $F(M^+)$ is the free monoid on M^+, $\oplus$ denotes concatenation, and $\nu_M(1) = \varepsilon$, while $\nu_M(m) = m$ for $m \neq 1$. The assignment sending a MRS $(A, \cdot, R)$ to the monoid $I(A, \cdot, R)$ extends to a strict 2-functor

$$I : \mathbf{NCRS}_2 \longrightarrow \mathbf{Mon},$$

while the construction of $G(M)$ from M extends to a strict 2-functor

$$G : \mathbf{Mon} \longrightarrow \mathbf{NCRS}_2.$$

Our first main result is the following.

Theorem 1. The pair (G, I) forms a biadjunction $G \dashv I$. More precisely, the unit $\eta : 1_{\mathbf{Mon}} \Rightarrow IG$ is the identity 2-natural transformation, and for every Noetherian confluent MRS A the counit $\varepsilon_A : GI(A) \longrightarrow A$ sends a word $a_1 \oplus \cdots \oplus a_n$ of irreducibles to the product $a_1 \cdots a_n$ in A.

Equivalently, for every monoid M and every Noetherian confluent MRS A, there is a natural equivalence of categories $\mathrm{Hom}_{\mathbf{NCRS}_2}(G(M), A) \simeq \mathrm{Hom}_{\mathbf{Mon}}(M, I(A))$. Thus, maps out of the canonical presentation $G(M)$ correspond, up to the natural 2-dimensional identification in $\mathbf{NCRS}_2$, to monoid homomorphisms out of M.

2 Generalized Tietze Transformations

We then study how different convergent MRS presenting the same monoid are related. For this we introduce *generalized elementary Tietze transformations* (GETTs) [1]. Types 1, 2, and 3 are direct analogues of the classical transformations: adding a redundant rewriting rule, deleting a redundant rewriting rule, and adjoining a new generator together with a defining rule.

The essential new feature is a type 4 transformation. Let $(A, \cdot, R)$ be a Noetherian confluent MRS and let $J \subseteq R$ be a coherent subset, meaning that J is confluent and that the collapsed system obtained by taking the J-irreducibles as the underlying monoid and transporting the remaining rules $R \setminus J$ is again Noetherian and confluent. We denote this collapsed system by $A_J = (A_J, \cdot_J, R_J)$. A type 4 GETT is therefore the passage from $(A, \cdot, R)$ to $(A_J, \cdot_J, R_J)$.

Our second main result is a generalized Tietze theorem.

Theorem 2. Let $(M, \cdot)$ be a fixed monoid, and let $(A, \cdot, R)$ and $(B, *, L)$ be Noetherian confluent MRS such that

$$I(A, \cdot, R) \cong (M, \cdot) \cong I(B, *, L).$$

If B is infinite of cardinality λ, then $(B, *, L)$ can be obtained from $(A, \cdot, R)$ by at most λ GETTs.

Equivalently,

$$I(A, \cdot, R) \cong I(B, *, L)$$

if and only if $(A, \cdot, R)$ and $(B, *, L)$ are connected by a possibly infinite sequence of GETTs.

The results show that two fundamental features of classical rewriting theory survive in the generalized setting: every monoid has a canonical convergent presentation, and all convergent presentations of a fixed monoid are related by Tietze-type moves.

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Being a Union-splitting is Decidable in Modal Logic

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Studies in modal logic often involve determining whether a logic has a certain property, such as consistency, Kripke completeness, the finite model property, etc. It is then natural to ask whether such properties are decidable. More specifically, we say that a property P is *decidable in* $\mathsf{NExt}\,\mathsf{K}$ (the lattice of all normal modal logics) if there is an algorithm that, given a formula φ, determines whether the logic $\mathsf{K} + \varphi$ has the property P.

Many properties have been shown to be undecidable for normal modal logics. The pioneering work by Thomason [9] showed the undecidability of Kripke completeness. This was followed by a series of works by Chagrov [3, 4, 5], who introduced a general method for showing undecidability. This method can be applied to show the undecidability of various properties, including the finite model property, first-order definability, decidability, and tabularity. See [10] and [2, Chapter 17] for a comprehensive overview and references.

It seems that Chagrov's method is so general that it applies to almost all meaningful properties of logics, showing that they are undecidable. Indeed, it was pointed out in [10] that "we know only two interesting decidable properties of finitely axiomatizable logics in $\mathsf{NExt}\,\mathsf{K}$: consistency and coincidence with K." These two properties are not very fascinating because there is only one inconsistent logic and only one logic that coincides with K.

However, in this talk, we show that the property of being a *union-splitting* in $\mathsf{NExt}\,\mathsf{K}$ is decidable, answering the open question [10, Problem 2] in the affirmative. A logic L is a *splitting* in $\mathsf{NExt}\,\mathsf{K}$ if there is a logic L' such that $L \not\subseteq L'$ and for any logic L'', either $L'' \subseteq L'$ or $L \subseteq L''$. A logic L is a *union-splitting* if it is the join of a set of splittings. According to Blok [1], a logic L is a union-splitting iff it is *strictly Kripke complete*, that is, for any logic $L' \neq L$, there is a Kripke frame that validates one of L and L' and refutes the other. In other words, L that can be separated from any other logic by a Kripke frame. So, we also obtain the decidability of strict Kripke completeness.

Theorem 1. *Being a union-splitting is decidable in* $\mathsf{NExt}\,\mathsf{K}$*. Consequently, strict Kripke completeness is decidable in* $\mathsf{NExt}\,\mathsf{K}$*.*

These two properties are much more non-trivial than consistency and coincidence with K as there are continuum many union-splittings and continuum many non-union-splittings in $\mathsf{NExt}\,\mathsf{K}$. The proof idea is to provide a characterization of union-splittings in terms of finite modal algebras, which shows that the class of non-union-splittings is recursively enumerable. Since the class of union-splittings is also recursively enumerable, we obtain the decidability.

Moreover, we observe that, for a formula φ, the following are equivalent:

1. $\mathsf{K} + \varphi$ is a union-splitting in $\mathsf{NExt}\,\mathsf{K}$ or the inconsistent logic,

2. the axiomatization problem for $\mathsf{K} + \varphi$ is decidable,

3. φ is a decidable formula.

Note that consistency is decidable as $\mathsf{NExt}\,\mathsf{K}$ has two co-atoms, i.e., maximal consistent logics [6]. Thus, the decidability of being a union-splitting implies that having a decidable axiomatization problem and being a (un)decidable formula are also decidable. The latter answers [2, Problem 17.3] for $\mathsf{NExt}\,\mathsf{K}$ in the affirmative.

Theorem 2. *It is decidable that, for a formula φ, whether the axiomatization problem for $\mathsf{K}+\varphi$ is decidable and whether φ is a decidable formula.*

This talk is based on [8] (see also [7, Chapter 5]).

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A Model Companion
for an Expansion of Abelian Lattice-Ordered Groups

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Lattice-ordered groups, or ℓ-groups (see [1] for an introductory reference), play a central role in the study of MV algebras and Łukasiewicz logic, as the category of MV algebras is equivalent to the category of abelian ℓ-groups with a strong order unit [3]. Additionally, they are found as the value groups of valued fields, the groups of divisibility of Bézout domains [1, Thm 11.2], as well as important reducts of rings of real-valued continuous functions on a Tychonoff space.

We define a first-order extension to abelian ℓ-groups, called **densely valued ℓ-groups**. These are triples $(\mathcal{G},\mathcal{L},P)$ where $\mathcal{G}$ is an abelian ℓ-group, $\mathcal{L}$ is a bounded distributive lattice, and $P:\mathcal{G}\to\mathcal{L}$ is a lattice morphism which is surjective onto $\mathcal{L}\setminus\{\bot\}$, and satisfying:

$$P(a)=P(na) \text{ and } 0\leqslant a \iff P(a)=\top$$

for all $a\in\mathcal{G}$ and $n\in\mathbb{N}$. We show that every densely valued ℓ-group is isomorphic to one of the form $(\mathcal{G},\bar{\mathcal{K}}(X),V(-\wedge 0)\cap X)$, where $X\subseteq\ell\text{-Spec}(\mathcal{G})$[1] is a dense spectral subspace, $\bar{\mathcal{K}}(X)$ is the set of closed constructible sets of X, and $V(a\wedge 0)\cap X$ is the set of $\mathcal{J}\in X$ with $a\wedge 0\in\mathcal{J}$. Further, this gives us that the obvious forgetful functor from densely valued ℓ-groups to ℓ-groups has a left adjoint, given by:

$$F(\mathcal{G})=(\mathcal{G},\bar{\mathcal{K}}(\ell\text{-Spec}(\mathcal{G})),V(-\wedge 0))$$
$$F(\phi)=(\phi,\bar{\mathcal{K}}(\ell\text{-Spec}(\phi))).$$

Using this, we provide a generalisation to the classical result that every abelian ℓ-group is an ℓ-subgroup of an ℓ-group of functions into a divisible ordered abelian group (DOAG). In particular, we show that every densely valued ℓ-group is embeddable in a densely valued ℓ-group of the form $(\Gamma^X,\mathcal{P}(X),\{-\geqslant 0\})$, where Γ is a DOAG, X a set, and $\{f\geqslant 0\}=\{x\in X \mid f(x)\geqslant 0\}$.

Then, we exposit a classical result due to Fuxing Shen and Volker Weispfenning (originally due to [6, Thm 2.1]; see [4, Thm 2.4.29] or [7, Thm 3.8] for more modern treatments), which tells us that if $(\mathcal{G},\mathcal{L},P)$ is a densely valued ℓ-group with $\mathcal{G}$ divisible and satisfying **patching** i.e.:

For all $a,b\in\mathcal{G}$ and $l,k\in\mathcal{L}$ such that $l\wedge k\leqslant P(a-b)\wedge P(b-a)$,
there exists some $g\in\mathcal{G}$ with $l\leqslant P(a-g)\wedge P(g-a)$ and $k\leqslant P(b-g)\wedge P(g-b)$

then for every densely valued ℓ-group formula ϕ, there is a bounded lattice formula χ and ℓ-group terms t_i such that for all $\bar{a}\in\mathcal{G}^n$ and $\bar{l}\in\mathcal{L}^m$:

$$(\mathcal{G},\mathcal{L},P)\models(\phi(\bar{a},\bar{l})\leftrightarrow\chi(P(t_1(\bar{a})),\ldots,P(t_n(\bar{a}))),\bar{l})).$$

[1] Here, we take ℓ-Spec$(\mathcal{G})$ to be the set of all prime ℓ-ideals including $\mathcal{G}$, so that the map ℓ-Spec$(-)$ is functorial.

In particular, this implies that the first-order theory of a divisible abelian ℓ-group is completely determined by the distributive lattice of its principal ℓ-ideals.

This then allows us to characterise fully the algebraically closed densely valued ℓ-groups in particular, we show that the former are those densely valued ℓ-groups $(\mathcal{G}, \mathcal{L}, P)$ with patching, $\mathcal{G}$ divisible, and $\mathcal{L}$ Boolean. Further, we show that the existentially closed densely valued ℓ-groups are the algebraically closed ones with $\mathcal{L}$ atomless. We remark that this is a first-order axiomatisation, and that this contrasts with the usual theory of ℓ-groups, for which no such axiomatisation exists (see [2, Corollary 1.9])

Finally, using Shen-Weispfenning and some elementary properties of atomless Boolean algebras [5], we are able to show that the first-order theory of existentially closed densely valued ℓ-groups is complete, and has quantifier elimination when we introduce a symbol for Boolean negation.

This work is part of my PhD thesis under the supervision of Marcus Tressl at the University of Manchester.

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A topos for étale-finite Heyting algebras

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In [6], Andrew Pitts proved that, for each Heyting algebra H, the inclusion $i\colon H \to H[x]$ has left and right adjoints. In the introduction of that paper, the author explains how such a result arose out of an attempt to disprove the following conjecture: for each Heyting algebra H there is an elementary topos $\mathcal{E}$, such that $\mathsf{Sub}_{\mathcal{E}}(1)$, the lattice of truth-values, is isomorphic to H. For complete Heyting algebras [5, Chapter IX.5] and Boolean algebras (by a construction of Peter Freyd, see [3, Exercise 9.11]) the answer has been known. In the early 2000s, Dimitri Pataraia began work on this problem; part of his analysis is outlined in a lecture given by Peter Johnstone [2] on the subject of "hochas" (higher-order cylindric Heyting algebras), and involved adapting methods related to the step-by-step construction of finitely generated free Heyting algebras [1] to adding higher-order quantifiers. Although there were claims in the community of a solution, Pataraia passed away without leaving a written proof.

In this paper, we provide two main results contributing to this research direction:

- We provide a generalization of Freyd's construction for a class of Heyting algebras (previously studied by Kuznetsov [4]) called here *étale-finite*, showing that all such algebras arise as lattices of truth-values of elementary toposes; this provides a new class of examples beyond Boolean algebras and complete Heyting algebras for which Pitts' question has a positive answer.
- We show that our methods cannot be extended beyond this class without non-trivial modifications: we identify a class of toposes, called here *finitely propositional*, which arise out of the above construction, and we provide an example of a Heyting algebra which cannot be the lattice of truth values of a finitely propositional topos.

Definition 1 (Strict p-morphism)**.** Let $f\colon X \to Y$ be an order-preserving map between Esakia spaces. We say that f is a *strict p-morphism* if for all $x \in X$ and $y \in Y$ with $f(x) \leq y$ there is a unique $x' \geq x$ such that $f(x') = y$.

Definition 2. Let H be a finite Heyting algebra, H' an arbitrary Heyting algebra, and $f\colon H \to H'$ a homomorphism. We say that H' is an *H-étale* Heyting algebra if $f^{-1}\colon \mathsf{Spec}(H') \to \mathsf{Spec}(H)$ is a strict p-morphism.

We say that a Heyting algebra H' is *étale-finite* if there is a finite Heyting algebra H such that H' is H-étale. We say that an Esakia space X is étale-finite if it is dual to an étale-finite Heyting algebra.

Remark 3. *It was shown in [4] that being H-étale is equivalent to (1) satisfying an H-law of excluded middle, namely $\bigvee_{h\in H} x \leftrightarrow h$, or to (2) belonging to the variety $\mathsf{Var}(H)_{h\in H}$ generated by the Heyting algebra H enriched with constants for every element in H.*

Example 4. Every Boolean algebra is 2-étale. For a non-Boolean and non-complete example, consider $B = \mathsf{FinCofin}(\omega)$, the Boolean algebra of finite and cofinite subsets; let $H = B + 1$, the algebra obtained by adding a new top element to B. This is a Heyting algebra, which is 3-étale, where 3 is the three element Heyting chain.

Definition 5 (Spectral local homeomorphism)**.** Let $f\colon Y \to X$ be a continuous and order-preserving map. We say that f is a *spectral local homeomorphism* if f is a local homeomorphism with respect to the spectral topology, i.e., for each $y \in Y$, there is a clopen upset $U \ni y$ such that $f[U]$ is a clopen upset, and $f{\restriction}_U\colon U \to f[U]$ is an order-homeomorphism.

We write $\mathbf{Esa}_{\mathrm{SLH}}$ for the category of Esakia spaces with spectral local homeomorphisms. Given any Esakia space X, we denote by $\mathbf{OEt}_{\mathrm{Es}}(X)$ the slice category $\mathbf{Esa}_{\mathrm{SLH}}/X$. Towards showing that this is a topos, the following is obtained without too much trouble:

*Presenting author.

Proposition 6. *For each étale-finite Esakia space X, the category $\mathbf{OEt}_{\mathrm{Es}}(X)$ is finitely complete. Moreover:*

1. *The identity $1_X : X \to X$ is the terminal object;*
2. *Equalizers and finite products are computed as in* **Set**, *with finite products being given by pullback;*
3. *For each object $f : Y \to X \in \mathbf{OEt}_{\mathrm{Es}}(X)$, we have $\mathsf{Sub}(Y) \cong \mathsf{ClopUp}(Y)$.*

In light of Proposition 6.(3), if H is a Heyting algebra, and X is its Esakia dual, then $\mathsf{Sub}(1_X) = \mathsf{ClopUp}(X)$, so, by Esakia duality, H will be the lattice of truth values of $\mathbf{OEt}_{\mathrm{Es}}(X)$. The main difficulty lies in constructing power objects:

Definition 7. Let $f : Y \to X$ be a spectral local homeomorphism. We define:

$$P_f(Y) := \{(x, K) : x \in X,\ K \subseteq f^{-1}[\uparrow x],\ K \text{ an upset}\},$$

where we write $(x, K) \preceq (x', K')$ iff $x \leq x'$ and $K' = K \cap f^{-1}[\uparrow x']$. We define the following for every clopen upset $V \subseteq Y$ and clopen $W \subseteq X$:

$$\Xi(V) := \{(x, K) \in P_f : K = V \cap f^{-1}[\uparrow x]\} \text{ and } \Theta(W) := \{(x, K) \in P_f : x \in W\}$$

and declare a subbasis of P_f to consist of all sets $\Xi(V)$ for $V \subseteq Y$ clopen upset, their complements, as well as all sets $\Theta(U)$ for $U \subseteq X$ clopen. We write $(P_f(Y), \preceq, \tau)$ for the ordered space obtained in this way. We let $p : P_f(Y) \to X$ be the projection which maps (x, K) to x.

The assumption of finiteness on the upsets allows us to prove that this space satisfies the Priestley separation axiom, and even the Esakia condition. By relating the space $(P_f(Y), \preceq, \tau)$ to some specific subspaces of the Vietoris space on Y, we can show that this topology is compact. In fact we show:

Proposition 8. *The space $(P_f(Y), p)$ is an object in $\mathbf{OEt}_{\mathrm{Es}}(X)$, and is the power object of (Y, f).*

We thus obtain:

Theorem 9. *For an étale-finite Heyting algebra H there is an elementary topos $\mathcal{E}$, namely $\mathbf{OEt}_{\mathrm{Es}}(X)$, such that $\mathsf{Sub}_{\mathcal{E}}(1) \cong H$.*

Definition 10. An elementary topos $\mathcal{E}$ is called *propositional* if every object is a colimit of subterminal ones. It is called *finitely propositional* if every object is a finite colimit of subterminal ones.

It follows from our construction that, for each H étale-finite, $\mathbf{OEt}_{\mathrm{Es}}(X)$ is finitely propositional. Now consider $H = \mathcal{P}^*_{\mathrm{cof}}(\omega)$, the Heyting algebra of cofinite subsets of natural numbers and the empty set. Then we show:

Theorem 11. *There is no finitely propositional topos $\mathcal{E}$ such that $\mathcal{P}^*_{\mathrm{cof}}(\omega)$ is the lattice of truth values of $\mathcal{E}$.*

We conclude by remarking on some possibilities to generalize the methods above beyond the scope of étale-finite Heyting algebras, and explain how one can use intuitions coming from the free algebras in the varieties $\mathsf{Var}(H)_{h \in H}$ as inspiration.

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Semiconical residuated lattices with an idempotent skeleton

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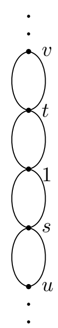
Residuated structures play an important role in the field of algebraic logic since they constitute the equivalent algebraic semantics, in the sense of Blok and Pigozzi, of substructural logics (see [1, 2]). These encompass many of the interesting nonclassical logics: intuitionistic logic, fuzzy logics, relevance logics, linear logics and also classical logic as a limit case. Thus, the algebraic investigation of residuated lattices is a powerful tool in the systematic and comparative study of such logics. While many deep results have been obtained in the last decades, the multitude of different kinds of residuated lattices and their rich theory makes their study fairly complicated, and at the present moment large classes of residuated lattices lack a structural description. Because of this, it is of utter importance to understand how one can decompose algebraic structures in simpler and more tractable substructures since it will give an insight in the comprehension of both residuated lattices and substructural logics.

In more detail, a *residuated lattice* is an algebra $\mathbf{A} = (A, \cdot, \backslash, /, \wedge, \vee, 1)$ of type $(2, 2, 2, 2, 2, 0)$ where: $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid, and the residuation law holds, i.e. $x \cdot y \leq z$ if and only if $y \leq x\backslash z$ if and only if $x \leq z/y$ for any $x, y, z \in A$. An element x of a residuated lattice $\mathbf{A}$ is said to be *conical* if for any $y \in A$, $x \leq y$ or $y \leq x$; we call a residuated lattice *conical* if the unit 1 is a conical element.

In this contribution we focus on identifying and studying in detail a wide class of conical residuated lattices taking inspiration from the work of Galatos and Fussner [3] where they describe the class of semiconic idempotent residuated lattices, i.e. the variety generated by conical residuated lattices with idempotent product. In this work, we drop the hypothesis of idempotency for the product obtaining a wider class of semiconical residuated lattices, and see how to use and generalize the ideas of [3] to describe such variety. In particular, the building blocks of the variety are conical residuated lattices having as a skeleton an idempotent chain. Intuitively, they can be viewed as a chain of pearls where every pearl is either a residuated lattice if it is below 1 or a partial residuated lattice with the reverse ordering if it is above 1 and the top of each pearl is a conical idempotent element. We give an illustrative picture,



where s, t, u, v are idempotent and conical elements.

We describe the properties needed by the "pearls" for the whole structure to be a residuated lattice. Then we consider the generated variety and we provide an axiomatization for it.

Moreover, we provide a decomposition result which allows to describe the above building blocks by means of simpler structures. In particular, we see that one needs a system $(\mathbf{S}, \{\mathbf{A}_s \mid s \in S\})$ where $\mathbf{S}$ is a chain of idempotent elements that intuitively will serve as the skeleton of the resulting structure and $\{\mathbf{A}_s \mid s \in \mathbf{S}\}$ is a collection of particular residuated structures indexed by the elements $s \in \mathbf{S}$ each of which intuitively represents a component of the resulting residuated lattice. The skeleton $\mathbf{S}$ serves as a guide to define the operations between components.

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SEMI-DIRECT PRODUCTS AND SIMILAR CONSTRUCTIONS OF RESIDUATED LATTICES

The commutative ordered monoid $(A, \cdot, 1, \leqslant)$ is ordered *naturally* if the order relation satisfies : $a \leqslant b$ if and only if there exists $c \in A$ such that $b \cdot c = a$. A *hoop* can be defined as naturally ordered commutative monoid possessing the binary operation $\to$ such that the adjointnes property

$$a \cdot b \leqslant c \text{ if and only if } a \leqslant b \to c$$

holds for all a, b, c.

Using a new type of function between two hoops (called *product morphism*) that satisfies

(pM1) $f(1) = 1$,
(pM2) $f(x) \cdot f(y) = f(x \cdot y) = f(x) \wedge f(y) = f(x \wedge y)$.

we define a new type of product by:

Theorem 1. *If* $\mathbf{A} = (A; \cdot, \to, 1)$ *and* $\mathbf{B} = (B; \cdot \to, 1)$ *are hoops and* $f\colon B \longrightarrow A$ *is a product morphism, then the algebra*

$$\mathbf{A} \ltimes_f \mathbf{B} = (\textstyle\sum_{x \in B}(f(x)], \cdot, \to, (1,1))$$

such that

$(\cdot)$ $(a, x) \cdot (b, y) := (a \cdot b, x \cdot y)$,
$(\to)$ $(a, x) \to (b, y) := (f(x \to y) \wedge (a \to b), x \to y)$

for any $(a, x), (b, y) \in \sum_{x \in B}(f(x)]$ *is a hoop. We say that* $\mathbf{A} \ltimes_f \mathbf{B}$ *is a* f*-product of the hoops* $\mathbf{A}$ *and* $\mathbf{B}$

We will show several results, primarily that

- this product satisfies a certain associativity,
- each finite hoop is the product of finite MV-chains,

and the following theorem:

Theorem 2. *If* $\mathbf{A}$ *is a finite hoop and* $F \in \mathbf{Fil}\,\mathbf{A}$ *then the mapping*

$$\psi\colon \mathbf{A}/F \longrightarrow \mathbf{F}$$

defined by $\psi(X) = t_X \vee l$ *is a product morphism and moreover it satisfies*

$$\mathbf{A} \cong \mathbf{F} \ltimes_\psi (\mathbf{A}/F).$$

Furthermore, this product satisfies the associative law in the following form:

Theorem 3. *If* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$ *are hoops then there exists an one-to-one correspondence between pairs of product morphisms* (α, β)*, such that*

$$\alpha\colon \mathbf{B} \longrightarrow \mathbf{A},\ \beta\colon \mathbf{C} \longrightarrow A \ltimes_\alpha \mathbf{B},$$

and pairs $(\overline{\alpha}, \overline{\beta})$*, such that*

$$\overline{\beta}\colon \mathbf{C} \longrightarrow \mathbf{B},\ \overline{\alpha}\colon \mathbf{B} \ltimes_\alpha \mathbf{C} \longrightarrow \mathbf{A}.$$

Moreover, mutually corresponding pairs satisfy:

$$(\mathbf{A} \ltimes_{\alpha} \mathbf{B}) \ltimes_{\beta} \mathbf{C} \cong \mathbf{A} \ltimes_{\overline{\alpha}} (\mathbf{B} \ltimes_{\overline{\beta}} \mathbf{C}).$$

This, together with the fact that finite simple hoops are precisely finite MV-strings, gives us the following representation.

Theorem 4. *(i) For any finite hoop* $\mathbf{A}$ *there exists a (finite) sequences of finite MV-chains* $(\mathbf{M}_{i_1}, \mathbf{M}_{i_2}, \ldots, \mathbf{M}_{i_n})$ *and appropriate product morphisms* $(f_1, \ldots, f_{n-1})$ *such that*

$$\mathbf{A} \cong \mathbf{M}_{i_1} \ltimes_{f_1} (\mathbf{M}_{i_2} \ltimes_{f_2} (\mathbf{M}_{i_3} \cdots \ltimes_{f_{n-2}} (\mathbf{M}_{i_{n-1}} \ltimes_{f_{n-1}} \mathbf{M}_{i_n}) \ldots)).$$

((i) For any finite hoop $\mathbf{A}$ *there exists a (finite) sequences of finite MV-chains* $(\mathbf{M}_{i_1}, \mathbf{M}_{i_2}, \ldots, \mathbf{M}_{i_n})$ *and appropriate product morphisms* $(g_1, \ldots, g_{n-1})$ *such that*

$$\mathbf{A} \cong (((\mathbf{M}_{i_1} \ltimes_{g_1} \mathbf{M}_{i_2}) \ltimes_{g_2} \mathbf{M}_{i_3}) \cdots \ltimes_{g_{n-2}} \mathbf{M}_{i_{n-1}}) \ltimes_{g_{n-1}} \mathbf{M}_{i_n}.$$

The previous results form the basis for several analogous constructions with similar outcomes in different classes of residuated lattices.

Semi-divisible rings

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The notion of *residuated lattices* emerged in the study of algebraic semantics for substructural logics, with the foundational contributions of Ward and Dilworth in the early 20^{th} century [8]. These structures consist of a partially ordered set equipped with a monoidal operation and a residual (implication-like) operation satisfying a natural adjointness property. Residuated lattices have found applications in logic, algebra, and computer science.

A fruitful connection between ring theory and residuated lattices arises when one considers the set of ideals of a ring R, ordered by inclusion and equipped with ideal multiplication. For many classes of rings, precisely the rings generated by idempotents, this structure forms a residuated lattice where the residual is defined by

$$I \to J = \{r \in R : rI \subseteq J\},$$

for ideals $I, J \subseteq R$. This construction has been studied in works such as [1], where the authors explore the conditions under which the set of ideals $Id(R)$ of a ring R is an MV-algebra. Several authors have studied conditions under which the set of (two-sided) ideals of a ring, ordered by inclusion, forms a special class of residuated lattices (see [1, 2, 4, 5]).
Among the aforementioned rings, *multiplication rings* have received considerable attention (see [3, 4]). In these rings, every ideal is a product of another ideal and a principal ideal, which facilitates the existence and behavior of the residual operation. These rings satisfy a strong residuated lattice condition called divisibility condition, given by

$$x \wedge y = x \odot (x \to y)$$

for x and y belonging to the residuated lattice L [4].

To generalize this framework, the concept of *semi-divisibility* was introduced by Turunen and Mertanen [7]. A residuated lattice is said to be *semi-divisible* if for all elements x, y in the lattice, the identity

$$(x^* \wedge y^*)^* = (x^* \odot (x^* \to y^*))^*$$

holds, where $x^* = x \to 0$. Analogously, a ring is called *semi-divisible ring* (or *SD-ring*) if its ideal lattice, equipped with residuals, satisfies this identity for all ideals. This weaker condition allows for a broader class of rings than multiplication rings, and it provides a natural algebraic abstraction within the context of residuated lattices. While multiplication rings and their ideal lattices have been extensively studied, semi-divisible rings have not received the same level of attention. Recent investigations have aimed to understand the structural properties of multiplication rings, including their behavior under common ring-theoretic constructions such as direct sums, direct products, and quotients. For example, it has been shown that multiplication rings, which are always semi-divisible, are closed under these operations. Whether semi-divisible rings enjoy similar closure properties is a question of both algebraic and logical significance.

In this talk, we aim to characterize such rings by discovering equivalent representation of the semi-divisibility axiom in the ring-theoretic set-up and also establishing representation-type results for these rings. We shall also provide explicit examples and counterexamples, and examine how semi-divisibility interacts with classical ring-theoretic properties such as multiplication, annihilator conditions, and Baer-type properties. In fact, we study the class of semi-divisible rings and find it to be closed under some of the main algebraic constructions. We show that the class of semi-divisible rings is closed under finite direct products, arbitrary direct sums, and homomorphic images. Moreover, we consider the relationship between the newly introduced class and some of the most popular classes of rings. For instance, we prove that the quotient of a Baer ring (see [6]) by a Baer ideal is a semi-divisible ring. These results enhance our understanding of the interplay between ring theory and residuated lattice theory and provide further evidence for the structural richness of semi-divisible residuated lattices derived from rings.

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Introduction to the model theory of modules

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Model theory of modules over a ring [5, 6] provides a natural bridge between mathematical logic, algebra and category theory. Its basic objects arise from the first–order language of modules and from the study of definable subgroups of modules. In this talk we recall some fundamental notions and results of the theory with particular emphasis on the case of finite dimensional algebras.

Let R be a ring. The language of right R–modules consists of symbols for addition, zero and scalar multiplication by elements of R. A central role in the theory is played by *pp–formulas* (positive primitive formulas). These are formulas of the form

$$\varphi(\bar{x}) := \exists \bar{y}\ (\bar{x}A + \bar{y}B = 0),$$

where A, B are matrices of appropriate sizes with entries in R.

One of the basic structural results of the theory describes the logical complexity of formulas in the language of modules.

Theorem 1 (Baur–Monk). *Every formula in the language of modules is equivalent to a Boolean combination of pp–formulas.*

This theorem shows that pp–formulas form the fundamental building blocks of the first–order theory of modules.

For every R-module M a pp–formula $\varphi(\bar{x})$ defines a subgroup $\varphi(M)$ of M^n, for some natural number n, called the *pp-definable subgroup*. The set of all pp-definable subgroups forms a modular lattice which contains essential information about the model theory of modules.

The semantic counterparts of pp–formulas are given by the *pure–injective modules*. Recall that a monomorphism of modules $f : M \to N$ is called *pure* if every pp–formula satisfied in M remains satisfied in N after applying f. A module M is called *pure–injective* if every pure embedding $M \to N$ splits. There are various equivalent algebraic ways of characterizing pure monomorphisms and pure-injective modules.

Pure–injective modules play in a sense role analogous to algebraically closed structures in classical model theory. They admit a strong decomposition property.

Theorem 2 (Gabriel). *Every pure–injective module decomposes uniquely, up to isomorphism and permutation of factors, as a direct sum of some indecomposable pure–injective modules and a pure-injective module without indecomposable direct summands.*

Pure-injective modules without indecomposable direct summands are called *super-decomposable*. Existence of such modules highly complicates decomposition theory of the pure-injectives.

The set of isomorphism classes of indecomposable pure–injective modules can be endowed with a natural topology, the *Ziegler topology*. The resulting topological space is called the *Ziegler spectrum* of the ring and provides a powerful geometric invariant of the module category.

When the ring R is a finite dimensional algebra over a field, model theory of modules interacts in a particularly rich way with the representation theory of algebras [1]. In this context, the Ziegler spectrum, pp–definable subgroups and functor categories encode important representation–theoretic information. In particular, invariants such as the Krull–Gabriel dimension of functor categories (see for example [3, 4]) can be studied using model–theoretic techniques. These methods make it possible to relate the Krull–Gabriel dimension, as well as the existence of super-decomposable pure-injective modules, with the *representation type* of a finite dimensional algebra. Research in this direction leads to the following two conjectures due to M. Prest [5]. Assume that A is a finite dimensional K-algebra over an algebraically closed field K.

Conjecture 1. *Algebra A is of domestic representation type if and only if the Krull-Gabriel dimension* $\mathrm{KG}(A)$ *of A is finite.*

Conjecture 2. *Algebra A is of domestic representation type if and only if there is no super-decomposable pure-injective A-module.*

The representation type is one the central concepts in representation theory, measuring the *complexity* of the whole module category. A *domestic representation type* is, in a sense, the least complicated one among the infinite representation type. It is also worth to note that the above conjectures are related, because the finiteness of Krull-Gabriel dimension $\mathrm{KG}(A)$ yields there is no super-decomposable pure-injective A-module [5].

Recent work [3, 4] shows that *Galois coverings of locally bounded categories* [2] can be effectively applied in the verification of Prest's conjectures. In the talk we briefly discuss one of the main results of [3] which shows that *Galois coverings do not increase the Krull-Gabriel dimension*, see [3, Theorem 3.9]. This result gives a rather handy method for constructing possible counterexamples to the first conjecture, see [3, Section 7]. During the talk we also show how our work relates with a general description from [3, Section 3] of the left and the right adjoint functors to the *pull-up functor* (also called the *restriction functor*) along a given K-linear functor.

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A Duality Theorem for Constructive K

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Constructive K *Constructive K* (CK) [1] is an intuitionistic variant of the classical normal modal logic K, obtained by extending intuitionistic logic with unary modalities $\Box, \Diamond$ satisfying

$$\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi, \qquad \Diamond\varphi \to \Diamond(\varphi \vee \psi) \quad \text{and} \quad \Box\varphi \wedge \Diamond\psi \to \Diamond(\varphi \wedge \psi),$$

and necessitation (if φ is a theorem then so is $\Box\varphi$). It arises proof-theoretically from restricting a standard sequent calculus for K to single conclusions. Interestingly, $\Diamond$ is monotone but not normal, i.e. $\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$ is not valid. The logic CK and its extensions have applications ranging from knowledge representation [4] to various flavours of constructive epistemic logic [8, 5] to modelling parallel computation [7] and evaluation [6].

Algebraic semantics Since CK is based on intuitionistic logic, its algebraic structures correspond to Heyting algebras with operators. A *CK-algebra* $\mathcal{A} = (A, \Box, \Diamond)$ consists of a Heyting algebra A and unary operators $\Box, \Diamond : A \to A$ satisfying

$$\Box\top = \top, \qquad \Box a \wedge \Box b = \Box(a \wedge b), \qquad \Diamond a \leq \Diamond(a \vee b) \quad \text{and} \quad \Box a \wedge \Diamond b \leq \Diamond(a \wedge b).$$

We denote the category of CK-algebras and homomorphisms by `CKAlg`.

Relational semantics The logic can be interpreted in frames of the form $\mathfrak{X} = (X, \leq, R)$, consisting of a nonempty set X, a preorder $\leq$ on X, and a binary relation R on X. In presence of a valuation that assigns proposition letters to upsets of $(X, \leq)$, we can interpret CK-formulas by interpreting the intuitionistic connectives in $(X, \leq)$ as usual, and

$$\begin{aligned} \mathfrak{M}, x \Vdash \Box\varphi \quad &\text{iff} \quad \forall y, z\ (x \leq y \text{ and } yRz \text{ imply } \mathfrak{M}, z \Vdash \varphi) \\ \mathfrak{M}, x \Vdash \Diamond\varphi \quad &\text{iff} \quad \forall y\ (x \leq y \text{ implies } \exists z \in X \text{ s.t. } yRz \text{ and } \mathfrak{M}, z \Vdash \varphi) \end{aligned}$$

Turning algebras into frames I The usual way of turning such a modal algebra $\mathcal{A}$ into a frame is by taking the collection of prime filters ordered by $\subseteq$ and R, where for prime filters p, q we define pRq iff $\Box a \in p$ implies $a \in q$ and $a \in q$ implies $\Diamond a \in p$, for all $a \in \mathcal{A}$.

However, for CK-algebras this does not work because this definition guarantees validity of $\Diamond(a \vee b) \to (\Diamond a \vee \Diamond b)$. To see this, let p be a prime filter such that $\Diamond(a \vee b) \in p$. Then it can be shown that p has an R-successor q that contains $a \vee b$. But since q is prime we must have $a \in q$ or $b \in q$, so by definition $\Diamond a \in p$ or $\Diamond b \in p$.

Turning algebras into frames II To circumvent this, we use *segments*. An $\mathcal{A}$-segment is a pair (p, Γ) consisting of a prime filter p and a set Γ of prime filters such that for any $a \in \mathcal{A}$:

$$\text{if } \Box a \in p \text{ then } a \in q \text{ for all } q \in \Gamma, \qquad \text{if } \Diamond a \in p \text{ then } a \in q \text{ for some } q \in \Gamma.$$

We can then define relations $\subsetneq$ and $R_{\mathcal{A}}$ on the set $\mathrm{SEG}_{\mathcal{A}}$ of all segments via

$$(p,\Gamma) \subsetneq (q,\Delta) \text{ iff } p \subseteq q, \qquad (p,\Gamma)R_{\mathcal{A}}(q,\Delta) \text{ iff } q \in \Gamma.$$

to obtain a frame $(\mathrm{SEG}_{\mathcal{A}}, \subsetneq, R_{\mathcal{A}})$. Now if $\Diamond(a \vee b) \in p$ we can construct two segments (p,Γ) and (p,Δ) such that no prime filter in Γ contains a and no prime filter in Δ contains b. This gives the desired non-validity of normality of diamonds.

While it is not necessary to use all segments to have "enough points" to invalidate unwanted formulas, using all segments provides a canonical choice. Interestingly, this means that the duality for CK is not based on Esakia duality.

Descriptive frames A *descriptive frame* is a tuple $(X,\leq,R,A)$ where $(X,\leq,R)$ is a frame and A is a collection of *admissible* upsets of $(X,\leq)$ such that:

(D0) A contains $\emptyset$ and X, and for any $a,b \in A$ the following sets are in A as well:

$$a \cap b, \qquad a \cup b, \qquad a \to b := \{x \in X \mid \forall y \geq x (y \in a \Rightarrow y \in b)\},$$
$$\boxdot a := \{x \in X \mid \forall y \geq x (yRz \Rightarrow z \in a)\}, \quad \Diamond a := \{x \in X \mid \forall y \geq x (\exists z (yRz \text{ and } z \in a))\};$$

(D1) for any $x,y \in X$, if $x \not\leq y$ then there exists $a \in A$ such that $x \in a$ and $y \notin a$;

(D2) for all $x,y,z \in X$, if $xRy \sim z$ then xRz;

(D3) if $B \subseteq A \cup -A$ has the finite intersection property, then $\bigcap B \neq \emptyset$;

(D4) if $x \in X$ and $U \subseteq X$ is such that for all $a \in A$:

- $x \in \boxdot a$ implies $U \subseteq a$,
- $x \in \Diamond a$ implies $U \cap a \neq \emptyset$,

then there exists a $x' \in X$ such that $x \sim x'$ and $R[x'] = U$.

Let `CKDescr` be the category of descriptive frames and functions that are bounded with respect to both relations and whose inverse image preserves admissible upsets.

Duality theorem There is a dual equivalence $\texttt{CKAlg} \equiv^{\mathrm{op}} \texttt{CKDescr}$ [3].

Variations Item (D4) reflects the choice that descriptive frames contain points for any possible combination of world and successor set, i.e. they are constructed by using all possible segments. But this is not the only option: variations on (D4) can be constructed mirroring, for example, the various types of segments used in the canonical model constructions in [2, Section VI], which are tailored to work well with various additional axioms.

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Structural completeness in basic hoops and BL-algebras

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A large class of substructural logics (i.e. logics that lack some structural rules) is given by the extensions of the *Full Lambek Calculus* $\mathcal{FL}$ (see [5] p. 76). All extensions of $\mathcal{FL}$ have a primitive connective $\mathbf{0}$ that denotes *falsum*; so if $\mathcal{L}$ is an extension of $\mathcal{FL}$ it makes sense to study its *positive* fragment $\mathcal{L}^+$. The language of $\mathcal{L}^+$ is obtained by the language of $\mathcal{L}$ by deleting $\mathbf{0}$ from the signature; the valid derivations in $\mathcal{L}^+$ are the valid derivations in $\mathcal{L}$, that contains only 0-free formulas.

An important subfamily of substructural logics over $\mathcal{FL}$ consists of logics that satisfy both exchange and weakening but lack contraction; in this case the primitive connectives can be taken as $\vee, \wedge, \rightarrow, \mathbf{0}, \mathbf{1}$ where of course $\mathbf{1}$ represents *truth*. The minimal substructural logic of this kind is denoted by $\mathcal{FL}_{ew}$ and hence all the substructural logics lacking contraction are an extension of it. These logics are *algebraizable* with an *equivalent algebraic semantics* (in the sense of Blok-Pigozzi[3]) that is at least a quasivariety of algebras. BL-*algebras* and their zero-free counterparts *basic hoops* (as equivalent algebraic semantics of Hajek's Basic Logic $\mathcal{BL}$ [7] and of its positive fragment $\mathcal{BL}^+$) are among the most investigated classes of algebras appearing in this framework.

Algebraizability in Blok-Pigozzi's fashion entails that the deducibility relation of an algebraizable logic is characterized by means of the algebraic equational consequence of its equivalent algebraic semantics. Therefore, interesting properties of algebraizable logics can be studied via algebraic means and structural completeness and hereditary structural completeness are among them.

A rule is *admissible* in a logic if, when added to its calculus, it does not produce new theorems. A logic $\mathcal{L}$ is *structurally complete* if every admissible rule of $\mathcal{L}$ is derivable in $\mathcal{L}$; it is *hereditarily structurally complete* if every finitary extension of $\mathcal{L}$ is structurally complete. In an algebraizable logic, these notions correspond to the associated quasivariety being *structural* and *primitive* respectively. A quasivariety Q is *structural* if for every subquasivariety $\mathsf{Q}' \subseteq \mathsf{Q}$, $\mathbf{V}(\mathsf{Q}') = \mathbf{V}(\mathsf{Q})$ implies $\mathsf{Q}' = \mathsf{Q}$ (where $\mathbf{V}(\mathsf{Q})$ is the variety generated by Q). A quasivariety Q is *primitive* if every subquasivariety of Q is structural.

In this contribution we tackle the problem of structural completeness in (quasi)varieties of BL-algebras and basic hoops.

BL-algebras and basic hoops are deeply connected to the varieties of Wajsberg algebras and Wajsberg hoops, the equivalent algebraic semantics of Łukasiewicz logic and its positive fragment respectively; indeed, every totally ordered basic hoop is isomorphic to the ordinal sum of a family of Wajsberg hoops, and every totally ordered BL-algebra is isomorphic to the ordinal sum of a family of Wajsberg hoops, the first of which is a Wajsberg algebra. Moreover, since the sum of two nontrivial Wajsberg hoops is never a Wajsberg hoop, the components of this sum are unique up to isomorphism ([1]). Given this uniqueness, whenever we have a totally ordered BL-algebra or basic hoop, we can speak of its *Wajsberg components*.

For this reason, the starting point for the results presented here is mainly from two articles: the seminal work by Gispert [6], where he characterizes every structural quasivariety of Wajsberg algebras, and the more recent work by Aglianò and Manfucci [2], where the authors characterize every structural and primitive quasivariety of Wajsberg hoops.

In order to present our results, let us define some useful totally ordered Wajsberg algebras: let $\mathbf{Ł}_n = \Gamma(\mathbb{Z}, n)$ and $\mathbf{Ł}_{n,k} = \Gamma(\mathbb{Z} \times_l \mathbb{Z}, (n,k))$ (the notation for these constructions is the one of Mundici [8]). We also denote their corresponding 0-free reduct, that are Wajsberg hoops, by $\mathbf{Ł}_n^+$ and $\mathbf{Ł}_{n,k}^+$; moreover let $\mathbf{C}_\omega$ be the infinite unbounded Wajsberg hoop given by the negative cone of $\mathbb{Z}$, with the operations defined in the obvious way.

In this contribution we present the following results:

Theorem 1. *1. Every locally finite variety of basic hoops is primitive;*

2. let V_n be the variety of basic hoops generated by $\mathbf{Ł}_n^+ \oplus \mathbf{C}_\omega$, then V_n is structural for every $n \in \mathbb{N}$;

3. a locally finite variety of BL*-algebras is primitive if and only if it is structural if and only if it is generated by totally ordered algebras whose first Wajsberg component is $\mathbf{Ł}_1$;*

4. let V_n be the variety of BL*-algebras generated by $\mathbf{Ł}_1 \oplus \mathbf{Ł}_n^+ \oplus \mathbf{C}_\omega$, then V_n is structural for every $n \in \mathbb{N}$.*

In order to prove the theorem above, we needed to find a characterization of free algebras in $\mathbf{V}(\mathbf{Ł}_n^+ \oplus \mathbf{C}_\omega)$, based on the one of $\mathbf{V}(\mathbf{Ł}_n \oplus \mathbf{C}_\omega)$ given by Busaniche and Cignoli in [4].

Moreover, we also managed to prove some negative results:

Theorem 2. *Let $\mathsf{Q} = \mathbf{Q}(\mathbf{A}_1, \dots, \mathbf{A}_m)$ be a quasivariety of* BL*-algebras (or basic hoops) such that for every $r \neq s$ it holds $\mathbf{A}_r \notin \mathbf{Q}(\mathbf{A}_s)$. If there exists j such that at least one of the following holds*

1. $\mathbf{A}_j$ is a BL*-chain whose decomposition in Wajsberg components is $\mathbf{A}_j \cong \mathbf{Ł}_n \oplus \bigoplus_{i>0} \mathbf{B}_i$ for some $n \neq 1$ and some family of Wajsberg chains $\{\mathbf{B}_i\}$,*

2. $\mathbf{A}_j$ is a BL*-chain whose decomposition is Wajsberg components is $\mathbf{A}_j \cong \mathbf{Ł}_{n,k} \oplus \bigoplus_{i>0} \mathbf{B}_i$ for some $n \neq 1$ and some family of Wajsberg chains $\{\mathbf{B}_i\}$,*

3. $\mathbf{A}_j$ is a BL*-chain (or* BH*-chain) whose decomposition in Wajsberg components is $\mathbf{A}_j \cong \bigoplus_{i=0}^t \mathbf{B}_i \oplus \mathbf{Ł}_{n,k}^+$ for some $n, k \neq 1$ and some finite family of Wajsberg chains $\{\mathbf{B}_i\}$,*

then Q *is not structural.*

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Symmetric Difference in Lattices With Complementation

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The symmetric difference in Boolean lattices can be defined in two equivalent ways:

$$x +_1 y := (x' \wedge y) \vee (x \wedge y'),$$
$$x +_2 y := (x \vee y) \wedge (x' \vee y').$$

This operation can also be introduced in any bounded *lattice with complementation* (i.e., a complemented lattice equipped with a fixed complementation operation), where these two definitions need not coincide. It is well known that the symmetric difference is associative in every Boolean lattice, and earlier results show that in orthomodular lattices it is associative if and only if the lattice is Boolean. For either of the two symmetric differences, these results extend to all bounded lattices with complementation.

Theorem 1. *Let* $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ *be a lattice with complementation. Then* $\mathbf{L}$ *is Boolean if and only if* $+_1$ *or* $+_2$ *is associative.*

There is also a simple two-variable identity involving only the symmetric difference that yields another if-and-only-if characterization of Boolean lattices.

Theorem 2. *Let* $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ *be a lattice with complementation. Then* $\mathbf{L}$ *is a Boolean algebra if and only if* $+$ *satisfies the identity*

$$(x + y) + y \approx x$$

where $+ = +_1$ *or* $+ = +_2$.

Since the two symmetric differences coincide in Boolean lattices, it is natural to ask whether this property characterizes Boolean lattices. Thus we consider the variety of lattices with complementation satisfying the identity

$$(x' \wedge y) \vee (x \wedge y') \approx (x \vee y) \wedge (x' \vee y'). \tag{1}$$

It turns out that lattices with complementation satisfying this identity need not be Boolean. Even if we strengthen the assumption by considering ortholattices (i.e., bounded involutive lattices with complementation satisfying the De Morgan laws), the conclusion still does not hold. However, finite examples suggest that such lattices are in some sense close to Boolean lattices (see Figure 1). This observation is only heuristic and would require a more detailed investigation, which we do not pursue here. Nevertheless, we can at least state the following result.

Theorem 3. *Let* $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ *be an ortholattice with the property that for all* $x, y \in L$, x *and* y *or* x *and* y' *are comparable. Then* $\mathbf{L}$ *satisfies identity* (1).

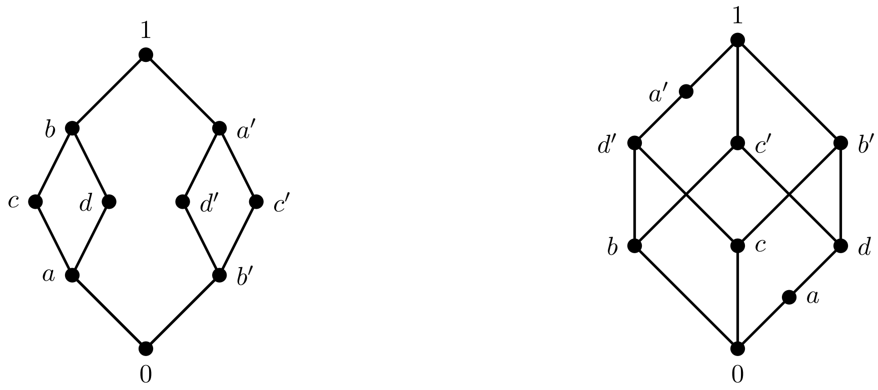

Figure 1: Ortholattices satisfying identity (1).

Theorem 4. *Let* $(L, \vee, \wedge, ', 0, 1)$ *be a non-trivial lattice with complementation satisfying identity* (1)*, and let* $a \in L$*. Then there does not exist an element* $b \in L$ *that is simultaneously a complement of both* a *and* a'*.*

Another direction of research is to consider the quasi-identity

$$x + y = 0 \;\Rightarrow\; x = y. \tag{2}$$

If $+ = +_1$ and a lattice with complementation satisfies the De Morgan laws and the quasi-identity (2), then it is both weakly orthomodular and dually weakly orthomodular. There are several further interesting questions and conjectures concerning the quasivariety of lattices with complementation satisfying (2). For example, are such lattices necessarily modular? How does the class of lattices that are both weakly and dually weakly orthomodular sit inside this quasivariety?

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Implicit operations in varieties of commutative monoids

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We assume that the reader is familiar with [3] (see also [1]). Recall that an *implicit operation* of a class of similar algebras K is a collection of first order definable partial functions on the members of K that is globally preserved by homomorphisms. For example, “taking inverses” can be viewed as a unary implicit operation of the class of all monoids because its graph on a given monoid is defined by the equation $xy \approx 1$ and monoid homomorphisms preserve existing inverses. As this example demonstrates, the implicit operations of a class K need not be given by terms of K. Therefore, it is natural to expand K by adding enough implicit operations to ensure the validity of the strong Beth definability property. The result of such an expansion of K is said to be a *Beth companion* of K. For example, the variety of Abelian groups is a Beth companion of the quasivariety of cancellative commutative monoids (see [1, Thm. 11.9(i)]). In the context of varieties, Beth companions are essentially unique when they exist, in the sense they are all term equivalent (see [1, Thm. 11.7]).

Our main result is a description of the varieties of commutative monoids with a Beth companion based on the following concept (see [8, p. 242]).

Definition 1.

(i) *A commutative monoid* $\mathbf{A}$ *is said to be* inverse *when for every* $a \in A$ *there exists* $b \in A$ *such that* $a = a^2b$.

(ii) *A variety of commutative monoids is* inverse *when so are its members.*

In view of Head’s classification of varieties of commutative monoids (see, e.g., [5, Thm. 5.1]), inverse varieties of commutative monoids admit the following transparent description.

Proposition 2. *A variety of commutative monoids is inverse if and only if it validates the equation* $x^{n+1} \approx x$ *for some* $n > 0$.

In this talk, we will establish the following result.

Theorem 3. *A variety of commutative monoids has a Beth companion if and only if it is inverse, in which case it is its own Beth companion.*

We recall that the variety CM of all commutative monoids lacks the strong Beth definability property because, in general, the implicit operation f of “taking inverses” cannot be *interpolated* by a family of monoid terms, in the sense that there is no family of monoid terms $\{t_i(x) : i \in I\}$ such that for all $\mathbf{A} \in \mathsf{CM}$ and $a \in \mathsf{dom}(f^{\mathbf{A}})$ there exists $i \in I$ such that $f^{\mathbf{A}}(a) = t_i^{\mathbf{A}}(a)$. On the other hand, inverses can be interpolated in every inverse variety K by a term of the form x^{n-1},

*Speaker.

because K validates the equation $x^{n+1} \approx x$ for some $n > 0$ by Proposition 2. It is tempting to conjecture that this is the reason why the sole varieties of commutative monoids with a Beth companion are the inverse ones. However, this impression should be dispelled for every proper variety of commutative monoids K possesses an implicit operation f, defined by the formula

$$\varphi(x, y) = (x^{n+1}y \approx x^n) \sqcap (y^2x \approx y) \text{ for some } n \in \mathbb{N},$$

that interpolates the implicit operation of "taking inverses" and, moreover, is *extendable*, in the sense that each $\mathbf{A} \in \mathsf{K}$ can be extended to some $\mathbf{B} \in \mathsf{K}$ in which $f^{\mathbf{B}}$ is total and extends $f^{\mathbf{A}}$. The latter makes it possible to expand K by adding f, thus resolving the problem of intepolating inverses. However, this is not enough to ensure the validity of the strong Beth definability property, unless K was an inverse variety from the start.

In order to explain what prevents the existence of a Beth companion for noninverse varieties of commutative monoids, we rely on the following concept [8].

Definition 4. *Given a subalgebra* $\mathbf{A}$ *of a member* $\mathbf{B}$ *of a variety* K*, the* dominion *of* $\mathbf{A}$ *in* $\mathbf{B}$ *relative to* K *is the set*

$$\begin{aligned}\mathsf{d}_{\mathsf{K}}(\mathbf{A}, \mathbf{B}) = \{&b \in B : g(b) = h(b) \textit{ for every pair of homomorphisms}\\ &g, h \colon \mathbf{B} \to \mathbf{C} \textit{ with } \mathbf{C} \in \mathsf{K} \textit{ such that } g{\restriction}_A = h{\restriction}_A\}.\end{aligned}$$

Isbell's Zigzag Theorem provides a description of dominions in the varieties of all monoids and all (resp. commutative) semigroups in terms of certain formulas, which we term *Isbell's formulas* (see [8, Thm. 2.3], [7, Thm. 1.1], and [6, Thm. 1.2]). Using elementary category theory tools and Head's classification of varieties of commutative monoids, we will establish the following.

Theorem 5. *Isbell's Zigzag Theorem holds for all varieties of commutative monoids.*

We will show that the Isbell's formula

$$\varphi(x_1, x_2, x_3, y) = \exists w, z((y \approx wx_2z) \,\&\, (wx_2 \approx x_1) \,\&\, (x_2z \approx x_3))$$

defines an implicit operation on every proper variety K of commutative monoids and, unless K is inverse, K cannot be expanded so that this implicit operation be interpolated by a family of terms. This impossibility proof relies on Grillet's characterization of finitely generated subdirectly irreducible commutative semigroups (see [4, Cor. IV.4.6]) and on Theorem 5. Our results have been collected in [2].

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Algebraic Semantics for First-Order Relevant Logics

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Abstract

The first-order extensions of relevant logics cannot be given a ternary relational semantics with a constant domain with the usual Tarskian interpretation of the quantifiers, which says that a universally quantified formula is verified at a point iff every instances of the formula is verified at that point in the model. This result was shown by [1]. As a result, many alternative approaches to giving semantics for first-order relevant logics have been developed. To highlight a couple: [2], which uses admissible propositions and a non-Tarskian interpretation of the quantifiers; the Lindenbaum-Tarski algebra for closed formulas of first-order **RQ**, omitting the quantifiers as operators, as in [3]; and the Mares-Goldblatt Structures of [5], which relate two algebras by an embedding. However, there has not been algebraic semantics for first-order relevant logics. We develop a algebraic semantics for a wide range of first-order relevant logics in the form of *Polyadic Relevant Algebras*.

Definition 1 (Relevant Algebras)**.** A *relevant algebra* is a tuple $\mathbf{A} = \langle A, \wedge, \vee, \circ, \rightarrow, \neg, 1\rangle$ where $\langle A, \wedge, \vee\rangle$ is a distributive lattice and (with $a \leq b$ as usual):

1. $a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$;
2. $(b \vee c) \circ a = (b \circ a) \vee (c \circ a)$;
3. $\neg(a \vee b) = \neg a \wedge \neg b$;
4. $\neg(a \wedge b) = \neg a \vee \neg b$
5. $1 \circ a = a$;
6. $a \circ b \leq c$ iff $a \leq b \rightarrow c$;

These correspond to the relevant logic **BM**, which is in some sense the weakest possible relevant logic with a ternary relational semantics. Extensions of **BM** and relevant algebras include the usual relevant logics in the literature, including **R**, which corresponds to the the subclass of relevant algebras that are De Morgan Monoids.

Polyadic relevant algebras are given first by a set I, which we can intuitively think of as a set of variables, and then the several operators. The first kind of operators are the $\forall_J$ and $\exists_J$ for $J \subseteq I$. The second kind of operators are substitution operators of the form S_σ, where σ is a mapping from I to I (written I^I). The polyadic relevant algebras are then of the form

$$\mathbf{A} := \langle A, \wedge, \vee, \circ, \rightarrow, \neg, 1, \langle \forall_J, \exists_J \mid J \subseteq I\rangle, \langle S_\sigma \mid \sigma \in I^I\rangle\rangle$$

In general, we will assume that J is always a finite subset of I and that σ are mappings that are finite in the sense that they are the identity mapping minus a finite subset of I. Of course, polyadic relevant algebras require many additional conditions in order to correspond to the right first-order relevant logics. First, from [4], we have some basic requirements for polyadic algebras. Where $O \in \{\sim, \wedge, \vee, \rightarrow, \circ\}$ and gO its arity:

- (PA$_0$) $S_\sigma 1 = 1$;
- (PA$_1$) $S_\delta x = x$;
- (PA$_2$) $S_\sigma(S_\tau x) = S_{\sigma\tau} x$;
- (PA$_3$) $S_\sigma O(x_1, \ldots x_{gO}) = O(S_\sigma(x_1), \ldots S_\sigma(x_{gO}))$;
- (PA$_4$) $S_\sigma Q_J x = S_\tau Q_J x$, when $\sigma J_* \tau$;
- (PA$_5$) $Q_J S_\sigma x = S_\sigma Q_{\sigma^{-1}J} x$, where σ is one-to-one on $\sigma^{-1}J$;

In addition, we require the following conditions (with (EC) only for logics with $\forall x(A \vee B) \to (A \vee \forall x B)$, where x is not free in A, and its dual):

(RA) The equalities defining relevant algebras;

(Q0) $x \leq y$ implies $S_\sigma x \leq S_\sigma y$;

(Q1) $\forall_J 1 = 1 = \exists_J 1$;

(Q2) $\forall_\emptyset x = x = \exists_\emptyset x$;

(Q3) $\forall_{J \cup K} x = \forall_J \forall_K x$;

(Q4) $\exists_{J \cup K} x = \exists_J \exists_K x$;

(Q5) $\forall_J x \leq x$;

(Q6) $x \leq \exists_J x$;

(Q7) $\forall_J (x \wedge y) = \forall_J x \wedge \forall_J y$;

(Q8) $\exists_J (x \vee y) = \exists_J x \vee \exists_J y$;

(Q9) $\forall_J \forall_J x = \forall_J x = \exists_J \forall_J x$;

(Q10) $\exists_J \exists_J x = \exists_J x = \forall_J \exists_J x$;

(Q11) $\forall_J (x \to y) \leq (\forall_J x \to \forall_J y)$;

(Q12) $\forall_J (x \to y) \leq (\exists_J x \to \exists_J y)$;

(Q13) $x \leq y$ implies $x \leq \forall_J y$, where $x = \forall_J x = \exists_J x$;

(Q14) $x \leq y$ implies $\exists x \leq y$, where $y = \forall_J y = \exists_J y$;

(Q15) $\forall_J (\forall_J x \otimes \forall_J y) = \forall_J x \otimes \forall_J y$, where $\otimes \in \{\to, \circ\}$, $x = \forall_J x = \exists_J x$, and $y = \forall_J y = \exists_J y$;

(EC) $\forall_J (x \vee y) = x \vee \forall_J y$; $\exists_J (x \wedge y) = x \wedge \exists_J y$; (where $x = \forall_J x = \exists_J x$);

This relevant algebra (without (EC)) corresponds to the first-order relevant logic **QBM**. Moreover, for the extensions of **QBM** extending the propositional base to the usual propositional relevant logics, there is a corresponding equality to add to polyadic relevant algebras such that the following is provable.

Theorem 1. A formula A is a theorem of a first-order relevant logic extending **QBM** iff $1 \leq t(A)$ in the class of locally finite polyadic relevant algebras (where t is a straightforward translation of the formula of first-order logics into terms of polyadic relevant algebras.)

We also have several representation theorems. First, we can use [4] to show that polyadic relevant algebras can be represented by *functional* polyadic relevant algebras. Second, we can show that polyadic relevant algebras are almost gaggles (as in [**?**]) minus the substitution operators. Nonetheless, we can represent polyadic relevant algebras by relational restructures wherein each $\forall_J$, $\exists_J$, and S_σ is modelled by a binary relation between points in a frame. Notably, this semantics does not require that the points in the canonical model are ω-complete: it only requires prime filters. That is, there is no requirement for a filter to contain $\forall_J p$ if it contains all 'instances' of p (through variables J).

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Knowledge on a Budget

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Topological Evidence Logic (TEL) is a recent development in epistemic logic that uses topological notions to model epistemic concepts such as justification and knowledge (e.g. see [2, 3, 4, 5]). In TEL, open sets in a topological space $\langle X, \mathcal{T} \rangle$ represent accessible *evidence*. A hypothesis $P \subseteq X$ is *justified* if it is supported by a dense open set, i.e. there is evidence $U \in \mathcal{T}$ such that $U \subseteq P$ (U supports P) and $U \cap V \neq \emptyset$ for all non-empty $V \in \mathcal{T}$ (U is consistent with all consistent evidence). Hypothesis P is *known* in a state $x \in X$ if it is supported by a dense open neighbourhood of x, i.e. there is $U \in \mathcal{T}$ as above and, moreover, $x \in U$ (U is truthful).

In practice, accessing evidence requires *resources* such as time, money, energy, or memory, and real-life agents operate within limited resource *budgets*. Consequently, we aim to develop a finer-grained, resource-aware notion of justification which allows us to reason not just about what is *observable* in principle, but what is *affordable* on a given budget.

In this talk, we present a variant of TEL that takes resource limitations into account. We introduce *semiring-annotated topological spaces* (short *seats*), structures that extend topological spaces $\langle X, \mathcal{T} \rangle$ with an annotation function $\mathcal{A}_K : \mathcal{T} \times X \to \mathcal{P}(K)$. Here $K = \langle K, \oplus, \odot, \mathbb{0}, \mathbb{1} \rangle$ is a semiring whose members represent resources, the operation $\oplus$ represents resource *choice* ($a \oplus b$ can be read as 'a or b') and $\odot$ represents resource *combination* ($ab := a \odot b$ can be read as 'a and b'). The intuition is that $\mathcal{A}_K(U, x)$ is the collection of resources $a \in K$ which are sufficient to access the evidence $U \in \mathcal{T}$ in state $x \in X$. The annotation $\mathcal{A}_K$ satisfies the following conditions:

(1) *Resource Strengthening*: $a \in \mathcal{A}_K(U, x) \;\Rightarrow\; ab, ba \in \mathcal{A}_K(U, x)$ for all $b \in K$.

(2) *Resource Choice*: $a, b \in \mathcal{A}_K(U, x) \;\Rightarrow\; a \oplus b \in \mathcal{A}_K(U, x)$.

(3) *Resource Combination*: $a \in \mathcal{A}_K(U, x)$ & $b \in \mathcal{A}_K(V, x) \;\Rightarrow\; a \odot b \in \mathcal{A}_K(U \cap V, x)$.

(4) *Evidence Weakening*: $a \in \mathcal{A}_K(U, x)$ & $U \subseteq V \;\Rightarrow\; a \in \mathcal{A}_K(V, x)$.

(5) *Available Tautologies*: $\mathbb{0} \in \mathcal{A}_K(X, x)$.

These reflect the above-mentioned readings of $a \odot b$ as 'using a together with b', of $a \oplus b$ as 'using a or b', of $U \subseteq V$ as 'evidence U is stronger than evidence V' and of $U \cap V$ as 'evidence U combined with evidence V'. Also note that (1)–(2) imply that $\mathcal{A}_K(U, x)$ is an *ideal* of K.

As an example, let $X = \{0, 1\}^\infty$ be the set of countable binary words (e.g. outputs of a sensor or a program), a directed-complete partial order under the prefix order, and let $O \subseteq \{0, 1\}^*$ be a set of finite binary words containing the empty word (the 'possible observations'). Let $\mathcal{T}_O$ be the topology on X generated by the basis $\{\uparrow w \mid w \in O\}$; open sets in this topology represent 'observable properties' of binary words (it is the Scott topology if $O = \{0, 1\}^*$). Intuitively, the resource spent to observe the word w is *time*, conveniently represented by the length $|w|$. The corresponding resource semiring is extended natural numbers $\mathbb{N}^\infty = \langle \mathbb{N} \cup \{\infty\}, \min, \max, \infty, 0 \rangle$.

*Speaker

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Our seat-based epistemic logic is defined as follows. Given a set *Prop* of propositional variables, a *K-model* is a seat $\langle X, \mathcal{T}, \mathcal{A}_K \rangle$ together with a valuation $\mathcal{V}\colon Prop \to 2^X$. These models interpret formulas of language $\mathfrak{L}_K$, obtained from *Prop* using $\neg$, $\wedge$, $\Box$ and F_a for $a \in K$. The satisfaction clauses are those familiar from topological semantics of modal logic [1], including $\boldsymbol{M}, x \models \Box\varphi$ iff $\exists U \in \mathcal{T}(x \in U \subseteq [\![\varphi]\!]_{\boldsymbol{M}})$, i.e. $[\![\Box\varphi]\!]_{\boldsymbol{M}} = Int([\![\varphi]\!]_{\boldsymbol{M}})$ extended with

$$\boldsymbol{M}, x \models \mathrm{F}_a\varphi \iff \exists U \in \mathcal{T}\big(U \subseteq [\![\varphi]\!]_{\boldsymbol{M}} \;\&\; a \in \mathcal{A}_K(U, x)\big) \quad .$$

Intuitively, $\Box\varphi$ expresses that 'there is truthful evidence for φ' and $\mathrm{F}_a\varphi$ means that 'there is evidence for φ which can be obtained using a'. Accordingly, the formula $\Box_a\varphi := \Box\varphi \wedge \mathrm{F}_a\varphi$ expresses that 'there is truthful evidence for φ which can be obtained using a'.

We provide (for countable K) a sound and strongly complete axiomatisation for the logic of all K-seats and for the logics of several seat classes defined by natural conditions, for example (6) $\mathbb{1}$-*Boundedness*: $\mathbb{1} \in \mathcal{A}_K(X, x)$, i.e. tautologous evidence is for free; (7) $\mathbb{0}$-*Boundedness*: $\mathbb{0} \in \mathcal{A}_K(U, x)$, i.e. every evidence is annotated; (8) *Uniformity*: $\mathcal{A}_K(U, x) = \mathcal{A}_K(U, y)$, i.e. the cost of evidence is independent of the state; (9) *Strongness*: $a \oplus b \in \mathcal{A}_K(U, x) \Rightarrow a, b \in \mathcal{A}_K(U, x)$.

Next we consider the language $\mathfrak{L}_{K\forall}$ expanded by the global modality A, allowing us to define budget-relative justification and knowledge modalities $\mathrm{B}^a_b\varphi := \mathrm{A}\Diamond_b\Box_a\varphi$ and $\mathrm{K}^a_b := \Box_a\varphi \wedge \mathrm{B}^a_b\varphi$.

$$\boldsymbol{M}, x \models \mathrm{B}^a_b\varphi \iff \exists U \in \mathcal{T}\Big(U \subseteq [\![\varphi]\!]_{\boldsymbol{M}} \;\&\; a \in \mathcal{A}_K(U, x) \;\&$$
$$\forall V \in \mathcal{T}\big(b \in \mathcal{A}_K(V, x) \;\&\; V \neq \emptyset \implies U \cap V \neq \emptyset\big)\Big)$$

That is, $\mathrm{B}^a_b\varphi$ states that one can obtain evidence U for φ using a such that U is consistent with all consistent evidence V that can be obtained using b (i.e. U is 'b-dense'). Intuitively, this means that φ is justifiable given a budget a for *finding* supporting evidence and a budget b for finding *counterarguments* against it. For $\mathrm{K}^a_b\varphi$ this evidence U has to be truthful. If the seat underlying $\boldsymbol{M}$ satisfies conditions (6)–(9) above (i.e. it is a bounded, uniform and strong seat), then $\mathrm{B}^{\mathbb{0}}_{\mathbb{0}}$ and $\mathrm{K}^{\mathbb{0}}_{\mathbb{0}}$ correspond to the justified belief and knowledge operators of TEL [3]. However, our setting is more general and able to model more fine-grained notions of belief and knowledge. For instance, it can be used to express that an agent is susceptible to misleading evidence: for a 'small' $\epsilon \in K$, the formula $\mathrm{B}^\epsilon_\epsilon\varphi \wedge \neg\Box\varphi$ means that there is evidence for φ that is cheap for the agent (i.e. little 'energy' is required for the agent to accept it) and consistent with all other cheap evidence, yet φ is not supported by any truthful evidence (e.g. φ is a hoax).

We provide sound and strongly complete axiomatisations for the $\mathfrak{L}_{K\forall}$-logics of several K-seat classes, including all seats and all bounded, uniform and strong seats.

In future research, we aim to determine the decidability and complexity of our seat-based logics and to develop extensions of our framework, e.g. to dynamic and multi-agent settings.

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On the universal theory of the free pseudocomplemented distributive lattice

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Definition 1. *A* pseudocomplemented distributive lattice *(also known as a distributive p-algebra) is an algebra* $\boldsymbol{A} = \langle A; \wedge, \vee, \neg, 0, 1\rangle$ *which comprises a bounded distributive lattice* $\langle A; \wedge, \vee, 0, 1\rangle$ *and a unary operation* $\neg$ *such that for every* $a \in A$ *the element* $\neg a$ *is the largest* $b \in A$ *such that* $a \wedge b = 0$*.*

Pseudocomplemented distributive lattices form a locally finite and finitely axiomatizable variety, which we denote by PDL. The importance of pseudocomplemented distributive lattices derives from the fact that they are the implication-free subreducts of Heyting algebras (see, e.g., [1, Proof of Thm. 2.6]). As such, pseudocomplemented distributive lattices can be viewed as the algebraic counterpart of the implication-free fragment of intuitionistic propositional logic.

The elementary theory of all finitely generated free pseudocomplemented distributive lattices was shown to be undecidable in [4]. In contrast, we show that the situation is different for the *universal theory* of all finitely generated free pseudocomplemented distributive lattices, which coincides with the universal theory of any infinitely generated free pseudocomplemented distributive lattice. Recall that a sentence is said to be *universal* when it is of the form $\forall x_1, \dots, x_n P$, where P is a quantifier-free formula.

Definition 2. *Let* K *be a class of algebras in the same language.*

(i) *The set of universal sentences valid in* K *is called the* universal theory *of* K *and will be denoted by* $\mathsf{Th}_\forall(\mathsf{K})$*. When* $\mathsf{K} = \{\boldsymbol{A}\}$*, we will write* $\mathsf{Th}_\forall(\boldsymbol{A})$ *as a shorthand for* $\mathsf{Th}_\forall(\mathsf{K})$*.*

(ii) K *is called a* universal class *when it can be axiomatized by a set of universal sentences. We denote by* $\mathbb{U}(\mathsf{K})$ *the least universal class containing* K*.*

We denote by $\boldsymbol{F}_\mathsf{V}(\kappa)$ the *free algebra* of a variety V with κ free generators, where κ is a positive cardinal. The following observations are folklore (see, e.g., [2, Thms. 2.5 and 2.7]).

Theorem 3. *Let* V *be a variety.*

(i) *For every infinite cardinal* κ *we have*

$$\mathsf{Th}_\forall(\boldsymbol{F}_\mathsf{V}(\aleph_0)) = \mathsf{Th}_\forall(\boldsymbol{F}_\mathsf{V}(\kappa)) = \mathsf{Th}_\forall(\{\boldsymbol{F}_\mathsf{V}(n) : n \in \mathbb{Z}^+\}).$$

(ii) *Let* V *be a finitely axiomatizable and locally finite variety of finite type. If* $\mathsf{Th}_\forall(\boldsymbol{F}_\mathsf{V}(\aleph_0))$ *is recursively axiomatizable, then it is also decidable.*

*Speaker.

Our main tools are Priestley duality for PDL [5] and the dual description of its free objects [3, 6]. For our purposes the finite version of Priestley duality for PDL suffices. For a poset X, let $\max X$ denote the set of maximal elements of X. A map $p\colon X \to Y$ between posets is said to be a *weak p-morphism* when it is order preserving and for every $x \in X$ and $y \in \max Y$,

$$\text{if } p(x) \leqslant y, \text{ there exists } z \in \max{\uparrow}x \text{ such that } p(z) = y.$$

Theorem 4. *The category of finite pseudocomplemented distributive lattices and homomorphisms is dually equivalent to that of finite posets and weak p-morphisms.*

Our first goal is a dual characterization of the finite subalgebras of $\boldsymbol{F}_{\mathsf{PDL}}(\aleph_0)$.

Definition 5. *A poset X with minimum $\bot$ is said to have a* free skeleton *when:*

(i) *for every $x \in X$ and nonempty $Y \subseteq \max{\uparrow}x$ there exists an element $s_{x,Y} \in {\uparrow}x$ such that $Y = \max{\uparrow}s_{x,Y}$;*

(ii) *for every $x \in X$ and nonempty $Y, Z \subseteq \max{\uparrow}x$, we have that $Y \subseteq Z$ implies $s_{x,Z} \leqslant s_{x,Y}$;*

(iii) *for every $x \in X$ and nonempty $Y \subseteq \max X$, we have that $\max{\uparrow}x \subseteq Y$ implies $s_{\bot,Y} \leqslant x$.*

We say that a finite $\boldsymbol{A} \in \mathsf{PDL}$ has a free skeleton when its dual poset does. The following theorem is our main result, describing the models of $\mathsf{Th}_\forall(\boldsymbol{F}_{\mathsf{PDL}}(\aleph_0))$.

Theorem 6. *A finite $\boldsymbol{A} \in \mathsf{PDL}$ embeds into $\boldsymbol{F}_{\mathsf{PDL}}(\aleph_0)$ if and only if it has a free skeleton. Moreover, the class of models of $\mathsf{Th}_\forall(\boldsymbol{F}_{\mathsf{PDL}}(\aleph_0))$ is*

$$\{\boldsymbol{A} \in \mathsf{PDL} : \textit{every finite subalgebra of } \boldsymbol{A} \textit{ has a free skeleton}\}.$$

Since PDL is a finitely axiomatizable class, we can consider a finite set of axioms Σ for PDL. Recall that the *atomic diagram* $\mathsf{diag}(\boldsymbol{A})$ of $\boldsymbol{A} \in \mathsf{PDL}$ is the set of equations and negated equations, in the language expanded with constants for the elements of $\boldsymbol{A}$, that hold in $\boldsymbol{A}$.

Theorem 7. *The theory $\mathsf{Th}_\forall(\boldsymbol{F}_{\mathsf{PDL}}(\aleph_0))$ is recursively axiomatizable by*

$$\Sigma \cup \{\neg\exists x_1, \dots, x_n \bigsqcap \mathsf{diag}(\boldsymbol{A}) : \boldsymbol{A} \in \mathsf{PDL} \textit{ is finite and lacks a free skeleton}\}.$$

A more explicit axiomatization will also be discussed. Theorems 3(ii) and 7 yield:

Theorem 8. *The theory $\mathsf{Th}_\forall(\boldsymbol{F}_{\mathsf{PDL}}(\aleph_0))$ is decidable.*

As a consequence, we obtain that the admissibility of multiconclusion rules in the implication-free fragment of intuitionistic propositional logic is decidable.

These results have been collected in the article [2].

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The Beth companion: making implicit operations explicit

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Let K be a class of algebras in the same language.

Definition 1. *An n-ary* partial function *on K is a family $f = \langle f^{\mathbf{A}} : \mathbf{A} \in \mathsf{K}\rangle$ of functions $f^{\mathbf{A}} \colon \mathsf{dom}(f^{\mathbf{A}}) \to A$ with $\mathsf{dom}(f^{\mathbf{A}}) \subseteq A^n$. A partial function f on K is said to be*

(i) implicit *when it is defined by a first order formula $\varphi(x_1, \dots, x_n, y)$ in the language of K; i.e.,*

$$\mathsf{dom}(f^{\mathbf{A}}) = \{\langle a_1, \dots, a_n\rangle \in A^n : \textit{there exists } b \in A \textit{ such that } \mathbf{A} \vDash \varphi(a_1, \dots, a_n, b)\}$$

and $f^{\mathbf{A}}(a_1, \dots, a_n)$ is the unique element $b \in A$ such that $\mathbf{A} \vDash \varphi(a_1, \dots, a_n, b)$;

(ii) an operation *when it is preserved by homomorphisms: for every homomorphism $h \colon \mathbf{A} \to \mathbf{B}$ with $\mathbf{A}, \mathbf{B} \in \mathsf{K}$ and $\langle a_1, \dots, a_n\rangle \in \mathsf{dom}(f^{\mathbf{A}})$ we have $\langle h(a_1), \dots, h(a_n)\rangle \in \mathsf{dom}(f^{\mathbf{B}})$ and*

$$h(f^{\mathbf{A}}(a_1, \dots, a_n)) = f^{\mathbf{B}}(h(a_1), \dots, h(a_n)).$$

Example 2. Let DL be the variety of bounded distributive lattices and φ the conjunction of equations

$$(x \wedge y \approx 0) \,\&\, (x \vee y \approx 1).$$

If $\mathbf{A} \in \mathsf{DL}$, then $\mathbf{A} \vDash \varphi(a, b)$ if and only if b is a complement of a in $\mathbf{A}$. Since complements in bounded distributive lattices are unique when they exist and are preserved by bounded lattice homomorphisms, we conclude that φ defines an implicit operation of DL.

Definition 3. *A class of algebras K has the* strong Beth definability property *when for each n-ary implicit operation f of K there exists a set $\mathcal{T}$ of n-ary terms in the language of K such that for all $\mathbf{A} \in \mathsf{K}$ and $\langle a_1, \dots, a_n\rangle \in \mathsf{dom}(f^{\mathbf{A}})$ there exists $t \in \mathcal{T}$ such that*

$$f^{\mathbf{A}}(a_1, \dots, a_n) = t^{\mathbf{A}}(a_1, \dots, a_n).$$

Intuitively, K has the strong Beth definability property when all implicit operations of K are made explicit by terms. Not all implicit operations can be made explicit. For instance, DL does not have the strong Beth definability property because there is no set of terms in the language of bounded lattices making the operation of taking complements explicit.

When a finitary logic is algebraized by a quasivariety K, the former has the Beth definability property iff all epimorphisms in K are surjective [1, Thm. 3.12], and has the projective Beth definability property iff K has the strong Beth definability property or, equivalently, when all monomorphisms in K are regular, (see [3, 4] for this correspondence in the case of modal and intuitionistic logic).

*Speaker.

Theorem 4. *A universal class* K *has the strong Beth definability property iff it has the* strong epimorphism surjectivity property*; i.e., for every homomorphism* $f\colon \mathbf{A} \to \mathbf{B}$ *with* $\mathbf{A}, \mathbf{B} \in \mathsf{K}$ *and* $b \in B - f[A]$ *there exists a pair of homomorphisms* $g, h\colon \mathbf{B} \to \mathbf{C}$ *with* $\mathbf{C} \in \mathsf{K}$ *such that* $g \circ f = h \circ f$ *and* $g(b) \neq h(b)$*.*

Our main contribution is a method to optimally expand a given class of algebras into one where all implicit operations can be made explicit.

Let $\mathcal{F}$ be a set of implicit operations of K. We denote by $\mathscr{L}_{\mathcal{F}}$ the language obtained by expanding the language of K with function symbols acting as names for the operations in $\mathcal{F}$. Whenever $\mathbf{A} \in \mathsf{K}$ has the property that $f^{\mathbf{A}}$ is a total function for each $f \in \mathcal{F}$, we can expand $\mathbf{A}$ to an algebra $\mathbf{A}[\mathscr{L}_{\mathcal{F}}]$ in the language $\mathscr{L}_{\mathcal{F}}$ by interpreting the new function symbols with the respective implicit operations in $\mathcal{F}$. Let $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ be the class of all such expansions.

Definition 5. *An implicit operation* f *of* K *is said to be* extendable *in* K *if every* $\mathbf{A} \in \mathsf{K}$ *can be embedded into some* $\mathbf{B} \in \mathsf{K}$ *such that* $f^{\mathbf{B}}$ *is a total function.*

Definition 6. *A* pp expansion M *of* K *is the class of subalgebras of* $\mathsf{K}[\mathscr{L}_{\mathcal{F}}]$ *for some set* $\mathcal{F}$ *of extendable implicit operations of* K *that are defined by primitive positive formulas (conjunctions of equations prenexed by existential quantifiers). If moreover* M *has the strong Beth definability property, we call it a* Beth companion *of* K*.*

Examples 7.

(i) The variety of Boolean algebras is a Beth companion of the variety of bounded distributive lattices.

(ii) The variety of Abelian groups is a Beth companion of the quasivariety of cancellative commutative monoids.

(iii) Every variety of Heyting algebras of depth at most 2 admits a Beth companion, whereas every variety generated by a finite Heyting chain of size at least 5 does not.

Theorem 8. *All Beth companions of a quasivariety are quasivarieties and are all term-equivalent.*

The properties of a class K are often significantly enhanced in its Beth companion M when it is *congruence preserving*; i.e., when $\mathsf{Con}_{\mathsf{M}}(\mathbf{A}) = \mathsf{Con}_{\mathsf{K}}(\mathbf{A}{\restriction}_{\mathscr{L}_{\mathsf{K}}})$ for every $\mathbf{A} \in \mathsf{M}$.

Theorem 9. *Let* K *be a relatively congruence distributive quasivariety such that the class of its relatively finitely subdirectly irreducible members is closed under nontrivial subalgebras. Every congruence preserving Beth companion of* K *is an arithmetical variety with the congruence extension property.*

Corollary 10. *Every Beth companion of a relatively filtral quasivariety is a discriminator variety.*

These results are collected in the manuscript [2], which is available online.

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Ultracontact algebras and stack systems*

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Boolean Contact Algebras (BCAs) are structures meant to study the phenomenon of nearness of regions in an abstract setting [4, 2, 1]. As it is normally done, the standard language of Boolean algebras is expanded with a binary predicate 'C' of *contact* and we interpret $x \mathrel{\mathsf{C}} y$ as the situation in which—in some sense—region x is in contact with region y. In a point-based setting, we would say that x and y share at least one point. The standard BCAs are subalgebras of the complete algebras of regular closed subsets of topological spaces. In this context, $x \mathrel{\mathsf{C}} y$ if $x \cap y$ (the set-theoretical intersection) is non-empty. If X is a topological space, $\mathrm{RC}(X)$ is its complete Boolean algebra of regular closed sets, S is a subalgebra of $\mathrm{RC}(X)$, and $x, y \in S$, then $\cap$ usually is not an operation of S, but the operation of the power set algebra of X. So the nearness of x and y is expressed by means of the operation from beyond S. In the abstract setting, C is assumed as primitive.

In this work, we aim to study an expansion of the notion of contact relations from pairs of regions to arbitrary collections. That is, we can easily imagine a ternary relation C_3 that holds among x, y, and z if all three regions are in contact in the sense that there is a location in space common to all three. Transferring the intuition to the point-based environment again, it reduces to the existence of at least one point that is in $x \cap y \cap z$. The ternary relation C_3 allows one to distinguish between the two configurations shown in Figure 1, which are indistinguishable from the point of view of binary contact relations, since each pair of regions among x, y, z shares a common point

This idea can be further generalized to arbitrary n-ary contact relations, as it was done in [3]. However, we aim at something more far-reaching, and in a way we want to study in one swoop all those possibilities, including infinite ones, by considering a family $\mathscr{K}$ of subsets of a Boolean algebra B, such that every set $M \in \mathscr{K}$ consists of regions sharing a common location, or having «a mutual point of contact». Allowing $\mathscr{K}$ to contain infinite subsets of B makes it possible, for example, to express that the family $\{(0, 1/n+1] \mid n \in \omega\}$ of subsets of the real interval $(0, 1]$ lacks a common point of contact; a property that cannot be captured by finitary contact relations.

Thus, we propose to study, under the name of *Ultracontact Algebras*, structures $\mathfrak{B} \coloneqq \langle B, \mathscr{K} \rangle$ where B is a Boolean algebra and $\mathscr{K} \subseteq 2^B$. The family $\mathscr{K}$, which we call *ultracontact*, satisfies the following constraints for all $F, G \subseteq B$:

(K0) $\emptyset \notin \mathscr{K}$.

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†Speaker
‡Paula Menchón (Tandil, Argentina) contributed to the development of ideas presented in this abstract.

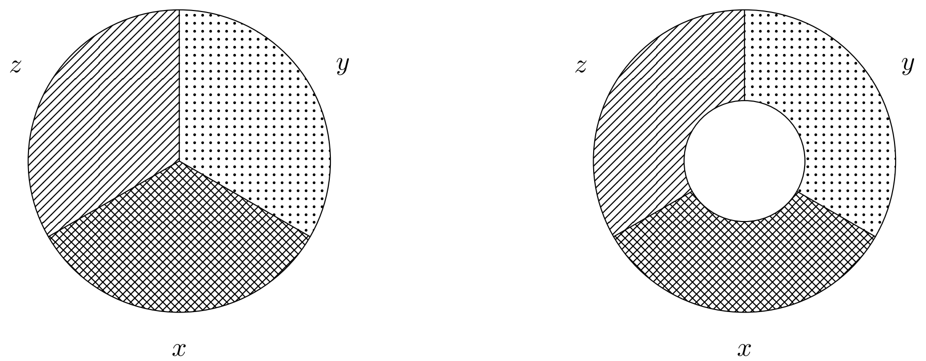

Figure 1: Configurations that are indistinguishable by binary contact relations

(K1) If $\mathbf{0} \in F$, then $F \notin \mathscr{K}$.

(K2) If $x \neq \mathbf{0}$, then $\{x\} \in \mathscr{K}$.

(K3) If $F \preceq G \neq \emptyset$ and $F \in \mathscr{K}$, then $G \in \mathscr{K}$.

(K4) If $F + G \in \mathscr{K}$, then $F \in \mathscr{K}$ or $G \in \mathscr{K}$.

Above, $\mathbf{0}$ is the zero of the underlying Boolean algebra B, and $F \preceq G$ (in words: F supports G) if for every $g \in G$ there is an $f \in F$ such that $f \leq g$.

In the talk, we would like to present basic findings on ultracontact algebras, also in relation to their equivalent characterization via the so-called *stack systems*, i.e., distinguished families of upper sets of Boolean algebras. We will also demonstrate how standard contact relations can be captured in the proposed framework, and how ultracontact algebras relate to known mathematical structures.

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Conrad Frames and Structural Transfer from ℓ-Groups to Prelinear Residuated Lattices

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A substantial body of research has established the central role of lattice-ordered groups (ℓ-groups) in logic. In earlier work [7], see also [8, Chapter 4], we initiated a Conrad-type approach to algebras of logic inspired by Paul Conrad's structural analysis of ℓ-groups.

The Conrad Program studies individual residuated lattices and classes of residuated lattices via the lattice-theoretic structure of their lattices of convex subalgebras. It was shown that fundamental structural results from the theory of ℓ-groups extend systematically to the variety of prelinear residuated lattices, namely those satisfying the identity $x\backslash e \approx e/x$ (e-cyclicity) together with the prelinearity laws $((x\backslash y)\wedge e)\vee((y\backslash x)\wedge e) \approx e$ and $((y/x)\wedge e)\vee((x/y)\wedge e) \approx e$.

A key outcome of this development was the recognition that the lattice of convex subalgebras is itself the fundamental structural object. This perspective rests on the work of [4, 5, 6], where it was shown that several central structural theorems in the theory of ℓ-groups depend solely on the order-theoretic properties of the lattice of convex ℓ-subgroups.

In this setting, the lattice of convex subalgebras forms what we call a *Conrad frame*: an algebraic distributive lattice whose compact elements form a sublattice and whose meet-irreducible elements constitute a root system (that is, a dual tree). The transfer of results from ℓ-groups to this wider class occurs in three distinct ways. First, some results are purely lattice-theoretic, depending only on structural properties of a Conrad frame; these extend automatically once the lattice of convex substructures is recognized as such. Second, certain results rely primarily on the Conrad frame structure but require additional algebraic features for their precise formulation. Third, some results depend essentially on the ℓ-group structure itself and cannot be derived from lattice-theoretic considerations alone.

To minimize terminology, the examples below illustrating the three categories are drawn from finite-valued ℓ-groups. An ℓ-group is *finite valued* if each element has only finitely many values, that is, convex ℓ-subgroups maximal with respect to not containing that element. These values correspond to the completely meet-irreducible elements of the lattice of convex ℓ-subgroups.

An instructive example of the first type is Conrad's characterization [2] of finite-valued ℓ-groups as those whose lattice of convex ℓ-subgroups is completely distributive. The main result of [4] showed that this characterization is purely lattice-theoretic by establishing it for arbitrary Conrad frames. Since the lattice of convex subalgebras of a prelinear residuated lattice forms a Conrad frame, such a residuated lattice is finite valued if and only if its lattice of convex subalgebras is completely distributive.

A central example of the second type is Conrad's characterization of small ordinal sums (small lexico-sums). We say that an ℓ-group G is an *ordinal extension* of an ℓ-group H if H is a convex ℓ-subgroup of G and the interval $(H, G]$ in the lattice $C(G)$ of convex ℓ-subgroups of G is a chain. An ℓ-group G is a *small ordinal sum* of o-groups H^1_γ $(\gamma \in \Gamma)$ if there exists a finite or infinite sequence $H^1 \subseteq H^2 \subseteq \cdots$ of convex ℓ-subgroups of G such that $\bigcup_{n=1}^{\infty} H^n = G$, and $H^1 = \sum_{\gamma\in\Gamma} H^1_\gamma$, while for $n > 1$, $H^n = \sum_{\gamma\in\Gamma_n} H^n_\gamma$, where each H^n_γ is either a nontrivial ordinal extension of a finite cardinal sum of two or more components H^{n-1}_β, or coincides with one of them. Conrad showed [1] that an ℓ-group G is a small ordinal sum of o-groups if and only if every positive element of G exceeds at most finitely many pairwise orthogonal positive elements.

This characterization admits a reformulation at the level of the lattice $C(G)$ of convex ℓ-subgroups: every upper bounded set of compact orthogonal elements is finite, a property depending only on the order structure of $C(G)$ [6]. Thus the existence of the corresponding join decomposition is governed entirely by the Conrad frame. However, a crucial distinction arises: in the ℓ-group setting the symbol $\sum_\gamma H_\gamma$ denotes a cardinal (direct) sum of convex ℓ-subgroups, and for orthogonal components this coincides with their join in $C(G)$. In an abstract Conrad frame one retains only the join $\bigvee_\gamma a_\gamma$, without any intrinsic notion of direct sum. Consequently, when this result is transferred to prelinear residuated lattices, the algebraic realization of the join must be addressed separately. This explains why the small ordinal sum theorem belongs to the second category of transfer phenomena rather than the first.

A particularly instructive example of the third type is torsion theory. A *torsion class* [3] of ℓ-groups is one closed under convex ℓ-subgroups, homomorphic images, and joins of convex ℓ-subgroups. The class of finite-valued ℓ-groups provides a basic example. While finite-valuedness is determined entirely by the order-theoretic structure of the lattice of convex ℓ-subgroups, the proof that finite-valued ℓ-groups form a torsion class relies essentially on the ℓ-group operations. This contrast raises the natural question whether finite-valued prelinear residuated lattices form a torsion class within the class of prelinear residuated lattices, a problem that lies beyond purely lattice-theoretic considerations.

From a broader logical perspective, Conrad frames provide a unifying order-theoretic infrastructure for all the phenomena discussed above. They isolate the structural content of convex subobject lattices independently of the ambient algebra, allowing purely lattice-theoretic results to transfer verbatim, structural decompositions to be reformulated at the level of joins and compact elements, and genuinely algebraic phenomena to be precisely identified as lying beyond order-theoretic abstraction. In this way, the passage from ℓ-groups to prelinear residuated lattices is mediated by an intrinsic frame structure that captures what is transferable and clarifies what requires additional algebraic input. Within this framework, torsion theory acquires a transparent order-theoretic formulation, while at the same time revealing the precise point at which the algebraic operations become indispensable.

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Interpolation in Two-Dimensional Cylindric Modal Logic

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Cylindric algebras are algebraic models of first-order logic [3]. Modal logic counter-parts of finitely dimensional cylindic algebras are n-dimensional products of the modal logic $\mathsf{S5}$ for $n \in \omega$ (see e.g., [2]). On the other hand, n-dimensional cylindric algebras (with the diagonal) correspond to cylindric modal logics (CML)[8]. Let $\mathsf{S5}^n$ denotes the n-dimensional product of $\mathsf{S5}$, and CML_n the cylindric modal logic corresponding to n-dimensional cylindric modal algebras. The lattices $\mathsf{NExt}(\mathsf{S5}^2)$ and $\mathsf{NExt}(\mathsf{CML}_2)$ have been investigated in [1], where it was shown that every normal extension of $\mathsf{S5}^2$ is finitely axiomatizable and has the FMP, while there exist continuum many non-finitely axiomatizable logics extending CML_2. Marx and Areces [6] showed that $\mathsf{S5}^2$ lacks interpolation. However, a full characterization of logics with the interpolation property in extensions of $\mathsf{S5}^2$ and CML_2 remained open.

In this work, we study the interpolation properties of two-dimensional cylindric modal logics via the (super)amalgamation property of two-dimensional cylindric algebras and the duality between frames and cylindric algebras. The amalgamation property has been shown to be equivalent to the *deductive interpolation property* (DIP) for a large class of logical systems (see e.g., [7]). The *Craig interpolation property* (CIP) for modal and intuitionistic logics via superamalgamation was studied in [4, 5].

Definition 1. *Let L be a two-dimensional CML. Then we say that L has the:*

1. Deductive interpolation property (DIP)*, if for any sets of variables $\overline{p}, \overline{q}, \overline{r}$ and formulas $\varphi(\overline{p}, \overline{q}), \psi(\overline{p}, \overline{r})$ with $\varphi(\overline{p}, \overline{q}) \vdash_L \psi(\overline{p}, \overline{r})$, there exists $\chi(\overline{p})$ such that $\varphi(\overline{p}, \overline{q}) \vdash_L \chi(\overline{p})$ and $\chi(\overline{p}) \vdash_L \psi(\overline{p}, \overline{r})$.*

2. Craig interpolation property (CIP)*, if for any sets of variables $\overline{p}, \overline{q}, \overline{r}$ and formulas $\varphi(\overline{p}, \overline{q}), \psi(\overline{p}, \overline{r})$ with $\vdash_L \varphi(\overline{p}, \overline{q}) \to \psi(\overline{p}, \overline{r})$, there exists $\chi(\overline{p})$ such that $\vdash_L \varphi(\overline{p}, \overline{q}) \to \chi(\overline{p})$ and $\vdash_L \chi(\overline{p}) \to \psi(\overline{p}, \overline{r})$.*

Definition 2. *Let $\mathcal{V}$ be a variety of cylindric algebras and $A, B_0, B_1 \in \mathcal{V}$. A tuple $T = (A, B_0, B_1, f_0, f_1)$ is called an* amalgamation triple *if $f_i : A \to B_i$ is an embedding for $i \in \{0, 1\}$. We say a tuple (C, g_0, g_1) is an* amalgam *of T in $\mathcal{V}$ if $C \in \mathcal{V}$ and $g_0 : B_0 \to C$ and $g_1 : B_1 \to C$ are embeddings such that $g_0 f_0 = g_1 f_1$. Moreover, we call an amalgam (C, g_0, g_1) a* superamalgam *if it satisfies the following: for all $j \in \{0, 1\}$, $b_j \in B_j$ and $b_{1-j} \in B_{1-j}$, if $g_j(b_j) \leq g_{1-j}(b_{1-j})$, then $b_j \leq f_j(a)$ and $f_{1-j}(a) \leq b_{1-j}$ for some $a \in A$. We say that $\mathcal{V}$ has the* (super)amalgamation property *if a (super)amalgam exists for every amalgamation triple.*

By general results on interpolation properties of polymodal logics (see, e.g., [9]), we have:

Theorem 3. *Let L be a two-dimensional CML and $\mathsf{V}(L)$ the variety of all L-algebras. Then*

L has the DIP (CIP) if and only if $\mathsf{V}(L)$ has the (super)amalgamation property.

An $\mathsf{S5}^2$-frame is a tuple (X, E_1, E_2) where $X \neq \varnothing$ and E_1, E_2 are equivalence relations on X such that $E_1 \circ E_2 = E_2 \circ E_1$. For all $\alpha, \beta, \gamma \leq \omega$, we write $\mathfrak{R}^{\gamma}_{\alpha,\beta}$ for the rooted $\mathsf{S5}^2$-frame (X, E_1, E_2) consisting of α many E_1-clusters and β many E_2-clusters, such that each $E_1 \cap E_2$-cluster has cardinality γ. Our main result (Theorem 4) provides a full characterization of the interpolation properties of extensions of $\mathsf{S5}^2$. It implies that there are exactly 18 extensions of $\mathsf{S5}^2$ with the CIP (respectively, the DIP).

Theorem 4. *Let $L \in \mathsf{NExt}(\mathsf{S5}^2)$. Then L has the CIP iff L has the DIP iff $L \in \mathsf{IP}$, where*

$$\mathsf{IP} = \{\mathsf{Log}(\mathfrak{R}^k_{1,m}), \mathsf{Log}(\mathfrak{R}^k_{m,1}) : k, m \in \{1, 2, \omega\}\} \cup \{\mathsf{Log}(\mathfrak{R}^1_{2,2}), \mathsf{Log}(\mathfrak{R}^1_{2,\omega}), \mathsf{Log}(\mathfrak{R}^1_{\omega,2})\}.$$

The proof of the main theorem comprises the following key steps:

1. We show that for any $L \in \mathsf{NExt}(\mathsf{S5}^2)$, if L has the DIP, then $\mathfrak{F}^*$ in Figure 1 is not a frame for L, since the co-amalgamation triple of frames depicted in Figure 1 does not have any co-amalgam.
2. Since $\mathsf{S5}^2$ is pre-locally tabular (see [1, Corollary 6.2.12]), we show that a logic $L \in \mathsf{NExt}(\mathsf{S5}^2)$ has the DIP (CIP) if and only if every amalgamation triple in $\mathsf{V}_{fsi}(L)$ has a (super)amalgam in $\mathsf{V}(L)$, where $\mathsf{V}_{fsi}(L)$ is the class of finite subdirectly irreducible elements in $\mathsf{V}(L)$.
3. Using the notions of depth and girth of logics in $\mathsf{NExt}(\mathsf{S5}^2)$ defined in [1], we show that if $L \in \mathsf{NExt}(\mathsf{S5}^2)$ has the DIP, then all of the E_1-depth, E_2-depth and girth of L must be $1, 2$ or $\aleph_0$. Moreover, we show that every $L \in \mathsf{NExt}(\mathsf{S5}^2)$ with the DIP must be one of the logics in IP.
4. Finally, we check that $\mathsf{Log}(\mathfrak{R}^1_{2,2}), \mathsf{Log}(\mathfrak{R}^1_{2,\omega}), \mathsf{Log}(\mathfrak{R}^1_{\omega,2})$ have the CIP and $\mathsf{Log}(\mathfrak{R}^k_{1,m}), \mathsf{Log}(\mathfrak{R}^k_{m,1})$ have the CIP for all $k, m \in \{1, 2, \omega\}$, which concludes the proof of Theorem 4.

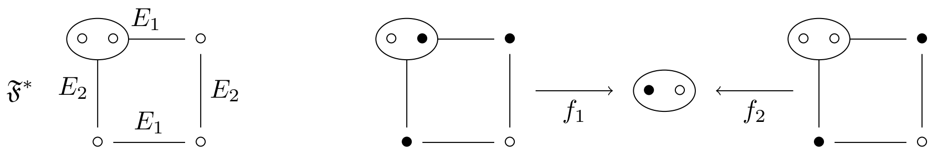

Figure 1: The frame $\mathfrak{F}^*$ and a co-amalgamation triple without amalgam

Using similar techniques, we obtain a full characterization of the interpolation properties in $\mathsf{NExt}(\mathsf{CML}_2)$:

Theorem 5. *There exists exactly 12 logics in $\mathsf{NExt}(\mathsf{CML}_2)$ with the DIP, of which 9 have the CIP.*

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Unicoherence in Locales

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In this talk, we generalize the concept of unicoherence to the context of frames. Unicoherence, originally introduced by Kuratowski, is a connectedness property that is well studied in classical topology and used to detect holes of a space. We extend the notion of unicoherence to locales and we then investigate its properties. In particular, we prove that many of the known characterizations of unicoherence for topological spaces extend to the setting of locales. Some of these characterizations interestingly involve separation properties for locales.

1 Unicoherence

Definition 1.1. A *continuum* in a locale L is a non-void closed connected sublocale $\mathfrak{c}(u) \subseteq L$.
A *region* in a locale L is a non-void open connected sublocale $\mathfrak{o}(u) \subseteq L$.

Proposition 1.2. *Let X be a topological space. The closed sublocale $\mathfrak{c}(X \setminus F)$ of $\mathcal{O}(X)$ associated to a closed subspace F of X is a continuum precisely when F is a continuum (in the classical topological sense).*

Definition 1.3. A locale L is *unicoherent* if whenever $L = H \vee K$ with $H, K \subseteq L$ continua, we have that $H \cap K$ is a continuum.

L is *open unicoherent* if whenever $L = A \vee B$ with $A, B \subseteq L$ regions, we have that $A \cap B$ is a region.

Proposition 1.4. *Let X be a topological space. The locale $\mathcal{O}X$ is unicoherent precisely when X is unicoherent.*

2 Main Results

Theorem 2.1. *Let L be a connected and locally connected locale. The following properties are equivalent:*

(*I*) *Whenever $X, Y \subseteq L$ with $X \neq \emptyset \neq Y$ and $\mathfrak{c}(u), \mathfrak{c}(v) \subseteq L$ are such that $\mathfrak{c}(u) \cap \mathfrak{c}(v) = \emptyset$ and neither $\mathfrak{c}(u)$ nor $\mathfrak{c}(v)$ separate any element of X (different from 1) from any element of Y (different from 1) in L, we have that $\mathfrak{c}(u) \vee \mathfrak{c}(v)$ does not separate X and Y in L;*

(*II*) *(Brouwer Property) If $\mathfrak{c}(u) \subseteq L$ is a continuum, then every component D of $\mathfrak{o}(u)$ has as boundary $\mathrm{bd}\,(D)$ a continuum;*

(*III*) *(Unicoherence) L is unicoherent;*

(*IV*) *If $\mathfrak{c}(u) \subseteq L$ is non-void and D_1, D_2 are disjoint components of $\mathfrak{o}(u)$ such that $\mathrm{bd}\,(D_1) = \mathrm{bd}\,(D_2)$ then $\mathrm{bd}\,(D_1)$ is a continuum;*

(V) *If* C *and* D *are disjoint complemented connected sublocales of* L *such that* $\mathrm{bd}\,(C) \subseteq \mathrm{bd}\,(D)$*, then* $\mathrm{bd}\,(C)$ *is connected;*

(VI) *If* $C \subseteq L$ *is simple, then* $\mathrm{bd}\,(C)$ *is connected;*

(VII) *If* R *is a simple region, then* $\mathrm{bd}\,(R)$ *is connected;*

$(VIII)$ *If* A *and* B *are disjoint regions such that* $\mathrm{bd}\,(A) = \mathrm{bd}\,(B)$*, then* $\mathrm{bd}\,(A)$ *is connected;*

(IX) *If* A *and* B *are regions such that* $\mathrm{bd}\,(A) \cap \mathrm{bd}\,(B) = \emptyset$*, then* $A \cap B$ *is connected;*

(X) *(Open unicoherence)* L *is open unicoherent.*

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Axiomatization of logics complete with respect to 3-element Semi-Heyting algebras

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Abstract

The main goal of this paper is to show logics complete with respect to 3-element semi-Heyting algebras. Thus addressing the open problem 14.3 from [1]. There are ten 3-element semi-Heyting algebras, one is a well-known 3-element Heyting algebra. For two of them some new results were proved [2]. This paper shows new results for the remaining seven of them. The topic is dictated by findings obtained in last 15 years by various authors. It was proven [3] that some semi-Heyting subvarieties are term-equivalent to Gödel algebras' varieties. Among subvarieties of semi-Heyting algebras some of them are also term-equivalent to connexive logics [2]. Due to importance of these categories of algebras, and corresponding logics, in modern investigation, the possibility of expanding intuitionistic logic to version, which contains these categories, is an interesting one. Although some 3-element semi-Heyting logics were already been shown to be deductively equivalent, there isn't any paper proving general result for all 3-element semi-Heyting algebras.

Our main definition is:

Definition 1 (Semi-Heyting Algebra)**.** *An algebra* $\boldsymbol{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ *is a Semi-Heyting Algebra if the following conditions hold:*

(SH1) $\langle L, \vee, \wedge, 0, 1\rangle$ *is a lattice with* $0, 1$.

(SH2) $x \wedge (x \to y) = x \wedge y$.

(SH3) $x \wedge (y \to z) = x \wedge [(x \wedge y) \to (x \wedge z)]$.

(SH4) $x \to x = 1$.

The main theorem is:

Theorem 1. *There exists excatly 2 logic, which are not deductively equivalent, which are algebraized by either L1 (or algebra term-equivalent to it) or L6 (or algebra term-equivalent to it).*

Two theorems used for proving the main theorem are:

Theorem 2. *Algebras L2, L3, L4, L9, L10 (as defined in [1]) are term-equivalent to L1.*

Theorem 3. *Algebras L5, L6, L7, L8 are term-equivalent and they are not term-equivalent to L1.*

To prove them we need to explain term-equivalence.

Definition 2 (Term-equivalence)**.** *The algebras* $\mathfrak{A} = (A, F_A)$ *and* $\mathfrak{B} = (A, F_B)$ *are* term-equivalent *if there are maps*

$$\tau : F_A \to \mathrm{Term}(F_B), \qquad \sigma : F_B \to \mathrm{Term}(F_A),$$

such that the following holds.

(1) If $f \in F_A$ is n-ary, then $\tau(f)$ is an n-ary F_B-term such that for all $a_1, \ldots, a_n \in A$,

$$f^{\mathfrak{A}}(a_1, \ldots, a_n) = \tau(f)^{\mathfrak{B}}(a_1, \ldots, a_n).$$

(2) If $g \in F_B$ is m-ary, then $\sigma(g)$ is an m-ary F_A-term such that for all $a_1, \ldots, a_m \in A$,

$$g^{\mathfrak{B}}(a_1, \ldots, a_m) = \sigma(g)^{\mathfrak{A}}(a_1, \ldots, a_m).$$

For the part of main Theorem it is useful to know the following lemma:

Lemma 1. *If $\mathfrak{A}$ and $\mathfrak{B}$ are term-equivalent, then congruences and universes of subalgebras of $\mathfrak{A}$ and $\mathfrak{B}$ are the same.*

The results, although similar in method to the one used in [2] were obtained independently through investigation with the help of prof. Zalan Gyenis.

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Semiring-Valued Logics with Twist-Structure Semantics

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Abstract

We propose semiring-based propositional logics via a dual-valuation twist-structure semantics, with paraconsistent as well as consistent variants. We compare several consequence relations on the semiring-valued twist-structures and sketch possible extensions.

Semirings [2] are pervasive in computer science because their two binary operations $+$ and $\cdot$ suffice to capture, abstractly, parallel *choice* vs. sequential *accumulation,* underlying applications that range from resource modeling to weighted automata to weighted argumentation. These applications motivate formal logics for semiring-annotated reasoning, developed, e.g., in *semiring semantics* [3]. An acknowledged problem of such logics is the treatment of *negation* and *implication*, which do not occur among the semiring operations; existing approaches typically restrict negation (e.g., to literals) or assume additional constraints on the semiring (e.g., natural order). Here we propose a semiring-based logic that includes negation without such restrictions, and we outline how implication might be treated within the framework.

To account for *negation* without imposing extra conditions on the underlying semiring, we use a (dual-valuation) *twist-structure* semantics [4]: each formula φ is assigned a pair $(t_\varphi, f_\varphi) \in S \times S$ representing *support-for* and *support-against* by values in a semiring S; negation just swaps the components, $(t_{\neg\varphi}, f_{\neg\varphi}) = (f_\varphi, t_\varphi)$; and the connectives $\vee, \wedge$ are defined as follows:

$$(t_{\varphi\vee\psi}, f_{\varphi\vee\psi}) = (t_\varphi + t_\psi, f_\varphi \cdot f_\psi), \quad (t_{\varphi\wedge\psi}, f_{\varphi\wedge\psi}) = (t_\varphi \cdot t_\psi, f_\varphi + f_\psi).$$

We call the resulting algebras $S^{\bowtie}$ in the signature $\{\vee, \wedge, \neg\}$ *semiring-based dual-valuation algebras* (DVAs). For semirings that are (distributive) lattices this subsumes the familiar product-bilattice semantics; in particular, the truth-connective reduct of the bilattice FOUR with $\neg$, used for FDE [1], is $\mathbf{2}^{\bowtie}$ over the Boolean semiring $\mathbf{2}$. Conceptually, the two-channel evaluation is natural since verification and refutation can be implemented by different procedures with different costs or witnesses, so negation need not correspond to any single map on S; dual valuation separates positive and negative evidence while remaining fully compositional, and has recently been mentioned as a promising direction in semiring semantics [3].

The dual-valuation semantics is in general paraconsistent, since both t_φ, f_φ may be nonzero. A consistent variant requires that at most one of the two channels succeeds, i.e., restricts valuations to the axial subalgebra $\{(t, f) \in S \times S \mid t = 0 \text{ or } f = 0\}$, as failure is represented by the semiring value 0. We call these subalgebras *orthogonal* dual-valuation algebras (oDVAs).

Semiring-based DVAs and oDVAs support a range of consequence relations with standard matrix semantics; here we only single out preservation of confirmability and of irrefutability.

Confirmability corresponds to the non-failing t-channel, therefore we define:

$$\Gamma \models^{t\neq 0} \psi \;\equiv_{\mathrm{df}}\; (\forall\varphi \in \Gamma)(t_\varphi \neq 0) \implies t_\psi \neq 0, \quad \text{for all valuations in } S^{\bowtie} \text{ over all semirings } S.$$

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Over DVAs, this logic is paranormal with no tautologies but validates familiar rules such as double negation, De Morgan, and $\wedge$-elimination (while many other rules fail, incl. adjunction). Since it has FOUR among its models, it is weaker than FDE. The logic is neither protoalgebraic nor self-extensional. The same condition over oDVAs yields a consistent (explosive) strengthening which still has no tautologies and still invalidates adjunction; since the oDVA over the Boolean semiring **2** is Kleene's 3-valued algebra, this logic is (strictly) weaker than K3.

Irrefutability preservation $\models^{f=0}$ is defined analogously for the f-channel failure ($f = 0$). Over DVAs it is paranormal, has no tautologies, and is strictly weaker than non-falsity preservation in FOUR; unlike $\models^{t\neq 0}$, it validates adjunction and the commutativity of $\wedge$ (but not of $\vee$). Over oDVAs it is paraconsistent, does have tautologies (e.g., LEM), and is strictly weaker than Priest's logic LP.

Beyond these, semiring-based DVAs and oDVAs admit further meaningful consequence relations, such as $\models^{t=1}$ (preserving "costless confirmation"), $\models^{t\in\mathbb{N}^+}$ ("costless confirmation with free choice"), $\models^{t=f=0}$ (preserving the "undecidable status"), and various strict–tolerant variants. Many such logics are rather weak over all semirings but become stronger over narrower classes (idempotent, ordered, etc.), yielding a rich family of semiring twist-logics. Additional variants result from symmetrized clauses for $t_{\varphi\wedge\psi}$ and $f_{\varphi\vee\psi}$, since the current definitions enforce a fixed evaluation order, adequate only in some settings. Further variations include expansions by propositional constants, such as $\mathbf{t} = (1, 0)$, and additional connectives, especially *implication*: every dual-valuation semiring logic already contains a pair of material implications $\neg\varphi\vee\psi$ and $\varphi \vee \neg\psi$, which support, e.g., contraposition and currying; however, they fail to validate basic inferential rules, including modus ponens and hypothetical syllogism. While some restricted classes of semirings (e.g., naturally ordered ones) allow better-behaved implications, a uniform option is to treat implication as internalized single-premise entailment and study sequent rules with first-degree implication ($\varphi \Rightarrow \psi$). Overall, the point is not to propose one canonical semiring-valued logic, but to initiate a broader program of investigating the family of dual-valuation semiring logics and their applications, presenting basic properties of $\models^{t\neq 0}$ and $\models^{f=0}$ as an initial exposition.

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Homological Algebra in Abelian Framed Bicategories

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Abstract

We introduce abelian framed bicategories, which are particular framed bicategories that are locally abelian, and show that they are suitable for developing homology and cohomology theories for directed structures. This means in particular that similar exact sequences as the relative homology and Mayer–Vietoris long exact sequences can be shown to hold. Also, for closed monoidal abelian framed bicategories, Künneth theorem holds as well. Finally, we prove embedding theorems similar to the Gabriel and Freyd–Mitchell theorems, for particular abelian framed bicategories, allowing to see those as bicategories of bimodules over algebras. This naturally links to the original motivation of this work, which was to generalize directed homology developed in the abelian framed bicategory of bimodules over (path) algebras.

Introduction Designing suitable homology and homotopy theories for directed objects, such as directed topological spaces, is a notorious difficult problem [**?**, **?**, **?**, **?**, **?**]. The non-reversibility of time in those models makes the needed algebra much more complex than in the classical case: to faithfully abstract this non-reversibility in time, the algebraic structures employed cannot be reversible (like abelian groups in classical algebraic topology, or ordered homology groups in [**?**]) or even cancellative (like Patchkoria's theory in cancellative monoids [**?**]), because non-homotopic directed paths can become homotopic when extended, see Fahrenberg's matchbox [**?**].

This means we have to move away from classical abelian categories as a suitable algebraic framework. Still, some form of local abelianity is needed: directed spaces are $(\infty, 1)$-categories by nature (see [**?**]), because if paths are not invertible, (higher) homotopies are. Consequently, over the non-invertible low-dimensional data lies a more classical, fully invertible world where abelian structures can be used. That is the direction some took to develop non-cancellative homology theories for directed spaces, such as homology as functor from a category of traces to the singular homology of trace spaces for directed spaces [**?**] or homology bimodules over a path algebra for precubical sets [**?**]. Those theories have a similar flavor: they look at the classical homology modules of trace spaces between two points and describe how those modules evolve by pre- and post-composing with additional traces. Written differently, they make traces act on the left and on the right of an object in an abelian category. Following this idea, we introduce abelian framed bicategories as a new suitable framework for developing homology theories for directed structures.

Framed bicategories, introduced by Shulman in [**?**], are a way to categorify the way a structure, the coefficients, act (on the left and on the right) on another structure and how those actions evolve by changing the coefficients by extension and restriction. They are given by a bifibration $B \to C$ where C is the category of coefficients while B is a category of structures on which coefficients can act, and for which the effect of restriction (resp. the extension) of coefficients is given by cartesian (resp. cocartesian) liftings. Then, abelian framed bicategories are framed bicategories for which the fibers above pairs of coefficients (that is the subcategories of structures on which fixed coefficients act) are abelian and for which the bicategorical structure preserves enough of the abelian structure. The archetypal examples of such are categories of bimodules over algebras, such as the ones used in [**?**] to develop the homology of precubical sets.

Definition 0.1. An **abelian framed bicategory** is a framed bicategory $\mathbb{D}$ that is locally abelian, i.e. all local categories $\mathcal{D}(A, B)$ are abelian, and such that horizontal composition yields additive functors $\mathcal{D}(A, B) \times \mathcal{D}(B, C) \to \mathcal{D}(A, C), (M, N) \mapsto M \odot N$ for all objects $A, B, C \in \mathbb{D}_0$.

We define (co)chain complexes and (co)homology in an abelian framed bicategories $\mathbb{A}$.

Definition 0.2. A **chain complex** in $\mathbb{A}$ is a chain complex in a local abelian category of $\mathbb{A}$, i.e., a chain complex in $\mathcal{A}(A, B)$ for some objects A and B. A **morphism** $(\alpha_i)_{i\geq 0}$ from a chain complex $(M_i)_{i\geq 0}$ in $\mathcal{A}(A, B)$ to a chain complex $(N_i)_{i\geq 0}$ in $\mathcal{A}(C, D)$ is a pair of vertical arrows $f : A \to C$ and $g : B \to D$ along with 2-cells $\alpha_i : M_i \underset{f}{\overset{g}{\Rightarrow}} N_i$ for all $i \geq 0$, such that $\alpha_i \partial_{i+1} = \partial_{i+1} \alpha_{i+1}$ for all $i \geq 0$.

Consider a functor $M : \mathcal{C} \to \mathrm{Ch}(\mathbb{A})$ into the category of chain complexes in $\mathbb{A}$.

Definition 0.3. The **homology** of an object X in $\mathcal{C}$ is the homology $H_*(X) : L(X) \nrightarrow R(X)$ of the chain complex $M_*(X)$ in the local abelian category $\mathcal{D}(L(X), R(X))$. This construction is functorial.

As in abelian categories, we can derive interesting exact sequences. For particular morphisms $X \to Y$ in $\mathcal{C}$ that we call relative pairs, there is a relative homology exact sequence, and a Mayer-Vietoris and Künneth theorems, involving extension of scalars:

Theorem 0.4. *For all relative pairs (X, Y), there is a long exact sequence for relative homology:* $\cdots \to H_{i+1}(X, Y) \to {}^X H_i(Y) \to H_i(X) \to H_i(X, Y) \to \cdots$.

Finally, we show embedding theorems, similar to the Gabriel and Freyd–Mitchell theorems, for certain abelian framed bicategories with additional structure, that we call **module-like**.

Theorem 0.5. *If $\mathbb{A}$ is module-like, there is a framed lax functor $F : \mathbb{A} \to {}_R\mathbb{M}\mathrm{od}_R$ into a concrete abelian framed bicategories of modules over R-algebras, with $R = \mathrm{End}(U_I)$, such that F_1 is fully faithful and locally an equivalence of categories.*

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Many-Valued Coalgerbaic Lindström's Theorem*

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Coalgebraic logic ever since proposed by Larry Moss has two approaches – one based on the concept of relation lifting and a single cover modality [8], and the other on the more usual syntax with modalities defined via predicate liftings [9]. Inspired by the works on Lindström theorem of modal logic done by M. de Rijke [3]and van Benthem[10], A. Kurz and Y. Venema proved in [6] three Lindström's theorems characterizing coalgebraic logic. They first proposed an abstract coalgebraic logic $\mathcal{L}_T^F$ proved that any abstract coalgebraic logic $\mathcal{L}$ extend from $\mathcal{L}_T^F$ satisfying invariance under bisimulation, Boolean closure, compactness, and expressively closed at ω is equivalent to $\mathcal{L}_T^F$. Nevertheless, their work is based on final-coalgebra approximation instead of submodels. S. Enquivist improved the result in [4] through proposing the definition of submodels in coalgebraic logic. Since many-valued coalgebraic modal logic has been widely studied recently[5], it is natural to investigate which logical properties characterize a many-valued coalgebraic modal logic. In this talk, we will prove a many-valued coalgebraic Lindström theorem.

Let's first assume that $\mathbf{A}$ is a MTL-chain and Λ be a set of predicate liftings $Hom(-, A^n) \Rightarrow Hom(T(-), A)$[2]. We let $\mathcal{ML}^{\Lambda}_{T,\mathbf{A}}$ be the language $\mathcal{L}_{\mathbf{A}}$ in [7]. The semantics is the same as Definition 4 in [7]. We then define an abstract $\mathbf{A}$-coalgebraic logic $\mathcal{L}^{\Lambda}_{T,\mathbf{A}}$ for the functor T and the set of predicate lifting Λ consisting of

- a set $\mathcal{L}(\Phi)$ of formulas for every finite set of propositional variables Φ, and
- for every finite set of propositional variables Φ, a satisfaction function

$$\Vdash_{\mathcal{L}_\Phi} : PCoalg(T_\Phi) \times \mathcal{L}(\Phi)) \to \mathbf{A}$$

  where $PCoalg(T_\Phi)$ is a set of pointed coalgebras (S, σ, s) with $s \in S$ such that

  – (Extension) For each Φ, $\mathcal{ML}^{\Lambda}_{T,\mathbf{A}}(\Phi) \subseteq \mathcal{L}(\Phi)$ and for all pointed T_Φ-coalgebras and all $\varphi \in \mathcal{ML}^{\Lambda}_{T,\mathbf{A}}(\Phi)$

$$(S, \sigma, s) \Vdash_{\mathcal{ML}^{\Lambda}_{T,\mathbf{A}}(\Phi)} \varphi \iff (S, \sigma, s) \Vdash_{\mathcal{L}_\Phi} \varphi$$

  Here, $\Vdash_{\mathcal{ML}^{\Lambda}_{T,\mathbf{A}}(\Phi)}$ denotes the satisfaction function defined as in [7]

  – (Monotonicity) For each $\Phi \subseteq \Phi'$, $\mathcal{L}(\Phi) \subseteq \mathcal{L}(\Phi')$;

  – (Occurrence) If $\varphi \in \mathcal{L}(\Phi)$ then there is a finite $\Phi_\varphi \subseteq \Phi$ such that for every Φ',

$$\varphi \in \mathcal{L}(\Phi') \iff \Phi_\varphi \subseteq \Phi'$$

  – (Relativization) For every formula $\varphi \in \mathcal{L}(\Phi)$ and $p \in \Phi$, there is $rel(\varphi, p) \in \mathcal{L}(\Phi)$ such that for every pointed model with $(S, \sigma) \upharpoonright p$ well-defined as a Λ-submodel where p is true and $\langle S, \sigma, s\rangle \Vdash p$, we have

$$\langle S, \sigma, s\rangle \Vdash_{\mathcal{L}(\Phi)} rel(\varphi, p) \iff \langle (S, \sigma) \upharpoonright p, s\rangle \Vdash_{\mathcal{L}(\Phi)} \varphi$$

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– (Inclusion) Let $\Phi \subseteq \Phi'$ and let $\varphi \in \mathcal{L}(\Phi)$. Then there is a formula $inc(\varphi, \Phi) \in \mathcal{L}(\Phi')$ such that
$$\langle S, \sigma, s\rangle \Vdash_{\mathcal{L}(\Phi')} inc(\varphi, \Phi) \iff \langle F_{\Phi,\Phi'}(S, \sigma), s\rangle \Vdash_{\mathcal{L}(\Phi)} \varphi;$$

– (**A**-closure) $\mathcal{L}(\Phi)$ is closed under $\{\wedge, \vee, \odot, \rightarrow, ([\lambda])_{\lambda\in\Lambda}\}$.

where the functor $F_{\Phi,\Phi'} : Coalg(T_\Phi) \rightarrow Coalg(T_{\Phi'})$ is defined via $(S, \sigma) \mapsto (S, (T_\Phi(X) \rightarrow T_{\Phi'}(X)) \circ \sigma)$There are four further assumptions we have to make for **A**-coalgebraic modal logic to prove Lindström theorem.

1. The set Λ is finite and separating;
2. The mapping $Spt_S : TS \rightarrow [[S, \mathbf{A}^n], \mathbf{A}]$ form a natural transformation $Spt : T \Rightarrow [[-, \mathbf{A}^n], \mathbf{A}]$;
3. For any T-coalgebra (S, σ) and any $S' \subseteq S$, there is a mapping $\sigma' : S' \rightarrow TS'$ such that (S', σ') is a Λ-submodel of (S, σ);
4. $\mathcal{ML}^\Lambda_{T,\mathbf{A}}$ is compact.

Based on the four assumptions, we can first prove

Theorem 1. *Let $\mathcal{L}^\Lambda_{T,\mathbf{A}}$ be an abstract **A**-coalgebraic logic which is compact, has finite-depth property, and is invariant for behavior equivalence. Then $\mathcal{ML}^\Lambda_{T,\mathbf{A}}$ is equivalent to $\mathcal{L}^\Lambda_{T,\mathbf{A}}$.*

To prove the second-version of Lindström's theorem, we first have the following lemma similar to [1] :

Proposition 1. *The relativization, compactness, and behavioral equivalence invariance property give us finite-depth property.*

Then one can easily derive the following corollary from the theorem and proposition stated above.

Corollary 1. *Let $\mathcal{L}^\Lambda_{T,\mathbf{A}}$ be an abstract **A**-coalgebraic logic extending from $\mathcal{ML}^\Lambda_{T,\mathbf{A}}$. If $\mathcal{L}^\Lambda_{T,\mathbf{A}}$ is compact, has relativization property, and is invariant for behavior equivalence, then $\mathcal{ML}^\Lambda_{T,\mathbf{A}}$ is equivalent to $\mathcal{L}^\Lambda_{T,\mathbf{A}}$.*

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A Unified Categorical Duality for Preminimal Negation and Beyond: Extending Compatibility Semantics Along Dunn's Negation Hierarchy

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Negation in non-classical logics admits a rich spectrum of interpretations, ranging from classical involutive negation to weaker forms that abandon double negation elimination, excluded middle, or contraposition. Dunn's compatibility semantics [5, 6] provides a powerful relational framework for this spectrum: negation is interpreted via an incompatibility (perp) relation between situations, where $\sim A$ holds at a point when every situation compatible with it fails to verify A. The algebraic structures corresponding to the weakest system in Dunn's kite of negations [5, 9] are the K_i *algebras*, that is, bounded distributive lattices equipped with a preminimal negation operator.

This paper [1] develops a systematic categorical study of K_i algebras and their dual relational structures. A central novelty is the introduction of *partial bounded morphisms* between compatibility frames. Unlike standard Kripke bounded morphisms, partial bounded morphisms correctly preserve and reflect the impossibility clause for negation. We demonstrate, through an explicit counterexample, that the dual of a K_i algebra homomorphism need not be a bounded morphism in the Kripke sense. These morphisms specialise the $\neg$-morphisms of Celani [3] and the continuous bounded morphisms studied for meet-hemiantimorphisms in [10].

By enriching compatibility frames with differentiation, tightness, and compactness conditions we obtain *descriptive compatibility frames*. The main result is a full dual equivalence between the category **DF** of descriptive compatibility frames (with partial bounded morphisms) and the category **KI** of K_i algebras (with homomorphisms), established via explicitly defined functors $F\colon \mathbf{DF} \to \mathbf{KI}$ and $G\colon \mathbf{KI} \to \mathbf{DF}$. This provides a precise categorical bridge between algebraic and relational semantics for logics with weak negation, and aligns with Celani's $\neg$-spaces framework [3], building on the foundations of modal logic [2] and duality theory for lattices with operators [10].

The framework is then extended systematically along Dunn's negation hierarchy [5, 9]. For each of the following classes of algebras, namely De Morgan algebras, Kleene algebras, pseudocomplemented lattices, Stone algebras, and orthoalgebras, we define corresponding compatible relational frames (D-frames, Kleene frames, Ps-frames, Stone frames, and ortho frames) within our setting. Categorical duality results are established in each case, and the resulting frames are compared with existing dual spaces from the literature [4, 7]. Embedding results of [8] for Kleene and Stone frames are applied to locate these structures within Dunn's hierarchy. In the case of ortho frames, where ortho negation on a distributive lattice is Boolean negation [5], the compatibility relation degenerates to the identity relation, and the duality specialises to Stone's classical duality.

The categorical duals obtained here are further compared with established topological dualities. Using the topological foundations of [7], the category of descriptive compatibility frames is shown to be equivalent to the category of $\neg$-spaces of Celani [3]. Similarly, the categorical duals of De Morgan, Kleene, and Stone algebras correspond, respectively, to the De Morgan spaces of [4], the Kleene spaces, and the SLD-spaces, confirming the coherence of the approach with the broader landscape of duality theory.

Overall, the paper demonstrates how diverse negational logics, from preminimal to Boolean negation, fit coherently into a single semantic architecture founded on compatibility relations. The unified perspective clarifies the interrelations among previously scattered duality results and illustrates the robustness of Dunn's relational approach to negation.

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Quad Rings and Quad Algebras: From Duality to Logical Perspective

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Abstract

We study a class of algebras equivalent to the class of rings, whose elements satisfy the quartic identity $x^4 = x$. Furthermore, we explore the duality between these algebras and certain associated topological spaces, and develop a corresponding four-valued logical framework.

1 Introduction

Boolean algebras have long been studied from two distinct but highly influential perspectives. The first stems from George Boole's formulation of two-valued logic in terms of Boolean algebras. The second major perspective arises from the celebrated duality of M. H. Stone [4], which establishes that the category of Boolean algebras and homomorphisms is dually equivalent to the category of zero-dimensional, compact, T_0 spaces (Stone spaces) and continuous maps. Moreover, Boolean algebras admit another well-known dual representation in terms of Boolean rings, whose elements satisfy the quadratic identity $x^2 = x$.

Theorem 1. *[4] Let $(B, \vee, \wedge, \prime, 0, 1)$ be a Boolean algebra. Then $(B, +, \cdot)$ is a Boolean ring with unity 1, where $x + y = (x \wedge y') \vee (x' \wedge y)$ and $x \cdot y = x \wedge y$ for all $x, y \in B$. Conversely let $(B, +, \cdot)$ be a Boolean ring with unity 1. Then $(B, \vee, \wedge, \prime, 0, 1)$ is a Boolean algebra where $x \vee y = x + y + x \cdot y$, $x \wedge y = x \cdot y$ and $x' = 1 + x$ for all $x, y \in B$.*

Motivated by this classical correspondence, we investigate a broader class of algebraic structures arising from rings that satisfy a higher-degree polynomial identity. Specifically, we consider rings with unity satisfying the identity $x^4 = x$. We refer to such rings as quad rings. These rings exhibit structural properties that allow them to support lattice-theoretic interpretations analogous to those present in Boolean rings. On the other hand, a quad algebra is an algebraic system $(L, \vee, \wedge, \prime, \neg, 0, 1)$, where $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice along with $\prime$ and $\neg$ as Boolean and De Morgan negations, respectively. In particular, we establish following equivalence between quad rings and quad algebras [3].

Theorem 2. *Let $(R, +, \cdot, 0, 1)$ be a quad ring. Then $\mathcal{L}(\mathcal{R}) := (R, \vee, \wedge, \prime, \neg, 0, 1)$ is a quad algebra, where for all $x, y \in R$,*

1. $x \vee y = x + y + x^2y + xy^2 + x^2y^2$,
2. $x \wedge y = x^2y + xy^2 + x^2y^2$,
3. $x' = 1 + x$,
4. $\neg x = 1 + x^2$.

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Theorem 3. *Let $\mathcal{L} := (L, \vee, \wedge, ', \neg, 0, 1)$ be a quad algebra. Then $\mathcal{R}(\mathcal{L}) := (L, +, \cdot, 0, 1)$ is a quad ring with unity 1, where for all $a, b \in L$*

1. $a + b = (a \wedge b') \vee (a' \wedge b)$,
2. $a \cdot b = (a \wedge \neg a \wedge (\neg b)') \vee ((\neg a)' \wedge b \wedge \neg b) \vee (a \wedge b \wedge (\neg b)') \vee (a \wedge (\neg a)' \wedge b) \vee (a' \wedge b' \wedge (\neg a)' \wedge (\neg b)')$.

Theorem 4. *Let $\mathcal{L} := (L, \vee, \wedge, ', \neg, 0, 1)$ be a quad algebra and $\mathcal{R} := (R, +, \cdot, 0, 1)$ be a quad ring with unity. Then, $\mathcal{R}(\mathcal{L}(\mathcal{R})) = \mathcal{R}$ and $\mathcal{L}(\mathcal{R}(\mathcal{L})) = \mathcal{L}$. Furthermore, a mapping $f : Q_1 \to Q_2$ is a quad algebra homomorphism if and only if it is a quad ring homomorphism. Thus, the categories QA and QR of quad algebras and quad rings with morphisms as corresponding homomorphisms, respectively are equivalent.*

Beyond the algebraic duality between quad rings and quad algebras, we also explore a topological perspective. Boolean algebras admit Stone duality with Stone spaces [4], while De Morgan algebras possess their own dual representation [1]. By combining and adapting the techniques used in these established dualities, we construct a corresponding topological framework suitable for quad algebras.

Definition 1. *A topological space $(X, \mathcal{T}, g)$ is called a quad space if*

1. *The topological space $(X, \mathcal{T})$ is compact, totally disconnected.*
2. *$g : X \to X$ is a continuous involution which is also a clopen map i.e., image of every clopen set is also a clopen set.*
3. *A mapping $f : X \to Y$ between two quad spaces $(X, \mathcal{T}, g)$ and $(Y, \overline{\mathcal{T}}, \overline{g})$ is a quad morphism if*
   (a) *f is continuous,*
   (b) *$\overline{g} \circ f = f \circ g$.*

Theorem 5. *The categories QA and QS of quad algebras and quad spaces, respectively are dually equivalent.*

From a logical standpoint, we further investigate the propositional logic $\mathcal{L}_{\mathcal{QA}}$ associated with quad algebras. The algebraic structure of quad algebras naturally suggests a semantics richer than the classical two-valued interpretation. Inspired by the work of J. M. Dunn [2], we introduce a four-valued semantic framework for $\mathcal{L}_{\mathcal{QA}}$ based on the four element diamond as quad algebra $\mathbf{4}=(\{0, a, b, 1\}, \vee, \wedge, ', \neg, 0, 1)$ where $0 < a, b < 1$ with $0' = 1$, $1' = 0$, $a' = b$, $b' = a$ and $\neg 0 = 1$, $\neg 1 = 0$, $\neg a = a$ and $\neg b = b$. Finally, we establish that the propositional system $\mathcal{L}_{\mathcal{QA}}$ is both sound and complete with respect to the four-valued semantics.

Theorem 6. *$\alpha \vdash_{\mathcal{L}_{\mathcal{QA}}} \beta$ if and only if $\alpha \models_{0,b,1} \beta$ for any $\alpha, \beta \in \mathcal{F}_{\mathcal{QA}}$.*

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We introduce and develop propositional *continuous intuitionistic logic* and propositional *continuous affine logic* through the study of two classes of algebras, and provide sequent-style deductive systems enjoying cut admissibility for these logics [3]. In order to prove a cut admissibility theorem for the logics presented in this paper, we will rely on Algebraic Proof Theory [2].

In order to embrace logics for constructive mathematics besides intuitionistic and classical continuous logic, we work with commutative residuated complete lattices, which are supposed integral. Our approach thus centres on AC-algebras (for affine continuous logic) and IC-algebras (for intuitionistic continuous logic). The former are the continuous analogues of commutative residuated complete lattices, the later are the continuous analogues of locales (see [1] and [4]). Elements of an AC-algebra $USC(\mathscr{L})$ are sup-preserving functions from $[0,1]$ to a commutative residuated complete lattice $\mathscr{L}$. We first study the link between an AC-algebra and its underlying residuated lattice. We then give a sufficient condition, based on "linearity", for a formula true in $[0,1]$ to be true in any AC-algebra. It leads us to an algebraic axiomatisation $\mathbf{T}$ of AC-algebras and IC-algebras in a language whose interpretation of symbols in $[0,1]$ are continuous functions. In a model of $\mathbf{T}$, there may be infinitesimals, and we can define a preorder by $a \preceq b \Leftrightarrow \forall n \in \mathbb{N}\ a \leq b \dot{+} \frac{1}{2^n}$. To remove the infinitesimals and obtain an Archimedean structure, we can quotient by the equivalence relation induced by this preorder. We prove the following theorem using Macneille completion.

Theorem A. *For all model A of $\boldsymbol{T}$, there exists a commutative residuated complete lattice $\mathscr{L}$ such that the Archimedean quotient of A embeds into $USC(\mathscr{L})$.*

Methods from Algebraic Model Theory enable us to find, from the axiomatisations, two cut-free deductive systems whose complete models are AC-algebras and IC-algebras and prove completeness of each class of algebras with respect to these systems. From completeness, we derive a cut-admissibility theorem for both systems.

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Axiomatizing logics of finite Gödel-Kripke models

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The semantics of Gödel Modal Logic is based on Kripke models whose accessibility relation and world-wise formulas take values in the standard Gödel algebra, generalizing naturally classical Kripke models. A notable restriction of this framework is obtained by considering classical Kripke frames, namely, models whose accessibility relation takes values only in $\{0, 1\}$.

An interesting feature of these logics is that, although their sets of theorems are decidable [1] and they enjoy the FMP, they generally fail to be complete with respect to finite models under the aforementioned semantics [6, 2]. Since this semantics is the most directly applicable to the modeling of reasoning schemes, it is natural to investigate the logic determined by its restriction to finite models (note that this does not amount to fixing a finite Gödel chain as the algebra of evaluation). These systems were studied in [5, 4], where cut-free sequent calculi for $\mathbb{KG}^{\omega}$ and $\mathbb{MG}^{\omega}$ were introduced, but complete deductive systems were still unknown.

In the present work, we provide complete axiomatizations for these systems, thereby closing that open problem from the literature. We also negatively settle open problems questions posed in [2] (and the analogous arising from [6]), concerning whether the natural extensions of the basic logics by the formulas respectively used to establish the failure of completeness with respect to witnessed models are themselves witnessed. Furthermore, we prove completeness results for the general logics with respect to classes of models that, while not finite, satisfy certain partial witnessing conditions.

Definition 1.1. A Gödel Kripke model is a tuple $\langle W, R, e\rangle$ with W a non-empty set, $R\colon W^2 \to [0, 1]$ and $e\colon W \times \mathcal{V} \to [0, 1]$, such that e is world-wise a propositional Gödel homomorphism from the set of formulas into $[0, 1]$, and

$$e(v, \Box\varphi) \coloneqq \bigwedge\{R(v, w) \to e(w, \varphi)\colon w \in W\} \qquad e(v, \Diamond\varphi) \coloneqq \bigvee\{R(v, w) \wedge e(w, \varphi)\colon w \in W\}$$

A model is crisp when $R\colon W^2 \to \{0, 1\}$. A formula φ is valid in a model if $e(v, \varphi) = 1, \forall v \in W$.

We let the logic MG be the set of theorems of all Gödel Kripke models ($\mathbb{MG}$), axiomatized in [3], and KG that of the crisp Gödel Kripke models ($\mathbb{KG}$), axiomatized in [7]. We will denote by $L_\Box$ or $L_\Diamond$ the corresponding mono-modal fragment (axiomatized in [2, 6]) [1]. It is well known that each of the above logics (as sets of theorems) is equivalent to the local entailment over the corresponding class of models.

A model is *witnessed* whenever for every v and φ there are $v_{\Box\varphi}$ and $v_{\Diamond\varphi}$ such that

$$e(v, \Box\varphi) = R(v, v_{\Box\varphi}) \to e(v_{\Box\varphi}, \varphi) \quad \text{and} \quad e(v, \Diamond\varphi) = R(v, v_{\Diamond\varphi}) \to e(v_{\Diamond\varphi}, \varphi)$$

We denote by $\mathbb{MG}^w$ and $\mathbb{KG}^w$ the classes of witnessed models in $\mathbb{MG}$ and $\mathbb{KG}$, respectively. It can be proven that, for a logic being a set of theorems (or local entailment), being complete with respect to a class of witnessed models is equivalent to being complete with respect to finite models of the class. We nevertheless refrain from using this latter formulation in order to avoid

[1] Observe that, from the same class of models, three logics arise, in this sense. We will instinctively say ”complete” w.r.t. the same class of models, for the bi-modal and for the mono-modal logics.

confusion with the fact that the above logics do enjoy the finite model property, although not with respect to the intended semantics considered previously.

It is known that $MG_\Diamond$ is complete w.r.t. $\mathbb{MG}^w$ [2], but that $KG_\Diamond$ is not complete w.r.t. $\mathbb{KG}^w$ [6], using the formula $(\Diamond p \to \Diamond q) \to (\neg\Diamond q \vee \Diamond(p \to q))$. On the other hand, none of KG, MG nor their mono-modal fragments with only $\Box$ are witnessed [2], using the formula $\Box\neg\neg p \to \neg\neg\Box p$. It was an open question from [2] whether $KG_\Box$ plus the axiom $\Box\neg\neg p \to \neg\neg\Box p$ was complete w.r.t. $\mathbb{KG}^w$. Similarly, it was not known if $KG_\Diamond$ plus the axiom $(\Diamond p \to \Diamond q) \to (\neg\Diamond q \vee \Diamond(p \to q))$ from [6] was complete w.r.t. $\mathbb{KG}^w$. We will answer negatively to these questions and provide an axiomatization for the logics of finite models.

Lemma 2.1. *$KG_\Box$ + $\Box\neg\neg p \to \neg\neg\Box p$ is not complete w.r.t. witnessed $\mathbb{KG}$-models.* [2]

Lemma 2.2. *$KG_\Diamond$ + $(\Diamond p \to \Diamond q) \to (\neg\Diamond q \vee \Diamond(p \to q))$ is not complete w.r.t. witnessed $\mathbb{KG}$-models.*

It is possible to check that the witnessing condition of $MG_\Diamond$ extends to MG. A model is called $\Diamond$-witnessed whenever it is witnessed for $\Diamond$ formulas.

Lemma 2.3. *MG is complete w.r.t. $\Diamond$-witnessed $\mathbb{MG}$-models.*

While $KG_\Diamond$ (and hence, KG) is known to not be witnessed (hence, not $\Diamond$-witnessed), we can nevertheless prove a partial witnessing completeness that is of great interest. A model is called $\langle\Diamond, \top\rangle$-witnessed whenever, for any formula φ and any world w, it holds that, if $e(v, \Diamond\varphi) = 1$, then there is some w such that $R(v, w) = 1$ and $e(w, \varphi) = 1$.

Lemma 2.4. *KG (and hence, $KG_\Diamond$) is complete w.r.t. $\langle\Diamond, \top\rangle$-witnessed $\mathbb{KG}$ models.*

Lastly, to axiomatize the logics arising from finite models, we propose the following axiom schemata, whose intuition lies in addressing the problems that the formulas from Lemmas 2.1 and 2.2 do not cover.

$$W_\Box := \Box(((p \to q) \to q) \wedge (q \to p)) \to ((\Box p \to \Box q) \to \Box q)$$
$$W_\Diamond := ((\Diamond p \to \Diamond q) \wedge ((\Diamond q \to \Diamond z) \to \Diamond z)) \to (\Diamond z \vee \Diamond((p \to q) \wedge ((q \to z) \to z))$$

Theorem 2.5. *$KG + W_\Box + W_\Diamond$ is complete w.r.t. $\mathbb{KG}^w$, and the same holds for its mono-modal fragments. $MG + W_\Box$ is complete w.r.t. $\mathbb{MG}^w$ (and the same happens with $MG_\Box + W_\Box$).*

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[2] And recall from [2] that $MG_\Box = KG_\Box$.

Automata on Graph Alphabets
Extended Abstract

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The theory of finite automata concerns itself with words over an alphabet Σ, that is, the free monoid on Σ together with concatenation and without further structure. There are, however, important applications which use alphabets which are structured in some sense, for example in database theory [6], model checking of systems with data [5,8], type checking [7], or concurrency theory [2].

Here we work with a particular type of structured data, namely an alphabet which is given as a (finite or infinite) directed graph. This constrains concatenation: two strings may only be concatenated if the end vertex of the first is equal to the start vertex of the second.

We give an example for motivation. In a concurrent setting, processes' behavior depends on whether or not they have access to shared resources. Figure 1 shows a simple graph alphabet designed to reflect this, where the action available depends on the state: in unsafe state, processes may do a, but after locking the resource (P), they can do b, until the resource is released (V). Note that the graph in Fig. 1 only constrains the *alphabet*; processes themselves may have arbitrarily complex behavior, but they are by design restricted to not do b while in unsafe state.

Another, more comprehensive example comes from higher-dimensional automata theory. We have in [1–3] introduced ST-automata: state-labeled automata on an alphabet of so-called starters and terminators. These are best understood as automata on an infinite graph alphabet where vertices denote events which are running and edges are starters and terminators.

A *graph alphabet* (V, Σ, d_0, d_1) consists of sets V and Σ together with source and target mappings $d_0, d_1 : \Sigma \to V$. We will generally omit d_0 and d_1 from the signature. Both V and Σ may be finite or infinite, as may the automata we define next.

Definition 1. *An* automaton *on graph alphabet* (V, Σ) *is a structure* $A = (Q, I, F, E, s, t, \mu, \lambda)$ *consisting of a set of states* Q *with initial and accepting states* $I, F \subseteq Q$*, a set of transitions* E *with source and target mappings* $s, t : E \to Q$*, and labelings* $\mu : Q \to V$*,* $\lambda : E \to \Sigma$*. We require that* $\mu(s(e)) = d_0(\lambda(e))$ *and* $\mu(t(e)) = d_1(\lambda(e))$ *for every* $e \in E$*.*

That is, the transitions of A are labeled with elements of Σ, but in a way consistent with the graph (V, Σ): a transition labeled a must emanate from a state labeled $d_0(a)$ and lead to a state labeled $d_1(a)$. We may understand the state labels as *types* which restrict the application of transitions.

A *path* in A is an alternating sequence $\pi = (q_0, e_1, q_1, \dots, e_n, q_n)$ of states and transitions such that $s(e_i) = q_{i-1}$ and $t(e_i) = q_i$ for all i; we naturally expand this notation to $s(\pi) = q_0$ and $t(\pi) = q_n$. The *label* of π is $\lambda(\pi) = (\mu(q_0), \lambda(e_1) \cdots \lambda(e_n), \mu(q_n))$.

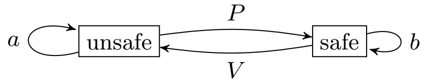

Figure 1: Alphabet for locking and releasing a resource.

We keep the labels of the start and target state of π in $\lambda(\pi)$ as we need typing information for concatenation. The shortest paths in A are the constant paths $\pi = (q)$ for $q \in Q$; the label of such a path π is $\lambda(\pi) = \mu(q)$. A path π is *accepting* if $s(\pi) \in I$ and $t(\pi) \in F$. The *language* of A is $L(A) = \{\lambda(\pi) \mid \pi \text{ accepting path in } A\}$.

We develop the beginnings of an automata theory for languages on graph alphabets. We show that they admit a Kleene theorem, relating rational and regular languages, and a Myhill-Nerode theorem, stating that languages are regular iff they have finite prefix or, equivalently, suffix quotient (subject to an easy condition on the graph). Further, automata on graph alphabets are determinizable and complementable, and minimal deterministic automata exist.

Finally, we also investigate a generalization of our setting to alphabets which form simplicial sets. One main key difference is that languages are no more freely generated. However, we conjecture that also in this setting, Kleene and Myhill-Nerode theorems hold.

This is an extended abstract of the preprint [4] with the same title.

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The Eliminability of Delta in Gödel Logics *

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Gödel logics G_V, (where set of truth values V is a closed subset of $[0,1]$ containing 0 and 1) form an essential class of intermediate logics, those that are stronger than intuitionistic logic yet weaker than classical logic. The language is standard (propositional, first-order) with countably infinite propositional variables A_i, connectives $\wedge$, $\vee$, $\rightarrow$, and the constants $\bot$ for "false" and $\top$ for "true"; Atomic formulas include propositional variables, and truth constants.

Definition 1. *A valuation $\mathcal{I}$ based on V is a function from the set of propositional variables into V given as follows:*

$$(1)\ \mathcal{I}(\bot) = 0, \quad (2)\ \mathcal{I}(\top) = 1, \qquad (3)\ \mathcal{I}(A \wedge B) = \min\{\mathcal{I}(A), \mathcal{I}(B)\},$$
$$(4)\ \mathcal{I}(A \vee B) = \max\{\mathcal{I}(A), \mathcal{I}(B)\}, \qquad (5)\ \mathcal{I}(\forall x A(x)) = \mathit{inf}\{\mathcal{I}(A(u))\ u \in U_{\mathcal{I}}\},$$
$$(6)\ \mathcal{I}(\exists x A(x)) = \mathit{sup}\{\mathcal{I}(A(u))\ u \in U_{\mathcal{I}}\}, \qquad (7)\ \mathcal{I}(A \supset B) = \begin{cases} \mathcal{I}(B), & \mathcal{I}(A) > \mathcal{I}(B), \\ 1, & \mathcal{I}(A) \leq \mathcal{I}(B). \end{cases}$$

A formula in Gödel logic is valid *iff the formula evaluates to 1 under every interpretation. The Gödel logic G_V is defined as the set of valid formulas. Note that the validity and 1-satisfiability are not dual in Gödel logic.*

The asymmetry between the truth values 0 and 1 in Gödel logics, stemming from continuity conditions at 1, motivates the introduction of the absoluteness operator Δ [1], which precisely identifies formulas evaluating to 1

$$\mathcal{I}(\triangle A) = \begin{cases} 1 & \text{if } \mathcal{I}(A) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1. *There is no connective $\triangle$ definable with other connectives and variables*

We present a method for eliminating the absoluteness operator $\triangle$ from propositional Gödel logics by adopting a restricted semantics, where all propositional atoms, except the constant for truth, are interpreted strictly below 1. Although $\triangle$ is not definable from other connectives in standard Gödel logics, we show that under the restricted semantics, every formula with $\triangle$ is equivalent to a disjunction of chains, thus eliminating $\triangle$ entirely.

The argument used for the propositional case does not extend to the first-order case. For example, when 1 is not isolated and does not belong to a perfect set, however, 0 is isolated or does belong to a perfect set, the first-order Gödel logic with $\triangle$ is not recursively enumerable, while the first-order Gödel logic without $\triangle$ is. This holds both for standard and restricted semantics. Consequently, in restricted semantics, first-order formulas with $\triangle$ are not (validity)

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equivalent to those without $\triangle$. Therefore there is not even an effective validity equivalence elimination of $\triangle$, and obviously no valid equivalence as in the propositional case. However, in the witnessed setting, all quantifier-shift principles are available and axiomatized, which allows us to extend the $\triangle$-elimination algorithm to witnessed Gödel logics.

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On the theory of finite Gödel Algebras

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A *Heyting algebra* is an algebra $\boldsymbol{A} = \langle A; \wedge, \vee, \to, 0, 1\rangle$ such that $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded lattice and the binary operation $\to$ satisfies the *residuation law*, i.e., for every $a, b, c \in A$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c.$$

A *Gödel algebra* is a Heyting algebra $\boldsymbol{A}$ such that for every $a, b \in A$ it holds that

$$(a \to b) \vee (b \to a) = 1.$$

Let us denote by HA the variety of all Heyting algebras and by GA the variety of all Gödel algebras. For axiomatizations of these varieties, see, e.g., [2, 3].

From a combination of classical results from Tarski and Burris we know that the only non-trivial variety of Heyting algebras with decidable elementary theory is the one of Boolean algebras (see [4]). In contrast to this result, K. Idziak and P. Idziak proved in [4] that a variety of Heyting algebras has decidable elementary theory of its finite members if and only if it is contained in GA. This naturally motivates the search for a comprehensive axiomatization of the elementary theory of finite Gödel algebras. Since finiteness is not expressible in first-order logic, one must instead identify first-order conditions that capture finitary behavior as closely as possible. In this respect, forests and the join-irreducible elements of an algebra play a central role.

Let $\boldsymbol{A}$ be an algebra with a bounded lattice reduct. Recall that an element $a \in A$ is *join irreducible* if $a \neq 0$ and, whenever there are $b, c \in A$ with $a = b \vee c$, then $a = b$ or $a = c$. We denote by $\mathsf{J}(\boldsymbol{A})$ the set of join irreducible elements of $\boldsymbol{A}$. We say a subset $X \subseteq A$ is *join dense* in $\boldsymbol{A}$ when for every $a \in A$ there is some $Y \subseteq X$ such that $a = \bigvee Y$. Given a poset $\mathbb{X}$ and $x \in X$ we denote by $\downarrow x$ the downset generated by x in $\mathbb{X}$, by $\mathbb{X}^\partial$ the dual of $\mathbb{X}$, and by $\mathsf{Up}(\mathbb{X})$ both the set of its upsets and the Heyting algebra $\langle \mathsf{Up}(\mathbb{X}); \cap, \cup, \to, \varnothing, X\rangle$, where $\to$ is defined as

$$U \to V := X - \downarrow (U \cap (X - V)),$$

for every $U, V \in \mathsf{Up}(\mathbb{X})$. Lastly, a *forest* is a poset $\mathbb{X}$ in which $\downarrow x$ is linearly ordered for every $x \in X$.

The connection between these notions and Gödel algebras is captured by the following well-known results.

Theorem 1. *Given a poset $\mathbb{X}$, the Heyting algebra $\mathsf{Up}(\mathbb{X})$ is a Gödel algebra if and only if $\mathbb{X}^\partial$ is a forest.*

Theorem 2 (Representation theorem for finite Gödel algebras)**.** *For every finite Gödel algebra $\boldsymbol{A}$, it holds that $\boldsymbol{A} \cong \mathsf{Up}(\langle \mathsf{J}(\boldsymbol{A}), \geqslant\rangle)$*

Corollary 3. *For any finite Gödel algebra $\boldsymbol{A}$, the poset $\langle \mathsf{J}(\boldsymbol{A}), \leqslant\rangle$ is a finite forest.*

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*Speaker.

Theorem 4. *Let* $\boldsymbol{A}$ *be a Gödel algebra for which* $\mathsf{J}(\boldsymbol{A})$ *is join dense. Then, the map*

$$\varepsilon_{\boldsymbol{A}} \colon \boldsymbol{A} \to \mathsf{Up}(\langle \mathsf{J}(\boldsymbol{A}), \geqslant \rangle)$$
$$a \mapsto \mathsf{J}(\boldsymbol{A}) \cap {\downarrow} a,$$

is an embedding.

Recall that, in every finite lattice, the set of join-irreducible elements is join dense. Hence, for every finite Gödel algebra $\boldsymbol{A}$, the set $\mathsf{J}(\boldsymbol{A})$ is join dense. As this fact can be expressed by a first order sentence, the same holds in every model of the theory of finite Gödel algebras. Consequently, every model $\boldsymbol{A}$ of this theory will be isomorphic to $\varepsilon_{\boldsymbol{A}}[\boldsymbol{A}]$ by Theorem 4.

This prompts the question of whether we can axiomatize the theory of finite Gödel algebras by imposing enough conditions on the behavior of join irreducible elements. For starters, these elements need to satisfy the theory of finite forests. Based on this, our first result is an axiomatization of the theory of finite forests inspired by the methods of [1] (see also [6] and [5]). We are currently trying to exploit this axiomatization to derive a concrete axiomatization of the full theory of finite Gödel algebras.

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A Semantics for Modal Language Based on Subset Approximation Structure

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Abstract

Rough set theory, introduced by Pawlak [?], provides a mathematical framework for addressing vagueness and incomplete information, and exhibits a close structural and semantic correspondence with modal logic. This connection has been studied extensively since the inception of rough set theory (see, e.g., [?, ?]). The classical Pawlak rough set model is defined on an approximation space (W, R), where W is a non-empty set of objects and R is an indiscernibility relation (an equivalence relation). Any subset X of W can be approximated by two sets $L_R(X)$ and $U_R(X)$, which are defined as

$$L_R(X) = \{w \in W : R(w) \subseteq X\},$$
$$U_R(X) = \{w \in W : R(w) \cap X \neq \emptyset\},$$

where $R(w) = \{u \in W : (w, u) \in R\}$. We call L_R and U_R the lower approximation operator and the upper approximation operator, respectively.

This model corresponds to the modal system $S5$, where the necessity and possibility modal operators capture the lower and upper approximation operators, respectively. Over the years, the Pawlak model has been generalized in various ways to accommodate different applications and situations. One notable generalization is the rough set model based on *subset approximation structures* (in short, SAS), in which approximation operators are defined relative to a family of interest subsets (see [?, ?]). An SAS is a triple $F = (W, \sigma, R)$, where W is a non-empty set, σ is a non-empty collection of subsets of W, and $R \subseteq W \times W$ is a binary relation on W.

The study in [?] investigates the notion of the possibility lower approximation operator and develops a modal logic framework to reason about it. The language of the logic contains two primitive unary modal operators, $\Box$ and $\Box_1$. It should be noted, however, that these operators do not directly correspond to the possibility lower approximation operator. Rather, they are employed to define an operator $\blacktriangle^p$, which represents the possibility lower approximation operator.

Additionally, the deductive systems developed in [?] include axioms involving $\Box$ and $\Box_1$, while the operator $\blacktriangle^p$ for the possibility lower approximation does not appear explicitly in the axiomatization. Consequently, these systems do not capture the characterizing properties of the possibility lower approximation operator. To address this gap, the article [?] adopts a simpler language, namely the basic modal language with the operator $\Box$. The proposed semantics is based on a more general structure called an approximation frame, which extends neighbourhood frames by incorporating a relation on the carrier set. SASs represents a subclass of approximation frames distinguished by certain additional conditions.

Definition 1. *An* approximation frame *is a tuple* (W, ρ, R) *such that*

- W *is a non-empty set of objects,*
- $\rho : W \to P(P(W))$ *is a neighbourhood function, and*
- $R \subseteq W \times W$.

The *possibility lower approximation operator* based on SASs proposed in [**?**] leads us to the following notion of lower approximation operator defined on approximation frames.

Definition 2. *Let* $\mathfrak{F} := (W, \rho, R)$ *be an approximation frame. The* possibility lower approximation operator $L^p_{\mathfrak{F}}$ *and* necessity upper approximation operator $U^n_{\mathfrak{F}}$ *are defined as follows. Let* $X \subseteq W$.

$$\begin{aligned}
&L^p_{\mathfrak{F}}(X) := \{x \in W : x \in [R]_S(X) \text{ for some } S \in \rho(x)\},\\
&U^n_{\mathfrak{F}}(X) := \{x \in W : x \in \langle R\rangle_S(X) \text{ for all } S \in \rho(x)\}, \text{ where}\\
&[R]_S(X) := \{z \in W : R(z) \cap S \subseteq X\}, \text{ and}\\
&\langle R\rangle_S(X) := \{z \in W : R(z) \cap S \cap X \neq \emptyset\}.
\end{aligned}$$

In this presentation, we introduce a semantics for the basic modal language based on the possibility lower approximation operator of subset approximation structures. We establish axiomatizations over various classes of approximation frames, expressiveness, and invariance results for this semantics. From a rough set perspective, this work provides a formal language for reasoning about the possibility lower approximation operator, while the axiomatization results yield characterizing properties of this operator. Details and proofs of all results can be found in our published work [**?**].

On pseudo-associative conjunctive operations on relations of rank 3

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Abstract

Systems of axioms and bases of identities for classes of relation algebras with conjunctive operations are found.

Any set of binary relations closed with respect to some collection of operations on relations forms an algebra called an *algebra of relations*. The theory of algebras of relations is an essential part of algebraic logic [1] and modern algebra. A.Tarski proposed an axiomatic approach to their study [2]. Later in 1970, B.M. Shein formulated the general concept of algebras of relations and found conditions for their axiomatizability [3]. Within the framework of the axiomatic approach the following problems naturally arise and are solved when some class of algebras of relations are considered:

1. *Find a system of axioms for this class.*
2. *Find a basis of identities for the variety generated by this class.*
3. *Does this class form a variety or a quasi-variety?*

The subject of our consideration will be binary operations on relations, that is, classes of groupoids of relations. The detailed motivation for these studies can be found in [4]. An additional incentive for such research is the possibility of using groupoids in cryptography [5].

As a rule, operations on relations are specified using logical formulas. These operations are called *logical* [5]. Let $Rel(U)$ be the set of all binary relations on a set U. For any formula $\varphi(z_0, z_1, r_1, r_2)$ of the first-order predicate calculus (without equality and constants) containing two free variables z_0, z_1 and two binary predicate symbols r_1, r_2, we can define a binary operation F_φ on $Rel(U)$ in the following way: $F_\varphi(\rho_1, \rho_2) = \{(u, v) \in U \times U : \varphi(u, v, \rho_1, \rho_2)\}$, where $\varphi(u, v, \rho_1, \rho_2)$ means that the formula φ is satisfied whenever z_0, z_1 are interpreted as u, v, and r_1, r_2 are interpreted as the ralations ρ_1, ρ_2 from $Rel(U)$. The operation $F^*(\rho_1, \rho_2) = F(\rho_2, \rho_1)$ is called *dual* to the operation F. The operation $F^\star(\rho_1, \rho_2) = (F(\rho_1^{-1}, \rho_2^{-1}))^{-1}$, where $^{-1}$ is the operation of relational inverse, is called *inverted* to the operation F. The operation $F^{*\star}(\rho_1, \rho_2) = F^{\star *}(\rho_1, \rho_2) = (F(\rho_2^{-1}, \rho_1^{-1}))^{-1}$ is called *conjugate* to the operation F. Groupoids with the operations F and $F^\star$ are isomorphic, while groupoids with the operations F and F^* are anti-isomorphic to each other. For this reason, it makes sense to combine these four operations into a single cluster, which can contain from one to four elements. Then, to solve Problems 1-3, it is sufficient to consider one of the cluster operations.

An operation on relations is called conjunctive if it is defined by a formula representing a conjunction of atoms [6]. The rank of an conjunctive operation is the number of atoms in the formula defining this operation. Conjunctive operations of rank 2 were considered in [6,7]. As a result of these studies, a number of identities typical of conjunctive operations were identified. In particular, these identities include the following ones:

$$((xy)z)w = (x(yz))w \ (1), \quad w((xy)z) = w(x(yz)) \ (1'),$$
$$x^2y = xy \ (2), \quad xy^2 = xy \ (2'), \quad (xy)z = (yx)z \ (3), \quad x(yz) = (xy)(yz) \ (3').$$

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[*]

These identities are natural generalizations of the well-known identities of associativity, idempotency, and commutativity. In semigroup theory, identity (3) [respectively, (3')] is called pseudo-commutativity on the left [on the right]. It is natural identity (2) [respectively (2')], to call pseudo-idempotency on the left [on the right], identity (1) [respectively, (1')] to call pseudo-associativity on the left [on the right]. Non-associative groupoids satisfying the identities (1) and (1') will be called a *pseudo-semigroup*.

By routine checking it established that there are 14 clusters of conjunctive operations of rank 3, three of which generate classes of pseudo-semigroups. The satisfiability of the corresponding identities was also verified using a computer program. The idea of the program algorithm is based on the results from [8].

Let us formulate the results for one of these operations $*$ that is defined as follows:

$$\rho_1 * \rho_2 = \{(u,v) \in V \times V : (u,u) \in \rho_1 \wedge (u,v) \in \rho_2 \wedge (v,u) \in \rho_2\}.$$

Let $R\{*\}$ (respectively, $R\{*,\subseteq\}$) be the class of all groupoids (partially ordered groupoids) isomorphic to groupoids (partially ordered groupoids) of relations with the operation $*$, let $Var\{*\}$ be the variety generated by $R\{*\}$, and let $Var\{*,\subseteq\}$ be the variety generated by $R\{*,\subseteq\}$ in the class of all partially ordered groupoids.

Theorem *The following statements are fulfilled:*

(1) *A partially ordered groupoid* $(A,\cdot,\leq)$ *belongs to the variety* $Var\{*,\subseteq\}$ *if and only if it satisfies the identities:* (1), $(1')$, (2), (3), *and* $x(yz) = (xy)(yz)$ (4), $xy \leq y$ (5).

(2) *A groupoid* $(A,\cdot)$ *belongs to the variety* $Var\{*\}$ *if and only if it satisfies identities* (1), $(1')$, (2), (3), (4).

(3) *The class* $R\{*,\subseteq\}$ *does not form a variety in the class of all partially ordered groupoids. A partially ordered groupoid* $(A,\cdot,\leq)$ *belongs to the class* $R\{*,\subseteq\}$ *if and only if it satisfies identities* (1), $(1')$, (2), (3), (4), (5) *and the quasi-identities:*

$$(x \leq yz \wedge x \leq uv) \Longrightarrow x \leq (yu)z \ (6), \quad (x \leq y(zw) \wedge x \leq yv) \Longrightarrow x \leq y(zv) \ (7).$$

(4) *The class* $R\{*\}$ *does not form a variety. A groupoid* $(A,\cdot)$ *belongs to the class* $R\{*\}$ *if and only if it satisfies identities* (1), $(1')$, (2), (3), (4) *and the quasi-identities:*

$$xy = uv \Longrightarrow (xz)y = (uz)v \ (8), \quad xy = uv \Longrightarrow x(zy) = u(zv) \ (9),$$

$$((xy)(zw) = (vt)s \wedge (uv)(ts) = (yz)w) \Longrightarrow (yz)w = (vt)s \ (10),$$

where variables y,z,v,t *in* (10) *can be empty symbols.*

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Localic distances and asymmetric metrization theorems

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The theory of metric locales, introduced in 1984 by A. Pultr [5], is formulated in terms of the notion of metric diameter. Denote by $\mathbb{R}_+^\infty$ the set of all nonnegative reals plus $+\infty$. A *diameter* on a locale L is a mapping $\delta\colon L \to \mathbb{R}_+^\infty$ such that

(D1) $\delta(0) = 0$,

(D2) $a \le b \;\Rightarrow\; \delta(a) \le \delta(b)$,

(D3) $a \wedge b \neq 0 \;\Rightarrow\; \delta(a \vee b) \le \delta(a) + \delta(b)$, and

(D4) for every $p > 0$, the set $U_p^\delta = \{a \in L \mid \delta(a) < p\}$ is a cover of L.

δ is said to be *admissible* if the system of covers $\mathcal{U}(\delta) = \{U_p^\delta \mid p > 0\}$ is admissible, that is,

(D5) $\forall a \in L,\; a = \bigvee\{x \in L \mid U_p^\delta x \le a$ for some $p > 0\}$.

The pair (L, δ) is then referred to as a *metric locale* (or *metric frame*).

This theory has proved very successful in proving metrization theorems for localic uniformities, such as e.g. the Uniform Metrization Theorem stating that a uniformity on a locale is induced by a metric if and only if it has a countable basis.

In this talk, we will consider the long-standing open problem of extending these metrization theorems to asymmetric uniformities (i.e. quasi-uniform locales). This requires an asymmetric approach to metrics, which asks for some notion of a *distance function*.

In 2016, with the aim of studying localic Banach spaces, S. Henry [1] actually introduced a *constructive* theory of metrics on a locale which are defined by a *distance function*. We will revisit Henry's approach and show that distance functions are completely described in terms of the entourages of the locale introduced in [2]; coincidentally, all the needed facts on entourages were already available in [2, 3]. Unlike the approach above based on diameters, here the symmetry condition appears explicitly in the definition. By relaxing it, we can then talk about *quasi-distances*, and our questions about metrization results in quasi-uniform frames can be addressed.

In order to make this idea more precise, let us recall that the localic product of a locale L by itself is given by the frame $L \oplus L$ of all c-ideals of L [4]: a *c-ideal* is a downset E of the cartesian product $L \times L$ (with the partial order given by the order in L) such that $A \times \{b\} \subseteq E \;\Rightarrow\; (\bigvee A, b) \in E$ and $\{a\} \times B \subseteq E \;\Rightarrow\; (a, \bigvee B) \in E$. In particular, every c-ideal contains the set $\mathsf{O} = (\{0\} \times L) \cup (L \times \{0\})$. Intersections of c-ideals are obviously a c-ideal and thus we have a complete lattice, which is indeed a frame. Most importantly, this frame is join-generated by the c-ideals $a \oplus b = {\downarrow}(a, b) \cup \mathsf{O}$. An *entourage* [2] in L is just an element $E \in L \oplus L$ such that $\bigvee\{a \in L \mid a \oplus a \subseteq E\} = 1$. Composition of entourages is given by the operator $E \circ F = \bigvee\{a \oplus c \mid a \oplus b \in E,\; b \oplus c \in F$ for some $b \neq 0\}$.

Now, let $\overleftarrow{\mathbb{R}_+^\infty}$ be the *frame of nonnegative extended upper reals* presented by generators $[0, p)$, with $p \in \mathbb{Q}_+$, subject to the relations

(R1) $[0,p) = \bigvee_{q<p}[0,q)$ for every $p \in \mathbb{Q}_+$.

Further, let $d\colon L \oplus L \to \overleftarrow{\mathbb{R}_+^\infty}$ be a localic map. The corresponding frame homomorphism $d^*\colon \overleftarrow{\mathbb{R}_+^\infty} \to L \oplus L$, left adjoint to d, must be defined on generators by some correspondence

$$[0,p) \overset{d^*}{\longmapsto} \Delta_p^d \in L \oplus L$$

such that $\Delta_p^d = \bigvee_{q<p} \Delta_q^d$. By the adjunction $d^* \dashv d$, $d(E) = \bigvee\{[0,p) \mid \Delta_p^d \subseteq E\}$ for every $E \in L \oplus L$ and $\Delta_p^d = \bigcap\{E \in L \oplus L \mid [0,p) \leq d(E)\}$ for every $p \in \mathbb{Q}_+$.

We write "$d(x \oplus y) < p$" to mean that $x \oplus y \subseteq \Delta_p^d$, that is, $(x,y) \in \Delta_p^d$. Then we have $\Delta_p^d = \bigvee\{x \oplus y \mid d(x \oplus y) < p\}$. We say that d is a *pre-distance* on L whenever

(M1) Δ_p^d is an entourage of L (i.e. $\bigvee\{x \in L \mid d(x \oplus x) < p\} = 1$) for every $p \in \mathbb{Q}_+$,

(M2) Δ_p^d is symmetric (i.e. $x \oplus y \subseteq \Delta_p^d$ iff $y \oplus x \subseteq \Delta_p^d$) for every $p \in \mathbb{Q}_+$, and

(M3) $\Delta_p^d \circ \Delta_q^d \subseteq \Delta_{p+q}^d$ for every $p, q \in \mathbb{Q}_+$.

This implies that for any pre-distance d, $\{\Delta_p^d \mid p \in \mathbb{Q}_+\}$ is a basis of symmetric entourages for a pre-uniformity $\mathcal{E}_d$ on L ([3, 4]). A pre-distance is a *distance* if it is *admissible* on L, that is,

(M4) $\forall a \in L,\ a = \bigvee\{x \in L \mid x \lhd_p^d a \text{ for some } p > 0\}$ (where $x \lhd_p^d a$ means that the join $\bigvee\{y \in L \mid (y,y) \in \Delta_p^d,\ y \wedge x \neq 0\}$ is less than or equal to a).

We then can show, among other results, that:

1. For each metric diameter δ on L, the localic map $d_\delta\colon L \oplus L \to \overleftarrow{\mathbb{R}_+^\infty}$ given by the frame homomorphism $d_\delta^*\colon \overleftarrow{\mathbb{R}_+^\infty} \to L \oplus L$, defined on generators by $[0,p) \longmapsto \Delta_p^\delta = \bigvee\{u \oplus v \mid \delta(u \vee v) < p\}$, is a distance on L.

2. Conversely, for any metric distance $d\colon L \oplus L \to \overleftarrow{\mathbb{R}_+^\infty}$, $\{\Delta_p^d \mid p \in \mathbb{Q}_+\}$ is a basis for an admissible (countable) entourage uniformity $\mathcal{E}_d$ on L. Since it is countable, there is a diameter $\delta_d\colon L \to \mathbb{R}_+^\infty$, given by $\delta_d(u) = \inf\{p \in \mathbb{Q}_+ \mid u \oplus u \subseteq \Delta_p^d\}$, whose associated uniformity is precisely $\mathcal{E}_d$.

3. The correspondences $\delta \mapsto d_\delta$ and $d \mapsto \delta_d$ are inverse to each other.

4. By dropping (M2), we can deal with metric *quasi-distances* and conclude that a quasi-uniformity is induced by a metric quasi-distance if and only if it has a countable basis.

5. Diameters and distances can be extended from elements to sublocales of L.

This talk is based on ongoing joint work with A. Pultr (Charles Univ., Prague) and M. Sioen (Vrije Univ. Brussel).

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Topological and differentiable aspects of Clifford semigroups

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Clifford semigroups represent a fundamental class of inverse semigroups that admit a structural decomposition as strong semilattices of groups.[1] Algebraically, a Clifford semigroup S is the disjoint union of its maximal semigroups $\{G_e\}_{e\in E(S)}$ indexed over the semilattice $E(S)$ of the idempotent elements; moreover, for every pair of idempotents such that $e \geq f$ there exists a homomorphisms $\varphi_{f,e}\colon G_f \to G_e$. S can be endowed with several topologies making the multiplication and inversion continuous. We focus on three topologies on Clifford semigroups: 1) the disjoint union topology yielding a *strong semilattice of topological groups* [9, 10]; 2) the *Yeager topology* [14] making a Clifford semigroup S compact, so as the semilattice $E(S)$ and each topological group G_e (with $e \in E(S)$); 3) *the Bowman topology* [4]: a specialization of the Yeager topology when $E(S)$ is a perfect semilattice (a compact Hausdorff Lawson semilattice [5]).

In this contribution, we explore the interplay between the algebraic nature and additional topological or differentiable constraints on Clifford semigroups. Our investigation focuses on three main pillars: the generalization of Hilbert's Fifth Problem to the semigroup setting, the construction of explicit compatible metrics for the Bowman topology, and the impact of C^1-regularity at idempotents on the global structure of the semigroup.

Hilbert's Fifth Problem for Clifford semigroups

The classical Hilbert's Fifth Problem asks whether a locally Euclidean topological group is a Lie group (see e.g. [12]). We extend this investigation to strong semilattice of topological groups, by shifting the focus to local conditions around the idempotents. We prove that if a topological Clifford semigroup S is *weakly locally Euclidean at the idempotents*, then S is of Lie type, meaning each maximal subgroup G_e is a finite-dimensional Lie group. Furthermore, we show that locally compact Clifford semigroups satisfying the No Small Subgroups (NSS) property are necessarily of Lie type, extending the classical results by Gleason [6] and Yamabe [13] to this broader algebraic context.

An explicit metric inducing the Bowman topology

While existence theorems for metrics on Clifford semigroups are well-known [2, 3], providing an explicit compatible metric is often challenging. We construct an explicit metric d on S (inducing the Bowman topology) by integrating the metric ρ of the semilattice $E(S)$ with the

[1] The construction has been later generalized in Universal algebra to Płonka sums [11].

metrics d_b of the groups G_b through a countable domain basis $\mathcal{B} = \{b_j\}_{j\geq 1}$ of $E(S)$. We further show how to provide a metric under the assumption that all the groups homomorphisms $\varphi_{f,e}$ are isometries.

Differentiability and C^1 structure at idempotents

Finally, we investigate the impact of C^1-regularity. Using a framework due to Holmes [7], we assume that S is a C^1-manifold and the multiplication is a C^1 map. Our main result shows that C^1-regularity at the idempotents forces $E(S)$ to be discrete in S. This implies that any "smooth" Clifford semigroup (in the Banach sense) is structurally a strong semilattice of Lie groups, effectively reducing the study of differentiable Clifford semigroups to the study of discrete collections of Lie groups linked by smooth homomorphisms.

During the presentation, we will also discuss how some of these results can be extended to the broader framework of Płonka sums of arbitrary compact Hausdorff topological algebras. This talk is based on [1] and ongoing work.

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Reasoning in the real world with BL multi-modal logic

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Abstract

Many real-world applications make use of temporal and spatial data. Reasoning with this kind of data is crucial to solve many problems, from monitoring, to scheduling, up to predicting future scenarios. Unsurprisingly, the literature is abundant in multi-modal temporal and spatial logic approaches leveraging modal operators to treat the relations between points in time (*Linear Temporal Logic* [**?**]) and space (*Compass Logic* [**?**, **?**]), or even time intervals (*Halpern and Shoham's Interval Temporal Logic* [**?**]) or areas in an image (*Lutz and Wolter's Logic of Topological Relations* [**?**]).

However, most of the time, these applications live in scenarios characterized by uncertainty and vagueness in the data due to many factors, including sensoring, discretization, and so on. Classically, this problem is tackled using continuous t-norms fuzzy logics such as *Gödel logic* (G) [**?**], *Lukasiewic logic* (Ł) [**?**], and *Product logic* (Π) [**?**].

With the aim of modeling real world scenarios through spatial and temporal data, but also taking uncertainty into account, we employ a many-valued extension for well-known temporal and spatial logics based on Basic Fuzzy Logic (BL) [**?**], introduced by Petr Hájek. In order to do so, we will leverage the notion of a many-valued linear order, allowing for the definition of a many-valued semantics of modal frames.

The framework can be summarized as follows. Let $\mathcal{P}$ be a set of propositional letters, $\mathcal{A}$ a complete BL-algebra, and $X_i, \ldots, X_n$ a set of modalities we are interested in (e.g., relations between points in space or time, or between time intervals). The well-formed formulas of the *Multi-Modal Logic BL-K_n* are obtained by the following grammar:

$$\varphi ::= \alpha \mid p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \langle X_i \rangle \varphi \mid [X_i]\varphi,$$

for $1 \leq i \leq n$, $p \in \mathcal{P}$, and $\alpha \in \mathcal{A}$. Then, given the same set of propositional letters $\mathcal{P}$, the same complete BL-algebra $\mathcal{A}$, and a non-empty set of *worlds* W, we define a *BL-Kripke frame* as an object $\widetilde{F} = \langle W, \widetilde{R}_1 \ldots, \widetilde{R}_n \rangle$, where each $\widetilde{R}_i \colon (W \times W) \longrightarrow \mathcal{A}$ is an *accessibility* function. A *BL-Kripke structure* is a BL-Kripke frame enriched with a *valuation function* $\widetilde{V} \colon (W \times \mathcal{P}) \longrightarrow \mathcal{A}$, and it is denoted by $\widetilde{M} = \langle \widetilde{F}, \widetilde{V} \rangle$. Given a well-formed formula φ, we compute its *value in $\widetilde{M}$ at w*, for some $w \in W$, by extending $\widetilde{V}$ to formulas as follows:

$$\begin{array}{lcl}
\tilde{V}(w, \alpha) & = & \alpha, \\
\tilde{V}(w, \varphi \wedge \psi) & = & \tilde{V}(w, \varphi) \cdot \tilde{V}(w, \psi), \\
\tilde{V}(w, \varphi \vee \psi) & = & \tilde{V}(w, \varphi) \cup \tilde{V}(w, \psi), \\
\tilde{V}(w, \varphi \rightarrow \psi) & = & \tilde{V}(w, \varphi) \hookrightarrow \tilde{V}(w, \psi), \\
\tilde{V}(w, \langle X_i \rangle \varphi) & = & \bigcup \{\widetilde{R}_i(w, s) \cdot \tilde{V}(s, \varphi)\}, \\
\tilde{V}(w, [X_i]\varphi) & = & \bigcap \{\widetilde{R}_i(w, s) \hookrightarrow \tilde{V}(s, \varphi)\}.
\end{array}$$

In multi-modal logic, worlds represent the basic entities we are modeling (e.g., points in time or space, time intervals, or areas in an image), and modalities inquiry on the relations between these entities (i.e., *given a point in time p in which a certain event is ocurring, is there another point in time happening later than p in which another event is ocurring?*).

In our framework, we constraint these entities to be defined over points on one (or more) linear order(s), so that it can be *manified*, modeling points that can be both *one after the other* (and viceversa) and *the same point* at the same time, possibly with different values, allowing to treat uncertainty. This many-valued linear order, or *BL-linear order*, is defined so that the following conditions apply for every 3 points x, y, and z:

$$\begin{array}{l}
\widetilde{=}(x,y) = 1 \text{ iff } x = y,\\
\widetilde{=}(x,y) = \widetilde{=}(y,x),\\
\widetilde{<}(x,x) = 0,\\
\widetilde{<}(x,z) \succeq \widetilde{<}(x,y) \cdot \widetilde{<}(y,z),\\
\text{if } \widetilde{<}(x,y) \succ 0 \text{ and } \widetilde{<}(y,z) \succ 0, \text{ then } \widetilde{<}(x,z) \succ 0,\\
\text{if } \widetilde{<}(x,y) = 0 \text{ and } \widetilde{<}(y,x) = 0, \text{ then } \widetilde{=}(x,y) = 1,\\
\text{if } \widetilde{=}(x,y) \succ 0, \text{ then } \widetilde{<}(x,y) \prec 1.
\end{array}$$

We also present an automated reasoning system, based on analytic tableau technique, allowing to solve classical problems like satisfiability, validity, equivalence and entailment with many-valued temporal and spatial logics compliant with the proposed framework. Furthermore, we define a relaxation of the satisfiability problem, namely α-satisfiability, to inquiry if a model evaluates a formula to a grade greater than a specific value α. This can be formalized as follows. A formula φ of BL-K_n is *α-satisfied* at world w in a BL-Kripke structure $\tilde{M}$ if and only if

$$\tilde{V}(w,\varphi) \succeq \alpha.$$

A formula φ is *α-satisfiable* if and only if there exists a structure and a world in which it is α-satisfied, and it is *satisfiable* if it is α-satisfiable for some $\alpha \in \mathcal{A}$, $\alpha \succ 0$; respectively, a formula is *α-valid* if it is α-satisfied at every world in every model, and *valid* if it is 1-valid.

This framework, together with the described reasoning tool, is also implemented in the Julia programming language as part of an open-source package for representing, reasoning, learning and post-hoc analysis from structured and unstructured data, namely *Sole.jl* [**?**, **?**].

Interpolation in the Inquisitive Hierarchy

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Inquisitive logic is a formal framework developed to study reasoning involving both questions and statements in a uniform setting [6, 3, 4]. Formally speaking, *Basic Inquisitive Logic*, henceforth **InqB**, is obtained by extending Intutitionistic Propositional Logic with each instance of the axiom schema (KP),

$$(KP) \quad (\neg\varphi \to \psi \vee \chi) \to (\neg\varphi \to \psi) \vee (\neg\varphi \to \psi),$$

as well as an instance of $\neg\neg p \to p$ for each $p \in Prop$, and then closing under modus ponens. Crucially, Inquisitive Logic and its extensions are not closed under uniform substitution and therefore do not constitute standard superintuitionistic logics. While Inquisitive Logic has largely been studied in the context of its applications, its mathematical foundations have also received attention. In [1], Bezhanishvili et al. introduced a non-standard algebraic semantics that overcomes the failure of uniform substitution and developed a duality theory that connected their semantics to the more typically used information-state based semantics, originally presented in [6].

One of the most notable results regarding Inquisitive Logic to come out of this line of research was that the lattice of extensions of Inquisitive Logic $\mathscr{E}(\mathbf{InqB})$, dubbed the Inquisitive Hierarchy [6], is dually isomorphic to $\omega+1$ [2]. Despite the finding that $\mathscr{E}(\mathbf{InqB})$ is dually isomorphic to $\omega+1$, and some results from the early days of inquisitive logic [6], little is known about $\mathscr{E}(\mathbf{InqB})$. In particular, there is little understanding of the distribution of commonly studied metalogical properties within this hierarchy. In this report on work in progress, we contribute to filling this gap by providing a complete characterization of the logics in $\mathscr{E}(\mathbf{InqB})$ possessing the Craig interpolation property, showing that—quite surprisingly—all axiomatic extensions of **InqB** have the Craig interpolation property. This further contributes to past work on interpolation in inquisitive logic in [5, 7].

The Craig interpolation property (CIP) and its variants are some of the most thoroughly studied properties of logical systems [8]. For logics with an implication connective $\to$, the CIP can be stated as follows:

> A logic L has CIP if whenever $\to \psi$ is a theorem of L, then there is a formula χ such that $\to \chi$ and $\chi \to \psi$ are both theorems of L and the variables occurring in χ are among the variables that occur in both and ψ.

For standard superintuitionistic logics, a classical result proved by L. Maksimova states that there are exactly eight (substitution invariant) extensions of Intuitionisitic Propositional Logic with the CIP [11]. She showed these results by first establishing the now well-known bridge theorem associating Craig interpolation for a superintuitionistic logic to the amalgamation property for its corresponding class of Heyting algebras.[1] Bridges between various amalgamation properties and associated interpolation properties are now among the most prominent methodologies for studying interpolation [9, 10].

However, the fact that logics in $\mathscr{E}(\mathbf{InqB})$ are not algebraizable in the usual sense presents a significant obstacle in applying the usual repertoire of techniques. The algebraic semantics of logics in

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[1] A class of algebras $\mathscr{K}$ has the *amalgamation property* (AP) if for all $A,B,C \in \mathscr{K}$, if $f_B : A \to B$ and $f_C : A \to C$ are injective homomorphisms, then there is an algebra $D \in \mathscr{K}$ and injective homomorphisms $g_B : B \to D$ and $g_C : C \to D$ such that $g_B \circ f_B = g_C \circ f_C$.

$\mathscr{E}(\mathbf{InqB})$ are defined with respect to KP-algebras, which are Heyting algebras that additionally satisfy $\top = (\neg a \to b \vee c) \to ((\neg a \to b) \vee (\neg a \to c))$. The complication arises because, in inquisitive logics, valuations $v\colon Prop \to H$ from propositional letters to a KP-algebra H are required to satisfy $v(p) = \neg\neg v(p)$. It was shown in [2] that this restriction on admissible valuations leads to a dual isomorphism between $\mathscr{E}(\mathbf{InqB})$ and the lattice $\Lambda^{Inq}(KP)$ of what we call *inquisitive-varieties*, i.e. varieties of KP-algebras $\mathscr{X}$ that are additionally closed under the operation $\mathscr{X}^{\uparrow} = \{H \mid \exists A \in \mathscr{X}(H_\neg = A_\neg \ \& \ A \leq H)\}$. Our first contribution is to extend the usual bridge theorem to this setting:

Theorem 1. *A logic* $\mathbf{L}$ *in* $\mathscr{E}(\mathbf{InqB})$ *has the Craig interpolation property if and only if the class of regularly generated algebras in* $V_{Inq}(\mathbf{L})$ *has the amalgamation property.*

Here $V_{Inq}(\mathbf{L})$ is the inquisitive variety of the logic $\mathbf{L} \in \mathscr{E}(\mathbf{InqB})$ and a *regularly generated Heyting algebra* is a Heyting algebra H that is generated by the set of its regular elements, $H_\neg = \{a \in H \mid \neg\neg a = a\}$.

In addition to the aforementioned bridge theorem, the second key ingredient to obtaining our main result consist in certain technical lemmas relating amalgamation in inquisitive varieties to categorical properties of Boolean algebras. Combining these results with the bridge theorem previously mentioned yields the main result of this study:

Theorem 2. *Every logic in* $\mathscr{E}(\mathbf{InqB})$ *has the Craig Interpolation Property.*

Beyond its significance to the study of inquisitive logics, we anticipate that this work will prove an early step in the study of interpolation in logics that are not invariant under substitution; see [12].

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Algebras and general frames for interpretability logics

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Abstract

Interpretability logics are a family of logics that extend the modal provability logic GL with an additional binary modal operator $\rhd$. The aim of interpretability logics is to formalize the notion of relative interpretability among arithmetical theories. The basic interpretability logic is the system **IL**, introduced by Visser [**?**], from which various extensions are obtained by adding axiom schemata. The standard relational semantics for interpretability logics, Veltman semantics, lacks some properties that we would like it to have. For example, the system **IL** is complete [**?**], but it is not strongly complete with respect to the class of all Veltman frames (for the exact same reason as GL). Moreover, some extensions of the system **IL** are incomplete with respect to the class of frames they define [**?**].

In this talk, we will define general frames for interpretability logics, somewhat analogously to general frames for modal logics [**?**]. Unlike the modal case, where general frames are defined on Kripke frames, we define the structure of *quasi-Veltman frames*, a triple $(W, R, \{S_w : w \in W\})$, where R is a binary relation on W which is irreflexive and transitive, and for every $w \in W$, S_w is a binary relation on $R[w]$ which is reflexive and transitive. This structure weakens the notion of Veltman frame, but it suitable for the definition of *general **IL**-frame*, which is a tuple $(W, R, \{S_w : w \in W\}, A)$, where $(W, R, \{S_w : w \in W\})$ is a quasi-Veltman frame and $A \subseteq \mathcal{P}(W)$ is a set od *admissible subsets* which is

1. closed under taking unions,
2. closed under taking complements,
3. closed under oprator $m_{\rhd}$, where

$$m_{\rhd}(X, Y) = \{w \in W : \forall u \in X \ (wRu \to \exists v \in Y \ (uS_wv))\},$$

such that the following holds

$$m_{\Box}((W \setminus m_{\Box}(X)) \cup X) = m_{\Box}(X),$$

where $m_{\Box}(X)$ is shorthand for $m_{\rhd}(W \setminus X, \emptyset)$. From general frames we obtain models by taking *admissible valuations*, $V : \mathrm{Prop} \to A$ (as opposed to $V : \mathrm{Prop} \to \mathcal{P}(W)$ with Veltman frames).

When studying this semantics, strong completeness theorems for **IL** and for its extensions hold. This may be proven using canonical model, which cannot be done for Veltman semantics, but can be done using maximally consistent sets and by modifnying the construction from [**?**] fo general **IL**-frames. We also prove that general **IL**-frames include Veltman frames as a special case and, therefore, generalize the existing semantics (similarly to general frames for modal logic).

Afterwards, we turn towards algebraic semantics by adding operators to Boolean algebras, which correspond to the modal operators in the language of interpretability logics (this is done in an analogous way to the modal case [**?**]), thus obtaining the following

$$(A, +, -, 0, f_{\rhd}),$$

where $(A, +, -, 0)$ is a Boolean algebra and $f_{\rhd}$ an operator satisfying the following conditions:

1. $f_\Diamond(0) = 0$,
2. $f_\Box(-f_\Box(x) + x) = f_\Box(x)$,
3. $f_\rhd(x, z) \cdot f_\rhd(y, z) = f_\rhd(x + y, z)$,
4. $-f_\Box(-x + y) + f_\rhd(x, y) = 1$
5. $-f_\rhd(x, y) + (-f_\rhd(y, z)) + f_\rhd(x, z) = 1$,
6. $-f_\rhd(x, y) + f_\Box(-x) + f_\Diamond(y) = 1$,
7. $f_\rhd(f_\Diamond(x), x) = 1$.

Above, $f_\Box(x)$ is shorthand for $f_\rhd(-x, 0)$ and $f_\Diamond(x)$ is shorthand for $-f_\Box(-x)$. These structures are called *interpretability algebras*. System **IL** and its extensions are complete with respect to the corresponding class of algebras, which is proven in [**?**]. Finally, we show how a general **IL**-frame $\mathcal{F}$ can be associated with an interpretability algebra $\mathcal{F}^*$ and, conversely, how an interpretability algebra $\mathcal{A}$ can be associated with a general frame $\mathcal{A}_*$, in such a way that the validity of formulas is preserved. This is done by utilizing both, construction of general frame associated with an algebra from [**?**], which uses ultrafilters and construction of canonical model. Furthermore, it may be proven that

$$(\mathcal{A}_*)^* \simeq \mathcal{A},$$

which is the main result and means that every interpretability algebra (up to isomorphism) arises from a general **IL**-frame (and one such frame is given by the ultrafilter construction).

This construction is, however, not sufficient to show results analogous to Jónsson-Tarski duality, due to the nature of the construction. These problems will be briefly outlined.

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Closure ℓ-monoids and their proof theory*

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A *closure ℓ-monoid* is an algebra $\mathbf{A} = \langle A, \cdot, \wedge, \vee, \Diamond, \boxminus, 1, \bot, \top\rangle$ such that $\langle A, \wedge, \vee, \bot, \top\rangle$ is a bounded distributive lattice, $\langle A, \cdot, \wedge, \vee, 1\rangle$ is an ℓ-monoid, $\bot$ is an absorbing element for monoid multiplication, and $\Diamond, \boxminus$ are unary operations axiomatised by the following equations:

(1) $x \leq \Diamond x$ (3) $\Diamond(x \vee y) \approx \Diamond x \vee \Diamond y$ (5) $\Diamond(x \cdot y) \leq \Diamond x \cdot \Diamond y$ (7) $x \leq \boxminus\Diamond x$

(2) $\Diamond\Diamond x \approx \Diamond x$ (4) $\boxminus(x \wedge y) \approx \boxminus x \wedge \boxminus y$ (6) $\Diamond\boxminus x \leq x$

Closure ℓ-monoids thus form a variety, denoted by $\mathfrak{LMC}$. It is straightforward to verify that, in any such algebra, $\langle \Diamond, \boxminus\rangle$ is a residuated pair with respect to the lattice order, $\Diamond$ is a closure operator, $\boxminus$ is a weak conucleus [2], and both operators preserve the lattice bounds. Closure ℓ-monoids can be naturally constructed as follows. Let $\langle \mathbf{M}, \preceq\rangle$ (with $\mathbf{M} = \langle M, \cdot, 1\rangle$) be a monoid equipped with a preorder satisfying the following variant of the Riesz Decomposition Property for ℓ-groups [1]:

$$\text{for all } a, b_1, b_2 \in M,\ a \preceq b_1 \cdot b_2 \text{ implies } a = a_1 \cdot a_2, \text{ with } a_1 \preceq b_1 \text{ and } a_2 \preceq b_2 \qquad (*)$$

Consider the inverse image operator $\Diamond_{\preceq}$ associated with $\preceq$ and its residual $\boxminus_{\preceq}$ (with respect to set-theoretic inclusion). These are unary operations on $\wp(M)$ defined as follows, for $A \subseteq M$:

$$\Diamond_{\preceq} A := \{b \mid \exists a(b \preceq a \curlywedge a \in A)\} \qquad \boxminus_{\preceq} A := \{b \mid \forall a(a \preceq b \Rightarrow a \in A)\}$$

Then the algebra $\wp(\mathbf{M})_+$ expanding the bounded ℓ-monoid $\wp(\mathbf{M}) = \langle \wp(M), \cdot, \cap, \cup, \{1\}, \emptyset, M\rangle$ (where subset multiplication is defined as usual) with $\Diamond_{\preceq}$ and $\boxminus_{\preceq}$ is a closure ℓ-monoid, with $(*)$ being the frame condition corresponding to equation (5).

In this talk, closure ℓ-monoids are examined in greater detail, building on the preceding considerations. Next, a Gentzen-style calculus for these algebras, called LMC, is introduced. LMC is a display-like, single-conclusion sequent system based on the division-free fragment of the Distributive Full Lambek Calculus [6, 8, 13], which includes monoid- and meet-semilattice-like structural operators. In particular, the structural meet enables the derivation of the distributivity laws for the lattice operations—an idea independently pioneered by Dunn [3] and Minc [11] in their work on the positive relevant logic R^+—and comes with a *global contraction rule*: contraction applies to complex structural terms rather than merely to formulas. In addition to this machinery, the syntax of LMC also includes a structural diamond, whose rules—together with those for the modal connectives—are taken from Moortgat's system NL($\Diamond$) [12]. The modal rules of LMC are displayed in Figure 1, where $\circ$ and $\langle - \rangle$ are, respectively, the structural multiplication and diamond. The rules K, T, and 4 govern the behaviour of the structural diamond: T and 4 are the counterparts of equations (1) and (2), respectively, while K describes the interaction between $\langle - \rangle$ and $\circ$, capturing equation (5). As for the remaining logical rules, $\Diamond$L converts the structural diamond into the corresponding logical connective, $\Diamond$R expresses the isotonicity of the diamond operator with respect to the lattice order, $\boxminus$L echoes equation (6), and $\boxminus$R reflects the residuation law. LMC is shown to be sound and complete for the variety $\mathfrak{LMC}$ and to admit cut elimination. The latter result is obtained as a corollary of a mix

*This abstract is based on selected results from the PhD thesis [4], including material presented in the preprint [5], co-authored with Alessandro Aldini and available on arXiv under the CC BY 4.0 International licence.

$$\frac{\Gamma\{\langle\Delta_1\rangle \circ \langle\Delta_2\rangle\} \succ \varphi}{\Gamma\{\langle\Delta_1 \circ \Delta_2\rangle\} \succ \varphi}\,\mathsf{K} \qquad \frac{\Gamma\{\langle\Delta\rangle\} \succ \varphi}{\Gamma\{\Delta\} \succ \varphi}\,\mathsf{T} \qquad \frac{\Gamma\{\langle\Delta\rangle\} \succ \varphi}{\Gamma\{\langle\langle\Delta\rangle\rangle\} \succ \varphi}\,\mathsf{4}$$

$$\frac{\Gamma\{\langle\varphi\rangle\} \succ \psi}{\Gamma\{\Diamond\varphi\} \succ \psi}\,\Diamond\mathsf{L} \qquad \frac{\Gamma \succ \varphi}{\langle\Gamma\rangle \succ \Diamond\varphi}\,\Diamond\mathsf{R} \qquad \frac{\Gamma\{\varphi\} \succ \psi}{\Gamma\{\langle\boxdot\varphi\rangle\} \succ \psi}\,\boxdot\mathsf{L} \qquad \frac{\langle\Gamma\rangle \succ \varphi}{\Gamma \succ \boxdot\varphi}\,\boxdot\mathsf{R}$$

Figure 1: Modal rules of LMC.

elimination theorem, established by adapting a technique originally devised for hypersequent calculi for fuzzy logics [9] and later employed in the cut elimination proof for the hypersequent calculus CSemFL for commutative, semilinear pointed residuated lattices [10]. Decidability of LMC is established by adapting Gentzen's classical argument for LK [7]. The proof combines cut elimination with normalisation procedures controlling backward applications of T and contraction. Finally, turning to applications, closure ℓ-monoids and LMC are shown to provide an algebraic/proof-theoretic framework for the structural analysis of finite-trace properties in concurrent systems, thereby allowing for a compositional account of safety, liveness, and related notions.

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Duality for quasi-Nelson algebras with recovery operators

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We develop a two-sorted Priestley duality for quasi-Nelson algebras enriched with recovery operators, corresponding to twist-algebras built from Heyting algebra factors enriched with a dual pseudo-complement operation. Our work joins a number of approaches that ultimately originate from H. Priestley's work on the representation of distributive lattices, namely: (i) Priestley's original duality, in particular its restriction to Heyting algebras and its extension (also due to Priestley) to pseudo-complemented lattices [7]; (ii) G. Bezhanishvili and R. Jansana's duality for distributive meet-semilattices [1, 2]; (iii) the framework for two-sorted dualities recently proposed by A. Jung and U. Rivieccio [9].

Quasi-Nelson algebras form a variety (QN), introduced in [10] and further investigated in a number of subsequent publications. The interest in quasi-Nelson algebras and their logic is manifold, and can be motivated from a number of perspectives, such as constructive logic, residuated lattices, order theory, and universal algebra. From a logical point of view, the quasi-Nelson formalism may be viewed as a framework that encompasses two outstanding constructive traditions: intuitionistic logic on the one hand, and on the other Nelson's logic with strong negation [6]. This means that one may take quasi-Nelson as a basic logical formalism from which both intuitionistic and Nelson logic can be obtained as axiomatic extensions.

Alternatively, one may view quasi-Nelson logic/algebras as one of the formalisms in the substructural family. Formally, quasi-Nelson algebras are precisely the commutative, integral, bounded (but not necessarily involutive) residuated lattices that satisfy the Nelson equation:

$$(x \Rightarrow (x \Rightarrow y)) \wedge ({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x)) \leq x \Rightarrow y \qquad \text{(Nelson)}$$

This latter point of view connects more directly with the recent papers [8, 4, 5], which investigate the effect of adding *recovery operators* to quasi-Nelson algebras, extending the approach of [3] beyond the involutive setting. These operators arise in the setting of *Logics of Formal Inconsistency* and *Formal Undeterminedness*, which are non-classical systems that do not validate two well-known principles of classical logic: respectively, the *principle of explosion* ($\varphi, {\sim}\varphi \vdash \bot$) and the *excluded middle* ($\vdash \varphi \vee {\sim}\varphi$). The main formal peculiarity of these logics is that both the above-mentioned principles are reintroduced in a "more gentle" form, precisely via the recovery operators. The unary *consistency* operator $\circ$ (we will also just call it "white operator") allows one to recover a more controlled form of explosion via the rule: $\varphi, {\sim}\varphi, \circ\varphi \vdash \bot$. Likewise, a "gentle excluded middle" is recovered via a black (*undeterminedness*) operator $\bullet$ that is required to satisfy the axiom: $\vdash \varphi \vee {\sim}\varphi \vee \bullet\varphi$. From these two basic operators and the negation, two more can be defined that express *inconsistency* (${\sim}\circ$) and *determinedness* (${\sim}\bullet$).

On algebraic models, recovery operators display an unusual behaviour in that, for instance, they are neither monotone nor antitone with respect to the natural order. However, following a seminal intuition from [3] and imposing some further restrictions, we may obtain straightforward

*Speaker.

characterizations in terms both of the lattice structure of the underlying algebras and of the twist representation (see the next section for the details). These restrictions amount to rules such as:

$$\frac{\sim(\varphi \wedge \sim\varphi \wedge \psi)}{\psi \Rightarrow \circ\varphi} \qquad \frac{\varphi \vee \neg\varphi \vee \psi}{\bullet\varphi \Rightarrow \psi}.$$

Such constraints determine the connectives dubbed in [4, 5], respectively, max-*consistency* and min-*undeterminedness* operator; further restrictions can be imposed to obtain operators satisfying alternative desiderata. Here, we focus on the algebraic models of logics obtained by adding the two basic operators – *quasi-Nelson algebras with recovery operators* – and in particular on their dual representation via Pristley-style duality.

Our duality is an instance of an approach developed in a more general setting in [9], essentially based on the availability, for the algebras of interest, of some structural representation that allows one to view them as *two-sorted* entities (in a sense that matches the standard one of many-sorted universal algebra). For quasi-Nelson algebras, the result we can rely on is the well-known twist representation, by which a quasi-Nelson algebra is decomposed into a pair of Heyting algebras connected by two maps. The duality obtained via this strategy is a (maybe deceivingly) simple one, which will result by seamlessly joining together the available results on the relevant algebras and operators (Heyting algebras, two-sorted lattices, meet-preserving maps, dual pseudo-complements). Considering the present contribution as a preliminary study (indeed the first, to our knowledge) of recovery operators from a dual perspective, we take this as a good sign as well as an indication that promising work lies further ahead.

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Sequent Calculi for Semiring Semantics

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Semiring provenance, proposed in 2007 by Green, Kravounarakis and Tannen [**?**] was introduced as a comprehensive framework to capture different forms of annotated databases. The central idea is to annotate the atomic facts in a database by the values in a naturally ordered commutative semiring, then extending these to database queries using the operations of the semiring. This defines the *positive logic* of a given semiring. More recently, Tannen and Grädel [**?**] proposed a new approach, *semiring semantics* which incorporates negative information by annotating literals subject to the *model-defining* restriction - exactly one of each literal pair takes value 0.

Definition A (commutative naturally ordered) semiring is an algebra $\mathcal{S} = (S, +, \cdot, 0, 1)$ with $0 \neq 1$ such that:

$(S, +, 0)$ and $(S, \cdot 1)$ are commutative monoids. $\qquad$ $\cdot$ distributes over $+$

The relation $s \leq t :\Leftrightarrow \exists r : s + r = t$ is a partial order. $\qquad$ $\forall s \in S\ 0 \cdot s = 0$

This has become a recent thriving research area whose major focus is the extent to which classical model theory extends to semiring semantics. However, the proof theory has received less attention with little understanding of appropriate proof systems for the majority of the main semirings. The lack of a defined implication in semirings points towards *sequent calculi* as a natural form for potential proof systems. Such calculi hold particular importance to *substructural logics*, effectively defining these non-classical logics as a group [**?**]. Semantically substructural logics are the logics whose algebraic models are *residuated lattices*; these in turn are closely linked to semirings as algebraic structures. Every residuated lattice has a semiring reduct and many of the main application semirings can be expanded to residuated lattices - often one well studied in the literature.

Absorptive polynomial ring $\mathbb{S}[X] \mapsto \mathbb{S}[X]^{\rightarrow} \in FL_{ew}$.

Min-max semiring $\mathbb{F} = ([0,1], max, min, 0, 1) \mapsto [0,1]_{\mathbb{G}}$ Standard Gödel algebra.

This opens up the potential for an effective development of sequent calculi for semirings via known results for their corresponding residuated lattice. In this talk, we present the early stages of just such an investigation focusing on the most straightforward technique - the *fragmentary approach.* For those semirings with a residuated lattice expansion, we may obtain a calculus for its positive logic by taking a fragment of an available calculus for its corresponding residuated lattice. Whether this is possible relies on having either a standard completeness result for the corresponding latttice or the semiring having a kind of freeness property. Moreover, when the original calculus has the *subformula property* we can can take the exact fragment as our calculus. The prototypical example of this approach is the absorptive polynomial semiring $\mathbb{S}[X]$. $\mathbb{S}[X]$ is the free absorptive semiring, where absorption is the requirement that $s + st = s$. It can be expanded with an implication connective; the resulting residuated lattice belongs to the variety of FL_{ew}-algebras. Moreover, the $[\vee, \cdot]$-reduct of any FL_{ew}-algebra is an absorptive semiring. A well known sequent calculi, here denoted FL for FL_{ew}-algebras comes from Ono [**?**]. This has the subformula property due to enjoying cut elimination, accordingly, the $[\vee, \cdot]$-fragment of this calculus provides a sound and complete proof system for positive $\mathbb{S}[X]$.

Notably, a calculus for the positive logics based on $\mathbb{S}[X]$ was already known which is equivalent to the $[\vee, \cdot]$-fragment of Ono's calculus for FL_{ew}. Our first new contributio is extending the fragment calculus for the positive logic to capture full semiring semantics complete with literals

Initial sequents

$$\phi \Rightarrow \phi \text{ (Id)} \qquad p, \neg p \Rightarrow \phi \ (\neg \Rightarrow) \qquad 0 \Rightarrow \phi \text{ (0)} \qquad \varnothing \Rightarrow 1 \text{ (1)}.$$

Structural rules

$$\frac{\Gamma \Rightarrow \varphi \qquad \varphi, \Pi \Rightarrow \gamma}{\Gamma, \Pi \Rightarrow \gamma} \text{ (Cut)} \qquad \frac{\Gamma, \Rightarrow \gamma}{\phi, \Gamma \Rightarrow \gamma} \text{ (w} \Rightarrow\text{)} \qquad \frac{\Gamma, \phi, \psi, \Pi \Rightarrow \gamma}{\Gamma, \psi, \phi, \Pi \Rightarrow \gamma} \text{ (e} \Rightarrow\text{)}$$

Logical rules

$$\frac{\phi, \Gamma \Rightarrow \gamma \qquad \psi, \Gamma \Rightarrow \gamma}{\phi \vee \psi, \Gamma \Rightarrow \gamma} \ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \vee \psi} \ (\Rightarrow \vee_1) \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \vee \psi} \ (\Rightarrow \vee_2)$$

$$\frac{\phi, \psi, \Gamma \Rightarrow \gamma}{\phi \cdot \psi, \Gamma \Rightarrow \gamma} \ (\cdot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \phi \qquad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \phi \cdot \psi} \ (\Rightarrow \cdot)$$

Figure 1: Sequent calculus for Absorptive Provenance Semiring

and model-defining negation. In the case of $\mathbb{S}[X]$ we can add a single initial sequent governing the behaviour of literal pairs on the left giving the calculus in figure 1 which we denote $FL[Lit]$. Completeness is checked directly.

Lemma (Sequent Calculus for $\mathbb{S}[X]$) For all sequents $\Gamma \Rightarrow \delta$ consisting of classical formulas in negation normal form $\vdash_{FL[Lit]} \Gamma \Rightarrow \delta$ iff $\Gamma \models_{\mathbb{S}[X]} \delta$.

For other semirings with a residuated lattice expansion the fragmentary approach cannot be applied so readily. On the easier side is the fuzzy semiring $\mathbb{F}$ whose corresponding residuated lattice is the standard Gödel algebra. In this case, whilst one can utilise available calculi for the Gödel algebra this results in a significantly overcomplicated system. Instead one can demonstrate semantically that the positive logic of $\mathbb{F}$, and indeed any semiring which interprets disjunction and conjunction as the usual lattice connectives, coincides with the positive fragment of classical logic. From there, one can extend to the full logic with negation using the same initial sequent addressing literal pairs on the left.

On the more difficult side are the Viterbi semiring $\mathbb{V}$ and the Łukasiewicz semiring $\mathbb{L}$. These can be expanded with an implication giving the standard product algebra and standard Łukasiewicz algebra respectively. Whilst calculi are available for both these residuated lattices they are non-standard; they both make some essential use of the implication connective and utilise an alternative definition of sequent validity which yields the familiar definition only for single-conclusion sequents. Nevertheless, we can account for these restrictions and define sound and complete calculi for the positive logics of the semirings. Handling negation however is more subtle, and in the remainder of the talk we discuss alternative approaches to develop calculi for these semirings that aim to incorperate negation effectively.

Automata in bicategories, with an eye on Probabilistic Metrics

Fosco Loregian

March 6, 2026

Abstract

This talk fits in a line of research [BLLL23b, BLLL23a, BFL$^+$23, Lor25] that studies abstract state machines through the lens of category theory. The subject has a long history, and for the last few years I have been busy reviving and extending the line of work sketched in [Gog72, Gog73, BK81, BK82, KR90, Bai72, Gui78] in terms of more sophisticated categorical concepts. In this presentation I will put particular emphasis on bi- and 2-categorical structures arising from *monoidal topology* [CT03, HST14], built from locally posetal bicategories of (T, V)-relations; these bicategories provide a common categorical setting for order-theoretic, topological, and metric structures by varying the two parameters (roughly speaking, a monad T on Set, and a quantale V). The goal is to develop a categorical framework in which (Mealy) automata are interpreted not merely in monoidal categories but in *bicategories* of generalized relations. This perspective recovers usual monoidal automata as particular instances, when the ambient bicategory has a single 0-cell, and it might provide a rich structural setting capable of capturing nondeterministic, topological, and metric aspects of computation within a unified, elegant mathematical language.

A central question in this line of work is whether a fragment of formal category theory can play a foundational role for ASM comparable to the role of functorial semantics in universal algebra or the role of topos theory in foundations of mathematics. Classical observations that categories of automata correspond to models of a simple Cat-enriched sketch support the idea that automata admit a neat, systematic (2-)algebraic description. Bicategorical methods allow this perspective to be refined by encoding computational processes as higher-dimensional categorical structures.

Another guiding principle is that *nondeterminism* can be understood as a lifting to a Kleisli category: in the classical setting, nondeterministic automata arise by passing to the Kleisli category of the powerset monad. Analogously, in a 2-dimensional context one should observe how the passage from (T, V)-categories to more general (T, V)-relations accounts for a notion of nondeterminism (with a topological, metric, or order-theoretic flavour automatically hardcoded).

As an instance of category of (T, V)-relations, I plan to study *probabilistic metric spaces*. Unlike classical metric spaces, PMSs assign to each pair of points a probability *distribution* describing the likelihood that their distance is bounded by a given value. These structures, originally introduced by Menger [Men42] and further developed using triangular norms [Wal43, SS83], provide a flexible generalization of usual metric geometry.

Monoidal topology offers a natural categorical interpretation of these spaces, the definition of which proved to be quite elusive. The interval $[0, 1]$ and $[0, \infty]$ are anti-isomorphic via an exponential–logarithmic correspondence. Using this identification, probabilistic metric spaces can be described as categories enriched in $[0, 1]$-enriched presheaves over $[0, \infty]$.

The general language of automata valued in a bicategory can then be applied to this specific instance of (T, V)-categories and (T, V)-relations.

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Double equivalential algebras

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Abstract

We study double equivalential algebras, defined as subreducts of double Heyting algebras with respect to the operations of equivalence and dual equivalence. We develop their basic properties and study the subvariety generated by the three-element chain, proving that it is semisimple.

1 Introduction

Double Heyting algebras are Heyting algebras whose dual lattices also form Heyting algebras. In these structures, in addition to the lattice operations and the constants 0 and 1, there are two further binary operations interpreted as relative pseudocomplements. Double Heyting algebras first appeared in the early 1970s [4, 5] and were later studied, among others, by R. Beazer, S. Ghilardi, L. Iturrioz and P. Köhler. In recent years, double Heyting algebras have been revisited in the doctoral dissertations of Taylor and Martins ([7], [2]), and more recently by Bezhanishvili, Martins and Moraschini [1].

The aim of this talk is to present basic properties of another class of subreducts, namely double equivalential algebras, i.e. algebras equipped with two binary operations: one corresponding to equivalence and the other to dual equivalence. These structures are $(\cdot, \circ)$-subreducts of double Heyting algebras, where $x \cdot y = (x \to y) \wedge (y \to x)$ and $x \circ y = (x - y) \vee (y - x)$, and where $-$ denotes the dual (co-Heyting) relative pseudocomplement.

The study of these structures presents several challenges. In particular, they are not Fregean, since they are not congruence orderable. Moreover, the variety of double equivalential algebras is not locally finite. For this reason, in what follows we consider the subvariety of double equivalential algebras generated by finite chains.

2 Definition and main results

Let $\mathbf{A} = (A, \cdot, \circ, 1, 0)$ be a $(\cdot, \circ)$-subreduct of a double Heyting algebra, that is, $\mathbf{A}$ is such that $(A, \cdot)$ is an equivalential algebra and $(A, \circ)$ is a dual equivalential algebra. Then $\mathbf{A}$ is called a **double equivalential algebra**. We denote the class of double equivalential algebras by $\mathcal{DE}$. This class is congruence-permutable but not congruence-distributive. Moreover, $\mathcal{DE}$ forms a quasivariety. Note that if $\mathbf{A} = (A, \cdot, \circ, 1, 0) \in \mathcal{DE}$, then $(A, \cdot, 0)$ is an equivalential algebra with zero and $(A, \circ, 1)$ is a dual equivalential algebra with zero (see [6, p. 2]). Moreover, $1 = x \cdot x$ and $0 = x \circ x$ for all $x \in A$. We introduce the following notation: $d(x) := (x \circ 1) \cdot 0$ and $g(x) := (x \cdot 0) \circ 1$.

We have the following theorem (these and the following results come from [3]):

Theorem 1. *Let $\mathbf{A} \in \mathcal{DE}$ and let $a, b \in A$. Then:*

1. $((a \circ 1) \cdot a) \cdot a = a \circ 1, \quad ((a \cdot 0) \circ a) \circ a = a \cdot 0,$
2. $(a \cdot (a \circ 1)) \cdot (a \circ 1) = a, \quad (a \circ (a \cdot 0)) \circ (a \cdot 0) = a,$

3. $((a \cdot 0) \circ a) \cdot 0 = 0, \quad ((a \circ 1) \cdot a) \circ 1 = 1,$

4. $((a \cdot 0) \circ a) \cdot a = (a \cdot 0) \cdot 0, \quad ((a \circ 1) \cdot a) \circ a = (a \circ 1) \circ 1,$

5. $(a \cdot 0) \cdot (a \circ 1) = a \cdot d(a), \quad (a \cdot 0) \circ (a \circ 1) = a \circ g(a).$

The following theorem describes how congruences behave for algebras in $\mathcal{DE}$.

Theorem 2. *Let* $\mathbf{A} \in \mathcal{DE}$*, and let* $\varphi \in \operatorname{Con} \mathbf{A}$*. Then:*

1. $x \cdot y \in 1/\varphi$ *iff* $x \circ y \in 0/\varphi$ *iff* $(x \circ y) \cdot 0 \in 1/\varphi$,
2. $x \in 1/\varphi$ *iff* $d(x) \in 1/\varphi$,
3. $0/\varphi = \{x \in A : x \cdot 0 \in 1/\varphi\} = \{x \circ y : x, y \in 1/\varphi\}$.

2.1 V(3)

Let $n \in \mathbb{N}$. We denote by $\mathbf{n}$ the $(\cdot, \circ)$-reduct of an n-element double Heyting algebra whose underlying set forms a chain.

Remark 3. If $\mathbf{A} \in V(\mathbf{n})$, then the following identities hold:

1. $(x \circ y) \cdot 0 = ((x \cdot y) \circ 1) \cdot 0$,
2. $(x \cdot y) \circ 1 = ((x \circ y) \cdot 0) \circ 1$.

Using the Foster–Pixley theorem, we prove the following theorem:

Theorem 4. *Let* $\mathbf{A} \in V(\mathbf{3})$ *with* $|A| < \infty$*. Then* $\mathbf{A} \in \mathrm{IP}(\mathbf{3}, \mathbf{2}, \mathbf{2}_{0=1})$.

From the above theorem and Quackenbush's theorem, we obtain the following corollary:

Corollary 5. *The only subdirectly irreducible algebras in* $V(\mathbf{3})$ *are* $\mathbf{3}$*,* $\mathbf{2}$*, and* $\mathbf{2}_{0=1}$.

Finally, we obtain the following theorem.

Theorem 6. *The variety* $V(\mathbf{3})$ *is semisimple.*

The final part of the talk will be the construction of the 1-generated free algebra:

Example 7. For every $n \geq 3$, the 1-generated free algebra in the variety $V(\mathbf{n})$ is (up to isomorphism) the $(\cdot, \circ)$-reduct of the double Heyting algebra $\mathbf{3} \times \mathbf{2}^2$.

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Coherent and ideal actions in ideally exact categories with an application to varieties of universal algebras

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Abstract

The notion of action is pervasive in algebra, as it formalises how algebraic structures act on and interact with other objects.

A central role is played by *internal actions*, where both the acting and the acted object belong to the same category. For instance, the categorical description of a group action as a functor

$$\alpha\colon G(*) \to \mathsf{Set},$$

where $G(*)$ is the group G viewed as a one-object groupoid and $\alpha(*) = X$, is equivalent to a group homomorphism

$$G \to \mathrm{Aut}(X),$$

with $\mathrm{Aut}(X)$ denoting the group of automorphisms of X.

In [?], the authors introduced the notion of internal action in semi-abelian categories [?]: an internal action of an object B on an object X is described as an algebra on X for a specified monad

$$B\flat(-)\colon \mathsf{C} \to \mathsf{C}.$$

In this talk, we introduce the notions of internal *coherent* action and internal *ideal* action, defined in [?] to generalise internal actions to the context of ideally exact categories [?].

Definition 1. [?] Let U be an ideally exact category, and let $F \dashv U$ be a monadic adjunction $\mathsf{V} \to \mathsf{U}$ with cartesian unit, where V is semi-abelian. Let moreover B be an object of U and X an object of V. A *relative U-action* $\xi\colon U(B)\flat X \to X$ is said to be:

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1. *Coherent* if $\xi \circ (U(\iota_B) \flat \, \mathrm{id}_X) = \xi_0$, where $\xi_0 \colon UF(0) \flat X \to X$ is the relative U-action associated with the canonical split epimorphism

$$UF(X) \underset{UF(\iota_X)}{\overset{UF(\tau_X)}{\rightleftarrows}} UF(0),$$

and where $\iota_B \colon 0 \to B$, $\iota_X \colon 0 \to X$, and $\tau_X \colon X \to 0$ are the initial and terminal maps.

2. *Ideal* if the corresponding split epimorphism

$$A \underset{s}{\overset{p}{\rightleftarrows}} U(B)$$

is ideal, that is, if there exist a split epimorphism

$$A' \underset{s'}{\overset{p'}{\rightleftarrows}} B$$

in U and an isomorphism $\sigma \colon U(A') \to A$ in V such that $p \circ \sigma = U(p')$ and $\sigma \circ U(s') = s$.

We prove that every ideal action is coherent, and we call *BAT* any ideally exact context with a *good theory of actions*, i.e., in which all coherent actions are ideal and all morphisms of such actions are ideal. Here, the acronym BAT is inspired by the notion of BIT-variety, where BIT stands for **B**uona (good, in Italian) **I**deal **T**heory, introduced by A. Ursini. Analogously, BAT stands for **B**uona **A**ction **T**heory.

Moreover, we show that if a variety U is obtained from a semi-abelian variety V by freely adding nullary operations together with suitable identities, then the ideally exact context determined by the associated free–forgetful monadic adjunction with cartesian unit is BAT if and only if a compatibility condition between coherent actions and the added identities is satisfied, see [**?**].

Our setting applies in particular to certain categories of interest in algebraic logic, including MV-algebras [**?**] and product algebras, and unital non-associative $\mathbb{F}$-algebras. Finally, we study the BAT context $\mathsf{Set}^{\mathrm{op}}$, the dual of the category of sets.

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Enriched Płonka sums*

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Płonka sums [6] are a celebrated construction for building algebras from semilattice-indexed direct systems of algebras of the same type (so-called semilattice direct systems). This technique has proved instrumental in the theory of regular varieties: it provides the fundamental tool for the representation of regularisations of strongly irregular varieties [7] and also constitutes a first paradigmatic example of a semilattice sum [8].

While it is well known that, from a strictly universal-algebraic perspective, the applicability of Płonka sums does not extend far beyond this specific setting, representation theorems for several important irregular varieties, as well as for many varieties satisfying conditions stronger than regularity, employ "Płonka-type" constructions. The latter bear striking similarities to Płonka sums while differing from them in significant respects. Examples include the representation of Polin algebras [5], Katriňák and Guričan's representation of pseudocomplemented semilattices [4], and varieties admitting De Morgan–Płonka sum decompositions [9]. In this work, we aim to identify a convenient unified framework encompassing both Płonka sums and the aforementioned constructions.

Our point of departure is a 1974 paper by Grätzer and Sichler [3], who introduced sums over Agassiz systems of algebras (Agassiz sums) as a generalisation of Płonka sums to the setting of *transitive* systems of summands with *arbitrary* indexing algebras. Shortly thereafter, this construction was studied by Graczyńska and Wroński [2], who obtained markedly different results concerning preservation of identities. Although these works show that Płonka sums arise as a special case of Agassiz sums, they do not provide other significant examples.

One readily sees that none of the above-mentioned Płonka-type representations can be successfully expressed as Agassiz sums. If one aims to properly capture these constructions, it becomes clear that Grätzer and Sichler's framework is, in some respects, too general, while in others, still too restrictive. On the one hand, the full generality of arbitrary indexing algebras is not required—semilattices with an additional unary operation suffice. On the other hand, rather than a single transitive relation determining the homomorphisms in Agassiz systems, it is necessary to consider *two* distinct relations and, correspondingly, two different classes of connecting functions between summands. We therefore introduce the notion of an *enriched Płonka sum* over an *E-system* (*enriched semilattice direct system*) of algebras and show that it is sufficiently flexible to cover all the representations under consideration.

Finally, we investigate the problem of determining which identities are preserved by our construction. More precisely, we consider varieties $\mathfrak{V}$ whose members admit representations as enriched Płonka sums of algebras from a given subvariety $\mathfrak{W}$, and that are generated by algebras possessing "well-behaved" representations (in a precise sense defined in this work). We show that, in such a context, the identities holding in $\mathfrak{W}$ that remain valid in $\mathfrak{V}$ are precisely those satisfying a generalisation of the regularity condition introduced in [3].

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Constructing the Object of Bialgebras in a 2-Category with PIE-Limits

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Abstract

We present a framework for 2-dimensional algebraic logic modeled within a 2-category $\mathcal{K}$ with PIE-limits. A signature pair (Σ, σ) is defined: the **2-signature** Σ specifies the structure of the background universe, and a Σ-model M selects 0-cells, 1-cells, and 2-cells in $\mathcal{K}$ aligned with Σ. If $\mathcal{K}$ is the 2-category **CAT** of categories, a σ-model m in M selects objects and morphisms in M in accordance with the **1-signature** σ.

For a bialgebraic σ-theory T, we show that the object M^T of T-bialgebras can be constructed entirely via strict PIE-limits (Products, Inserters, and Equifiers), allowing us to lift the construction $M \mapsto M^T$ from **CAT** to $\mathcal{K}$. This machinery leads to our main result: if a given Σ-model M in **CAT** consists of accessible categories, accessible functors and natural transformations, then for any bialgebraic σ-theory T, the resulting category M^T of T-bialgebras within M is an accessible category.

1 The Signature Pair (Σ, σ)

We introduce a logical signature Σ that allows us to talk about a Σ-model M within which bialgebraic structures are considered. We outline preliminary notions to talk about a bialgebraic theory T associated to Σ. The category M^T of T-bialgebras is then defined when M is a Σ-model in **CAT**. It turns out that our notion of a bialgebra is equivalent to Ulmer's notion [4].

Definition 1 (2-Signature and Background Universe)**.** *A* ***2-signature*** $\Sigma = (C, \mathcal{F}, \mathcal{T})$ *consists of a set C of category symbols; a graph $\mathcal{F} \to C^* \times C$ of functor symbols $B\colon c_1 \cdots c_n \to d \in \mathcal{F}$; and a set $\mathcal{T}$ of transformation symbols $\eta^{\bar{x}}\colon t_1 \to t_2$ between Σ-terms $t_1, t_2\colon c$ in a context $\bar{x}$.*[1] *A Σ-**model** M in a 2-category $\mathcal{K}$ with products assigns an object M_c in $\mathcal{K}$ to each $c \in C$, a 1-cell $M(B)\colon \prod_i M_{c_i} \to M_d$ to each functor symbol $B\colon \bar{c} \to d \in \mathcal{F}$, and a 2-cell $M(\eta)\colon M_{\bar{x}}(t_1) \Rightarrow M_{\bar{x}}(t_2)$ to each transformation symbol $\eta^{\bar{x}}\colon t_1 \to t_2$.*

By varying Σ, we capture distinct signatures for algebraic logic without structural equations. As examples, consider the following 2-signatures for the settings of a Cartesian monoidal category, monoidal category, a monad, an adjunction, and a distributive law, respectively:

$$\Sigma_{\text{Cart}} = \big(\{c\},\ \{\otimes\colon cc \to c,\ I\colon () \to c\},\ \{\delta^x\colon x \to x \otimes x,\ !^x\colon x \to I\}\big)$$
$$\Sigma_{\text{Mon}} = \big(\{c\}, \{\otimes\colon cc \to c, I\colon () \to c\}, \{(x \otimes y) \otimes z \rightleftarrows x \otimes (y \otimes z), x \otimes I \rightleftarrows x \leftrightarrows I \otimes x\}\big)$$
$$\Sigma_{\text{Mnd}} = \big(\{c\},\ \{T\colon c \to c\},\ \{\mu^x\colon T(T(x)) \to T(x),\ \eta^x\colon x \to T(x)\}\big)$$
$$\Sigma_{\text{Adj}} = \big(\{c, d\},\ \{L\colon c \to d,\ R\colon d \to c\},\ \{\eta^x\colon x \to R(L(x)),\ \varepsilon^y\colon L(R(y)) \to y\}\big)$$
$$\Sigma_{\text{Dist}} = \big(\{c\},\ \{S\colon c \to c,\ T\colon c \to c\},\ \{\mu_S,\ \eta_S,\ \mu_T,\ \eta_T,\ \gamma^x\colon S(T(x)) \to T(S(x))\}\big)$$

[1] We define C^* to be the free monoid over C.

Definition 2 (1-Signature)**.** *A signature pair* (Σ, σ) *consists of a 2-signature* $\Sigma = (C, \mathcal{F}, \mathcal{T})$ *and a **1-signature*** σ *relative to* Σ*, where* $\sigma = (S, F)$ *has a typed set* $S \to C$ *of sorts and a set* F *of function symbols* $f^{\bar{s}} \colon t_1 \to t_2$ *between* Σ*-terms* t_1 *and* t_2 *within context* $\bar{s} \in S^*$ *of sorts. We define the **term graph*** G_σ *whose vertices are* Σ*-terms* t *within some context* $\bar{s} \in S^*$ *of sorts, and whose arrows are recursively generated using the functor, transformation and function symbols of the signature pair. A set* T *of parallel paths within* G_σ *is called a **bialgebraic*** σ***-theory**.*

A σ*-model* m *within a* Σ*-model* M *in the 2-category* **CAT***, consists of an object* m_s *in* M_{c_s} *for each sort* $s \in S$ *and a morphism* $m(f) \colon M_{\bar{s}}(t_1) \to M_{\bar{s}}(t_2)$ *for each* $f^{\bar{s}} \colon t_1 \to t_2 \in F$*. For a parallel path* $p, q \colon t_1 \to t_2$ *in* G_σ*, we say that* m *satisfies the equation* $p \approx q$ *if the equation of morphisms* $m(p) = m(q)$ *holds. A* σ*-model* m *is called a* T*-bialgebra, if* m *satisfies all the equations in* T*. We denote the category of* T*-bialgebras in* M *as* M^T*.*

2 Main Results

For a signature pair (Σ, σ) with a Σ-model M in **CAT** and a bialgebraic σ-theory T, we show that the category M^T can be constructed using PIE-limits (Products, Inserters, Equifiers). This allows us to lift the construction of $M \mapsto M^T$ from **CAT** to an arbitrary PIE-complete 2-category $\mathcal{K}$. As **Acc** of accessible categories, accessible functors, and natural transformations is closed under PIE-limits in **CAT** [1], we have an immediate corollary:

Corollary 1 (Accessibility of Bialgebras)**.** *Let* (Σ, σ) *be a signature pair and* T *a* σ*-theory. If* M *is a* Σ*-model in* **CAT** *such that every category* M_c *is accessible and every functor* $M(B)$ *is accessible componentwise, then the category of* T*-bialgebras* M^T *is an accessible category.*

Corollary 1 also holds in the setting $\mathbf{CAT}_\mathcal{V}$ of $\mathcal{V}$-enriched categories and its $\mathcal{V}$-accessible categories, $\mathcal{V}$-accessible functors and $\mathcal{V}$-natural transformations, where $\mathcal{V}$ is a locally presentable symmetric monoidal closed category [2]. This immediately shows that categories of bialgebras, such as the Hopf algebras within an accessible symmetric monoidal category, form an accessible category generalizing results in [3]. We will conclude the talk on how certain properties, like representable exactness conditions and Kan extensions, can be lifted and strictly created from the base Σ-model M in $\mathcal{K}$ to the object M^T of T-bialgebras. When $\mathcal{K} = \mathbf{CAT}_\mathcal{V}$, this result permits general conditions on M and T so that M^T is a locally presentable $\mathcal{V}$-category.

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On Bochvar algebras and regular double Stone algebras

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In this work we establish some categorical relationship between the quasivariety of Bochvar algebras and the variety of regular double Stone algebras. Bochvar algebras provide the equivalent algebraic semantics of external weak Kleene logics, which can be semantically presented as 3-valued systems with an infectious truth-value and equipped with a classical recapture operator J_2 ([1, 6]). The class of Bochvar algebras BCA was introduced in [4] and its structure theory has recently been developed in [3], employing the technique of Płonka sums ([10, 11]). In particular, every Bochvar algebra has an involutive bisemilattice reduct, where the structure of the latter is known to be a Płonka sum of Boolean algebras. This shows that Bochvar algebras can be represented as expansions of involutive bisemilattices in which the Boolean algebra corresponding to the bottom fibre plays a central structural role in that it contains as interval algebras isomorphic copies of all the fibres of the sum. Elaborating on this intuition, [2] proved that every Bochvar algebra can be represented as so-called Bochvar systems, that are pairs consisting of a Boolean algebra together with a designated meet-subsemilattice with unit, and they extend the result to a proper categorical equivalence between the respective categories.

On the other hand, regular double Stone algebras have a longer and well-established history. Among other things, by splitting classical negation into two distinct unary operations, one enforcing non-contradiction and the other enforcing excluded middle, the variety RDSA of regular double Stone algebras provides a natural algebraic framework for the study of partial information. It is term-equivalent to the variety of MV_3 algebras [7] and arises naturally in rough set theory [9], where it captures the interaction between lower and upper approximations. Beyond their connections with many-valued logics, recent studies have also highlighted the role of regular double Stone algebras in the investigation of quantum logics [5]. In particular, they can be seen as modelling "sharp contexts" in generalizations of orthomodular lattices, in much the same way as Boolean algebras represent classical contexts within orthomodular theory. A crucial result for our purposes is [8], which shows that the category of regular double Stone algebras is equivalent to a category whose objects are Boolean algebras with a designated filter. We refer to these structures as filtered Bochvar systems.

Our contribution consists of three main results:

(*i*) we provide a new proof of the categorical equivalence for Bochvar algebras, using a modified twist product construction in place of the Płonka sum representation;

(*ii*) we show that the functor witnessing this equivalence can be decomposed into a standard twist product and a translation map;

(*iii*) we prove that the algebraic category of regular double Stone algebras is equivalent to a full subcategory of the category of Bochvar algebras, and that every Bochvar algebra arises as the colimit of its regular double Stone subalgebras.

We introduce a modified notion of twist product that is able to generate, from any Bochvar system, a Bochvar algebra. Moreover we prove that every Bochvar algebra has a representation via this twist product of a certain Bochvar system, which is unique up to isomorphism. This bijective correspondence is lifted to a pair of functors witnessing the equivalence of the relative categories. The modified twist product is helpful in the individuation of certain subsets of points of any Bochvar algebra that have a major role in its description: sharp, dense and dually dense elements. In particular, sharp elements form the largest Boolean subalgebra of any Bochvar algebra (i.e. its bottom fibre), while dense elements form a meet-subsemilattice with unit of the former Boolean subalgebra.

Next we show how to translate the language of BCA to that of RDSA. This translation can be turned into a functor from the category of regular double Stone algebras into that of Bochvar algebras. Moreover we prove that the functor that transforms a filtered Bochvar system in a Bochvar algebra is the composition of the standard twist product (in the sense of [8]) with the mentioned translation. Therefore these particular Bochvar algebras are term-reducts of regular double Stone algebras.

As a consequence, we prove that the category of regular double Stone algebras is a full subcategory of that of Bochvar algebras. Furthermore, the connection between the two categories is stronger than that and is not limited to the special case of Bochvar algebras arising from filtered Bochvar systems. Finally, we prove that any 1-generated subalgebra of a Bochvar algebra arises as the translation of a regular double Stone one, which allows us to establish that every Bochvar algebra is the colimit of a diagram of regular double Stone algebras.

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A representation theory of arboreal categories

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Arboreal categories, introduced by Abramsky and Reggio [1] (see also [5]), axiomatise common features of many model-comparison games important in finite model theory, such as the Ehrenfeucht–Fraïssé games, bisimulation games, pebble games, etc. This axiomatic setting has been useful for generalising a number of results in the field [2].

As part of the data when specifying an arboreal category $\mathscr{A}$ is a choice of a suitable factorisation system $(\mathscr{Q}, \mathscr{M})$ of quotients and embeddings. This leads to the notion of *paths*, which is fundamental for the whole theory. Paths are the objects of $\mathscr{A}$ whose poset of $\mathscr{M}$-subobjects forms a finite linear chain. In fact, they are the building blocks of arboreal categories akin to compact objects of locally finitely presentable and accessible categories [3], that is, paths are dense in $\mathscr{A}$.

Embeddings of paths $P \rightarrowtail X$ represent plays in the model-comparison game that the arboreal category encodes. Therefore, defining *bisimilarity* $X \sim Y$ for objects in $\mathscr{A}$ as a back-and-forth equivalence on embeddings of paths encodes the winning condition in the game or, equivalently, the equivalence in the corresponding logic. For example, let $\mathsf{Str}(\sigma)$ be the category of structures in a relational signature σ, with morphisms being functions which preserve the relations. Then, one defines a category $\mathcal{T}_k(\sigma)$ of σ-structures with a 'compatible tree order' of height $\leq k$ and a functor

$$F\colon \mathsf{Str}(\sigma) \to \mathcal{T}_k(\sigma)$$

which maps a structure $X \in \mathsf{Str}(\sigma)$ to the unravelling of X according to the Ehrenfeucht–Fraïssé game, that is, FX is the tree of plays in X. It follows that two structures X, X' are logically equivalent with respect first-order formulas with quantifier rank $\leq k$ if, and only if, $FX \sim FX'$. The same construction works for the bisimulation games, pebble games, etc. whereby providing a link with modal logic, the k-variable fragment of (infinitary) first-order logic, and so on. Moreover, by replacing bisimilarity with other relations specified using embeddings and paths, we represent further restrictions of these fragments to the primitive positive, counting, existential, and positive fragments [2].

Although, the setting of arboreal categories is quite flexible for working syntax-free with weak fragments of logics (usually with logics not admitting compactness), only ad-hoc methods are known when constructing new examples of arboreal categories. On the other hand, locally finitely presentable categories admit fairly general methods in this regard, e.g. the Gabriel–Ulmer duality [4].

In this work we introduce a representation theory for arboreal categories which are *locally posetal*, that is, they have only a set of paths (up to isomorphism) and are *concrete over trees*. The latter condition is equivalent to the statement that there is at most one quotient between any two paths. All examples of arboreal categories arising from games that we have encountered so far are locally posetal.

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For any locally posetal arboreal category $\mathscr{A}$ we define a functor

$$\Phi_{\mathscr{A}} \colon \mathbb{N}^{op} \to \mathbf{Pos}$$

where $\Phi_{\mathscr{A}}(n)$ is the poset of (equivalence classes of) paths of height n ordered by the morphism order, i.e. $P \leq Q$ if there is a quotient $P \twoheadrightarrow Q$, and the function $\Phi_{\mathscr{A}}(n \leq m) \colon \Phi_{\mathscr{A}}(m) \to \Phi_{\mathscr{A}}(n)$ sends a path of height m to its prefix of height n.

Conversely, for any functor

$$\Phi \colon \mathbb{N}^{op} \to \mathbf{Pos}$$

where $\Phi(0)$ has the least element, we build a category $\mathscr{A}_\Phi$ as the category of labelled trees $(T, t \colon T \to \Phi)$ where t is a natural transformation from the tree T viewed as a presheaf $\mathbb{N}^{op} \to \mathbf{Set}$ with discrete order on fibres. Morphisms $f \colon (T,t) \to (R,r)$ in $\mathscr{A}_\Phi$ are defined as lax morphisms with respect to the pointwise poset orders, i.e. tree morphisms $f \colon T \to S$ such that $t \leq r \cdot f$.

We show that every locally posetal arboreal category is represented this way.

Theorem 1. *For any locally posetal arboreal category $\mathscr{A}$, we have $\mathscr{A} \cong \mathscr{A}_{\Phi_{\mathscr{A}}}$ and, for any functor $\Phi \colon \mathbb{N}^{op} \to \mathbf{Pos}$ such that $\Phi(0)$ has the least element, $\Phi \cong \Phi_{\mathscr{A}_\Phi}$.*

As an immediate application of this theorem we obtain that, for a locally posetal arboreal category $\mathscr{A}$,

1. the chosen factorisation system on $\mathscr{A}$ must be the (epi, regular mono) factorisation system,
2. $\mathscr{A}$ is order-enriched,
3. $\mathscr{A}$ has free amalgamations and one can define a 'quotient' of every object of $\mathscr{A}$ by its largest bisimulation relation, thus obtaining an object of (model-theoretic) types,
4. in case $\mathscr{A}$ has a unique path of height 0, the subcategory of $\mathscr{A}$ of pathwise embeddings (which is an important class of morphisms) is a presheaf topos.

There is also a version of the Grothendieck adjunction between tree diagrams $T \to \mathscr{A}_p$ (where $\mathscr{A}_p$ denotes the subcategory of paths) and certain presheaves $\mathscr{A}_p^{op} \to \mathbf{Set}$ that leads to a Gabriel–Ulmer duality for arboreal categories. Furthermore, on the horizon there seems to be an extension of Theorem 1 to arboreal categories which might not be concrete over trees. This, however, requires us to represent arboreal categories as certain functors of type $\Phi \colon \mathbb{N}^{op} \to \mathbf{Cat}$.

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Modally definable polyhedra

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Abstract

The aim of the paper is to prove an equivalent of Goldblatt-Thomason's theorem for classes of precompact polyhedra, i.e. subsets of the affine space whose closure is a union of simplices.

We adapt from the semantics from [1] on bounded polyhedra: Given a polyhedron P, we have an evaluation map from the standard modal language $\mathcal{L}$ to the set of subpolyhedra, and the closure ($\Diamond$) gets interpreted as the closure in $\mathbb{R}^n$, thus identifying "open" subpolyhedra with polyhedra whose complement is a closed subpolyhedron.

Polyhedra in the literature are seen as realizations of simplicial complexes, i.e. pairs (V, Σ), where V is a set of "vertices" and $\Sigma \subseteq \mathcal{P}(V)$ is a set closed downward under inclusion, representing our simplices, and closed subpolyhedra correspond to realizations of subcomplexes.

While simplicial complexes have good properties, the "downward closure" only allows us to give a correspondence between subcomplexes and closed subpolyhedra.

To solve this issue we embed the category of simplicial complexes into a category of more loose structures, that we call "precomplexes", where we drop the condition that Σ must be closed downward. Each precomplex K embeds into a complex $\overline{K}$ (its downward-closure) and up to PL-homeomorphism there exists a unique polyhedron $|K|$ such that $\overline{|K|} = |\overline{K}|$.

The result is a class of objects that is purely combinatorial and where each object has an associated poset $(\Sigma, \subseteq)$, but realizes freely to open and closed polyhedra, allowing us to swiftly translate conditions between the topological, combinatorial and posetal perspective.

What we obtain is a chain of results resembling what was found in [2]

Theorem.

1. *Any class of polyhedra is equivalent to the realization of a class of simplicial precomplexes closed under subdivision.*
2. *A class of (finite) simplicial precomplexes is modally definable if and only if it's closed under disjoint unions, open simplicial images, and open subcomplexes*
3. *A class of polyhedra is modally definable if and only if it's closed under disjoint unions, open PL-maps and open subpolyhedra.*

(where "open" refers to the notion of openness for polyhedra cited above)
As simplicial precomplexes (and their realization) carry homotopical data, we can easily obtain a negative result on homological properties:

4. The homology chains of a modally definable class of simplicial complexes must be closed under direct sums and images

Thus excluding the definability of any class of the kind "polyhedra with isomorphic n^{th} homology group"

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Theorems of alternatives, algebraically

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The interplay between the theories of ℓ-groups and (left or right, and bi-) orderable groups is well-known, and has been explored extensively in the literature of both fields (see, e.g., [1, 5, 4]). One notable example is the correspondence between the validity of equations in the varieties of abelian ℓ-groups and abelian groups, a version of Gordan's theorem that was established in [7] by extending partial orders on free abelian groups to total orders. The correspondence states that an inequation $0 \leq t_1 \vee \cdots \vee t_n$, where $t_1, \ldots, t_n$ are group terms, is valid in the variety of abelian ℓ-groups if, and only if, an equation $0 \approx \lambda_1 t_1 + \cdots + \lambda_n t_n$ is valid in the variety of abelian groups for some $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all 0.

A similar result for the variety of Sugihara monoids follows from a theorem of Avron [2] that relates derivability in the 'R-mingle' logic RM to derivability in its multiplicative fragment: an inequation $1 \leq t_1 \vee \cdots \vee t_n$, where $t_1, \ldots, t_n$ are multiplicative terms, is valid in the variety of Sugihara monoids if, and only if, $1 \leq \lambda_1 t_1 + \cdots + \lambda_n t_n$ is valid in this variety for some $\lambda_1, \ldots, \lambda_n \in \{0, 1\}$ not all 0. As shown in [6], results of this kind belong to a broader family of statements that can be understood as 'theorems of alternatives' for substructural logics, relating the validity of inequations in certain varieties of residuated lattices to the validity of residuated monoid inequations in the variety. In this work, we revisit these theorems from an algebraic perspective, establishing a connection to compatible preorders on commutative pomonoids.

The basic structures of our investigations are *involutive commutative residuated pomonoids* (for short, *icr-pomonoid*), consisting of a partially ordered commutative monoid $\langle A, \leq, \cdot, 1\rangle$ equipped with an involution $\neg$ satisfying $a \cdot b \leq c \iff \neg c \cdot b \leq \neg a$ for all $a, b, c \in A$. We also define $a + b := \neg(\neg a \cdot \neg b)$, $a \to b := \neg a + b$, $0 := \neg 1$, and, inductively, for $n \in \mathbb{N}$, $0a := 0$, $a^0 := 1$, $(n+1)a := na + a$, and $a^{n+1} = a^n \cdot a$. If the partial order of an icr-pomonoid $\langle A, \leq, \cdot, \neg, 1\rangle$ is a lattice order with meet and join operations $\wedge$ and $\vee$, we call the algebra $\langle A, \wedge, \vee, \cdot, \neg, 1\rangle$ an *involutive commutative residuated lattice* (for short, *icr-lattice*). Subdirect products of icr-lattices where the lattice order is total are called *semilinear*. (See [8] for details.)

A *preorder* $\preceq$ on an icr-pomonoid $\mathbf{A}$ is a reflexive and transitive binary relation on A that

(i) extends the partial order $\leq$, i.e., $\leq \subseteq \preceq$;

(ii) is compatible with $\cdot$, i.e., $a \preceq b$ implies $ca \preceq cb$ for all $a, b, c \in A$;

(iii) is compatible with $\neg$, i.e., $a \preceq b$ implies $\neg b \preceq \neg a$ for all $a \in A$.

We write $a \prec b$ to denote $a \preceq b$ and $b \not\preceq a$, and call $\preceq$ *total* if $a \preceq b$ or $b \preceq a$ for all $a, b \in A$.

Let $\mathbf{A}$ be any icr-pomonoid. A preorder $\preceq$ on $\mathbf{A}$ is uniquely determined by its *positive cone* $P_{\preceq} := \{a \in A \mid 1 \preceq a\}$, since for all $a, b \in A$,

$$a \preceq b \iff a \to b \in P_{\preceq}.$$

Moreover, we can give a precise description of the subsets of A that form positive cones of preorders on $\mathbf{A}$. Let us call $F \subseteq A$ a *filter submonoid* of $\mathbf{A}$ if F is an upset of $\langle A, \leq\rangle$ containing 1, and whenever $a, b \in F$, also $ab \in F$. Clearly, the positive cone $P_{\preceq}$ of any preorder $\preceq$ on $\mathbf{A}$ is a filter submonoid. Conversely, if $P \subseteq A$ is a filter submonoid, then $a \preceq^P b :\iff a \to b \in P$ defines a preorder on $\mathbf{A}$ satisfying $P_{\preceq^P} = P$. We may therefore henceforth identify preorders on $\mathbf{A}$ with filter submonoids of $\mathbf{A}$.

The following lemma may be understood as an algebraic version of a local deduction theorem for axiomatic extensions of the logic BCI (see, e.g., [3]).

Lemma 1. *Let P be a preorder on an icr-pomonoid $\mathbf{A}$ and $a \in A$. Then the preorder generated by $P \cup \{a\}$ is $\{c \in A \mid a^n \to c \in P$ for some $n \in \mathbb{N}\}$.*

Let $\mathbf{T}(X)$ and $\mathbf{T}_m(X)$ denote the term algebras over a set X of generators for the languages of icr-lattices and icr-pomonoids, respectively. For any variety $\mathcal{V}$ of icr-lattices, we obtain a corresponding substructural logic (consequence relation) by defining for any $\Sigma \cup \{s\} \subseteq T(X)$:

$$\Sigma \vDash_{\mathcal{V}} s \; :\Longleftrightarrow \; \text{for any homomorphism } h \text{ from } \mathbf{T}(X) \text{ to some } \mathbf{A} \in \mathcal{V}, \; 1 \leq h(t) \text{ for all } t \in \Sigma \implies 1 \leq h(s).$$

Now let $\mathcal{V}$ be any variety of semilinear icr-lattices. Let $\mathbf{F}(X)$ denote the free algebra of $\mathcal{V}$ over a set X, and let $\mathbf{F}_m(X)$ be the icr-pomonoid formed by the $\{\cdot, \neg, 1\}$-subreduct of $\mathbf{F}(X)$ generated by X, partially ordered by the restriction of the lattice order of $\mathbf{F}(X)$. For convenience, we write t for both $t \in T_m(X)$ and the corresponding member of $F_m(X)$.

Lemma 2. *The following are equivalent for any $\Sigma \cup \{t_1, \ldots, t_n\} \subseteq T_m(X)$:*

(1) *There is no total preorder on $\mathbf{F}_m(X)$ containing Σ and omitting $t_1, \ldots, t_n$.*

(2) $\Sigma \vDash_{\mathcal{V}} t_1 \vee \cdots \vee t_n$.

Let us say next that $\mathcal{V}$ admits a *theorem of alternatives* if the following are equivalent for any $\Sigma \cup \{t_1, \ldots, t_n\} \subseteq T_m(X)$:

(A) $\Sigma \vDash_{\mathcal{V}} t_1 \vee \cdots \vee t_n$.

(B) $\Sigma \vDash_{\mathcal{V}} \lambda_1 t_1 + \cdots + \lambda_n t_n$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{N}$ not all 0.

Theorem 3. *A variety of semilinear icr-lattices $\mathcal{V}$ admits a theorem of alternatives if, and only if, it satisfies $0 \leq 1$, $1 \leq x \vee \neg x$, and for each $n \in \mathbb{N}$, the equation $(nx)^k \leq m(x^n)$ for some $m \in \mathbb{N}^+, k \in \mathbb{N}$.*

This result applies to, for example, the varieties of abelian ℓ-groups, Sugihara monoids, and semilinear icr-lattices satisfying $x^n \approx nx$ for each $n \in \mathbb{N}^+$.

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Superamalgamation for modal lattices via non-distributive dualities

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We study the Craig interpolation property for (not necessarily distributive) modal logics through the lens of superamalgamation and duality. This formalism was introduced in [2].

Definition 1. A *modal lattice* is a tuple $(A, \top, \bot, \wedge, \vee, \Box, \Diamond)$ such that $(A, \top, \bot, \wedge, \vee)$ is a bounded lattice, and $\Box$ and $\Diamond$ are unary operators satisfying:

$$\top \approx \Box\top, \qquad \top \approx \Diamond\top,$$
$$\Box a \wedge \Box b \approx \Box(a \wedge b), \qquad \Diamond a \vee \Diamond b \leq \Diamond(a \vee b),$$
$$\Diamond a \wedge \Box b \leq \Diamond(a \wedge b).$$

Definition 2. Let K be a variety of modal lattices. A *V-formation* is a tuple of lattices $K, L_1, L_2 \in \mathsf{K}$ and embeddings $h_i : K \to L_i$, as in Figure 1a

A variety K of modal lattices has the *amalgamation property* if for every V-formation, there is a lattice M, and injective homomorphisms $p_i : L_i \to M$ such that $p_1 \circ h_1 = p_2 \circ h_2$, or in other words, such that the diagram in Figure 1b commutes. The lattice M is called an *amalgam*.

(a) V-formation (b) Amalgam

Figure 1: Amalgamation of lattices

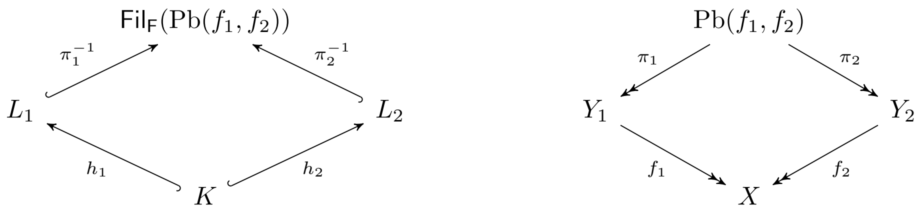
A variety K of modal lattices has the *superamalgamation property* if for every V-formation, there is an amalgam such that $p_1(a) \leq p_2(b)$ for $a \in L_1$ and $b \in L_2$ implies the existence of $c \in K$ with $a \leq h_1(c)$ and $h_2(c) \leq b$. The lattice M is called a *superamalgam*.

Theorem 3. *Let K be a variety of modal lattices, and let $\mathcal{L}(\mathsf{K})$ be the logic of K. If K has superamalgamation, then $\mathcal{L}(\mathsf{K})$ has the Craig interpolation property.*

Theorem 4. *The variety of modal lattices has the superamalgamation property.*

Proof sketch. This is done by using the duality developed in [2]. Let K, L_1, L_2, h_1, h_2 be a V-formation. We consider the corresponding dual structures (modal L-spaces), as presented in Figure 2b. Consider the pullback $\mathrm{Pb}(f_1, f_2) := \{(x, y) \mid f_1(x) = f_2(y)\}$ together with the surjective projections $\pi_i : \mathrm{Pb}(f_1, f_2) \to L_i$.

The pullback $\mathrm{Pb}(f_1, f_2)$ is in general not a modal L-space, but one can show that it is a modal L-frame, and that the projections are modal. We omit the technical details, see [1, Theorems



(a) Amalgam of lattices (b) Coamalgam of L-frames

Figure 2: Amalgamation diagram and its dual

3.12 and 4.7] for the full proof. It follows that the complex algebra $\mathsf{Fil}_{\mathsf{F}}(\mathrm{Pb}(f_1, f_2))$ is an amalgam. Using the separation property of L-spaces, one shows that it is a superamalgam. ■

Remark 5. Jónsson previously proved that the variety of lattice has superamalgamation (see [3] or [4] for a proof). Jónsson's proof is arguably simpler – however, it does not adapt to the modal case. For more details on this we refer to [1, Remarks 2.14 and 2.15].

Recall that *weak positive modal logic* is the logic of modal lattices [2]. As a direct consequence of Theorems 3 and 4, we obtain the following.

Theorem 6. *Weak positive modal logic has the Craig interpolation property.*

We conclude by stating that the above result could be generalized to a number of other weak modal logics, in particular, to those axiomatized by the analogues of the modal axioms $\mathsf{T}, \mathsf{4}, \mathsf{B}, \mathsf{5}, .2$, see [1, Theorem 4.17]. The result on .2 is of a particular interest as its classical analogue, the modal logic $\mathsf{K}.2$, lacks the Craig interpolation property [5, Section 5.6.2].

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Varieties of distributive ℓ-pregroups

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A *lattice-ordered pregroup* (ℓ-pregroup) is an algebra $(L, \wedge, \vee, \cdot, ^\ell, ^r, 1)$ such that $(L, \wedge, \vee)$ is a lattice, $(L, \cdot, 1)$ is a monoid, multiplication preserves the lattice order $\leq$ and for every $a \in L$,

$$a^\ell a \leq 1 \leq aa^\ell \text{ and } aa^r \leq 1 \leq a^r a.$$

Lattice-ordered pregroups can be seen as a generalization of lattice-ordered groups (ℓ-groups) which have been extensively studied [1, 8]. Indeed, ℓ-groups correspond exactly to the ℓ-pregroups that satisfy $x^\ell \approx x^r$ and in this case x^ℓ is the group inverse operation. On the other hand ℓ-pregroups are a special case of *pregroups* defined similarly to ℓ-pregroups but without demanding that its underlying order is a lattice. Pregroups where introduced in the context of mathematical linguistics [2, 3, 9].

An ℓ-pregroup is called *distributive* if its lattice reduct is distributive. The variety DLP of distributive ℓ-pregroups was studied in depth in [4] where a Holland-style representation theorem is obtained and shown that DLP has a decidable equational theory.

An ℓ-pregroup is called *n-periodic* for $n \in \mathbb{N}$ if it satisfies the equation $x^{\ell^n} \approx x^{r^n}$. As noted above, 1-preiodic ℓ-pregroups correspond exactly to ℓ-groups. We denote the variety of n-periodic ℓ-pregroups by LP_n. In [6] it was shown that every periodic ℓ-pregroup is distributive. Moreover, in [5] a representation theorem for periodic ℓ-pregroups is obtained and it is shown that the equational theory of LP_n is decidable for each $n \in \mathbb{N}$.

Let $f\colon \mathbf{P} \to \mathbf{Q}$ and $g\colon \mathbf{Q} \to \mathbf{P}$ be maps between posets. We say that g is a *residual* for f and f is a dual residual for g if for all $p \in P$, $q \in Q$,

$$f(p) \leq q \iff p \leq g(q).$$

The residual and dual residual of a map f are unique if they exist and we denote them by f^r and f^ℓ, respectively. Inductively, we define the nth-order residual if it exists, by $f^{r^1} = f^r$ and $f^{r^{n+1}} = (f^{r^n})^r$ and analogously we define the nth-order dual residual of f.

For a chain $\mathbf{\Omega}$ we denote by $\mathrm{F}(\mathbf{\Omega})$ the set of maps on $\mathbf{\Omega}$ that have residuals and dual residuals of every order. This set gives rise to a distributive ℓ-pregroup $\mathbf{F}(\mathbf{\Omega}) = (F(\mathbf{\Omega}), \wedge, \vee, \circ, ^\ell, ^r, id_\Omega)$, where $\wedge$ and $\vee$ are defined point-wise, $\circ$ is functional composition, and id_Ω is the identity map on Ω. The subset $\mathrm{F}_n(\mathbf{\Omega})$ of $\mathbf{F}(\mathbf{\Omega})$ of the maps that satisfy $f^{r^\ell} = f^{r^n}$ forms an n-periodic subalgebra $\mathbf{F}_n(\mathbf{\Omega})$ of $\mathbf{F}(\mathbf{\Omega})$. It is shown in [4] that every distributive ℓ-pregroup embed into $\mathbf{F}(\mathbf{\Omega} \overrightarrow{\times} \mathbb{Z})$ for some chain $\mathbf{\Omega}$ and that $\mathsf{DLP} = \mathbb{V}(\mathbf{F}(\mathbb{Z}))$. In [5] it was shown that every n-periodic ℓ-pregroup embeds into $\mathbf{F}_n(\mathbf{\Omega} \overrightarrow{\times} \mathbb{Z})$ and that $\mathsf{LP}_n = \mathbb{V}(\mathbf{F}_n(\mathbb{Q} \overrightarrow{\times} \mathbb{Z}))$. However, $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is a proper subvariety of LP_n which is also decidable [5]. Moreover, $\mathsf{DLP} = \bigvee_{n \in \mathbb{N}} \mathsf{LP}_n = \bigvee_{n \in \mathbb{N}} \mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$. In [7] it was shown that the variety $\mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$ is finitely axiomatizable for every $n \geq 1$ and further structural results about periodic ℓ-pregroups were obtained, establishing a deep connection between the structure of a periodic ℓ-pregroup and its subalgebra of invertible elements.

Many of the structural results for periodic ℓ-pregroups fail for non-periodic distributive ℓ-pregroups, e.g., there are distributive ℓ-pregroups where the only invertible element is the identity. This poses the question whether there is a non-periodic proper subvariety of DLP.

In this work we answer this question.

Theorem. *Every proper subvariety of* DLP *is* n*-periodic for some* $n \geq 1$.

Consequently, the subvariety lattice of DLP decomposes into a disjoint union of bounded intervals each consisting of all properly n-periodic varieties, for different $n \in \mathbb{Z}^+$. The top of such an interval is the variety LP_n and the bottom is characterized in [7] as $\bigvee_{q\in P} \mathbb{V}(\mathbf{F}_q(\mathbb{Z}))$, where $P = \{p_1^{k_1}, \ldots, p_l^{k_l}\}$ and $n = p_1^{k_1} \cdots p_l^{k_l}$ is the prime decomposition of n. Therefore, the study of the subvariety lattice of DLP reduces to the study of these n-periodic intervals and the relations between them. Moreover, as a corollary of our theorem we also obtain an alternative proof for the decidability of the equational theory of DLP: to decide whether an equation ε holds in DLP one can simultaneously try to prove ε from the axioms of DLP and try to prove that ε implies n-periodicity for some $n \in \mathbb{N}$.

To prove the theorem we rely heavily on the representation theorem for distributive ℓ-pregroups and two generation results for DLP. By [4, 5] the (one-generated) subalgebra $\mathbf{F}_{\mathsf{fs}}(\mathbb{Z})$ of $\mathbf{F}(\mathbb{Z})$ consisting of the finite support members of $\mathbf{F}(\mathbb{Z})$ generates DLP and for any infinite subset $I \subseteq \mathbb{N}$, $\mathsf{DLP} = \bigvee_{n\in I} \mathbb{V}(\mathbf{F}_n(\mathbb{Z}))$.

We show that every non-periodic distributive ℓ-pregroup $\mathbf{L}$ generates DLP. For this we consider $\mathbf{L}$ as a subalgebra of $\mathbf{F}(\mathbf{\Omega} \overrightarrow{\times} \mathbb{Z})$ for some chain $\mathbf{\Omega}$. There are two cases: either $\mathbf{L}$ has a non-periodic element or $\mathbf{L}$ has n-periodic elements for infinitely many n. In the latter case, using the results form [7] we can show that $\mathbb{V}(\mathbf{L})$ contains $\mathbf{F}_n(\mathbb{Z})$ for infinitely many n and thus $\mathbb{V}(\mathbf{L}) = \mathsf{DLP}$.

Otherwise, $\mathbf{L}$ has a non-periodic element and there is more work to do. The idea is to 'simulate' the generator $\mathbf{c}$ of $\mathbf{F}_{\mathsf{fs}}(\mathbb{Z})$ defined by $\mathbf{c}(0) = 1$ and $\mathbf{c}(x) = x$ for $x \in \mathbb{Z} \setminus \{0\}$. To do this we first analyze non-periodic members of $\mathbf{F}(\mathbb{Z})$ and show that each of them 'simulates' $\mathbf{c}$. Using this we consider non-periodic members of $\mathbf{F}_n(\mathbf{\Omega} \overrightarrow{\times} \mathbb{Z})$ and show that they either consist of infinitely many periodic components of different periodicity or they simulate $\mathbf{c}$. In both cases it follows that $\mathbb{V}(\mathbf{L}) = \mathsf{DLP}$.

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A Nested Approach to Relevant Logic

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Relevant logics form a family of non-classical logics introduced to address the paradoxes of material and strict implication [?]. They admit a modular presentation, obtained by starting from a minimal set of axioms and inference rules and then considering systematic extensions. In this setting, the *base positive relevant logic* B^+ (cf. [?, ?]), namely the negation free fragment of B, plays a central role. However, the proof theory of B^+ remains comparatively underdeveloped. Existing proof-theoretic treatments of B^+ mainly rely on labelled frameworks, such as labelled sequent calculi, tableaux, and natural deduction systems, e.g. [?, ?, ?]. To the best of our knowledge, there is no label-free calculus for this logic. This has stymied the proof-theoretic study of metalogical properties—such as interpolation—for B^+ and other weak relevant logics.

In order to overcome these challenges, we contribute to this line of research by introducing a novel proof system for B^+ (see Figure 1), based on a generalized form of nested sequents. Nested sequent calculi provide a fundamental proof-theoretic framework for logics based on Kripke-style relational semantics, including intuitionistic logic [?], modal logic [?] and their combinations [?, ?, ?]. In these systems, accessibility relations are binary and the nesting structure is unary, with each block containing a single sequent. In this work, we generalize the nested approach to relevant (modal) logics based on Routley-Meyer semantics, where implication is interpreted via a ternary relation, reflecting the relational dual of the operation interpreting implication on algebraic models. To this end, we introduce *binary relevant blocks* as primitive syntactic constructs that directly reflect the ternary structure of Routley-Meyer frames. We develop a nested sequent calculus for B^+, and establish completeness by a purely syntactic proof of cut-admissibility.

Our work shows that nested sequent calculi provide a robust proof-theoretic framework, applicable not only to logics based on binary relational semantics but also to relevant logics characterized by more complex relational structures.

Definition 1. *Nested sequents are generated by the following grammar:*

$$\Gamma :: \Lambda, \Pi \qquad \Lambda :: A_1^\bullet, \ldots, A_n^\bullet, \langle \Lambda_{11} \,/\!\!/\, \Lambda_{12} \rangle, \ldots, \langle \Lambda_{k1} \,/\!\!/\, \Lambda_{k2} \rangle \qquad \Pi :: A^\circ \mid \langle \Lambda \,/\!\!/\, \Gamma \rangle \mid \langle \Gamma \,/\!\!/\, \Lambda \rangle$$

Our main results are captured in the following theorems.

Theorem 2. *A formula A is valid if and only if it is provable in* $\mathbf{C}_{\mathsf{B}^+}$.

Theorem 3. *The following cut rule is admissible in* $\mathbf{C}_{\mathsf{B}^+}$*:*

$$\frac{G^*\{\Sigma^*, A^\circ\} \qquad G\{\Sigma, A^\bullet\}}{G\{\Sigma\}}\ (\mathsf{cut})$$

In subsequent work, we will deploy our novel proof-theoretic environment to study interpolation and other metalogical properties in B^+ and logics admitting similar proof-theoretic formalisms.

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Initial sequents:

$$\dfrac{}{G\{\Sigma, p^\bullet, p^\circ\}}\ (\mathsf{id})$$

Logical rules:

$$\dfrac{G\{\Sigma, A^\bullet, B^\bullet\}}{G\{\Sigma, A \wedge B^\bullet\}}\ \wedge^\bullet \qquad \dfrac{G\{\Sigma, A^\circ\} \quad G\{\Sigma, B^\circ\}}{G\{\Sigma, A \wedge B^\circ\}}\ \wedge^\circ$$

$$\dfrac{G\{\Sigma, A^\bullet\} \quad G\{\Sigma, B^\bullet\}}{G\{\Sigma, A \vee B^\bullet\}}\ \vee^\bullet \qquad \dfrac{G\{\Sigma, A^\circ\}}{G\{\Sigma, A \vee B^\circ\}}\ \vee^\circ \quad \dfrac{G\{\Sigma, B^\circ\}}{G\{\Sigma, A \vee B^\circ\}}\ \vee^\circ$$

$$\dfrac{G^*\{\Sigma^*, A \to B^\bullet, \langle \Phi_1^*, A^\circ /\!/ \Phi_2^* \rangle\} \quad G\{\Sigma, A \to B^\bullet, \langle \Phi_1 /\!/ \Phi_2, B^\bullet \rangle\}}{G\{\Sigma, A \to B^\bullet, \langle \Phi_1 /\!/ \Phi_2 \rangle\}}\ \to^\bullet \qquad \dfrac{G\{\Sigma, \langle A^\bullet /\!/ B^\circ \rangle\}}{G\{\Sigma, A \to B^\circ\}}\ \to^\circ$$

Structural rules:

$$\dfrac{\Sigma, \langle A^\bullet, \Phi_1 /\!/ A^\bullet, \Phi_2 \rangle}{\Sigma, \langle A^\bullet, \Phi_1 /\!/ \Phi_2 \rangle}\ (\mathsf{lift})$$

Figure 1: $\mathbf{C}_{\mathsf{B}+}$, a nested sequent calculus for B^+

Basic zero-dimensional spaces: A unifying framework for continuity and openness

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Articles such as [4], [3], and [2] have demonstrated that several results in point-free topology are consequences of more general results in the non-complete setting of Heyting semilattices.

In this talk, we will present ongoing work that continues this line of research. Our approach is motivated by a suggestion at the end of [3] to investigate the category of so-called *basic zero-dimensional* (*closure*) *spaces* (*b0-spaces*, for short). This category is denoted by **B0ds** and its objects are triples $\mathcal{S} = (X, \mathcal{C}, \mathcal{B}d)$, where

(BS1) X is a set,

(BS2) $\mathcal{B}d$ is a closure system on X, distributive as a lattice, and

(BS3) $\mathcal{C}$ is a subset of $\mathcal{B}d$ in which every element $C \in \mathcal{C}$ has a complement $\neg C$ in $\mathcal{B}d$,

such that every $M \in \mathcal{B}d$ is of the form

$$M = \bigcap_{i \in I} (C_i^1 \vee \neg D_i^1 \vee \cdots \vee C_i^{n_i} \vee \neg D_i^{n_i}),$$

for some $C_i^1, D_i^1, \ldots, C_i^{n_i}, D_i^{n_i} \in \mathcal{C}$.

The elements of $\mathcal{B}d$ are called *basic domains* and its bottom element is denoted by O. Although $\mathcal{C}$ does not need to be a closure system, we will refer to its elements as *basic closed* (*domains*) and to their complements in $\mathcal{B}d$ as *basic open* (*domains*).

The main motivating example is the case of locales: every locale L yields the b0-space

$$(L, \mathfrak{c}L, \mathsf{S}(L))$$

where $\mathfrak{c}L$ is the coframe of closed sublocales of L and $\mathsf{S}(L)$ is the coframe of all sublocales of L. We will also see how b0-spaces can be constructed from any Heyting semilattice.

The morphisms $f\colon \mathcal{S} \to \mathcal{T}$ of b0-spaces in the category **B0ds** are the *basic continuous maps*, that is, the maps between the underlying sets satisfying the following conditions:

(BC1) $f^{-1}[\mathsf{O}] = \mathsf{O}$.

(BC2) There exists a map $f^{\mathfrak{c}}_{\leftarrow} : \mathcal{C}(\mathcal{T}) \to \mathcal{C}(\mathcal{S})$, such that, for every $B \in \mathcal{C}(\mathcal{S})$ and $C \in \mathcal{C}(\mathcal{T})$,

$$f[B] \subseteq C \quad \text{iff} \quad B \subseteq f^{\mathfrak{c}}_{\leftarrow}[C].$$

(BC3) For every $C \in \mathcal{C}(\mathcal{T})$, $\neg f^{\mathfrak{c}}_{\leftarrow}[C] \subseteq f^{-1}[\neg C]$.

*Speaker.

(BC4) For every $C_i,\ D_i \in \mathcal{C}(\mathcal{T})$, $\bigvee\limits_{i=1}^{n} (f^{\mathfrak{c}}_{\leftarrow}[C_i] \vee \neg f^{\mathfrak{c}}_{\leftarrow}[D_i]) \subseteq f^{-1}[\bigvee\limits_{i=1}^{n} (C_i \vee \neg D_i)]$.

All the following categories are embedded in **B0ds**:

- **Locales and localic maps**.
- **Heyting semilattices and ℓ-morphisms** [3].
- **Zero-dimensional topological spaces and continuous maps**.
- **Closure spaces** [1] **and, in particular, topological spaces** (the embedding in this case is full).

Our main aim is to confirm that **B0ds** indeed provides a useful unifying framework for studying continuity and openness of maps, extending the modelling of continuous maps and open continuous maps in pointfree topology and Heyting semilattices. Working within the context of basic zero-dimensional spaces not only makes the results far more general but also illuminates the phenomena behind the proofs better.

We will survey our main results, namely:

1. Basic zero-dimensional spaces exhibit particularly good behaviour: the system of their substructures is always a coframe.
2. Furthermore, every basic continuous map admits a b0 analogue of the localic pre-image, called the *basic preimage*. The basic preimage is always a coframe homomorphism.
3. An extension to **B0ds** of the localic Joyal-Tierney theorem for open localic maps. Applied to Heyting semilattices, it gives an alternative proof of the Joyal-Tierney type-theorem for Heyting semilattices in [3].
4. Another feature of the category of basic zero-dimensional spaces is a "two for the price of one" duality principle:

   If a statement holds in **B0ds**, *then also its dual statement* (*obtained by swapping "basic open" and "basic closed"*) *holds.*

Applying these results to the category of Heyting semilattices and ℓ-morphisms, we overcome two constraints of this category identified in [3]: the standard system of "generalized subspaces" (the so-called *nuclear ranges*) is far from being a coframe, and there is no non-complete analogue for preimages of localic maps. This work also explains why open and closed localic maps behave so differently.

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A Directed Refinement of Constructive Ordinals in Algebraic Set Theory

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Abstract

Within the framework of Algebraic Set Theory (AST), as developed by Joyal and Moerdijk, ordinals may be defined categorically as the initial ZF-algebra with an inflationary successor operation. This construction coincides with the notions of ordinal in constructive set theory (CZF) and in Homotopy Type Theory (HoTT). In addition, Joyal and Moerdijk introduce Tarski ordinals, a particular initial ZF-algebra whose successor operation is monotone. This construction provides a constructive proof of Tarski's fixed point theorem and the resulting ordinals are directed.

In this work, we introduce and analyse a variant of directed ordinals obtained as the initial ZF-algebra with an inflationary successor whose strict downsets are closed under binary joins. We show the existence of this initial object and study its structural properties. The resulting construction gives a directed refinement of the usual notion of constructive ordinal within the framework of Algebraic Set Theory.

Introduction

Ordinals in constructive mathematics can be treated in various ways, one of which is formulated within the framework of Algebraic Set Theory (AST). Algebraic Set Theory, introduced by Joyal and Moerdijk [1], provides a categorical framework for studying set-theoretic structures in the form of ZF-algebras. A ZF-algebra $(A, \leq, s)$ consists of a partially ordered class $(A, \leq)$ in which all small suprema exist, together with a successor operation $s\colon A \to A$. The initial ZF-algebra, or the free ZF-algebra on the empty set, coincides with the cumulative hierarchy of sets V, ordered by inclusion, with suprema given by unions and successor s defined by $s(x) = \{x\}$.

We may also require the successor operation to be inflationary, i.e. $x \leq s(x)$ for all x. The initial ZF-algebra with inflationary successor coincides with the class of the hereditarily transitive sets. We explain that in this way it provides a categorical formulation of the notion of ordinal in constructive set theory (CZF) [2], which also coincides with the type-theoretic notion, as shown in [3].

Joyal and Moerdijk also consider the free partially ordered class with weakly directed suprema and a monotone successor. This structure turns out to be a ZF-algebra, known as the *Tarski ordinals*, as it provides a constructive proof of Tarski's fixed point theorem. Moreover, it can be characterized as the initial ZF-algebra on a successor that preserves binary joins. These ordinals are directed in relation to the strict order $<$ defined by

$$x < y \text{ iff } s(x) \leq y, \tag{1}$$

in the sense that whenever $\alpha, \beta < \gamma$, there exists an ordinal $\delta < \gamma$ such that $\alpha \leq \delta$ and $\beta \leq \delta$.

Main Contribution

Directedness is important in the constructive setting, since filtered colimits are better behaved in transfinite operations, and trichotomy, which classically guarantees directedness, is no longer available [4]. In this work, we therefore introduce and analyse a variant of directed ordinals in Algebraic Set Theory, which use an inflationary rather than monotone successor. We briefly discuss why some approaches fail before turning to our chosen one: ZF-algebras equipped with an inflationary successor, together with an additional requirement that for any α the strict downset[1]

$$\Downarrow \alpha = \{\, \beta \mid \beta < \alpha \,\},$$

with respect to the induced strict order $<$ given by (1) are closed under binary joins, i.e. for all $\beta, \gamma < \alpha$ we have $\beta \vee \gamma < \alpha$.

As discussed above, the ordinals in constructive set theory are given by the class of hereditarily transitive sets T^2, ordered by inclusion, with union as supremum and successor $s(x) = x \cup \{x\}$. This is precisely the initial ZF-algebra with inflationary successor, which we denote by N.

We define a recursive operation $*$ on T^2 in the following way:

$$x_1 * x_2 = x_1 \cup x_2 \cup \{y_1 * y_2 : y_1 \in x_1, y_2 \in x_2\}.$$

We consider the subclass $G \subseteq T^2$ consisting of those transitive sets that are hereditarily closed under $*$. We show that G, equipped with the inherited order and successor from N, and with joins obtained by first taking the finite closure under $*$ and then taking the union, forms a ZF-algebra. Moreover, we obtain the following result:

Theorem 1. *$J = (G, \subseteq, s)$ is the initial object in the category of ZF-algebras with inflationary successor and strict downsets closed under binary joins.*

Now, since we have that $x, y < \alpha$ implies $x \vee y < \alpha$, these ordinals are directed, like the Tarski ordinals. Rather than imposing the stronger condition of monotonicity on the successor, we worked with an inflationary successor, as in the usual notion of constructive ordinal. The additional closure condition on strict downsets under binary joins therefore provides a directed refinement of constructive ordinals in the framework of Algebraic Set Theory.

The construction coincides with a notion of directed ordinal previously introduced by Paul Taylor in a slightly different form [4]. Our approach differs in presentation and is developed independently within a predicative framework, in which we establish the freeness characterization in detail. Classically, the construction recovers the usual notion of ordinal.

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[1] Since we are working in predicative and constructive foundations (CZF), this is a priori a down*class* rather than a set. However, we show that for every α, the collection $\Downarrow \alpha$ can in fact be proven to form a set.

The lattice of smooth sublocales as a Bruns-Lakser completion

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For a frame L, the collection $\mathsf{S}(L)$ of sublocales is a coframe, that is, the order-theoretic dual of a frame. For every $a \in L$ we have its *closed* and *open* sublocale, given by

$$\mathfrak{c}(a) = \uparrow a \qquad \text{and} \qquad \mathfrak{o}(a) = \{\, a \to x \mid x \in L \,\},$$

respectively, which are complements of each other in $\mathsf{S}(L)$. The *locally closed* sublocales are those of the form $\mathfrak{c}(a) \cap \mathfrak{o}(b)$. Some subfamilies of $\mathsf{S}(L)$ have gained attention in the literature recently. On the one hand, we have the collection of *smooth sublocales* (see [1]):

$$\mathsf{S}_b(L) = \left\{ \bigvee_{a \in A,\, b \in B} \mathfrak{c}(a) \cap \mathfrak{o}(b) \;\middle|\; A, B \subseteq L \right\}$$

This forms a complete Boolean algebra (indeed, the Booleanization of $\mathsf{S}(L)$) and it has several applications in point-free topology (e.g., the naturality of the construction as a maximal essential extension [6], its role as a discretisation of a locale for modeling discontinuous localic maps [10], its role as the T_D–hull of a frame [8], stemming from its connection with Funayama's work [7], etc). Closely related is the frame $\mathsf{S}_c(L)$ of joins of closed sublocales (cf. [11]),

$$\mathsf{S}_c(L) = \left\{ \bigvee_{a \in A} \mathfrak{c}(a) \;\middle|\; A \subseteq L \right\},$$

which embeds into $\mathsf{S}_b(L)$ as a subframe. In general $\mathsf{S}_c(L)$ need not be Boolean (it is precisely for subfit frames), and actually it was recently shown in [2] that it need not even be a coframe.

Neither $\mathsf{S}_c(-)$ nor $\mathsf{S}_b(-)$ is functorial on all frame morphisms. This leads to the problem of characterising those morphisms that *lift*, which has recently attracted attention (see, e.g. [5]).

Recently, Suarez [12] solved the problem of characterising frame maps that lift to $\mathsf{S}_c(-)$. The characterization may be obtained by considering $\mathsf{S}_c(L)$ as a Bruns-Lakser completion. The *Bruns–Lakser completion* [9] of a join-semilattice is the collection of its *admissible upper sets*, namely those closed under admissible meets (i.e. the existing meets that distribute over all binary joins). The following is a known result.

Theorem 1. *A map of join-semilattices $f : S \to T$ lifts to their Bruns–Lakser completions if and only if it maps admissible families to admissible families, and preserves admissible meets.*

*Presenting author.

It is known (cf. [4]) that $\mathsf{S}_c(L)$ is the Bruns–Lakser completion of a frame L. It follows that the characterization of morphisms that lift (obtained in a different way in [12]) can be got as an instance of this general theorem. In this talk, we show that the same general principle can be used to solve the problem for $\mathsf{S}_b(-)$, following our recent work [3]. For that, we write

$$\mathsf{LC}(L) := \{(a,b) \in L \times L \mid a \leq b,\ b \to a = a\},$$

and equip $\mathsf{LC}(L)$ with an appropriate order making it a join-semilattice anti-isomorphic to the meet-semilattice of locally closed sublocales. We then prove our main result:

Theorem 2. *The Bruns–Lakser completion of* $\mathsf{LC}(L)$ *is isomorphic to* $\mathsf{S}_b(L)$*.*

Using this description, the lifting problem for $\mathsf{S}_b(-)$ becomes an instance of the general functoriality criterion for Bruns–Lakser completions.

Theorem 3. *Let* $f\colon L \to M$ *be a frame morphism. Then* f *lifts to a frame morphism* $\overline{f}\colon \mathsf{S}_b(L) \to \mathsf{S}_b(M)$ *if and only if* f *is* locally exact *(i.e. the induced map* $\mathsf{LC}(L) \to \mathsf{LC}(M)$ *preserves admissible families and their meets).*

Time permitting, we will also discuss several classes of frame morphisms that lift and some of the open problems of the theory.

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Implicit operations in reduced commutative rings

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We assume familiarity with the theory of implicit operations and Beth companions reviewed in [3] (see also [1] for a systematic presentation), as well as with the concept of dominions due to Isbell (see [4]) and basic notions of field theory (see, e.g., [5]), although we include a quick overview of the most important concepts below.

We recall from [1] that an n-ary *implicit operation* f on a class of similar algebras K is a family of partial functions $\langle f^{\mathbf{A}} : \mathbf{A} \in \mathsf{K}\rangle$ with $f^{\mathbf{A}} \colon \mathsf{dom}(f^{\mathbf{A}}) \to A$ that is preserved by homomorphisms in the sense that for all homomorphisms $h \colon \mathbf{A} \to \mathbf{B}$ between members $\mathbf{A}, \mathbf{B}$ of K and $\langle a_1, \ldots, a_n\rangle \in \mathsf{dom}(f^{\mathbf{A}})$ we have that

$$\langle h(a_1), \ldots, h(a_n)\rangle \in \mathsf{dom}(f^{\mathbf{B}}) \quad \text{and} \quad f^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) = h(f^{\mathbf{A}}(a_1, \ldots, a_n)).$$

A *pp expansion* of a quasivariety K is a class of algebras obtained by "adding" a family of implicit operations to K and a Beth companion of K is a pp expansion of K with the strong epimorphism surjectivity property.

In this talk (based on [2]), we focus on the class of reduced commutative rings and show that many of its structural deficiencies (such as the lack of an equational axiomatization, the failure of the amalgamation property, and the existence of nonsurjective epimorphisms) can be fixed by moving to its Beth companion. Moreover, we derive a tangible description of dominions in this context.

In the following, rings are considered as algebras in the language $\{+, -, \cdot, 0, 1\}$. A commutative ring in which every nonzero element a has a multiplicative inverse a^{-1} will be called a *field*. Given $n \in \mathbb{N}$, the notation na for an element a in some ring stands for the result of summing n times the element a.

The class RCR of *reduced commutative rings* forms a quasivariety axiomatized relative to the variety of commutative rings by the quasiequation $x^2 \approx 0 \to x \approx 0$. Equivalently, it can be described as the class of all the commutative rings that can be embedded into a direct product of fields (see, e.g., [6, 3.14 p. 23]).

Definition 1. *A field* $\mathbf{A}$ *is called* weakly rooted *if either it is of characteristic* 0 *or it is of prime characteristic* p *and for every* $a \in A$ *there exists* $b \in A$ *such that* $b^p = a$. *In the latter case,* b *is unique and will be called the* weak p-root *of* a *and denoted by* $\sqrt[p]{a}$.

Examples of weakly rooted fields, besides those of characteristic 0, include all finite fields and all algebraically closed fields.

For every prime number p consider the following formulas:

$$\mathsf{inv}(x, y) = (x^2 y \approx x) \sqcap (xy^2 \approx y) \quad \text{and} \quad \exists\mathsf{root}_p(x, y) = \exists z(\mathsf{inv}(p1, z) \sqcap (y^p \approx x - (p1)xz)).$$

*Speaker.

Proposition 2. *The formulas* $\mathsf{inv}(x, y)$ *and* $\exists\mathsf{root}_p(x, y)$ *define implicit operations* g *and* f_p*, respectively, of* RCR *for every prime* p*. Moreover,* $g^{\mathbf{A}}$ *and* $f_p^{\mathbf{A}}$ *are total for every prime* p *and weakly rooted field* $\mathbf{A}$ *and defined for every* $a \in A$ *as follows:*

$$g^{\mathbf{A}}(a) = \begin{cases} 0 & \text{if } a = 0; \\ a^{-1} & \text{else,} \end{cases} \qquad f_p^{\mathbf{A}}(a) = \begin{cases} \sqrt[p]{a} & \text{if } \mathbf{A} \text{ is of characteristic } p; \\ 0 & \text{else.} \end{cases}$$

By the previous proposition, we can expand every weakly rooted field with the set of implicit operations $\{g\} \cup \{f_p : p \text{ prime}\}$. The result of such an expansion will be called an *implicitly closed field* and every algebra in this expanded language that can be embedded into a direct product of implicitly closed fields will be called an *implicitly closed meadow*.

Theorem 3. *The class* ICM *of implicitly closed meadows is the Beth companion of* RCR *and thus has the strong epimorphism surjectivity property. Moreover, it is a discriminator variety with the amalgamation property.*

The rich structure theory of ICM (see Theorem 3) constitutes a significant improvement compared to that of RCR, which is a proper quasivariety that lacks both the amalgamation and the epimorphism surjectivity property.

Finally, we can apply our results to derive a tangible description of dominions in this context.

Definition 4. *Given a quasivariety* K *and* $\mathbf{A} \leq \mathbf{B} \in \mathsf{K}$*, the* dominion $\mathsf{d}_{\mathsf{K}}(\mathbf{A}, \mathbf{B})$ *of* $\mathbf{A}$ *in* $\mathbf{B}$ *relative to* K *is the set of all* $b \in B$ *such that* $g(b) = h(b)$ *for every pair of homomorphisms* $g, h \colon \mathbf{B} \to \mathbf{C}$ *with* $\mathbf{C} \in \mathsf{K}$ *such that* $g{\restriction}_A = h{\restriction}_A$*.*

It is an immediate consequence of the definition that a quasivariety K has the strong epimorphism surjectivity property if and only if $\mathsf{d}_{\mathsf{K}}(\mathbf{A}, \mathbf{B}) = A$ for all $\mathbf{A} \leq \mathbf{B} \in \mathsf{K}$.

For all $\mathbf{A} \in \mathsf{RCR}$ and prime ideals I of $\mathbf{A}$, let $\mathsf{icf}_I(\mathbf{A})$ be the unique implicitly closed field whose field reduct is the algebraic closure of the fraction field of the integral domain $\mathbf{A}/I$.

Theorem 5. *For every class* $\mathsf{K} \subseteq \mathsf{RCR}$ *that contains all fields and for all* $\mathbf{A} \leq \mathbf{B} \in \mathsf{K}$ *the dominion of* $\mathbf{A}$ *in* $\mathbf{B}$ *relative to* K *can be described as follows.*

$$\begin{aligned} \mathsf{d}_{\mathsf{K}}(\mathbf{A}, \mathbf{B}) = \{b \in B : & \text{ for every pair } I \text{ and } J \text{ of prime ideals of } \mathbf{B}, \\ & \langle b + I, b + J\rangle \text{ belongs to the subalgebra of } \mathsf{icf}_I(\mathbf{B}) \times \mathsf{icf}_J(\mathbf{B}) \\ & \text{generated by } \{\langle a + I, a + J\rangle : a \in A\}\}. \end{aligned}$$

This description becomes particularly simple when K is the class F of all fields, as for all $\mathbf{A} \leq \mathbf{B} \in \mathsf{F}$ the dominion $\mathsf{d}_{\mathsf{F}}(\mathbf{A}, \mathbf{B})$ is the least subfield of the algebraic closure of $\mathbf{B}$ that contains A and, moreover, is closed under weak prime roots.

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Defeasible Reasoning on Concepts*

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This paper contributes to a novel line of research on *defeasible* reasoning about concepts within the Formal Concept Analysis (FCA) framework. FCA provides a semantic environment for a broad class of lattice-based logics (LE-logics), which can be understood as logics of formal concepts [**?**, **?**]. A *model* for LE-logic is a tuple $\mathbb{M} = (\mathbb{P}, V)$, where $\mathbb{P} = (A, X, I)$ is a *formal context* and V is a *valuation* that maps LE-formulas to *formal concepts* determined by $\mathbb{P}$. For any object $a \in A$, feature $x \in X$, and formula $\phi \in \mathcal{L}$, we define relations $\Vdash$ and $\succ$ as follows: $\mathbb{M}, a \Vdash \phi$ iff a is in the extension of $V(\phi)$ and $\mathbb{M}, x \succ \phi$ iff x is in the intension of $V(\phi)$. We define *monotone* consequence (or entailment) relation $C_1 \vdash C_2$ between concepts, interpreted as concept inclusion: C_1 is a subconcept of C_2, that is, every object in the extension of C_1 belongs to the extension of C_2, or equivalently, every feature in the intension of C_2 belongs to the intension of C_1.

We generalize the KLM framework [**?**] to formalize three distinct defeasible entailment relations, $\mid\!\sim_A$, $\mid\!\sim_X$, and $\mid\!\sim_{AX}$, between concepts in FCA. For any concepts C_1 and C_2, the sequent $C_1 \mid\!\sim_A C_2$ is interpreted as "the typical objects of C_1 are in C_2", $C_1 \mid\!\sim_X C_2$ as "the typical features of C_2 are in C_1", and $C_1 \mid\!\sim_{AX} C_2$ as "the typical objects of C_1 have all the typical features of C_2." We introduce axiomatization systems **CC**, **CC**$^{\text{op}}$, and **EBC** for these relations by adapting classical systems for defeasible reasoning to the conceptual setting. Each system extends lattice-based propositional logic with suitable axioms and rules (displayed below). Their Loop-extensions **CCL**, **CCL**$^{\text{op}}$, and **EBCL** are obtained by adding (Loop), its dual[1] , or both, respectively.

CC	**CC**$^{\text{op}}$	**EBC**
$\phi \mid\!\sim_A \phi$	$\phi \mid\!\sim_X \phi$	**CC**+ **CC**$^{\text{op}}$
$\dfrac{\phi\vdash\psi \quad \psi\vdash\phi \quad \phi \mid\!\sim_A \chi}{\psi \mid\!\sim_A \chi}$	$\dfrac{\phi\vdash\psi \quad \psi\vdash\phi \quad \chi \mid\!\sim_X \phi}{\chi \mid\!\sim_X \psi}$	$\dfrac{\phi\vdash\psi \quad \psi\vdash\phi \quad \phi \mid\!\sim_{AX} \chi}{\psi \mid\!\sim_{AX} \chi}$ $\dfrac{\phi\vdash\psi \quad \psi\vdash\phi \quad \chi \mid\!\sim_{AX} \phi}{\chi \mid\!\sim_{AX} \psi}$
$\dfrac{\phi\vdash\psi \quad \chi \mid\!\sim_A \phi}{\chi \mid\!\sim_A \psi}$	$\dfrac{\phi\vdash\psi \quad \psi \mid\!\sim_X \chi}{\phi \mid\!\sim_X \chi}$	$\dfrac{\phi \mid\!\sim_A \psi}{\phi \mid\!\sim_{AX} \psi}$ $\dfrac{\psi \mid\!\sim_X \phi}{\psi \mid\!\sim_{AX} \phi}$
$\dfrac{\phi \mid\!\sim_A \psi \quad \phi \mid\!\sim_A \chi}{\phi\wedge\psi \mid\!\sim_A \chi}$	$\dfrac{\psi \mid\!\sim_X \phi \quad \chi \mid\!\sim_X \phi}{\chi \mid\!\sim_X \psi\vee\phi}$	$\dfrac{\phi \mid\!\sim_A \psi \quad \phi \mid\!\sim_{AX} \chi}{\phi\wedge\psi \mid\!\sim_{AX} \chi}$ $\dfrac{\psi \mid\!\sim_X \phi \quad \chi \mid\!\sim_{AX} \phi}{\chi \mid\!\sim_{AX} \psi\vee\phi}$
$\dfrac{\phi\wedge\psi \mid\!\sim_A \chi \quad \phi \mid\!\sim_A \psi}{\phi \mid\!\sim_A \chi}$	$\dfrac{\chi \mid\!\sim_X \psi\vee\phi \quad \psi \mid\!\sim_X \phi}{\chi \mid\!\sim_X \phi}$	$\dfrac{\phi\wedge\psi \mid\!\sim_{AX} \chi \quad \phi \mid\!\sim_A \psi}{\phi \mid\!\sim_{AX} \chi}$ $\dfrac{\chi \mid\!\sim_{AX} \psi\vee\phi \quad \psi \mid\!\sim_X \phi}{\chi \mid\!\sim_{AX} \phi}$

$$\frac{\phi_0 \mid\!\sim_A \phi_1 \quad \dots \quad \phi_{n-1} \mid\!\sim_A \phi_n \quad \phi_n \mid\!\sim_A \phi_0}{\phi_0 \mid\!\sim_A \phi_n} \quad \text{(Loop)}$$

The semantics of these defeasible consequence relations are obtained by generalizing cumulative models and their subclasses from [**?**]. Let $P \subseteq \mathcal{U}$ and $\prec$ be a binary relation on $\mathcal{U}$. P is said to be *smooth* if for any $t \in P$, either there exists s minimal in P s.t. $s \prec t$, or t itself is minimal in P. A (resp. *dual*) *pointed polarity-based model* is a tuple $\mathbb{M}_a = (\mathbb{M}, a)$ (resp. $\mathbb{M}_x = (\mathbb{M}, x)$),

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[1]Dual Loop rule is defined by replacing $\mid\!\sim_A$ with $\mid\!\sim_X$ in the rule Loop.

where $a \in A$ (resp. $x \in X$). Let $\mathcal{U}$ (resp. $\mathcal{V}$) be a set of (resp. dual) pointed polarity-based models, and $\mathcal{M}_A = (S_A, l_A, \prec_A)$ (resp. $\mathcal{N}_X = (S_X, l_X, \prec_X)$) be a tuple, where S_A (resp. S_X) is a non-empty set of *states*, $l_A : S_A \to \mathcal{P}(\mathcal{U})$ (resp, $l_X : S_X \to \mathcal{P}(\mathcal{V})$) is a map that assigns to each state a set of (resp. dual) pointed polarity-based models, and $\prec_A$ (resp. $\prec_X$) is a binary relation on S_A (resp. S_X). For any $\phi \in \mathcal{L}$ and $s \in S_A$ (resp. $s \in S_X$), $s \models_A \phi$ (resp. $s \models_X \phi$) if and only if for all $\mathbb{M}_a \in l_A(s)$ (resp. $\mathbb{N}_x \in l_X(s)$), $\mathbb{M}, a \Vdash \phi$ (resp. $\mathbb{N}, x \succ \phi$). $\mathcal{M}_A$ (resp. $\mathcal{N}_X$) is said to be a *conceptual cumulative model* (resp. *dual conceptual cumulative model*) if the set $\widehat{\phi}_A = \{s \mid s \in S_A, s \models_A \phi\}$ (resp. $\widehat{\phi}_X = \{s \mid s \in S_X, s \models_X \phi\}$) is smooth for any $\phi \in \mathcal{L}$. A conceptual bi-cumulative model is a tuple $\mathcal{B} = (\mathcal{M}_A, \mathcal{N}_X)$, where $\mathcal{M}_A$ (resp. $\mathcal{N}_X$)) is a (resp. dual) conceptual cumulative model.

A conceptual cumulative model $\mathcal{M}_A$ is said to be a *conceptual preferential model* if l_A assigns every state to a single pointed polarity based model. A conceptual cumulative model (resp. conceptual preferential model) $\mathcal{M}_A$ is said to be *conceptual cumulative ordered model* (resp. *conceptual preferential ordered model*) if $\prec_A$ is a strict partial order. We can define the corresponding subclasses of dual conceptual cumulative models in an analogous manner.

Every (resp. dual) conceptual cumulative model $\mathcal{M}_A$ (resp. $\mathcal{N}_X$) and a conceptual bi-cumulative model $\mathcal{B} = (\mathcal{M}_A, \mathcal{N}_X)$ define *consequence relations* as follows: $\phi \mid\!\sim_{\mathcal{M}_A} \psi$ (resp. $\phi \mid\!\sim_{\mathcal{N}_X} \psi$) if and only if for any s minimal in $\widehat{\phi}_A$ (resp. $\widehat{\psi}_X$), $s \in \widehat{\psi}_A$ (resp. $s \in \widehat{\phi}_X$) and $\phi \mid\!\sim_{\mathcal{B}_{AX}} \psi$ if and only if for any $s_1 \in S_A$ minimal in $\widehat{\phi}_A$ and $s_2 \in S_X$ minimal in $\widehat{\psi}_X$, if $\mathbb{M}_a \in l_A(s_1)$, $\mathbb{N}_x \in l_X(s_2)$ are based on the same polarity-based model, then aIx.

We can now generalize the completeness results shown in [**?**] to the conceptual defeasible reasoning systems **CC**, $\mathbf{CC}^{\mathrm{op}}$, and **EBC**. In particular, these logical systems are sound and complete with respect to the corresponding class of models described in the following table.

Logical Systems	Models
CC (resp. **CCL**)	Conceptual Cumulative (resp. Ordered) Models
$\mathbf{CC}^{\mathrm{op}}$ (resp. $\mathbf{CCL}^{\mathrm{op}}$)	Dual Conceptual Cumulative (resp. Ordered) Models
EBC (resp. **EBCL**)	Conceptual Bi-cumulative (resp. Ordered) Models

Table 1: Soundness and Completeness

However, unlike the classical setting, in a conceptual setting we can show that the reasoning system **CC** (resp. **CCL**) is complete w.r.t. the class of conceptual preferential (resp. ordered) models, whereas the reasoning system $\mathbf{CC}^{\mathrm{op}}$ (resp. $\mathbf{CCL}^{\mathrm{op}}$) is complete w.r.t. the class of dual conceptual preferential (resp. ordered) models.

The future directions of this research line include studying the complexity of the reasoning systems discussed, extending this framework to defeasible reasoning systems incorporating Rational Monotonicity and ranked preference models, and exploring connections to AGM belief revision in a conceptual setting.

Coequivalence relations and descent in modal and superintuitionistic logic

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In algebraic logic it has long been a subject of interest to correspond categorical properties of classes of models, on one hand, to logical properties of calculi, especially definability properties. In [**Ghilardi2002**], Ghilardi and Zawadowski introduced the concept of a *coequivalence relation*, a formula $\rho(\overline{p}_1, \overline{p}_2)$ which, in some context $\tau(\overline{p})$, is proved to be an equivalence relation by a logic L. Coequivalence relations in a logic L correspond to r-equivalence relations in the category $\mathsf{Alg}(L)^{op}_{fp}$. Coequivalence formulas equivalent to formulas of the form,

$$\tau(\overline{p}_1) \wedge \tau(\overline{p}_2) \wedge \bigwedge_{i=1}^{m} \psi_i(\overline{p}_1) \leftrightarrow \psi_i(\overline{p}_2)$$

are said to *separate variables.* Separating coequivalence relations correspond to *effective* r-equivalence relations, i.e., those that arise from a quotient.

Definition 1. A logic L is said to have the *coequivalence separation property* (CoSP) if all of its coequivalence relations separate variables.

This property stands in analogy with Beth definability: whereas projective Beth says that a term proved to be functional can be made explicit, coequivalence separation says that equivalence classes in the logic can be picked out by formulas explicitly. Categorically, the conjunction of these two properties corresponds to the Barr-exactness of the category $\mathsf{Alg}(L)^{op}_{fp}$. Model theoretically, it is related to questions of uniform elimination of imaginaries [**poizattheoriegalois**], and it has also been considered in the context of database theory [**Benedikt2024InterfacesTD**].

Ghilardi and Zawadowski [**Ghilardi2002**] showed that IPC fails to have the CoSP. Work of the second author [**barrcoexactness**] implies that several systems of modal logic, including the well-known system S5, fail to have the CoSP. At the same time, Ghilardi and Zawadowski asked whether for systems such as IPC one has that $\mathsf{Alg}(L)^{op}_{fp}$ is *almost Barr-exact.*

Given $\mathbf{C}$ a category with pullbacks and $p : E \to B$ a morphism in $\mathbf{C}$, there is a pair of adjoint functors

$$p_! \dashv p^* : \mathbf{C}/B \longrightarrow \mathbf{C}/E$$

with p^* pulling back along p and with $p_!$ composing with p from the left. We denote the category of algebras for the monad induced by this adjunction by $\mathsf{Des}(p)$, the category of *descent data* for p. We denote by $\Phi^p : \mathbf{C}/B \to \mathsf{Des}(p)$ the comparison functor.

Definition 2. A morphism p is an *effective descent morphism* if Φ^p is an equivalence of categories.

In other words, effective descent morphisms allow for an algebraic description of the B-indexed families by means of the E-indexed families in $\mathbf{C}$. Effective descent morphisms are always regular epimorphisms. Moreover, if $\mathbf{C}$ is Barr-exact, the vice versa holds.

Definition 3. A regular category $\mathbf{C}$ is said to be *almost Barr-exact* if all regular epimorphisms are effective descent morphisms.

An equivalent condition for regular categories is that all equivalence relations arising out of descent data are effective. These properties have been studied in category theory [**Janelidze˙Sobral˙Tholen˙2003**], locale theory [**PLEWE1997**], and algebra [**Zangurashvili2013**]. We introduce a logical counterpart:

For ease of notation, given $\alpha(\overline{p})$, we write $\mathsf{Ext}(\alpha) = \{\beta(\overline{p}, \overline{q}) : \beta \vdash \alpha\}$.

Definition 4. Let $\gamma(\overline{p}, \overline{q}, \overline{r}) \in \mathsf{Ext}(\beta(\overline{p}, \overline{q}))$. A sequence $\overline{\xi}$ of formulas over the variables $\overline{p}, \overline{q}_0, \overline{q}_1, \overline{r}$, namely $\xi_r(\overline{p}, \overline{q}_0, \overline{q}_1, \overline{r})$ for each $r \in \overline{r}$, is called a sequence of *local transition terms* if the following hold:

1. (γ-compatibility): $\gamma(\overline{p}, \overline{q}_0, \overline{r}) \wedge \beta(\overline{p}, \overline{q}_1) \vdash_L \gamma(\overline{p}, \overline{q}_1, \overline{\xi})$.
2. (Identity): For each $r \in \overline{r}$, $\gamma(\overline{p}, \overline{q}, \overline{r}) \vdash_L r \leftrightarrow \xi_r(\overline{p}, \overline{q}, \overline{q}, \overline{r})$.
3. (Cocycle): For each $r \in \overline{r}$, $\gamma(\overline{p}, \overline{q}_0, \overline{r}) \wedge \beta(\overline{p}, \overline{q}_1) \wedge \beta(\overline{p}, \overline{q}_2) \vdash_L \xi_r(\overline{p}, \overline{q}_0, \overline{q}_2, \overline{r}) \leftrightarrow \xi_r(\overline{p}, \overline{q}_1, \overline{q}_2, \overline{\xi})$.

Definition 5. Let $\alpha(\overline{p})$, $\beta(\overline{p}, \overline{q}) \in \mathsf{Ext}(\alpha)$ and $\gamma(\overline{p}, \overline{q}, \overline{r}) \in \mathsf{Ext}(\beta)$, equipped with a sequence of local transition terms $\overline{\xi}$. The *local coequivalence relation* associated is the γ-coequivalence relation

$$\rho_{\gamma,\overline{\xi}} = \gamma(\overline{p}_1, \overline{q}_1, \overline{r}_1) \wedge \gamma(\overline{p}_2, \overline{q}_2, \overline{r}_2) \wedge \bigwedge_{p \in \overline{p}} (p_1 \leftrightarrow p_2) \wedge \bigwedge_{r \in \overline{r}} (\xi_r(\overline{p}_1, \overline{q}_1, \overline{q}_2, \overline{r}_1) \leftrightarrow r_2).$$

Definition 6. A logic L has the *local coequivalence separation property* (LCoSP) if for each $\alpha(\overline{p})$, $\beta(\overline{p}, \overline{q}) \in \mathsf{Ext}(\alpha)$, $\gamma(\overline{p}, \overline{q}, \overline{r}) \in \mathsf{Ext}(\beta)$ and a sequence of local transition terms $\overline{\xi}$, the local coequivalence relation $\rho_{\gamma,\overline{\xi}}$ separates variables.

Our basic bridge theorem is the following:

Theorem 7. *For L a logic with the Beth property, L has the local coequivalence separation property if and only if in $\mathsf{Alg}(L)^{op}_{fp}$ all descent equivalence relations are effective. If L has the right-uniform Maehara interpolation, then this is additionally equivalent to $\mathsf{Alg}(L)^{op}_{fp}$ being an almost Barr-exact category.*

Using this bridge theorem and that of Ghilardi and Zawadowski, we prove the following results concerning the CoSP and the LCoSP:

Theorem 8. *The following systems fail to have the* CoSP*:* S5, IPC, LC.

Using a characterization of S5 free objects due to [**Bass1958**], and the Birkhoff adjunction, which represents them as $\mathsf{Fam}(\mathbf{FinSet}^{s}_{\neq\emptyset})$, families of non-empty finite sets with surjective maps, we prove the following:

Theorem 9. *The system* S5 *has the* LCoSP.

In the setting of S5 we also look at the *categorical Galois theory* [**janelidzeborceaux**] of S5 frames. By exploiting the adjunction employed in our representation, we show:

Theorem 10. *In* S5 *with respect to the Birkhoff adjunction, the morphisms of Galois descent are exactly the étale maps, i.e. the maps where the restriction to each cluster is an isomorphism.*

On the lattice of subvarieties of equational states

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Recent research in many-valued logics has focused on the study of probability theory within the framework of Łukasiewicz logic with the addition of a modal operator $\Box$ to the language. This approach has become particularly fruitful even beyond this specific field of research (see, for example, [5] or [9] for more traditional treatments of modalities in fuzzy logics). The main results can be found in [4], for probability on Boolean events, and [2] and [3], where the logic $\mathrm{FP}(\text{Ł},\text{Ł})$ is defined in order to reason about probability on vague events. In this framework, the truth value of a formula of type $\Box(\varphi)$ represents the probability of φ. All of these logics are two-layered, which makes it difficult to provide an adequate one-sorted algebraic semantics. This problem can be solved using the two-sorted variety of *equational states*, introduced in [6].

Definition 1 ([6, Definition 3.1]). *An equational state is a two-sorted algebra $\mathsf{A} \coloneqq (\mathbf{A}_1, \mathbf{A}_2, s)$, where both $\mathbf{A}_1$ and $\mathbf{A}_2$ are MV-algebras and $s\colon \mathbf{A}_1 \to \mathbf{A}_2$ is an operation (called the* state-operation*), such that $s(1_{A_1}) = 1_{A_2}$, $s(\neg a) = \neg s(a)$ and $s(a \oplus b) = s(a) \oplus s(\neg a \wedge b)$.*

The class of equational states is an algebraic semantics for $\mathrm{FP}(\text{Ł},\text{Ł})$ by [8, Proposition 6 and Theorem 8]. As a consequence of this result, we can study probability using the standard tools of Universal Algebra. In particular, the following complete characterization of subdirectly irreducible equational states is given in [7].

Theorem 1. *An equational state $\mathsf{A} \coloneqq (\mathbf{A}_1, \mathbf{A}_2, s)$ is subdirectly irreducible if and only if one of the following conditions holds:*

1. *$\mathbf{A}_2 = \{ 0 \}$ and $\mathbf{A}_1$ is a subdirectly irreducible MV-algebra,*
2. *$\mathbf{A}_2$ is a subdirectly irreducible MV-algebra and $s(x) = 0_{A_2}$ if and only if $x = 0_{A_1}$.*

The ultimate goal of this work is the complete characterization of all subvarieties of equational states starting from the previous result, in analogy with Komori's classification for subvarieties of MV-algebra (see [1, Theorem 8.4.4]). We first prove the following property.

Theorem 2. *For any equational state A, its lattice of congruences is distributive.*

This result is noteworthy, since as a corollary we can prove that a two-sorted version of Jónsson's Theorem holds for the variety of equational states. This implies that we can effectively determine the inclusion order between subvarieties generated by some classes of equational states. An example is given by finitely generated subvarieties $\mathbb{V}(\mathbf{A}_1, \mathbf{A}_2, s)$, where $(\mathbf{A}_1, \mathbf{A}_2, s)$ is finite, using the characterization of finite subdirectly irreducible equational states. To do so, we notice that by [1, Proposition 3.6.5] all finite MV-algebras are isomorphic to $\text{Ł}_{k_1} \times \cdots \times \text{Ł}_{k_n}$, where $\text{Ł}_k \coloneqq \{ 0, 1/k, \ldots, (k-1)/k, 1 \}$ for any positive natural k. We denote the i-th atom by a_i. Moreover, since by Theorem 1 the second sort is trivial or subdirectly irreducible, we derive that in the second case $\mathbf{A}_2 \cong \text{Ł}_k$. From this, we derive the following result.

Proposition 1. *Let* $\mathsf{A} \coloneqq (\mathbf{A}_1, Ł_k, s)$ *be a finite equational state. Then* A *is subdirectly irreducible if and only if* $\mathbf{A}_1 \cong Ł_{k_1} \times \cdots \times Ł_{k_n}$, $m_i \neq 0$ *for all* $1 \leq i \leq n$ *and* $\sum_{i=1}^{n} m_i k_i = k$, *where* $s(a_i) = m_i/k$.

To conclude, we study other classes of subdirectly irreducible equational states where the first sort is finitely generated. The main idea is to see the state-operation as a restriction of a state on the ℓ-group $\Gamma^{-1}(\mathbf{A}_1)$, where Γ^{-1} is the inverse of Mundici's equivalence (see [1, Section 7.1] and [6, Lemma 2.2]). Since states between ℓ-groups are unital positive group homomorphisms, if $\Gamma^{-1}(\mathbf{A}_2)$ is a subgroup of $\mathbb{R}^I$, the condition of having a positive group homomorphism can be expressed using a set of inequalities. In particular, the condition of group homomorphism can be expressed using a (possibly infinite) system of equations $E_{s,\mathbf{A}_1,\Gamma^{-1}(\mathbf{A}_2)}$. This method allows us to classify subdirectly irreducible equational states for some choices of the second sort, such as an infinite subalgebra of $[0,1]$, or, as it is done in the following result, for the Chang MV-algebra $\mathbf{C} \coloneqq \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1,0))$. We say that $x \in \mathbf{A}$ has the *downward* (respectively, *upward*) *finite chain property* if all chains between 0_A and x (respectively, between x and 1_A) are finite. We denote the set of elements with the downward finite chain property by $\mathbf{A}^{\downarrow}$. Moreover, we write $x \in (\mathbf{A}^{\downarrow})_n$ whenever $x \in \mathbf{A}^{\downarrow}$ and n is the length of a longest chain between 0_A and x.

Theorem 3. *An equational state of type* $(\mathbf{A}, \mathbf{C}, s)$ *is subdirectly irreducible if and only if*

1. *all* $x \in \mathbf{A}$ *have the downward or upward finite chain property;*
2. *the system of equations* $E_{s,\mathbf{A},\mathbb{Z}\times\mathbb{Z}} \cup \{ s(1_A) = (1,0) \}$ *is satisfied;*
3. *if* $x \in (\mathbf{A}^{\downarrow})_n$ *then* $s(x) \geq (0, n-1)$.

This is a joint work in progress with Nick Galatos and Serafina Lapenta.

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Modal Logics of Zariski-constructible and Semialgebraic Sets

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It was shown in [**Euclidean˙hierarchy**] that the modal logic of the closure algebras $\mathcal{CH}(\mathbb{R}^n)$ of chequered sets in $\mathbb{R}^n$ – which are finite unions of Cartesian products of open intervals and points – is $L(F^n)$, the logic of the nth power of the two-fork. In contrast, it follows from the classic McKinsey-Tarski theorem ([**McTarski**]) that the modal logics of the closure algebras on the power set $\mathcal{P}(\mathbb{R}^n)$ are all **S4**.

We establish two new Euclidean hierarchies by investigating the modal logics of the following subalgebras of $\mathcal{P}(\mathbb{R}^n)$:

$\mathcal{C}(\mathbb{A}^n)$**,** the closure algebra generated by algebraic sets in $\mathbb{R}^n$, i.e. sets $X \subseteq \mathbb{R}^n$, for which a set $A \subseteq \mathbb{R}[x_1, \ldots, x_n]$ of polynomials exists such that $X = \{\vec{x} \in \mathbb{R}^n \mid p(\vec{x}) = 0 \text{ for all } p \in A\}$. It turns out that elements of $\mathcal{C}(\mathbb{A}^n)$ coincide with the constructible sets of the affine spaces $\mathbb{A}^n$, i.e. $\mathbb{R}^n$ with the Zariski topology.

$\mathcal{SA}(\mathbb{R}^n)$**,** the closure algebra of semialgebraic sets in $\mathbb{R}^n$, i.e. definable sets (with parameters) in the L_{or}-structure $(\mathbb{R}, 0, 1, +, \cdot, <)$ over the language of ordered rings.

Our findings are summarized in figure 1, which shows the lattice of modal logics of the closure algebras $\mathcal{C}(\mathbb{A}^n)$ and $\mathcal{SA}(\mathbb{R}^n)$.

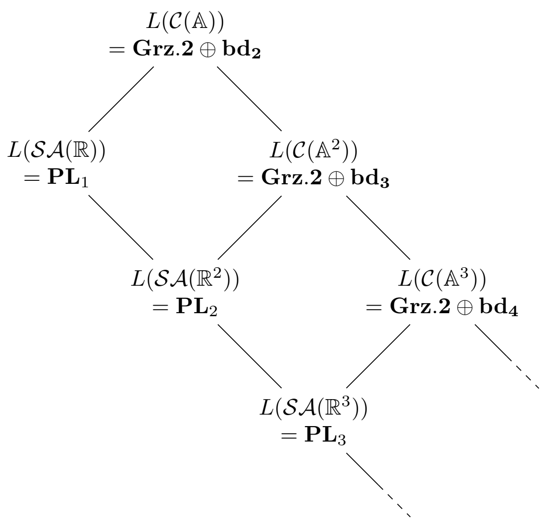

Figure 1:

Stratifications. We use stratifications to determine the logics of these closure algebras. Stratifications can be viewed as surjective interior maps $\pi : X \twoheadrightarrow I_\pi$ from a space onto a finite poset with the Alexandrov topology, the decomposition space (cf. [**lukaswaas**]). Let $\mathcal{C}(X)$ denote the Grz algebra of constructible sets, i.e. the Boolean envelope of the topology. Then, both $\mathcal{C}(\mathbb{A}^n)$ and $\mathcal{SA}(\mathbb{R}^n)$ are locally finite subalgebras of $\mathcal{C}(\mathbb{R}^n)$. For $A \subseteq \mathcal{C}(X)$, we speak of an A-stratification if all preimages $\pi^{-1}(i) \in A$ for $i \in I_\pi$. We use the following method to determine the modal logics of $\mathcal{C}(\mathbb{A}^n)$ and $\mathcal{SA}(\mathbb{R}^n)$:

Lemma 1 (Theorem 3.38 in [**MA**])**.** *Let $A \leq \mathcal{C}(X)$ be a locally finite subalgebra. Then, the logic of A is the logic of the decomposition spaces of all A-stratifications of X.*

Logic of Zariski-constructible sets. We show that the Grz algebras $\mathcal{C}(\mathbb{A}^n)$ validate the formula $\Diamond\Box p \to \Box\Diamond p$ and the axiom of bounded depth $\mathbf{bd}_{n+1}$. It turns out that this is an axiomatization for the logic $L(\mathcal{C}(\mathbb{A}^n))$.

Theorem 2 (Corollary 5.29 in [**MA**])**.** $L(\mathcal{C}(\mathbb{A}^n)) = \mathbf{Grz.2} \oplus \mathbf{bd}_{n+1}$.

The reason why $\mathcal{C}(\mathbb{A}^n)$ validates $\Diamond\Box p \to \Box\Diamond p$ is ultimately that $\mathbb{A}^n$ are irreducible (a.k.a. hyperconnected); and validation of $\mathbf{bd}_{n+1}$ can be traced back to $\mathbb{A}^n$ having Krull dimension n.

It is known that $\mathbf{Grz.2} \oplus \mathbf{bd}_{n+1}$ is complete with respect to graded trees of height $n-1$ with an extra top element. In order to prove Theorem 2, we show that all such topped trees can be realized as decomposition spaces of stratifications of $\mathbb{A}^n$. Specifically, we assign polynomial maps to the points of these frames and build a stratification from their graphs.

We think that generalizing away from the affine spaces and instead considering the closure algebras of constructible sets in spectra of commutative rings could foster connections between modern algebraic geometry and modal logic.

Logic of semialgebraic sets. The modal logics $\mathbf{PL}_n$ are the polyhedral logics of the n-simplex and were axiomatized in [**convex**] in form of intermediate logics. We show that they are also the logics of semialgebraic sets in $\mathbb{R}^n$.

Theorem 3 (Corollary 5.65 in [**MA**])**.** $L(\mathcal{SA}(\mathbb{R}^n)) = \mathbf{PL}_n$.

A central step in proving Theorem 3 is to show that $\mathbf{PL}_n$ is the logic of $\mathcal{SA}(\mathbf{B}^n)$, the logic of the closure algebra of semialgebraic subsets of the closed n-dimensional unit ball. We use the semialgebraic Hauptvermutung (the result that semialgebraic polyhedra are PL homeomorphic from [**HV**]) to generalize this to arbitrary compact semialgebraic sets. It turns out that their modal logic is always polyhedral:

Theorem 4 (Corollary 5.60 in [**MA**])**.** *Let $C \subset \mathbb{R}^n$ be a compact semialgebraic set. Then, there exists a polyhedron P such that $L(\mathcal{SA}(C))$ is the polyhedral logic of P. Furthermore, such P is unique up to PL homeomorphism.*

The results of Theorem 3 and Theorem 4 can be understood in the way that polyhedral logic does not just apply to polyhedra but also to a larger class of topological spaces.

Generalizing the d-nucleus to stably continuous frames

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An ℓ-ideal I of a Riesz space is a *d-ideal* provided

$$a \in I \text{ and } a^d = b^d \Longrightarrow b \in I,$$

where a^d is the set of elements disjoint from a (see, e.g., [Lux73, HdP80]). For each ℓ-ideal I, there is a least d-ideal dI containing I, and the correspondence $I \mapsto dI$ is a nucleus, whose fixpoints are precisely the d-ideals.

Martínez and Zenk [MZ03] initiated the study of the corresponding nucleus on an arbitrary arithmetic frame, which they termed the *d-nucleus*. Given such a frame L and denoting the pseudocomplement of $a \in L$ by a^*, we have:

$$da = \bigvee\{k^{**} \mid k \text{ is compact and } k \leq a\}.$$

They showed that d is the largest nucleus below the double negation nucleus that preserves directed joins. In other words, using the terminology of [Esc99], d is the largest *Scott continuous* nucleus below the double negation nucleus.

The d-nucleus and the corresponding sublocale dL were further studied in [DI13, DS19, BKM23]. The study of the spectrum of maximal d-elements was initiated in [Bha19]. In general, the maximal d-spectrum is T_1 and it is compact provided L has a unit. However, it need not be Hausdorff [BBM25]. This is surprising since in the classical setting of d-ideals and d-subgroups, the corresponding space is always Hausdorff. In general, it remains open precisely which compact Hausdorff spaces arise as maximal d-spectra of arithmetic frames.

We provide a positive answer to this question in the zero-dimensional case. Using Priestley duality for arithmetic frames as developed in [BM25], we show that every Stone space is homeomorphic to the maximal d-spectrum of an arithmetic frame (in fact, of a Stone frame). In addition, we obtain an alternate description of the Gleason cover of a compact Hausdorff space in terms of compact d-elements.

We then generalize the d nucleus to the setting of *stably continuous frames*. While the classical d-nucleus is defined in terms of compact elements of an arithmetic frame, in the broader setting of continuous frames, one works instead with the *way-below relation* $\ll$ (see, e.g., [GHK+03]). Replacing compact elements by the way-below relation, we define

$$\underline{d}a = \bigvee\{b^{**} \mid b \ll a\}.$$

We show that in the more general setting of stably continuous frames, $\underline{d}$ plays the same role as d does for arithmetic frames:

- For a stably continuous frame L, $\underline{d}$ is the largest Scott-continuous nucleus below the double negation nucleus.

*Speaker.

- If L happens to be arithmetic, then $\underline{d}$ coincides with d.

Using Priestley duality for stably continuous frames [BM23] we describe the $\underline{d}$-nucleus and identify the corresponding space of points, thereby generalizing the description of [BBM25]. Finally, we prove that every compact Hausdorff space is homeomorphic to the maximal $\underline{d}$-spectrum of a stably continuous frame (in fact, of a compact regular frame). Whether every compact Hausdorff space is homeomorhic to the maximal d-spectrum of an arithmetic frame remains an interesting open problem.

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Clonoids over finite vector spaces and related computational problems

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Abstract

A *clone* is a set of finitary operations on a given set that contains all projections and is closed under functional composition. Clones are a fundamental object in universal algebra, as they describe the set of term operations of any algebraic structure. Thus, starting with Post's classification of clones on a two-element set [5], there has been a rich history of structural results on finite algebras based on clone theory, see e.g. [6].

In recent years, generalizations of clones, such as *minions* and *clonoids*, have received increasing attention, especially for their connection with computational complexity theory.

Definition 1 (Clonoid)**.** Let $\mathcal{A}$ be a clone on a set A, and $\mathcal{B}$ be a clone on a set B. We then say that $\mathcal{C} \subseteq \mathcal{O}_{A,B} = \cup_{n\in\mathbb{N}} B^{A^n}$ is a *clonoid from* $\mathcal{A}$ *to* $\mathcal{B}$ (or $(\mathcal{A},\mathcal{B})$-*clonoid*, for short) if, for all $n, k \in \mathbb{N}$

- $f_1, \dots, f_n \in \mathcal{C}^{(k)}, g \in \mathcal{B}^{(n)} \Rightarrow g \circ (f_1, \dots, f_n) \in \mathcal{C}^{(k)}$,
- $f \in \mathcal{C}^{(k)}, g_1, \dots, g_k \in \mathcal{A}^{(n)} \Rightarrow f \circ (g_1, \dots, g_k) \in \mathcal{C}^{(n)}$.

In this paper, we are interested in classifying clonoids, for which $\mathbf{B}$ is a module. Such clonoids often appear naturally when studying the term operations of Mal'cev algebras with a central congruence.

There is already a series of classification results in the literature. The clonoids between groups of prime order $\mathbb{Z}_p$ and $\mathbb{Z}_q$ were completely classified by Kreinecker [3] (in the case that $p = q$) and the first author, for $p \neq q$ [1]. Interestingly, in the latter case, there are only finitely many clonoids that are all generated by their unary functions (while for $p = q$ one obtains infinitely many). The most recent and most general such finiteness result is by Mayr and Wynne [4], and it applies to all cases where $\mathbf{A}$ has a distributive lattice of submodules. This leads us to the following Conjecture:

Conjecture 2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite modules. Then, there are only finitely many clonoids from* $\mathbf{A}$ *to* $\mathbf{B}$*, if and only if* $\mathbf{A}$ *and* $\mathbf{B}$ *are of coprime order.*

The first main contribution of our paper is to confirm Conjecture 2 if $\mathbf{A}$ is a finite vector space.

Theorem 3. *Let* $\mathbf{F}^k$ *be the* k*-dimensional vector space over a finite field* $\mathbf{F}$*, and let* $\mathbf{B}$ *be a* $\mathbf{R}$*-module such that* $|F|$ *is invertible in* $\mathbf{R}$*. Then every clonoid* $\mathcal{C}$ *from* $\mathbf{F}^k$ *to* $\mathbf{B}$ *is generated by its* k*-ary part, i.e.,* $\mathcal{C} = \langle \mathcal{C}^{(k)} \rangle_{\mathbf{F}^k,\mathbf{B}}$.

We furthermore show that the arity k is optimal, i.e., not every clonoid from $\mathbf{F}^k$ to $\mathbf{B}$ is generated by their $k-1$-ary functions.

Theorem 4. *Let* $\mathbf{A}$ *be a finite* $\mathbf{R}_A$*-module and* $\mathbf{B}$ *be a finite* $\mathbf{R}_B$*-module. Assume that the full clonoid* $\mathcal{O}_{\mathbf{A},\mathbf{B}}$ *is generated by the* m*-ary functions, for some* m*. Then* $m \geq \frac{\log |A|}{\log |R_A|}$.

If we assume $\mathbf{B}$ to be finite, we can prove the following result:

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Corollary 5. *Let $\mathbf{F}^k$ be the k-dimensional vector space over a finite field $\mathbf{F}$, and let $\mathbf{B}$ be a finite module of coprime order. Then*

1. *there are only finitely many clonoids from $\mathbf{F}^k$ to $\mathbf{B}$.*
2. *every clonoid from $\mathbf{F}^k$ to $\mathbf{B}$ is finitely related by an at most $|F|^{k\times k}$-ary relation.*

In the case where $\mathbf{B}$ is coprime to the vector space $\mathbf{A}$, we are moreover able to give an explicit description of the lattice of all clonoids (Theorem 6). In particular, this answers [4, Question 1.2], which asked for a classification of clonoids if $\mathbf{A}$ is the group $(\mathbb{Z}_p)^2$, for some prime p.

Theorem 6. *The lattice of clonoids from $\mathbf{F}^k$ to $\mathbf{B}$ is isomorphic to $\prod_{i=0}^{k} \mathrm{Sub}(\mathbf{M}_{i,k}(\mathbf{F},\mathbf{B}))$. Moreover, for $j \leq k$, the sublattice $\prod_{i=0}^{j} \mathrm{Sub}(\mathbf{M}_{i,k}(\mathbf{F},\mathbf{B}))$ corresponds to the clonoids consisting of functions that are generated by their j-ary part.*

By combining our result with [4], we are further able to confirm Conjecture 2 for all modules $\mathbf{A} = \mathbf{F}_1^{k_1} \times \cdots \times \mathbf{F}_n^{k_n} \times \mathbf{D}$ as $\mathbf{F}_1 \times \cdots \times \mathbf{F}_n \times \mathbf{R}$-module, where $\mathbf{D}$ is a distributive $\mathbf{R}$-module, and all $\mathbf{F}_i$ are fields.

Lastly, let us mention that clonoids between affine algebras were also (sometimes implicitly) used to discuss some computational problems of Mal'cev algebras.

In particular, we have applied our results to the *subpower membership problem* of certain algebras. The subpower membership problem ($\mathrm{SMP}(\mathbf{A})$), for a fixed finite algebra $\mathbf{A}$, is the computational problem of determining whether a given partial operation $f\colon A^n \to A$ can be extended to a term operation of $\mathbf{A}$. In [2], it was asked whether the subpower membership problem is always polynomial time solvable for algebras with few subpowers. Using our clonoid results we prove that the subpower membership problem of a large class of 2-nilpotent algebras is in P.

Theorem 7. *Let $\mathbf{A} = \mathbf{U} \otimes \mathbf{L}$ be a finite 2-nilpotent Mal'cev algebra in a finite language, such that $\mathbf{U}$ is polynomially equivalent to a vector space. Then $\mathrm{SMP}(\mathbf{A})$ is solvable in polynomial time.*

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Hemi-Nelson algebras

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In this work we focus on unifying ideas derived from several algebraic varieties connected to intuitionistic logics, as for instance Heyting algebras [1], semi-Heyting algebras [11], subresiduated lattices [7], Nelson algebras [14], semi-Nelson algebras [5] and subresiduated Nelson algebras [8].

Hemi-implicative lattices [3] are defined as lattices with a greatest element, which will be denoted by 1, endowed with a binary operation $\to$ satisfying the equation $x \to x = 1$ and the inequality $x \wedge (x \to y) \leq y$. The previous inequality can be replaced by the following quasi-equation:

$$\text{If } z \leq x \to y \text{ then } z \wedge x \leq y.$$

A bounded distributive hemi-implicative lattice is defined as a hemi-implicative lattice whose underlying lattice is distributive and has a least element [1]. Many varieties of interest for algebraic logic are subvarieties of the variety of bounded distributive hemi-implicative lattices. Some relevant examples of these algebras are Heyting algebras, subresiduated lattices, RWH-algebras [4], semi-Heyting algebras and Hilbert bounded distributive lattices (i.e., Hilbert algebras [2, 6] whose natural order defines a bounded distributive lattice).

Nelson's constructive logic with strong negation, which was introduced and studied in [9] (see also [10, 13, 14]), is a non-classical logic that combines the constructive approach of positive intuitionistic logic with a classical (i.e. De Morgan) negation. The algebraic models of this logic are called Nelson algebras. The class of Nelson algebras, which is a variety, has been studied since at least the late 1950's (firstly by Rasiowa; see [10] and references therein). In the end of the 1970's it was proved, independently by Fidel and Vakarelov, that every Nelson algebra can be represented as a special binary product (here called a twist structure) of a Heyting algebra.

The main goal of the talk is to extend the twist construction in the framework of bounded distributive hemi-implicative lattices, thus obtaining a new variety, whose members will be called hemi-Nelson algebras. More precisely, an algebra $\langle T, \wedge, \vee, \to, \sim, 0, 1\rangle$ of type $(2, 2, 2, 1, 0, 0)$ is said to be a *hemi-Nelson algebra* (h-Nelson algebra for short) if $\langle T, \wedge, \vee, \sim, 0, 1\rangle$ is a Kleene algebra and the following conditions are satisfied for every $x, y, z \in T$:

1. $x \to x = 1$,
2. $x \wedge (x \to y) \leq x \wedge (\sim x \vee y)$,
3. $\sim (x \to y) \to (x \wedge \sim y) = 1$,
4. $(x \wedge \sim y) \to \sim (x \to y) = 1$,

[1] Bounded distributive hemi-implicative lattices were introduced and studied in [12] under the name of weak implicative lattices.

5. $(x \wedge y \wedge (x \rightarrow y)) \rightarrow (x \wedge (x \rightarrow y)) = 1$,
6. $(x \wedge (x \rightarrow y)) \rightarrow (x \wedge y \wedge (x \rightarrow y)) = 1$,
7. $(x \rightarrow y) \vee (z \wedge \sim z) = (x \vee (z \wedge \sim z)) \rightarrow (y \vee (z \wedge \sim z))$.

We will show that every hemi-Nelson algebra can be represented as a twist structure of a bounded distributive hemi-implicative lattice and prove that given an arbitrary hemi-Nelson algebra, there exists an order isomorphism between the lattice of its congruences and the lattice of its h-implicative filters (which are a kind of implicative filters). We will also prove that there exists an equivalence between the algebraic category of bounded distributive hemi-implicative lattices and the algebraic category of centered hemi-Nelson algebras, where a centered hemi-Nelson algebra is defined as a hemi-Nelson algebra endowed with a center, i.e., a fixed element with respect to the involution (this element is necessarily unique). Finally, we will apply some results developed throughout the present work in order to investigate certain aspects of semi-Nelson algebras and subresiduated Nelson algebras respectively.

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Dual presentations of frames

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Frame presentations are a very useful tool in pointfree topology that can be used to specify locales. In this talk we will discuss a variant of frame presentations that is more convenient for some purposes.

As an example of a standard presentation, the frame of opens of a discrete space X can be presented by taking a generator $[=x]$ for each $x \in X$ and the relations:

- $[=x] \wedge [=y] \leq [\![x=y]\!]$ for $x, y \in X$ (i.e. $[=x] \wedge [=y] \leq 0$ for $x \neq y$)
- $1 \leq \bigvee_{x \in X} [=x]$.

In the resulting frame, the generator $[=x]$ corresponds to the singleton $\{x\}$.

Frame presentations have a set of generators G, an indexing set of relations R and a pair of maps $R \rightrightarrows \langle G \rangle$ describing the relations themselves, where $\langle G \rangle$ denotes the free frame on G. By the universal property of the free frame on R, these maps correspond to frame homomorphisms $\langle R \rangle \rightrightarrows \langle G \rangle$. The presented frame $\langle G \mid R \rangle$ is then simply the coequaliser of these frame homomorphisms.

In the category of locales, a presentation can be expressed as an equaliser

$$\bullet \hookrightarrow \mathbb{S}^G \rightrightarrows \mathbb{S}^R, \tag{$*$}$$

where $\mathbb{S}$ is Sierpiński space and the power $\mathbb{S}^G$ denotes the product $\prod_{g \in G} \mathbb{S}$. This is because $\mathbb{S}$ is the locale corresponding to the free frame on one generator and the free frame functor preserves coequalisers.

In [2] and [3] I introduced a generalisation of frame presentations where the *sets* of generators and relations are replaced by *locales*. When G and R are locally compact, these are obtained by simply reinterpreting the powers $\mathbb{S}^G$ and $\mathbb{S}^R$ in ($*$) as exponential objects. When G and R are discrete locales, these recover standard presentations, but otherwise these allow for greater flexibility in specifying locales.

This talk concerns a different restriction of generalised presentations where we require G and R to be compact Hausdorff locales. (Note that it is the generators and relations that are compact Hausdorff, not the presented locale.) They could be called *compact Hausdorff* or *dual* presentations, since the theory of compact Hausdorff locales is in a sense dual to that of sets.

When working with compact Hausdorff presentations it is convenient to work with closed sublocales (or ‘closeds’) instead of opens. Analogously to our example presentation above, a compact Hausdorff locale X has a dual presentation which can be understood as consisting of a closed generator $[=x]$ for each $x \in X$ and relations

- $[=x] \wedge [=y] \leq [\![x=y]\!]$ for $x, y \in X$
- $1 \leq \bigvee_{x \in X} [=x]$,

where we note that the join is indexed by a compact Hausdorff locale, not a set. The closed generators again correspond to singletons $\{x\}$ in X.

Of course, we are not really considering a set of generators indexed by points of X, but something more abstract that takes the topology into account. Nevertheless, I will show how to make sense of descriptions like the one above.

As one application, we can describe a locale $[2^{\mathbb{N}} \rightharpoondown X]$ of partial surjections from the Cantor space $2^{\mathbb{N}}$ to a compact Hausdorff locale X. This can be shown to be a nontrivial locale, giving a substantial generalisation of the Alexandroff–Hausdorff theorem [1] without any cardinality restrictions. Further applications can be found in [4].

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Tense Information Logic with Incomparable Fusion and Overlap

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Tense information logic (TIL) [3] is a modal logic interpreted on posets of information states with two binary modalities: $\langle sup \rangle$ for informational addition and $\langle inf \rangle$ for informational overlap, interpreted via *least upper bound* and *greatest lower bound*, respectively. Recently, TIL was shown to be decidable and complete [1]. A natural question is what happens if we drop uniqueness of these bounds and evaluate $\langle sup \rangle$ and $\langle inf \rangle$ via *minimal upper bounds* and *maximal lower bounds* instead, modelling incomparable fusions and overlaps.

For modal information logic (MIL), the $\langle sup \rangle$-only fragment of TIL, Knudstorp [2] showed that this change is indistinguishable to the modal language: validities coincide under least- and minimal-upper-bound semantics. In this abstract we extend that result to the tense setting.

Language and semantics. We write $\mathcal{L}_T$ for the propositional modal language over **Prop** with Boolean connectives and the two binary modalities $\langle sup \rangle$ and $\langle inf \rangle$.

A (Kripke) poset model for $\mathcal{L}_T$ is a triple $\mathfrak{M} = (W, \leq, V)$, where W is a set, $\leq$ is a partial order and V is a valuation $V : \mathbf{Prop} \to \mathcal{P}(W)$. Boolean clauses are standard. For the binary modalities we set:

$$\begin{aligned} \mathfrak{M}, x \Vdash \langle sup \rangle \varphi\psi \ \text{ iff } \ \exists y, z \in W : \mathfrak{M}, y \Vdash \varphi,\ \mathfrak{M}, z \Vdash \psi,\ x = \sup\{y, z\}, \\ \mathfrak{M}, x \Vdash \langle inf \rangle \varphi\psi \ \text{ iff } \ \exists y, z \in W : \mathfrak{M}, y \Vdash \varphi,\ \mathfrak{M}, z \Vdash \psi,\ x = \inf\{y, z\}. \end{aligned}$$

Write $\Vdash_M$ for the interpretation obtained by replacing $x = \sup\{y, z\}$ with $x \in \text{mub}\{y, z\}$ and $x = \inf\{y, z\}$ with $x \in \text{mlb}\{y, z\}$, where $\text{mub}\{y, z\}$ and $\text{mlb}\{y, z\}$ denote the sets of minimal upper bounds and maximal lower bounds of $\{y, z\}$, respectively. Let *TIL* (resp. $TIL_{\text{min-max}}$) be the set of $\mathcal{L}_T$-formulas valid on all poset models under $\Vdash$ (resp. under $\Vdash_M$).

Results At first glance, it seems likely that changing the interpretation of the modalities will change the logic: in a poset where $\{y, z\}$ has an upper bound x that is not the supremum, one can have $x \Vdash_M \langle sup \rangle \varphi\psi$, while $x \nVdash \langle sup \rangle \varphi\psi$. We will nevertheless prove the following theorem.

Theorem 1.

$$TIL = TIL_{min\text{-}max}$$

Proof idea. For $TIL \subseteq TIL_{\text{min-max}}$, the soundness argument for the axiomatization of TIL given in [1] transfers almost without change from sup/inf to min/max semantics.

For the converse, we reason contrapositively. If $\varphi \notin TIL$, fix a poset model $\mathfrak{M}$ with $\mathfrak{M}, w \nVdash \varphi$ for some w. We construct an extension $\mathfrak{M}' = (W', \leq', V') \supseteq (W, \leq, V)$ such that:

(1) $\mathfrak{M}$ is a sup/inf p-morphic image of $\mathfrak{M}'$, and in particular, for all $x \in W$, $\mathfrak{M}, x \Vdash \psi \Leftrightarrow \mathfrak{M}', x \Vdash \psi$.

(2) In $\mathfrak{M}'$, the sup / inf semantics and the min/max semantics coincide, that is:

$$\forall x' \in W'\ \forall \psi \in \mathcal{L}_T :\ \mathfrak{M}', x' \Vdash \psi \ \Leftrightarrow\ \mathfrak{M}', x' \Vdash_M \psi.$$

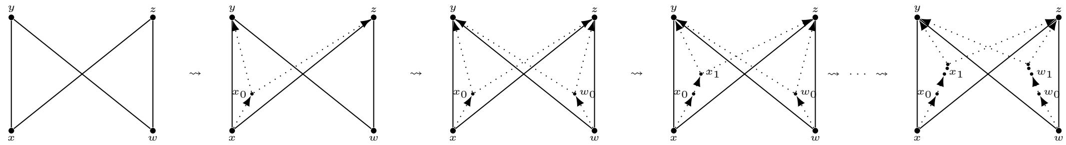
*This work formed part of my master's thesis under the supervision of Søren Brinck Knudstorp and Nick Bezhanishvili. I am especially grateful to Søren for comments on a draft of this abstract.

Thus $\mathfrak{M}', w \nVdash_M \varphi$, hence $\varphi \notin TIL_{\text{min-max}}$, which proves the inclusion. $\square$

For condition (2), we note that in any poset, $x = \sup\{y, z\}$ implies $x \in \text{mub}\{y, z\}$, and $x = \inf\{y, z\}$ implies $x \in \text{mlb}\{y, z\}$. We also want the converses to hold in $\mathfrak{M}'$: that for all $x', y', z' \in W'$, $x' \in \text{mub}\{y', z'\}$ implies $x' = \sup\{y', z'\}$, and $x' \in \text{mlb}\{y', z'\}$ implies $x' = \inf\{y', z'\}$. For this we distinguish two kinds of defects: mub-defects and mlb-defects.

Definition 2. *A triple $(x, y, z) \in W^3$ in a poset $(W, \leq)$ is an* mub-defect *iff $x \in \text{mub}\{y, z\}$ but $x \neq \sup\{y, z\}$. The corresponding notion for lower bounds is an* mlb-defect*. It is defined by replacing "minimal upper bound/supremum" with "maximal lower bound/infimum".*

To resolve, say, an mlb-defect, we cannot make x the infimum of $\{y, z\}$ without violating (1). Instead we add an infinitely ascending chain of lower bounds below x in the extension, as depicted (for the case when x is a maximal lower bound but not the greatest lower bound of $\{y, z\}$):



This process is made precise by the following lemma:

Lemma 3 (mlb-defect repair)**.** *For any mlb-defect (x, y, z) in $(W, \leq)$ there is an extension $(W', \leq') \supseteq (W, \leq)$ and a surjective sup/inf p-morphism $f : W' \to W$ such that (i) f restricts to id_W, (ii) all existing binary suprema and infima of elements of W are preserved, and (iii) $x \notin \text{mlb}'\{y, z\}$. A similar lemma removes mub-defects.*

Each copy $x_0, x_1, \ldots$ must satisfy the same $\mathcal{L}_T$-formulas as x. Since the forward- and past-looking diamonds are definable in $\mathcal{L}_T$, formulas can navigate finite $\leq$ / $\geq$-zigzags, the naive copying step indicated above need not preserve truth. We address this by duplicating the entire frame whenever we duplicate a world, placing each copy directly below the original, which is the main departure from the MIL case [2]. A second complication is that, after adding a new lower bound of $\{y, z\}$, a point that was previously their infimum need no longer be minimal among lower bounds. The new point must therefore be placed below every point in the least upset containing $\{y, z\}$ that is closed under binary infima, and similarly for new upper bounds.

Both factors are built into the repair lemmas. Iterating them yields a limit-model $\mathfrak{M}_\infty$ with no defects, of which $\mathfrak{M}$ is the p-morphic image. This concludes the proof.

Conclusion Theorem 1 transfers the known sound and complete axiomatization of TIL and its decidability [1] to $TIL_{\text{min-max}}$.

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Bunched implication logic is undecidable

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Abstract

We show that theoremhood in bunched implication logic is undecidable; equivalently, the equational theory of the variety of BI-algebras is undecidable. Actually, we prove undecidability for a whole interval of logics/varieties that also includes the variety of GBI-algebras (non-commutative versions of BI-algebras).

The logic of bunched implications (BI logic) was introduced in the late 90's by O'Hearn and Pym [5]. This logic has a logical fragment (that is intuitionistic) and a dynamic fragment (that is substructural), hence it provides two notions of implication on the same structure. BI logic has applications in dynamic memory allocation and pointer arithmetic and is related to separation logic; see [4] for a survey. BI logic was initially defined via a sequent calculus, where sequents (referred to as *bunches*) can have two structural connectives on the left-hand side. BI logic can alternatively, and equivalently, be presented via Kripke-style semantics and via algebraic semantics, BI algebras. The natural non-commutative generalization of the latter is provided by GBI algebras.

(Algebraic semantics) A *GBI-algebra* is an algebra of the form $\mathbf{A} = (A, \wedge, \vee, \to, \top, \bot, \cdot, \backslash, /, 1)$ where $(A, \wedge, \vee, \to, \top, \bot)$ is a Heyting algebra, $(A, \cdot, 1)$ is a monoid, and $\backslash, /$ are the left and right divisions of $\cdot$; that is, for all $x, y, z \in A$,

$$x \cdot y \leq z \quad \text{iff} \quad y \leq x\backslash z \quad \text{iff} \quad x \leq z/y.$$

A *BI algebra* is a GBI algebra that satisfies commutativity ($xy = yx$). We denote the corresponding varieties by GBI and BI.

The decidability of the equational theory of BI algebras has been claimed by various authors, but the arguments suffer from logical gaps. In contrast, in [3] we prove that the equational theory is actually undecidable via encoding an undecidable problem: acceptance in And-branching counter machines.

(And-branching counter machines) We describe a simple model of computation that is equivalent to that of a Turing machine. An ACM is a structure $\mathtt{M} = (Q, R, I, q_f)$, where Q is a set of *states*, $q_f \in Q$, $R = \{r_1, \ldots, r_k\}$ is the set of *register tokens*, and I is a set of instructions of the form $q\backslash q'r$ (increment), $qr\backslash q'$ (decrement) and $q\backslash(q_1 \vee q_2)$ (fork), where $q, q', q_1, q_2 \in Q$ and $r \in R$.

A *configuration* is an element of the free commutative monoid $(Q \cup R)^*$ of the form $qr_1^{n_1} \cdots r_k^{n_k}$, where $q \in Q$ and $n_1, \ldots, n_k \in \mathbb{N}$; here r^n denotes the n-fold product of r.

We consider the free commutative semigroup $(J, \vee)$ over $(Q \cup R)^*$, so the elements of J admit a normal form up to commutativity: $x_1 \vee \cdots \vee x_m$, where $m \in \mathbb{Z}^+$ and $x_1, \ldots, x_m \in (Q \cup R)^*$.

On J we define the *one-step computation* relation $\rightsquigarrow_1$ to be the smallest relation that includes $d \rightsquigarrow_1 d'$, for all $d\backslash d' \in I$, and is closed under multiplication and join. So, for example, if $q\backslash q'r_1 \in I$, then $q \rightsquigarrow_1 q'r_1$, but also $qr_1r_2 \rightsquigarrow_1 q'r_1r_1r_2$ and $qr_1r_2 \vee q'r_1r_3 \rightsquigarrow_1 q'r_1r_1r_2 \vee q'r_1r_3$. We define $\rightsquigarrow$ to be the reflexive transitive closure of $\rightsquigarrow_1$.

We note that there exists an ACM $\mathtt{M}_{\mathrm{Und}}$ such that: it is undecidable, given a configuration c of $\mathtt{M}_{\mathrm{Und}}$, whether $c \rightsquigarrow q_f \vee \cdots \vee q_f$. As a result, the undecidability of (G)BI will follow from the soundness and completeness of the encoding that we describe in the next lemma.

(Soundness) Given an ACM $\mathtt{M} = (Q, R, I, q_f)$, we write $\overline{q_f}$ for any finite (non-idempotent) join $q_f \vee \cdots \vee q_f$ of q_f's and we define the terms:

$$i := \bigwedge I, \qquad t := (q_f/i) \to q_f, \qquad \theta := (t/\top) \wedge 1.$$

Theorem 1. *If $c \rightsquigarrow \overline{q_f}$ then* GBI $\models \theta c \leq q_f$*, for every ACM (Q, R, I, q_f) and configuration c.*

The proof proceeds by simple algebraic manipulations within the theory of GBI algebras. The converse of Theorem 1 proceeds in two steps, represented by the two theorems that follow.

(Expanded computation) To capture the (potentially) higher expressivity of GBI relative to ACMs, we consider an expanded (when compared to $\rightsquigarrow$) notion of computation which we denote by $\preccurlyeq$. The sets $(Q \cup R)^*$ and J, considered in connection to $\rightsquigarrow$, are now replaced by M and K (for $\preccurlyeq$).

Let M be the absolutely free $\{\circ, \varepsilon, \curlywedge, \epsilon\}$-algebra over $Q \cup R$ and $(K, \vee)$ be the free commutative semigroup over M; so, $M \subseteq K$ and the elements of K admit a normal form up to commutativity: $x_1 \vee \cdots \vee x_m$, where $m \in \mathbb{Z}^+$ and $x_1, \ldots, x_m \in M$.

We write $\preccurlyeq$ for the least $\{\circ, \curlywedge, \vee\}$-compatible relation on J that contains $\rightsquigarrow$, the semilattice axioms for $\curlywedge$ (i.e., $x \curlywedge y \preccurlyeq y \curlywedge x$, $x \preccurlyeq x \curlywedge x$, $x \curlywedge y \preccurlyeq x$), $\circ$-associativity ($(x \circ y) \circ z \cong x \circ (y \circ z)$), $\circ$-commutativity ($x \circ y \preccurlyeq y \circ x$), and the identity axioms ($x \circ \varepsilon \cong x \cong \varepsilon \circ x$, $x \curlywedge \epsilon \cong x$), where $x \cong y$ is short for ($x \preccurlyeq y$ and $y \preccurlyeq x$), for $x, y \in W$.

Theorem 2. *If* GBI $\models \theta c \leq q_f$ *then $c \preccurlyeq \overline{q_f}$, for every ACM* M *and configuration c.*

The theorem shows that $\preccurlyeq$ reflects validity in GBI and its proof uses the theory of residuated frames [1] for the distributive setting [2].

Theorem 3. *If $c \preccurlyeq \overline{q_f}$ then $c \rightsquigarrow \overline{q_f}$, for every ACM* M *and configuration c.*

The proof of the theorem is reminiscent of contraction elimination arguments in proof theory.

Corollary 4. *The equational theory of* GBI *is undecidable. The same holds for* BI *and for every variety between them.*

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Revisiting the localic Kuratowski-Mrówka theorem

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Background

The Polish mathematician S. Mrówka published in [2] the famous result that a topological space X is compact iff, for any topological space Y, the projection $\pi_Y : X \times Y \to Y$ is closed. This is now referred to as the *Kuratowski-Mrówka theorem* (KM theorem, for short). The KM theorem shows that compactness (using the open-cover definition) need not be viewed as a property internal to a space; instead it can be viewed as an external (or extrinsic) property using morphisms. In my Master's thesis ([3]) one of the goals was to expound upon the work in [4] in which the localic (or pointfree) KM theorem was realised.

Localic compactness

Definition. A *cover* of a frame A is a subset $U \subseteq A$ for which $\bigvee U = 1$. A frame A is *compact* iff every cover of A contains a finite subcover.[1]

Recall that when we pass over from spaces to frames we need to 'reverse all arrows'; thus frame coproducts are the pointfree analogue of products of spaces. For frames A and B, we will denote by $A \oplus B$ the coproduct of A with B; of course, $A \oplus B$ is equipped with canonical 'frame injections':

$$A \xrightarrow{\iota_A} A \oplus B \xleftarrow{\iota_B} B.$$

For each $a \in A$ and $b \in B$ we write $a \oplus b := \iota_A(a) \wedge \iota_B(b)$; then the set $\{a \oplus b \mid a \in A, b \in B\}$ is a base[2] for $A \oplus B$. The authors of [4] defined the notion of the *unit decomposition property* (briefly, *UD*):

Definition. A frame A is said to satisfy *UD* iff, for any frame B and each decomposition of the unit $1_{A\oplus B} = \bigvee_{i\in I}(a_i \oplus b_i)$, the set $\{\bigwedge_{i\in K} b_i \mid K \subseteq I \text{ for which } \bigvee_{i\in K} a_i = 1\}$ is a cover of B.

In [4] it was shown that *every compact frame satisfies UD*; however, this result hinges on an important lemma first discovered in [1]. The so-called *Kříž-Pultr lemma* was proved in [1, Theorem 3.9]:

Lemma. *Let A be a compact frame, and let B be a nontrivial frame (i.e. $0_B \neq 1_B$). Set $1_{A\oplus B} = \bigvee_{i\in I}(a_i \oplus b_i)$. Then there exists a finite $K \subseteq I$ for which $\bigvee_{i\in K} a_i = 1$ and $\bigwedge_{i\in K} b_i \neq 0$.* ⋄

In order to obtain the above lemma, the authors of [1] define (and make use of) objects called "semitrees"; moreover, the Axiom of Choice is implicitly used there. We will show that both semitrees and Choice can in fact be *avoided*.

[1] Compactness of frames is a conservative property in the sense that a topological space X is compact iff its frame of opens ΩX is compact.

[2] A *base* for a frame A is a subset $M \subseteq A$ for which, for each $a \in A$, $a = \bigvee_A\{m \in M \mid m \leq a\}$.

Closed maps

Recall that, for a frame A, the sublocales of the form $\mathfrak{c}(a) := \uparrow a$ (for $a \in A$) are said to be *closed*.

Definition. A localic map $f : A \to B$ is *closed* iff, for each $a \in A$, $\mathfrak{c}(f(a)) \subseteq f[\mathfrak{c}(a)]$. We will say a frame homomorphism $h : B \to A$ is *closed* iff h's right adjoint h_* (a localic map) is closed.

We will find the following characterisation of closed frame homomorphisms to be useful:

Proposition. *A frame homomorphism $h : B \to A$ is closed iff, for each $a \in A$ and $b, b' \in B$,*

$$h(b') \leq a \vee h(b) \implies b' \leq h_*(a) \vee b.$$

◇

In the spirit of [4] we will show that

Proposition. *Let A be a frame satisfying UD. For any frame B, the coproduct injection $\iota_B : B \to A \oplus B$ is closed.* ◇

Finally, we will show that

Theorem. *For a frame A, the following are equivalent:*

(i) A is compact.

(ii) A satisfies UD.

(iii) For any frame B, the coproduct injection $\iota_B : B \to A \oplus B$ is closed.

◇

It should be noted that although we avoid Choice in our proof; our approach is not entirely constructive – we make use of Excluded Middle.

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T_D-coreflection pointfreely and its generalization to other separation axioms

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The T_D-coreflection of a space

The T_D separation axiom in topology is intimately related to the Skula topology (the topology generated by locally closed sets). Indeed, a topological space X is T_D if and only if its Skula topology is discrete. Banaschewski and Pultr [BP10] studied the T_D separation axiom from the pointfree perspective. They obtained a duality between T_D-spaces and T_D-spatial frames, from which they derived that the category of T_D-spaces is coreflective when the notion of morphism is suitably restricted. The coreflector sends a space X to the subspace of locally closed points, which is the largest Skula-discrete subspace of X:

Theorem 1 ([BP10])**.** *For every topological space X, the largest Skula-discrete subspace of X is the T_D-coreflection of X.*

This idea of *discretization* is not easy to express pointfreely (see, e.g., [BPP19]). In locale theory, discreteness of a locale is regarded to be equivalent to the locale being Boolean. However, the notion of "largest Boolean sublocale" does not make sense since Boolean sublocales are very far from being closed under joins (it is a standard result that every sublocale is a join of Boolean ones; see, e.g., [PP12, Sec. 10.5]). Because of this, the T_D-coreflection cannot be expressed in the language of frames (see, e.g., [BRSWW25, Rem. 5.28]).

The T_D-reflection of an MT-algebra

Our aim is to show that the notion of discretization can be expressed pointfreely in the richer language of McKinsey–Tarski algebras [BR23]. From this we deduce a pointfree analogue of Theorem 1.

Definition 2. A *McKinsey–Tarski algebra*, or an *MT-algebra* for short, is a complete Boolean algebra equipped with an interior operator $\Box : M \to M$.

The algebra M is viewed as an abstract powerset, and the fixpoints $\mathcal{O}(M)$ of $\Box$ play the role of open sets. Morphisms of MT-algebras are morphisms of complete Boolean algebras which preserve the opens. Since M is regarded as an abstract powerset, its elements may be identified with subspaces, unlike what happens in the formalism of frames. Defining a discrete subspace is then easily achievable:

Definition 3. An element a of an MT-algebra M is *discrete* provided $b \leq a$ implies $b \in \mathcal{O}(M)$.

*Speaker.

The join d of all discrete elements of M is the largest discrete element of M, and we call the MT-algebra

$$[0, d] = \{a \in M \mid a \leq d\} = \{a \in M \mid a \text{ is discrete}\}$$

the *discretization* of M. The category of discrete MT-algebras is reflective when the notion of morphism is suitably restricted:

Theorem 4. *For every MT-algebra M, its discretization is the discrete reflection of M.*

We define the *Skula topology on M* by setting the new opens to be the joins of locally closed elements. The corresponding new interior is:

$$\Box_D a = \bigvee\{x \leq a \mid x \text{ is locally closed}\}.$$

Then M is T_D exactly when the Skula topology is discrete, i.e., when $\Box_D$ is the identity. The *Skula-discretization* of M is the discretization of $(M, \Box_D)$. We thus arrive at the following pointfree version of Theorem 1.

Theorem 5. *For every MT-algebra M, its Skula-discretization is the T_D-reflection of M.*

This reflection, when dualized, yields the subspace inclusion of Theorem 1, showing that MT-algebras capture T_D-coreflection pointfreely.

T_i-coreflections of MT-algebras

The above idea extends to other separation axioms. Indeed, for each $i \in \{0, 1, 2\}$, there is a distinguished set of elements, which we call T_i*-elements*, such that an MT-algebra is T_i if and only if its T_i-elements are join dense [BR23]. In other words, there exists an interior $\Box_i$ such that an MT-algebra is T_i if and only if $\Box_i$ is discrete. This gives rise to the discretization with respect to $\Box_i$, which we call the T_i*-discretization*. We then have:

Theorem 6. *Let $i = 0, 1, 2$. For every MT-algebra M, its T_i-discretization is the T_i-reflection of M.*

Restricting to spatial MT-algebras, yields the following corollary for topological spaces:

Corollary 7. *Let $i = 0, 1, 2$. For every topological space X, its subspace of T_i-isolated points is the T_i-coreflection of X.*

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Easy direct limits property in some classes of algebras

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The direct limit construction belongs to the basic tools used in universal algebra. Despite its frequent application, a systematic account of properties of this construction is rare in the literature. Moreover, G. Grätzer [1], on page 161, specified problems concerning the direct limit operator that have not yet been solved. The question of what can be obtained from a single algebra by the direct limit construction is very basic in this context and little explored.

Let $\mathcal{A}$ be an algebra. We denote by $\underrightarrow{\mathbf{L}}\mathcal{A}$ the class of all isomorphic copies of direct limits which can be obtained from $\mathcal{A}$ and we denote by $\mathbf{R}\mathcal{A}$ the set of all retracts of $\mathcal{A}$. Then $\mathbf{R}\mathcal{A} \subseteq \underrightarrow{\mathbf{L}}\mathcal{A}$. In general, an algebra from $\underrightarrow{\mathbf{L}}\mathcal{A}$ need not be a retract of $\mathcal{A}$, see [5].

We say that $\mathcal{A}$ is *with easy direct limits* if every algebra from $\underrightarrow{\mathbf{L}}\mathcal{A}$ is isomorphic to a retract of $\mathcal{A}$. Retracts of an algebra with easy direct limits form a direct limit closed class, and this is the next motivation for dealing with direct limit families in which only one algebra occurs.

All axiomatic direct limit closed classes of the first order logic are described, Theorem 6 of [9]. The question of whether there is a description of axiomatic direct limit closed classes generally, even in the case of elementary classes, is open [4]. The system of all direct limit closed classes ordered by inclusion has been examined for general algebras [3], for cyclically ordered groups [8] and for mono-unary algebras [2].

Let us mention several earlier results about algebras with easy direct limits. It was shown in [7] that every finite algebra is with easy direct limits. In [5], an uncountable class of infinite unary algebras with easy direct limits was identified. Many of them do not even contain a finite subalgebra. It was proved in [5] that every mono-unary algebra with easy direct limits is countable; however, a criterion for this property remains unknown.

The goal of the talk is to present several examples of classical algebraic structures, such as groups and rings, which are (or not) algebras with easy direct limits, and to demonstrate that the question of whether an algebra is with easy direct limits can sometimes be resolved by means of a mono-unary algebra defined on the same underlying set.

Let (A, h) be a mono-unary algebra. Below we present the construction of the algebra $(A, h)^{\diamond}$, which is needed for the formulation of Theorem 1. Recall first that (A, h) is called a *line* if it is isomorphic to the mono-unary algebra of all natural numbers with the successor operation.

Let I be a nonempty set. For each $i \in I$ let (B_i, h) be a connected mono-unary algebra. We denote by $\sum_{i\in I}(B_i, h)$ a mono-unary algebra which is a disjoint union of algebras (B_i, h), $i \in I$.

Let $(A, h) = \sum_{i\in I}(B_i, h)$. One of the following two cases occurs:

1. the algebra (B_i, h) contains no cycle,

2. there is $k \in \mathbb{N}$ such that (B_i, h) contains a cycle of length k.

We denote by (C_i, h) a mono-unary algebra which is a line in the first case and mono-unary algebra which is a cycle of length k in the second case. We put

$$(A,h)^{\diamond} = \sum_{i\in I}(C_i, h).$$

Theorem 1. *Let $\mathcal{A} = (A, F)$ and f be a unary term such that f is an endomorphism of the algebra $\mathcal{A}$. If $\mathcal{A}$ is an algebra with easy direct limits, then the mono-unary algebra $(A, f)^{\diamond}$ is isomorphic to a retract of (A, f).*

Corollary 1. *Let $\mathcal{A} = (A, F)$ and f be a unary term such that f is an endomorphism of the algebra $\mathcal{A}$. Then*

1. *there exists $\mathcal{B} = (B, F) \in \underrightarrow{\mathbf{L}}\mathcal{A}$ such that f is bijective on B,*
2. *if $\mathcal{A}$ is an algebra with easy direct limits, then there exists $\mathcal{B} = (B, F) \in \mathbf{R}\mathcal{A}$ such that f is bijective on B.*

To see that the above results are meaningful, we will focus our attention on unary terms which are endomorphisms of algebras. The identity mapping is a term operation which is an endomorphism of every algebra. We will show that, within the class of rings with unity of characteristic zero, the identity mapping is the only mapping that is both a term operation and an endomorphism. We will also show that, in abelian groups and mono-unary algebras, every term operation is an endomorphism.

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Galois theory of difference and differential schemes

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History

Classical Picard-Vessiot theory is Galois theory of differential field extensions $(L, \delta_L)/(K, \delta_K)$ generated by solutions of linear differential equations. There is an analogous theory for linear difference equations.

Janelidze realised that Picard-Vessiot theory fits into the framework of his categorical Galois theory through the adjunction

$$\begin{array}{c} \delta\text{-Rng}^{\mathrm{op}} \\ S \downarrow \dashv \uparrow C \\ \mathrm{Rng}^{\mathrm{op}} \end{array}$$

between the opposite categories of differential rings and rings, where

$$S(R, \delta_R) = \mathrm{Const}(R, \delta_R), \quad C(R) = (R, 0).$$

He emphasised in [1] and [2] that the key object, the morphism of *relative Galois descent*, is associated to the *Picard-Vessiot ring* $(A, \delta)/(K, \delta)$ and not the field extension $(L, \delta)/(K, \delta)$, and that the Picard-Vessiot Galois group agrees with the categorical Galois group.

Motivated by a question by Akira Masuoka in 2023, we realised that the categorical Galois theory approach does not recover the classical Picard-Vessiot Galois correspondence between intermediate differential field extensions and Zariski closed subgroups of the Galois group, and that we must also consider *quasi-projective* quotients of $\mathrm{Spec}(A)$, and expand our theory to general differential schemes.

The immediate obstacle to being able to invoke categorical Galois theory is the fact that the natural functor $C : \mathrm{Sch} \to \delta\text{-Sch}$ does not admit a left adjoint.

Galois theory of differential schemes.

In a paper with Noohi [3], we resolve the problem by defining a *categorical quotient* of a differential scheme (X, δ_X) to be a morphism $(X, \delta_X) \to (Q, 0) = C(Q)$ universal from X to C. If such a scheme Q exists, we call it the *categorical scheme of leaves* of X and denote it

$$\pi_0(X).$$

In order to develop a theory of descent in differential algebraic geometry, we adopt a novel approach and consider differential schemes as *precategory actions.* To a differential scheme (X, δ_X), we can therefore associate an internal precategory $\mathbb{X}$ in Sch,

$$X_2 \rightrightarrows X_1 \leftrightarrows X_0.$$

We observe that $\pi_0(X)$ can be interpreted as the object of connected components of $\mathbb{X}$.

In algebraic geometry, descent usually works only for very specific indexed data on schemes, so we work with a chosen pseudofunctor

$$\mathscr{P} : (\mathrm{Sch})^{\mathrm{op}} \to \mathbf{Cat},$$

and extend it in a natural way to a pseudofunctor on differential schemes

$$\delta\text{-}\mathscr{P} : \delta\text{-}\mathrm{Sch}^{\mathrm{op}} \to \mathbf{Cat}.$$

We define (X, δ_X) to be *categorically simple* with respect to $\mathscr{P}$, provided that $\pi_0(X)$ exists and is *universal* for $\mathscr{P}$.

Proposition. *A morphism $f : (X, \delta_X) \to (Y, \delta_Y)$ is of effective descent for δ-$\mathscr{P}$, if the associated precategory morphism (f_0, f_1, f_2) satisfies that f_0 is of effective descent for $\mathscr{P}$, f_1 is descent for $\mathscr{P}$ and f_2 is weak descent for $\mathscr{P}$.*

Definition. A morphism $f : (X, \delta_X) \to (Y, \delta_Y)$ is *pre-Picard-Vessiot* with respect to $\mathscr{P}$, if

(i) f is a morphism of effective descent for δ-$\mathscr{P}$;

(ii) X, $X \times_Y X$, $X \times_Y X \times_Y X$ are simple for $\mathscr{P}$.

We say that f is *Picard-Vessiot* if, for a suitable $\mathscr{P}$, in addition to (i), it satisfies

(ii') X is simple and f is self-split.

If f is pre-Picard-Vessiot for $\mathscr{P}$, we can apply π_0 to its kernel-pair groupoid $\mathbb{G}_f$ and so we define the *Galois precategory*

$$\mathrm{Gal}[f] = \pi_0(\mathbb{G}_f).$$

If f is Picard-Vessiot, then $\mathrm{Gal}[f]$ is an internal *groupoid*..

Theorem. *A pre-Picard-Vessiot morphism f for $\mathscr{P}$ induces an equivalence*

$$\mathrm{Split}_C(f) \simeq \mathscr{P}^{\mathrm{Gal}[f]}$$

between the category of objects of δ-$\mathscr{P}(Y)$ that are C-split by f and the category of $\mathscr{P}$-actions of the precategory $\mathrm{Gal}[f]$*. If f is Picard-Vessiot, the latter is the category of $\mathscr{P}$-actions of the groupoid* $\mathrm{Gal}[f]$.

Galois theory of difference schemes

We have shown that

Differential Galois theory = (precategorical) descent + categorical Galois theory.

so the template above will work for any type of structure that can be seen as lax precategory actions, and that includes *difference schemes.* In a forthcoming work with Rui Prezado, we develop the categorical framework necessary for the more general theory of descent, and obtain an analogous difference Galois theory.

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Relative (co-)Yoneda Lemma for Categories Indexed over a Small-Generated Site

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Introduction. It is well known that a (Grothendieck) topos can be regarded as a category of sets through the use of a typed language (the so-called Mitchell-Bénabou language) which has a canonical interpretation (the so-called Kripke-Joyal semantics) in the topos. This allows a large part of mathematics to be developed internally to any topos, and therefore independently of classical set theory. It applies in particular to the "small" part of category theory, as the latter is by definition internal to the category of sets. However, category theory also contains a "large" part, which is essential in many applications, including topos theory. In order to develop category theory over an arbitrary base topos, one thus needs to go beyond the internal language. This can be done through the use of indexed categories, or fibrations. The study of the latter was initiated in the seventies by several authors, notably Bénabou and Paré-Schumacher ([PS78]). More recently, the relevance of indexed categories and fibrations as foundations of relative topos theory has been highlighted by Caramello-Zanfa ([CZ21]). Their approach has proven especially well-suited for expressing natural phenomena such as those that occur in geometric situations, in contrast to the one developed by category theorists such as Lawvere, Diaconescu and Johnstone which relies mainly on the internal language. Nevertheless, as Pitts pointed out in the eighties ([Pit85]), the development of a formal language and metatheorems, analogous to those mentioned above, which would allow suitable results of ordinary category theory to be automatically transposed to the relative setting, has yet to be fully worked out.

Main results. We develop some fundamental notions and results of the theory of categories indexed over a small-generated site, and we build a dictionary relating them to their equivalents in ordinary category theory. The first part of our work is based on a generalization of that of Paré-Schumacher ([PS78]), namely:

- We define the indexed category of indexed functors between two categories indexed over a small-generated site. It is well-defined under suitable assumptions, notably if all the fibers of the domain are small and the codomain is a stack for the given topology on the base category. In particular, we define the indexed category of indexed presheaves on an essentially small indexed category, in the sense of Caramello-Zanfa ([CZ21]), and we build an explicit equivalence between its category of "points" and the Giraud topos of the (essentially small) indexed category;

- We develop the theory of indexed (co)limits of indexed functors between categories indexed over a small-generated site, and we study to which extent they can be computed in terms of ordinary (co)limits. We define a natural notion of indexed (co)completeness, which we characterize in concrete terms. As a byproduct, we uncover a generalization of the classical Beck-Chevalley condition for categories indexed over a cartesian category.

The second part of our work differs from that of Paré-Schumacher with regard to the notion of indexed local smallness. Indeed, we associate to any (essentially small) indexed category a collection of indexed representable presheaves which defines an indexed Yoneda functor to the corresponding indexed category of indexed presheaves. We then prove an indexed version of the Yoneda lemma, as well as an indexed version of the co-Yoneda lemma, meaning a representation theorem of indexed presheaves as indexed colimits of indexed representable presheaves. These results are linked to ordinary ones by the equivalence mentioned on the first point above. In particular the second one corresponds to the classical theorem on the representation of sheaves in a (Giraud) topos as a colimit of representable sheaves in said topos. We also investigate how our results relate to the ones of Paré-Schumacher, as follows:

- On the one hand, we show that when the base category is a topos, the property of an indexed category being locally small, in the sense of Paré-Schumacher, is equivalent to being "separated" for the canonical topology on the topos. Moreover, we show that our definition of indexed representable presheaves for such a "separated" indexed category coincides with the one of Paré-Schumacher;
- On the other hand, we show that we can associate, in a canonical way, to any stack over a small-generated site a stack over the canonical site of the corresponding topos. This operation is in fact part of an equivalence of 2-categories, which yields in particular a correspondence between the indexed representable presheaves of a stack over a small-generated site and the ones of the associated stack over the canonical site.

These two observations imply that our approach and that of Paré-Schumacher agree at least at the level of stacks. Our general definition for an arbitrary indexed category relies implicitly on the existence of a "stackification" 2-functor, which allows to associate, in a canonical way, to any category indexed over a small-generated site a stack over said site.

Applications and further research directions. These results can serve as a basis for new conceptual approaches to various topics. Indeed, they benefit from a high degree of generality, while offering great technical flexibility. For example, they provide a natural way to work with internal locales in a topos. By extension, but on a larger scale, these results could be of interest for synthetic domain theory, meaning domain theory relative to an arbitrary base topos. They are also essential for relative topos theory, as they can help to conceptually reinterpret certain key concepts, as well as to transpose important ordinary notions to the relative setting. In this regard, we plan in the future to introduce a formal type-based language, whose semantics would correspond to indexed constructions, in order to automate the transposition as much as possible as well as to recover all of the concepts and results mentioned above in synthetic terms.

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Categorical Investigations of Properties of Algebras*

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A variety of algebras can be represented by the category of algebras over a monad. Inspired by Mal'cev type classifications – defining semantic properties as structural conditions – we endeavour to define properties of varieties as conditions on the monad they correspond to.

Our study[1] is housed within the category of finitary monads over sets, $\mathbf{Mnd_{fin}}(\mathbf{Set})$. We characterise the properties of varieties, 'pointed' and 'minimal', as orthogonality classes in $\mathbf{Mnd_{fin}}(\mathbf{Set})$, and we characterise the properties 'Mal'cev', 'additive', 'abelian', 'congruence modular', and 'idempotent' as algebraic-injective (or -projective) classes in $\mathbf{Mnd_{fin}}(\mathbf{Set})$. These characterisations, we argue, help us gather the ingredients required to provide a category-theoretic proof of results from universal algebra that classify minimal idempotent varieties [Kea00], and varieties in which every algebra is free [Giv78, KKS16].

More broadly, this work surveys and unifies learnings from categorical universal algebra and demonstrates the strength of category-theoretic tools in characterisation and definability efforts. We aim to provide building blocks and frameworks for further research at the intersection of universal algebra and category theory, while also providing explicit proofs of connections between diagrams in $\mathbf{Mnd_{fin}}(\mathbf{Set})$, $\mathbf{Cat}/_{\mathbf{Set}}$, and the properties of interest.

We build on the following work: [KV07] studies the concept of dense morphisms of monads and characterises relevant properties of $\mathbf{Mnd_{fin}}(\mathbf{Set})$, like being locally finitely presentable. [BR08] and [Ped95] produce a categorical perspective of Mal'cev conditions, especially with respect to co-sieves, presentablity and monadicity. A recent paper [BP26] studies Mal'cev conditions in higher dimensions and provides a good history of this work. Furthermore, we learn from and translate analyses of specific algebraic properties in [Gum83, Kea00, BJ03, VW14, KKS16, Pos22].

We now mention a few results and corresponding definitions and constructions. In a category, an object X is *orthogonal* to a morphism $f : A \to B$ if for every map $g : A \to X$, there exists a unique map $u : B \to X$ such that $u.f = g$ [AR94, Definition 1.32]. Given two monads $\mathbb{T} = (T, \eta^T, \mu^T)$ and $\mathbb{S} = (S, \eta^S, \mu^S)$, a *morphism of monads* $\alpha : \mathbb{T} \Rightarrow \mathbb{S}$ is a natural transformation between their endofunctors $\alpha : T \Rightarrow S$ that preserves the unit and multiplication. That is, $\eta^S = \alpha.\eta^T$ and $\alpha.\mu^T = \mu^S.\alpha\alpha$, where $\alpha\alpha = \alpha_S.T(\alpha) = S(\alpha).\alpha_T$ [KV07, Section 2.1].

A category is *pointed* if it has a zero object (an object that is both initial and terminal). If a variety of algebras is pointed, it has precisely one constant symbol in its signature. Given a finitary monad $\mathbb{T}$ on **Set** and its Eilenberg-Moore category of algebras $\mathsf{Alg}(\mathbb{T})$, we show that:

Theorem 1. $\mathsf{Alg}(\mathbb{T})$ *is pointed iff* $\mathbb{T}$ *is orthogonal to the morphism* $\mathbb{I} \Rightarrow \mathbb{M}$*, where* $\mathbb{M}$ *is the maybe monad*[2] *and* $\mathbb{I}$ *is the identity endofunctor monad.*

A variety of algebras is *minimal* if it has no non-trivial subvarieties. In $\mathbf{Mnd_{fin}}(\mathbf{Set})$, a morphism of monads $\alpha : \mathbb{T} \Rightarrow \mathbb{S}$ is *dense* iff the induced set-preserving functor $\alpha_* : \mathsf{Alg}(\mathbb{S}) \Rightarrow \mathsf{Alg}(\mathbb{T})$ is fully-faithful [KV07, Definition 3.2]. A dense morphism is *trivial* if it is of the form $\mathbb{T} \Rightarrow \mathbb{T}$, or $\mathbb{T} \Rightarrow \mathbb{1}$, where $\mathbb{1}$ is the terminal endofunctor monad that sends every set to the singleton set.

*Joint work with Ivan Di Liberti (University of Gothenburg, Sweden).

[1]This work is part of an ongoing Master's thesis at University of Gothenburg to be defended in June 2026.

[2]The endofunctor of this monad sends a set X to the disjoint union $X \sqcup \{*\}$. It's algebras are pointed sets.

Theorem 2. $\mathsf{Alg}(\mathbb{T})$ *is minimial iff* $\mathbb{T}$ *is orthogonal to the class of all non-trivial dense morphisms in* $\mathbf{Mnd_{fin}}(\mathbf{Set})$.

The next theorem uses Lemma 1.1 from [Pos22], which also states that in categories of the form $\mathsf{Alg}(\mathbb{T})$, the properties 'additive' and 'abelian' coincide. We consider *algebraic injectivity* in the sense of Section 2.3 of [Bou19], as a structural relative to orthogonality.

Theorem 3. $\mathsf{Alg}(\mathbb{T})$ *is additive iff* $\mathbb{T}$ *carries the structure of algebraic injective objects over the morphism* $\mathbb{I} \Rightarrow \mathbb{A}$*, where* $\mathbb{A}$ *is the finitary monad whose algebras are abelian groups.*

Given sets A, B we can construct a finitary monad on **Set**, $\langle\langle A, B\rangle\rangle$, whose endofunctor on any finite set n has the value $\langle\langle A, B\rangle\rangle(n) = \mathbf{Set}(A^n, B)$ [KV07, Section 2.11]. Morphisms of monads $\mathbb{T} \Rightarrow \langle\langle X, X\rangle\rangle$ correspond uniquely to $\mathbb{T}$-algebra structures on X [KV07, Proposition 2.12]. Consider the pullback construction $\{\{f, f\}\}$ from Proposition 2.12-2.13 of [KV07], and *algebraic projectivity* as the dual notion to algebraic injectivity.

Theorem 4. $\mathsf{Alg}(\mathbb{T})$ *is idempotent iff* $\mathbb{T}$ *carries the structure of algebraic projective objects over the class of morphisms of the form* $\pi^f_{(-)} : \{\{f, f\}\} \Rightarrow \langle\langle -, -\rangle\rangle$*, where* $f \in \mathbf{Set}[1, -]$*, and* $\pi^f_{(-)}$ *is retrieved from the pullback.*

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Relative Points and Completions of Indexed Categories

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The central aim of this work is to provide a geometric perspective on the stackification construction for indexed categories over a site, notably allowing one to compute it more easily. This construction is crucial in algebraic geometry, where stacks in groupoids are used to represent moduli spaces [4]. We are more generally interested in stacks in categories, as they encompass a number of important cases such as the canonical stack associated to a relative topos [3].

The standard construction of the sheaf associated to a presheaf relies on the double application of a functor usually denoted by $(-)^+$. For indexed categories, $(-)^+$ must be applied three times rather than two [5], which makes its computation rather heavy.

For a topological space, the sheafification process also admits a geometric interpretation via *espaces étalés*: it can be realized as the composite of the two functors that constitute the "presheaf-bundle adjunction". This was extended in [3] to an arbitrary site thanks to a relative 2-adjunction between the 2-category $\mathbf{Cat}_{\mathcal{C}}^{J}$ of $\mathcal{C}$-indexed categories which are J-small and the Grothendieck toposes over $\mathbf{Sh}(\mathcal{C}, J)$, restricting to an equivalence between sheaves and étale toposes:

$$\begin{array}{ccc} \mathbf{Cat}_{\mathcal{C}}^{J} & \underset{\Gamma}{\overset{\Lambda}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & \mathbf{Top}^{\mathrm{co}}_{/\,\mathbf{Sh}(\mathcal{C},J)} \\ \uparrow & & \uparrow \\ \mathbf{Sh}(\mathcal{C}, J) & \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} & \mathbf{Et}_{/\,\mathbf{Sh}(\mathcal{C},J)} \end{array}.$$

This adjunction fits into a broader framework that concerns the transfer of notions and results from the 1-categorical setting of presheaves and sheaves to the 2-categorical setting of indexed categories and stacks, and from the absolute case over **Set** to the relative case over any Grothendieck topos $\mathbf{Sh}(\mathcal{C}, J)$, as summarized in the table below.

Terminal category $\mathbf{1}$	Site $(\mathcal{C}, J)$
Topos of sets **Set**	Sheaf topos $\mathbf{Sh}(\mathcal{C}, J)$
Category $\mathcal{D}$ (small)	Indexed category $\mathbf{D}\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ (J-small)
Presheaf topos $\widehat{\mathcal{D}} = \mathbf{Cat}(\mathcal{D}^{\mathrm{op}}, \mathbf{Set})$	Giraud topos $\Lambda(\mathbf{D}) \simeq \mathbf{Cat}_{\mathcal{C}}(\mathbf{D}^{\mathrm{op}}, \mathbf{S})$
Grothendieck topos $\mathcal{E}$	Relative topos $f\colon \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$
Category of pts $\mathbf{Pt}(\mathcal{E}) = \mathbf{Top}(\mathbf{Set}, \mathcal{E})$	Stack of rel. pts $\Gamma(f)^{\mathrm{op}} = \int_U \mathbf{Top}_{/\,\mathbf{Sh}(\mathcal{C},J)}(\mathbf{Sh}(\mathcal{C}, J)_{/U}, \mathcal{E})$
Pro-completion $\mathbf{Pro}(\mathcal{D}) \simeq \mathbf{Pt}(\widehat{\mathcal{D}})^{\mathrm{op}}$	Indexed pro-completion $\mathbf{Pro}_{(\mathcal{C},J)}(\mathbf{D}) \simeq \Gamma\Lambda(\mathbf{D})$
Karoubi envelope $\mathbf{Kar}(\mathcal{D}) \simeq \mathbf{Pt}^{\mathrm{ess}}(\widehat{\mathcal{D}})^{\mathrm{op}}$	Indexed Karoubi envelope $\mathbf{Kar}_{(\mathcal{C},J)}(\mathbf{D}) \simeq \Gamma^{\mathrm{ess}}\Lambda(\mathbf{D})$

Table 1: From absolute notions to relative ones

Recall that Λ sends a J-small indexed category $\mathbf{D}$ to its *Giraud topos* – i.e. the topos of relative presheaves on $\mathbf{D}$. This is the topos of sheaves on the Grothendieck category $\mathcal{D} = \int_U \mathbf{D}(U)$ with respect to the smallest Grothendieck topology J_p which makes the projection functor $p\colon \mathcal{D} \to \mathcal{C}$ a comorphism of sites towards $(\mathcal{C}, J)$, with the structure geometric morphism

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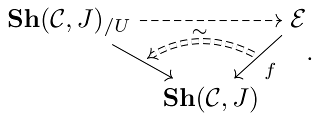
$\mathbf{Sh}(\mathcal{D}, J_p) \to \mathbf{Sh}(\mathcal{C}, J)$ induced by it. On the other hand, Γ maps a relative topos $f\colon \mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ to the opposite of its stack of *relative points*. The fiber of this stack over an object U of $\mathcal{C}$ is the category of the dashed geometric morphisms filling a diagram



The question of describing the stackification of an indexed category in terms of the adjunction $(\Lambda \vdash \Gamma)$ was already posed in [3]. Wei [6] showed that the stackification operation can be realized as the composite of the two functors forming a modified version of this adjunction, obtained by replacing indexed categories (resp. stacks, toposes) with indexed 2-categories (resp. 2-stacks, 2-toposes), but did not address the case of the original adjunction.

The problem is non-trivial since, unlike the case of presheaves, applying $\Gamma\Lambda$ to an indexed category $\mathbf{D}$ yields a stack *richer* than expected. When $\mathcal{C}$ is the category $\mathbf{1}$ with one object and its identity, this stack reduces to the ordinary pro-completion of $\mathcal{D}$ [1]. In this work we show, among other things, that the same phenomenon holds for any category $\mathcal{C}$ equipped with a Grothendieck topology J, provided that ordinary cofiltered limits are replaced by J-locally cofiltered indexed limits. Our main result is the following:

Theorem. *(i) For any J-small indexed category $\mathbf{D}$, $\Gamma\Lambda(\mathbf{D})$ is equivalent to the indexed pro-completion of $\mathbf{D}$.*

(ii) This operation defines a pseudomonad $\mathbf{Pro}_{(\mathcal{C},J)}$ on $\mathbf{Cat}^J_{\mathcal{C}}$ with a modification

$$\mathbf{Pro}_{(\mathcal{C},J)} \underset{\mathbf{Pro}_{(\mathcal{C},J)}(L)}{\overset{L_{\mathbf{Pro}_{(\mathcal{C},J)}}}{\underset{\Longrightarrow}{\overset{\Longrightarrow}{\Downarrow m}}}} (\mathbf{Pro}_{(\mathcal{C},J)})^2$$

making it a co-KZ-pseudomonad (where L is its unit).

(iii) The idempotent pseudomonad associated with this co-KZ-pseudomonad is isomorphic to the inverter of m, and yields the indexed Karoubi envelope of $\mathbf{D}$.

Moreover, it is equivalent to the full sub-indexed category $\Gamma^{\text{ess}}\Lambda(\mathbf{D})$ (in fact, a sub-stack) of $\Gamma\Lambda(\mathbf{D})$ spanned by the relatively essential points of the Giraud topos of $\mathbf{D}$.

(iv) The stackification of $\mathbf{D}$ embeds in $\Gamma^{\text{ess}}\Lambda(\mathbf{D})$ as the full sub-indexed category on the locally matching families of representable relative points.

Point (iii) of the theorem improves the observation of Bunge [2] that relatively essential points span a stack with Karoubian fibers and are local retracts of representable relative points.

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Conceptual completeness for geometric logic via ultraconvergence spaces

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The aim of this talk, based on [**?**], is to present a (strong) conceptual completeness theorem for geometric logic, generalizing Makkai-Lurie's analogous result for coherent logic [**?**, **?**]. Towards this goal, we will introduce *ultraconvergence spaces*, a joint generalization of ultracategories and topological spaces inspired by Barr's description of the latter as *relational β-algebras*.

Background. A conceptual completeness theorem, for a logic, is a *reconstruction* result for syntax starting from semantics. More precisely, it amounts to recovering a theory in the logic from its class of models, and it typically requires to consider additional structure on the latter. For instance, in the context of classical propositional logic, where theories can be identified with their Lindenbaum-Tarski algebras, this takes the form of Stone's representation theorem for Boolean algebras: every Boolean algebra B is isomorphic to the algebra of continuous maps $\mathrm{Mod}(B) \to \mathbf{2}$, once the set $\mathrm{Mod}(B) \coloneqq \mathbf{BoolAlg}(B, \mathbf{2})$ is endowed with the *Stone topology*.

In [**?**], Makkai introduced *ultracategories* in order to prove a conceptual completeness theorem for first-order coherent logic. Intuitively, an ultracategory is a category endowed with abstract *ultraproducts*: the usual notion of ultraproducts then makes the category of models of a coherent theory into an ultracategory. This structure is what allows for the reconstruction theorem to go through: in Lurie's reformulation [**?**], identifying coherent theories with their *classifying toposes*, we have that a coherent topos $\mathcal{E}$ is equivalent to the category of functors $\mathrm{pt}(\mathcal{E}) \to \mathbf{Set}$ preserving ultraproducts in an appropriate sense.

The previous result crucially relies on *Łoś's theorem*: an ultraproduct of models of a coherent theory $\mathbb{T}$ is itself a model of $\mathbb{T}$. Categorically, this can be expressed by saying that, for each ultrafilter ν on a set I, the ultraproduct functor

$$\mathbf{Set}^I \xrightarrow{\prod_{i:\nu}(-)} \mathbf{Set},$$

witnessing **Set** as an ultracategory, is *coherent*, i.e. it preserves finite limits, regular epimorphisms, and finite unions of subobjects. Crucially, it does not preserve arbitrary unions of subobjects, meaning that Łoś's theorem fails for *geometric* logic. Thus, Makkai-Lurie's reconstruction result cannot be immediately extended to geometric theories.

Ultraconvergence spaces. The key intuition allowing us to extend Makkai-Lurie's theorem comes from [**?**]. In the (small and) discrete case, ultracategories recover compact Hausdorff spaces, which by Manes' theorem [**?**] coincide with the algebras for the ultrafilter monad $\beta \colon \mathbf{Set} \to \mathbf{Set}$. This fact, originally noticed by Lurie, is especially evident in Rosolini's definition of ultracategories [**?**] as pseudoalgebras for the *ultracompletion* pseudomonad $\mathbb{B} \colon \mathbf{CAT} \to \mathbf{CAT}$: for a category C, objects of $\mathbb{B}C$ are formal ultraproducts (or *ultrafamilies*) in C, i.e. tuples $(c_i)_{i \in I}$ of objects of C together with an ultrafilter ν on the index set I, which an algebra functor $\mathbb{B}C \to C$ maps to 'actual' ultraproducts in C.

*Presenting author

In [**?**], Barr extended Manes' description of compact Hausdorff spaces as β-algebras to all topological spaces by allowing for *relational β-algebras*: that is, lax algebras for a (lax) monad extending β to the 2-category **Rel** of sets and relations. In other words, an arbitrary topology on X is described by a suitable relation $\beta X \nrightarrow X$ – i.e. a function $X \times \beta X \to \mathbf{2}$ – which we can think of as describing a notion of convergence between ultrafilters on X and points of X. In these terms, a topology is compact and Hausdorff if and only if the convergence relation is actually a function $\beta X \to X$, intuitively computing 'limits' of ultrafilters.

The role of relations, between categories, is played by *profunctors*: following Barr's blueprint, we introduce ultraconvergence spaces [**?**, **?**] as a profunctorial generalization of ultracategories. Intuitively, these categorify topological spaces by replacing a convergence relation $X \times \beta X \to \mathbf{2}$ with a profunctor $C^{\mathrm{op}} \times \beta\!\!\beta C \to \mathbf{Set}$ which associates, to an object c and an ultrafamily $(d_i)^{\nu}_{i\in I}$ in C, a set of *ultra-arrows* $c \multimap (d_i)^{\nu}_{i\in I}$ acting as 'witnesses' of the convergence of $(d_i)^{\nu}_{i\in I}$ to c. In the same spirit we introduce *continuous maps* of ultraconvergence spaces, categorifying the topological notion, and suitable 2-cells between them, obtaining a 2-category **UltSp**.

Conceptual completeness. Based on [**?**], we show how ultraconvergence spaces allow for a conceptual completeness theorem for geometric logic. Indeed, the category $\mathrm{Mod}(\mathbb{T})$ of models of a geometric theory $\mathbb{T}$ in a signature Σ carries a canonical structure of ultraconvergence space, where ultra-arrows $M \multimap (N_i)^{\nu}_{i\in I}$ are Σ-structure homomorphisms from the model M into the ultraproduct $\prod_{i:\nu} N_i$, even when the latter fails to be a model of $\mathbb{T}$ itself.

Identifying $\mathbb{T}$ with its classifying topos $\mathcal{E}$, the ultraproduct $\prod_{i:\nu} N_i$ of an ultrafamily $(N_i)^{\nu}_{i\in I}$ of points of $\mathcal{E}$ can be identified with the functor

$$\mathcal{E} \xrightarrow{\langle N_i\rangle_{i\in I}} \mathbf{Set}^I \xrightarrow{\prod_{i:\nu}(-)} \mathbf{Set}$$

which need not be itself a point of $\mathcal{E}$: in these terms, Σ-structure homomorphisms $M \to \prod_{i:\nu} N_i$ correspond simply to natural transformations $M \Rightarrow \prod_{i:\nu} N_i$, making $\mathrm{pt}(\mathcal{E})$ into an ultraconvergence space. Our theorem reads then as follows.

Theorem. *Let $\mathcal{E}$ be a topos with enough points. Then, $\mathcal{E} \simeq \mathbf{UltSp}(\mathrm{pt}(\mathcal{E}), \mathbf{Set})$.*

Time permitting, we will comment on [**?**], joint work of the presenting author with Quentin Aristote, where ultraconvergence spaces are recovered algebraically by extending the pseudomonad $\beta\!\!\beta$ to profunctors.

Hyperdoctrines are double functors

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The specification of a logical doctrine can be stratified in three levels: the choice of the underlying type theory of *contexts*, the designation of the *semantics* for predicates over such contexts, and an *interpretation map* connecting these pieces of data (subject to compatibility conditions). Different flavours of logic amount to the specification of different contexts, semantics, or properties of the interpretation map. In this talk, we will argue that double categories give a natural framework in which such parameters can be adjusted to obtain new logics.

We will focus on *regular hyperdoctrines*, which are powerful enough to talk about relations. It is well-known that pseudo functors from bicategories of spans are equivalent to Beck-Chevalley bifibrations [3], and therefore capture the relationships underlying the adjunctions suitable as semantics for existential quantification. This was further expanded upon in the context of double categories by Dawson, Paré and Pronk [4, 6]. However, these works do not take into account *Frobënius reciprocity*, which can be understood as a compatibility condition between connectives and existential quantification. In a simple form, the main result of the talk is that this, too, can be captured in double-categorical terms:

Theorem [14]:*(Generalised) regular hyperdoctrines correspond to those lax symmetric monoidal double functors* $\mathbb{S}\mathrm{pan}(\mathcal{C})^{\mathrm{op}} \to \mathbb{Q}\mathrm{t}(\mathrm{Cat})$ *whose laxators are companion commuter transformations.*

Examples of generalised regular hyperdoctrines include:

- All classical regular hyperdoctrines;
- Doctrines for which only a restricted form of Beck-Chevalley holds (logically, substitution and quantification are only permitted along certain special terms);
- Doctrines for which the Frobënius property holds with respect to connectives other than conjunction (e.g., the monoidal products used in quantitative logic/quantales [10]);
- Doctrines (as in dependent type theory) for which predicates are not just posets (e.g., are monoidal categories [7]).

Connecting hyperdoctrines to double functors is useful for compositionality results in the context of logic, and suggests that they might be conceivable as 2-algebraic structures (in the 2-monadic or $\mathcal{F}$-sketch sense). Moreover, there's a well-developed general theory of systems based on double categories [12], and this result allows one to import regular hyperdoctrines as systems themselves. As an application, we can recover a form of graphical regular logic suitable for modelling specifications of systems (e.g., port-plugging systems) that compose operadically [8].

If time permits, I will mention how these ideas suggest a new notion of *regular double hyperdoctrine*, where 2-dimensional contexts and semantic environments are allowed (in the form of double categories), and how these ideas relate to a general approach to coalgebraic modal logic.

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TACL 2026 ABSTRACT

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Title: Studying Primes of an Algebraic Frame without the FIP

Abstract: Let L be a frame. L is an *algebraic* frame if every element of L is join generated by its compact elements, $\mathfrak{K}L$. L is said to satisfy the *finite intersection property* or *FIP* if $\mathfrak{K}L$ is a lattice. An algebraic frame satisfying the FIP is known as an *arithmetic* frame or an M-frame.

It is well-known that given any commutative ring with identity, the frame of radical ideals of the ring is a coherent frame, and hence an M-frame. Other examples include the frame of convex lattice-ordered subgroups of any lattice-ordered group. On the other hand, if we want to study algebraic frames with the FIP condition, then we need to consider the frame of down-sets of any poset.

In an M-frame L, we study certain important collection of prime elements, namely, $Min(L)$ (minimal prime elements), $Max_d(L)$ (maximal d-elements), and $Max(L)$ (maximal elements) of L. We also study these as topological spaces, each endowed with the hull-kernel (or Zariski) topology as subspaces of $Spec(L)$. Furthermore, an interest has emerged in studying the "inverse topology" on a collection of primes. Some topological results are:

(1) $Min(L)$ is a zero-dimensional, Hausdorff space.
(2) $Min(L)^{-1}$ is a compact, T_1 space.
(3) $Max_d(L)$ is a compact, T_1 space.
(4) $Max(L)$ is a compact, T_1 space.

This talk will focus on these subspaces of $Spec(L)$ for algebraic frame L, with and without the condition of FIP. In a recent dissertation ([7]), the author has established that FIP on an algebraic frame is a sufficient, but not a necessary, condition to establish many results on $Min(L)$. A more general condition, called CAB (Compact Absoluteness property), can be used. This new condition has opened the door to studying other prime spaces, such as $Max_d(L)$, in a general algebraic frame without the FIP. There are many open questions to answer, both for $Min(L)$ and for $Max_d(L)$, when L is an algebraic frame without satisfying the FIP.

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Quantale-Enriched Universal Algebra: Completeness*

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Birkhoff's completeness theorem for equational logic was generalised to inequational logic for partially ordered algebras by [5]. More recently, quantitative equational logic has become an important topic of research [3, 4]. Here we generalize to the quantale-enriched setting. Moreover, we do this by refactoring the classical proof as presented in [7] into a more abstract one which has both the classical result and the quantale-enriched one as a special case.

Let us quickly indicate some standard preliminaries. $\mathrm{Term}_\Sigma(V)$ is the set (or absolutely free algebra) of all terms generated from a signature Σ of finitary operation symbols over a countably infinite set of variables V. For an algebra A and an equation θ we write $A \models \theta$ to say that the equation holds for all valuations $V \to A$ and $A \models \Theta$ extends this to sets of equations Θ. The quotient $\mathrm{Term}_\Sigma(V) \to A/\Theta$ is defined as the colimit (specifically: the coequalizer) of A with respect to a parallel pair that subsumes all substitution instances of all equations in Θ. The first important lemma aligns satisfiability with injectivity [2, 7]:

Lemma 1: $A \models \Theta$ if and only if A is injective wrt the quotient $\mathrm{Term}_\Sigma(V) \to \mathrm{Term}_\Sigma(V)/\Theta$.

From this we obtain that $\Theta \models \theta$ if and only if θ is in the kernel of $\mathrm{Term}_\Sigma(V) \to \mathrm{Term}_\Sigma(V)/\Theta$.

For general categorical reasons, taking the quotient (a colimit) and the kernel (a limit) form a Galois adjunction. It follows that "kernel after quotient" is a closure operator. We then get a completeness theorem if we are able to give a syntactic proof system that describes this closure operator:

Lemma 2: $\Theta \vdash \theta$ iff θ is in the kernel of $\mathrm{Term}_\Sigma(V) \to \mathrm{Term}_\Sigma(V)/\Theta$.

Theorem: $\Theta \vdash \theta$ iff $\Theta \models \theta$.

In the remainder of this abstract, we show that the same proof carries over to the quantale-enriched setting if we adopt the appropriate notions of kernel and quotient. The notions of signature and terms remain unchanged. The carrier of an algebra will be replaced by a Q-category, defined as follows:

- Q is a commutative quantale with order $\sqsubseteq$, join $\bigvee$, product $\cdot$, unit e;
- a Q-category is a set A with a 'distance function' $A(a,a') \in Q$ satisfying $e \sqsubseteq A(a,a)$ and $A(a,a') \cdot A(a',a'') \sqsubseteq A(a,a'')$ for all $a,a',a'' \in A$;
- a Q-functor is a function $f : A \to B$ with $A(a,a') \sqsubseteq B(f(a),f(a'))$ for all $a,a' \in A$.

The paradigmatic example is the Lawvere quantale of the non-negative reals $[0,\infty]$ with the opposite order and addition as product. Q-categories are Lawvere metric spaces, which include metric spaces, but do not need to have symmetric distances. Q-functors are non-expanding functions. Note that truth corresponds to zero, which is the top-element.

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To emphasize the intuitions coming with this example, we sometimes call Q-categories also 'generalized metric spaces' and Q-functors 'non-expanding functions'.

Algebras then are generalized metric spaces with non-expanding operations. But what becomes of the equations?

Instead of having equations $t = t'$ we have now Q-indexed inequalities $t \leq_q t'$ for $t, t' \in \mathrm{Term}_\Sigma(V)$ and $q \in Q$.

Proof system. Q-enriched inequational logic has axioms and rules:

$$\text{(Relax)}\ \frac{t \leq_q t'}{t \leq_{q'} t'}\ (q \sqsupseteq q') \quad \text{(Refl)}\ \frac{}{t \leq_e t} \quad \text{(Trans)}\ \frac{t \leq_q t' \quad t' \leq_{q'} t''}{t \leq_{q \cdot q'} t''}$$

$$\text{(Cont)}\ \frac{(t \leq_{q_i} t')_{i \in I}}{t \leq_{\bigvee_{i \in I} q_i} t'} \quad \text{(Cong)}\ \frac{t_1 \leq_{q_1} t'_1 \quad \ldots \quad t_n \leq_{q_n} t'_n}{\sigma(t_1, \ldots, t_n) \leq_{q_1 \cdot \ldots \cdot q_n} \sigma(t'_1, \ldots, t'_n)} \quad \text{(Subst)}\ \frac{t \leq_q t'}{s^\sharp(t) \leq_q s^\sharp(t')}$$

(Relax) is a form of weakening, (Refl) and (Trans) are analogous to equational logic but note that (Trans), the cut, now multiplies the truth-values in the conclusion, as does the congruence rule. (Cont) is a form of continuity and is infinitary if Q is infinite. (Subst) is as in the classical case.

The proof system arises from the following category theoretic analysis. In the classical case, quotients are surjections and the congruence relation by which a quotient quotients is given by its kernel pair. Moreover, quotients can be constructed categorically by coequalizers. Taking kernel pairs and taking coequalizers forms a Galois adjunction and equational logic is a syntactic characterization of the induced closure operator on relations.

In the Q-enriched case, quotients are bijections[1] and the congruence relation by which a bijection quotients is given by a certain limit which we call Q-comma (as it is a Q-indexed version of Lawvere's comma construction) and which was called the Q-nerve in [1]. The Q-comma construction results in a so-called relational presheaf [6] and the proof system above characterizes the relational presheaves that arise as Q-commas. Moreover, quotients can be constructed categorically by so-called Q-coinserters, a construction that was used in [1] to give an explicit description of enriched left Kan extensions between categories of Q-categories. As in the classical case, taking Q-commas and taking Q-coinserters forms a Galois adjunction and the proof system above is a syntactic characterization of the induced closure operator on relational presheaves.

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A Non-Distributive Duality for Finite Pointed Brouwerian Algebras and their Algebras of Fractions

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In [?], Galatos and Přenosil define a method of embedding any commutative (ℓ-)bimonoid into a complete commutative complemented (ℓ-)bimonoid, it's the *algebras of fractions*. While the construction is stated at the level of (ℓ-)bimonoids, many familiar algebraic structures, particularly those admitting residuation, can be translated into the context of (ℓ-)bimonoids, making the construction broadly applicable. One variety highlighted in [?] is that of *pointed Brouwerian algebras*.

Definition 1. A *pointed Brouwerian algebra* is a tuple $\mathbf{B} := (B, \wedge, \vee, \rightarrow, \top, \mathfrak{p})$ where:

1. $(B, \wedge, \vee)$ is a distributive lattice,
2. $a \leq \top$ for all a,
3. $c \leq a \rightarrow b \iff a \wedge c \leq b$ for all a, b, c, and
4. $[\mathfrak{p}, \top]$ is a Boolean lattice.

Any pointed Brouwerian algebra gives rise to a commutative (ℓ-)bimonoid via $x \cdot y = x \wedge y$ and $x + y = (\mathfrak{p} \rightarrow (x \wedge y)) \wedge (x \vee y)$. Applying the algebra of fractions construction in this setting yields a categorical equivalence between pointed Brouwerian algebras and idempotent involutive commutative residuated lattices satisfying $x \approx (1 \wedge x) \cdot (\mathfrak{p} \vee x)$. The equivalence is witnessed by the functor $\mathbb{F}$ whose action on the underlying set B of a pointed Brouwerian algebra $\mathbf{B}$ is defined as $\mathbb{F}(B) := \{\pi(a, b) \mid (a, b) \in B^2\}$ where π is defined as

$$\pi(a, b) = \big((a \rightarrow b) \rightarrow (a \wedge \mathfrak{p}),\ a \rightarrow b\big).$$

While powerful, this construction is complex, which can make it unwieldy in practice. If we restrict to *semilinear* pointed Brouwerian algebras then the algebra of fractions is always a Sugihara monoid, we can see a hope of simplification by moving into a topological context. In [?] Galatos and Fussner establish a clean categorical equivalence between the spaces corresponding to semilinear pointed Brouwerian algebras (bRS spaces) and Sugihara monoids (Sugihara spaces) by building on the earlier work of Galatos and Raftery [?, ?].

A bRS space is an Esakia space whose underlying poset is a forest, together with a distinguished set D of minimal points, also called a pointed Esakia space [?]. The equivalence with Sugihara monoids is realised by a *reflection* functor $\mathbb{R}$: given a bRS space E, each point $b \in E \backslash D$ acquires a reflected counterpart b', with $b' \leq c'$ if and only if $c \leq b$ in E. This reflection over the set D of minimal points is an *Urquhart space* [?] that is dual to the algebra of fractions, yielding a duality that is considerably more transparent than the algebraic construction.

However, as soon as one abandons semilinearity, the algebra of fractions need not be distributive, placing it beyond the reach of classical Urquhart duality. The loss of semi-linearity also causes the corresponding bRS space to cease to be forest.

We extend the Galatos–Fussner duality to all finite pointed Brouwerian algebras by salvaging the core notion of reflection even when downsets fail to be chains by tracking the local structure around each point. For each a we still create a reflected point a', but the structure of the reflection is based on whether a is conical. Concretely, $b' \leq a'$ if and only if $a \leq b$ and a is conical in $\downarrow b$ (that is related to every element in $\downarrow b$). We can think of this as non-conical points are somehow breaking the tree-like structure we would expect and so end up growing off to the side.

To complete the duality on the topological side we replace the usual downset functor with a *prime downset* functor $\mathbb{J}$. Concretely, d' lies in the prime downset of c' if either $d' \leq c'$, or $a_i' < c'$ for every non-conical $a_i \leq d$ in the original space. Together, the generalised reflection $\mathbb{R}$ and the prime downset functor $\mathbb{J}$ complete the duality square

$$\begin{array}{ccc} \mathbb{F}(\mathbf{B}) & \xleftarrow{\;\mathbb{J}\;} & \mathbb{R}(\mathbf{E}) \\ {\scriptstyle\mathbb{F}}\uparrow & & \uparrow{\scriptstyle\mathbb{R}} \\ \mathbf{B} & \longrightarrow & \mathbf{E} \end{array}$$

for all finite pointed Brouwerian algebras $\mathbf{B}$ with dual Esakia space $\mathbf{E}$. This yields a topological equivalence that genuinely extends the result of Galatos and Fussner in the finite case, mirrors the algebraic equivalence of [**?**], and remains considerably more clear than the algebra of fractions construction itself.

On the structure of involutive partially ordered monoids as models of multiplicative fragments of linear logic

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The fragment of classical linear logic restricted to contain only the multiplicative connectives is known to have the finite model property, and its entailment is NP-complete [7]. Algebraic models of non-commutative multiplicative linear logic are partially ordered algebras, where the partial order is generated by its two-sided sequent calculus [1, 2], and the multiplication is associative and order-preserving. For the lack of lattice connectives in the signature, the underlying order does not always form a semi-lattice, and interesting posets, including in particular certain unions of chains, arise this way. We aim at structural understanding of involutive, and therefore residuated, partially ordered monoids — algebraic models of non-commutative, non-cyclic multiplicative linear logic.

An involutive partially ordered monoid (ipo-monoid) is an algebra $\mathbf{A} = (A, \leq, \cdot, \sim, -, 0)$, such that $\leq$ is a partial order, $\cdot$ is associative, and the linear negation rules hold:

$$\text{(lin)} \qquad x \leq y \iff x \cdot {\sim} y \leq 0 \iff -y \cdot x \leq 0.$$

We can then define the monoid constant as $1 = {\sim}0 = -0$, and the two residuals of the multiplication arise from the following residuation laws:

$$\text{(ires)} \qquad x \cdot y \leq z \iff x \leq -(y \cdot {\sim} z) \iff y \leq {\sim}(-z \cdot x).$$

Adding the cyclicity property ($x \cdot y \leq 0 \Leftrightarrow y \cdot x \leq 0$), we obtain algebraic models of cyclic multiplicative linear logic CyMLL, where the two negations coincide. If full commutativity is added, we obtain models of MLL, or the multiplicative fragment of the contraction-free relevance logic RW.

Although ipo-monoids are very general, involution in the form of (lin) imposes restrictions on the underlying posets: (i) every connected component of the order $\leq$ is up-directed and down-directed, hence for finite algebras every connected component is bounded; (ii) the equivalence relation $\leftrightsquigarrow$ on a poset that has each connected component as an equivalence class is a congruence, and the quotient algebra is a group with the discrete order. Every (finite) ipo-monoid is based on a poset composed of a family of directed (bounded) connected components, indexed by a group $\mathbf{G}$ with the unit e, governing in which components the operations are computed. If $0 \leq 1$ in $\mathbf{A}$, then the unital component A_e containing the monoidal constant 1 itself is a subalgebra of the whole algebra, and it often determines how the rest of the algebra behaves. Particularly nice examples arise by considering algebraic models of relevant logic RM [4]: Finite ipo-monoids with multiplicative reducts of Sugihara monoids as their unital component additionally satisfy a periodic form of mingle/contraction axiom

$$\text{(n-pM)} \qquad x^{n+1} = x$$

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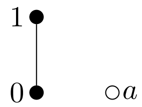
for n being the order of the governing group, which puts x^n in the unital component. One of the simplest examples of such algebras arises as a conuclear image of the product algebra $\mathbb{Z}_2 \times \mathbf{2}$. It is the following three-element ipo-monoid $\mathbf{2}_{\mathbb{Z}_2,\{e\}}$ where $A_e \cong \mathbf{2}$, $a = {\sim}a$ acts as an infectious value, and $a^2 = 0$ and $a^3 = a$:



Besides the structural theory of some classes of ipo-monoids, we will also discuss inequational bases for the (partially ordered) varieties generated by $\mathbf{2}_{\mathbb{Z}_2,\{e\}}$ and by $\mathbb{Z}_2 \times \mathbf{2}$.

As the first structural result, we describe commutative ipo-monoids with a unital component that is a Boolean algebra: let $\mathbf{A}$ be a such a commutative ipo-monoid with a unital component $\mathbf{B}$ and denote the group $\mathbf{A}/{\leftrightsquigarrow}$ by $\mathbf{G}$. Then $\mathbf{A}$ is a conuclear image of $\mathbf{G} \times \mathbf{B}$ and all components of $\mathbf{A}$ are isomorphic to principal ideals of $\mathbf{B}$ (the conuclear map on $\mathbf{G} \times \mathbf{B}$ is defined by $\delta((g, x)) = (g, \top_g \cdot \top_{g^{-1}} \cdot x)$.)

Next we will describe ipo-monoids with multiplicative reducts of Sugihara monoids as their unital component: Let $\mathbf{A}$ be such an ipo-monoid with $A_e \cong S_k$ where every element is of a finite order. Then $\mathbf{A}$ is a disjoint union of chains, each with $\leq k$ elements, and $\cdot$ determined by subgroups of $\mathbf{A}/{\leftrightsquigarrow}$. Moreover, $\mathbf{A}$ is cyclic and balanced ($x \cdot {\sim}x = -x \cdot x$). We can obtain this insight without assuming finite order, if $\mathbf{A}$ is balanced. Moreover, every commutative ipo-monoid $\mathbf{A}$ with $A_e \cong \mathbf{S}_k$ and $\mathbf{G} = \mathbf{A}/{\leftrightsquigarrow}$ is a conuclear image of $\mathbf{G} \times \mathbf{S}_k$.

We can alternatively use Płonka sums adapted from those described in [6, 3] to construct commutative ipo-monoids with multiplicative reducts of Sugihara monoids as their unital component: namely, Płonka sums over a linear join-semilattice of positive elements of the ipo-monoid in question, using the conuclear images of $\mathbf{G} \times \mathbf{B}$ with B a boolean algebra described above as fibers.

Part of our result closely relates to work on unilinear residuated lattices [5, 10]. Some of the results were discovered with the help of Prover9/Mace4 [8].

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Craig Interpolation in the Logic of Temporal Linear Structures

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Abstract

In this paper, we study the Craig interpolation property for a temporal logic defined on sequences of finitely many finite linear structures linked by surjective bounded morphisms. We also investigate normal extensions of this logic.

In [1], the authors provided a sound and complete axiomatization of the temporal logic of sequences of finitely many finite linear structures linked by surjective bounded morphisms.

Finite linear structures, i.e., finite sets equipped with a strict linear ordering, naturally arise as representations of a discrete and bounded flow of time. Many domains of everyday practice involve such structures to represent sequences of consecutive steps or states. Examples include time-series data, scene analysis, and sequential design patterns in programming, such as the chain-of-responsibility pattern. A familiar illustration is a movie represented as a sequence of individual frames ordered in time.

In many situations it is natural to group consecutive elements of a sequence into conceptually meaningful units that inherit the temporal ordering of the original structure. Moreover, this grouping can be repeated finitely many times, producing a hierarchy of ordered structures. For instance, individual movie frames may be grouped into episodes, episodes into scenes, and scenes into acts. The structure of episodes inherits the temporal order of the frames, and likewise the structures of scenes and acts inherit the ordering from lower levels. Such hierarchical temporal organizations can be naturally modeled by sequences of finite linear structures connected by surjective monotone maps that capture transitions between successive levels of abstraction.

A **TES** (Temporal Event Structure) is a structure

$$(F_1, \dots, F_n, <_1, \dots, <_n, f_1, \dots, f_{n-1})$$

where $(F_i, <_i)$ are finite strict linear orders, and $f_i : F_i \twoheadrightarrow F_{i+1}$ are surjective monotone maps, where monotone means $f_i(a) \leq_{i+1} f_i(b)$ for all $a \leq_i b$. Let $F := \bigcup_{i=1}^{n} F_i$, $< := \bigcup_{i=1}^{n} <_i$, and $f := \bigcup_{i=1}^{n-1} f_i$.

The language $\mathcal{L}$ is given by

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \text{⍐}\phi \mid \text{⍗}\phi \mid \text{⍇}\phi \mid \text{⍈}\phi,$$

where p ranges over propositional symbols. The remaining logical connectives are defined as usual.

The language $\mathcal{L}$ is interpreted over a **TES** $(F, <, f)$ such that $\boxminus$ and $\boxminus$ are interpreted over $(F, <, >)$, while $\boxdot$ and $\boxdot$ are interpreted over (F, f, f^{-1}). Let $\mathbf{T}_n$ denote the class of all **TES**s with fixed n.

The logic of $\mathbf{T}_n$ is defined in a standard way.

$$\mathsf{Log}(\mathbf{T}_n) = \{\varphi \mid F \models \varphi \text{ for every } F \in \mathbf{T}_n\}.$$

Next, we recall the definition of the Craig interpolation property, which is one of the central properties of logical systems. A logic L has the *Craig interpolation property* if whenever $L \vdash \varphi \to \psi$, there exists a formula θ such that $L \vdash \varphi \to \theta$, $L \vdash \theta \to \psi$, and $\mathrm{Var}(\theta) \subseteq \mathrm{Var}(\varphi) \cap \mathrm{Var}(\psi)$.

The following theorem shows that the logic of **TES** frames has Craig Interpolation Property. The proof employs algebraic and model-theoretic techniques developed in [3] and [2].

Theorem 1. *The logic* $\mathsf{Log}(\mathbf{T}_n)$ *has the Craig Interpolation Property.*

Next, we study normal extensions of $\mathsf{Log}(\mathbf{T}_n)$. A normal modal logic L_2 is called a *normal extension* of a modal logic L_1 if $L_1 \subseteq L_2$ and L_2 is closed under uniform substitution, modus ponens, and the rule of necessitation. For a given logic L, the set of its normal extensions forms a lattice and is denoted by $\mathrm{NExt}(L)$.

Let us define a relation on the class K of Kripke frames as follows:

$$F_1 \leq F_2 \quad \text{iff} \quad F_2 = \mathrm{HS}(F_1),$$

where H and S denote the class operations of taking homomorphic images and substructures, respectively. It is easy to show that, the class $\mathbf{T}_n$ forms a countable antichain with respect to this relation. The following result is a straightforward consequence of this fact.

Theorem 2. *There are uncountably many elements in* $\mathrm{NExt}(\mathsf{Log}(\mathbf{T}_n))$.

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Completions and Functional Representation of Monadic Orthocomplemented Lattices

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Abstract

An *orthocomplemented lattice* (henceforth, an *ortholattice*) is a bounded lattice $\langle A; \wedge, \vee, 0, 1 \rangle$ equipped with an additional operation ${}^{\perp} \colon A \to A$, known as an *orthocomplementation*, that is an order-inverting period two complementation. A *monadic ortholattice* is an ortholattice A equipped with a closure operator $\exists \colon A \to A$, known as a *quantifier*, whose closed elements form a sub-ortholattice. Janowitz [4] first considered quantifiers on orthomodular lattices, and Harding [2] studied them, as well as cylindric ortholattices, for their connections to von Neumann algebras, in particular, to subfactors.

In this talk, I will discuss some recent results obtained by myself, Harding, and Peinado [3] in which we show that the variety of monadic ortholattices is closed under MacNeille and canonical completions. In each case, the completion of a monadic ortholattice A is obtained by forming an associated dual space X_A that is a monadic orthoframe. This is a set equipped with an orthogonality relation and an additional binary relation satisfying certain conditions. For the MacNeille completion, X_A is formed from the non-zero elements of A, and for the canonical completion, X_A is formed from the proper filters of A. In either case, the corresponding completion is obtained by embedding A into the complete monadic ortholattice of bi-orthogonally closed subsets of X_A whose quantifier is defined through the binary relation of X_A.

I will then discuss some recent results obtained by myself and Lin [5] within the functional representation theory of monadic ortholattices. Let X be a non-empty set and let A be a complete ortholattice. The *full functional monadic ortholattice* determined by X and A is defined over the function space A^X whose ortholattice operations are defined point-wise and whose associated quantifier is defined by $(\exists f)(x) = \bigvee\{f(x) : x \in X\}$. A monadic ortholattice is called a *functional monadic ortholattice* if it is a subalgebra of a full functional one. Using tools from the theory of MacNeille completions, super-amalgamations, and directed limits along the lines of [1], we show that every monadic ortholattice is isomorphic to a functional one.

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Arithmetic polytopes, ordered groups, and affine monoids: a duality theorem

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Integral polyhedral geometry plays a central rôle in Łukasiewicz logic. The algebraic counterpart to Łukasiewicz logic is provided by Chang's MV-algebras, and finitely presented MV-algebras are representable as algebras of piecewise-affine functions on rational polyhedra, each affine component having integer coefficients [**?**]. In fact, it is a theorem that the category of finitely presented MV-algebras is dually equivalent to the category of rational polyhedra and piecewise-affine maps with integer coefficients [**?**, **?**]. The building blocks of the latter category are rational polytopes and affine maps between them whose scalar components, in coordinates, have integer coefficients. The purpose of the work presented here is to investigate this category and its opposite in their own right.

Choquet's theory in functional analysis, as extended e.g. in [**?**] from ordered real vector spaces to ordered Abelian groups, provides a useful analogy for such an investigation. In [**?**, Chapters 6 and 7], Goodearl describes a dual adjunction between the category of partially ordered Abelian groups equipped with a strong order unit, and the category of compact convex sets in real linear topological spaces. In one direction, an ordered group is sent to the collection of its *states*—unit-preserving monotone group homomorphisms to the unital additive ordered group $\mathbb{R}$ of real numbers with unit 1. In the opposite direction, a compact convex set is sent to the Banach space of all continuous $\mathbb{R}$-valued affine functions on that set, equipped with pointwise order and with the function constantly equal to 1 as unit.

To account for the arithmetic structure of our rational polytopes, we restrict attention to rational affine spaces $\mathbb{Q}^n$, to those affine maps which preserve the integer lattice $\mathbb{Z}^n \subseteq \mathbb{Q}^n$, and to rational-valued states. The finite dimension of our polytopes and of their ambient spaces will allow us to do away with all topological notions.

An affine map $\mathbb{Q}^n \to \mathbb{Q}^m$ is *$\mathbb{Z}$-affine* if it descends to a function $\mathbb{Z}^n \to \mathbb{Z}^m$. A *polytope* (in $\mathbb{Q}^n$) is the convex hull of a finite non-empty subset of $\mathbb{Q}^n$. For polytopes $P \subseteq \mathbb{Q}^n$, $Q \subseteq \mathbb{Q}^m$, a function $P \to Q$ is *$\mathbb{Z}$-affine* if it extends to some $\mathbb{Z}$-affine map $\mathbb{Q}^n \to \mathbb{Q}^m$. The *category of arithmetic polytopes*, written $\mathsf{P}_{\mathbb{Z}}$, has polytopes as objects and $\mathbb{Z}$-affine maps as morphisms.

A *partially ordered Abelian group* $(A, +, 0)$ is an Abelian group equipped with a partial order $\leqslant$ that is translation-invariant with respect to addition (for all $x, y, t \in A$, $x \leqslant y$ implies $x + t \leqslant y + t$). An element $a \in A$ is *positive* if $0 \leqslant a$, and the *positive cone* of A is the monoid $A^+ := \{a \in A \mid 0 \leqslant a\}$. A (*strong order*) *unit* of A is a positive element $1 \in A$ such that for all $a \in A$ there exists $n \in \mathbb{N}$ with $a \leqslant n1$. The category $\mathsf{A}^1_\leqslant$ has partially ordered Abelian groups with unit (hereafter, *unital ordered groups*) as objects, and unit-preserving (henceforth, *unital*) monotone group homomorphisms as morphisms. For $P \subseteq \mathbb{Q}^n$ a polytope, the set $\mathrm{Aff}_{\mathbb{Z}}(P, \mathbb{Q})$ of $\mathbb{Z}$-affine maps $P \to \mathbb{Q}$ endowed with the pointwise order, together with the function 1_P constantly equal to 1 as a unit, is a unital ordered group. We call a unital ordered group *polytopal* if it is isomorphic in $\mathsf{A}^1_\leqslant$ to such a group of $\mathbb{Z}$-affine maps, for some polytope P. The category $\mathsf{G}^1_\leqslant$ is the full subcategory of $\mathsf{A}^1_\leqslant$ whose objects are polytopal groups. There is a

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contravariant functor[1] $\mathrm{Aff}_{\mathbb{Z}}(-,\mathbb{Q})\colon \mathsf{P}_{\mathbb{Z}} \dashrightarrow \mathsf{G}^1_{\leqslant}$, which acts on morphisms by precomposition.

We next construct a contravariant functor in the opposite direction. A (*rational*) *state* of a unital ordered group A is a morphism $A \to \mathbb{Q}$ in $\mathsf{A}^1_{\leqslant}$; we write $\mathrm{St}(A,\mathbb{Q})$ for the set of states of A. An *atom* of A is an element $a \in A^+$ such that $a \neq 0$, and for each $b \in A$, $0 \leqslant b \leqslant a$ implies $b = 0$ or $b = a$; we write αA for the set of atoms of A. For a polytopal group A, we prove αA is a finite generating set both of the monoid A^+ and of the group A. Hence, each member of $\mathrm{St}(A,\mathbb{Q})$ is uniquely determined by its values at αA. Using this fact, $\mathrm{St}(A,\mathbb{Q})$ can be identified with a polytope in $\mathbb{Q}^{\alpha A}$. In turn, this yields a contravariant functor $\mathrm{St}(-,\mathbb{Q})\colon \mathsf{G}^1_{\leqslant} \dashrightarrow \mathsf{P}_{\mathbb{Z}}$ which acts on arrows by precomposition.

Theorem. *The contravariant functors*

$$\mathsf{P}_{\mathbb{Z}} \underset{\mathrm{St}(-,\mathbb{Q})}{\overset{\mathrm{Aff}_{\mathbb{Z}}(-,\mathbb{Q})}{\rightleftarrows}} \mathsf{G}^1_{\leqslant}$$

are a dual equivalence between the category of arithmetic polytopes and the category of polytopal groups.

If time allows, we will address further research that can be pursued in various directions starting from the Theorem.

The dual category $\mathsf{G}^1_{\leqslant}$ of polytopal groups exhibits close ties with the theory of affine monoids (finitely generated commutative monoids that are torsion-free and cancellative [**?**]) and toric varieties [**?**]. We discuss an intrinsic characterisation of the polytopal groups as ordered groups. Further, we discuss an equivalence of categories between $\mathsf{G}^1_{\leqslant}$ and a certain category of pointed affine monoids. The universal-algebraic aspects of this equivalence, including its relationship with the variety of MV-algebras, are a worthwhile research direction.

There is a rich literature on natural dualities induced by functors represented by a dualising object. Neither $\mathrm{Aff}_{\mathbb{Z}}(-,\mathbb{Q})$ nor $\mathrm{St}(-,\mathbb{Q})$ are representable functors. The construction of a conceptually satisfactory framework in which appropriate structures on the set of rational numbers $\mathbb{Q}$ induce a dual adjunction that extends the dual equivalence in the Theorem is a meaningful undertaking.

[1] In the duality-theoretic context of this work, we prefer to explicitly use contravariant functors rather than covariant ones and opposite categories. We denote them by dashed arrows.

Quantale-enriched proof theory

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We introduce weighted proper display calculi for basic quantale-enriched logic and some of its axiomatic extensions and language expansions, for which we prove soundness, completeness, conservativity, the subformula property, and cut elimination, generalising the Belnap-style metatheorem. All inference rules we consider are closed under uniform substitution and have no side conditions. The resulting proof-theoretic environment enjoys a high degree of modularity, paving the way for the systematic study of a neighbouring family of non-classical propositional and first-order quantale-enriched logics.

From a proof-theoretic standpoint, the main novelty is that the syntactic consequence relation is *weighted*, namely it is parametrised by terms interpreted as elements of a quantale. In the present work we focus on the case in which the quantale is *non-commutative*, allowing weights to encode directional or resource-sensitive information. In principle, this feature enhances the expressivity of (display) sequent calculi and facilitates a more fine-grained proof-theoretic analysis.

The semantics we adopt is polarity-based, providing a natural relational environment for interpreting sequents and their associated weights. Within this setting, weighted sequents admit a categorical interpretation in terms of enriched structures. In particular, a $\mathcal{V}$-enriched category (see, for instance [?, Section 1.2] for background) is defined relative to a monoidal category $\mathcal{V}$. It consists of a class of objects together with, for each pair of objects x, y, a *hom-object* $C(x, y) \in \mathcal{V}$. The hom-objects satisfy the usual requirements that identity arrows $I \to C(x, x)$ exist, where I is the unit object of $\mathcal{V}$, and that compositions $C(x, y) \otimes C(y, z) \to C(x, z)$ exist and are associative and unital. A quantale-enriched category is a special case in which $\mathcal{V}$ is a quantale (see [?, Chapter 2] for background on quantales).

As is customary in categorical semantics, we take hom-objects as the interpretation of sequents, which in our framework are *weighted*, thereby generalising and refining standard proof theory. Weighted sequents can be understood in several ways: they may represent the distance between premises and conclusions, the degree to which conclusions follow from premises, or the confidence that an agent assigns to the logical consequence relation. In the latter case, one could further parametrise sequents with the names of agents, thereby connecting with the modelling of resources and epistemic or doxastic attitudes.

These ideas resonate with developments in formal linguistics and distributional semantics. The slogan "You shall know a word by the company it keeps!" [?] aptly captures the distributional hypothesis [?], according to which the meaning of a word is closely related to the distribution of the words occurring around it. Within this framework, words are typically represented as vectors in high-dimensional spaces [?] or as elements of metric spaces. This perspective connects directly to the training of large language models, whose goal is to predict the next word in a sentence by assigning probabilities to possible continuations. In this context, enriched categorical structures have also been considered [?], with applications including measuring semantic similarity between words, sentences, and documents, as well as paraphrasing, question answering, and summarization.

By contrast, Lambek's program on type-logical grammar [?] takes a logical perspective, treating logic as a test of grammaticality: a sentence is grammatically well-formed if and only if it corresponds to a derivable sequent in a given logic. Applications of this approach range from parsing to reasoning tasks. More recently, several approaches have sought to integrate these two traditions (see, e.g., [**?**]).

We believe that *weighted sequents*, together with their interpretation in quantale-enriched categorical structures and polarity-based semantics, provide a key technical ingredient that could reshape this line of research and foster a deeper integration between logical and distributional approaches. To this end, we plan to develop the theory of *weighted labelled display calculi* together with their corresponding free categorical semantics.

Another application we aim to pursue is inherently proof-theoretic and concerns normalization. One may ask how far a given derivation is from its normal form. In this context, it is natural to assign a weight to the end sequent of a proof reflecting the distance from normal form, for instance by counting the number of reduction steps required to reach it. This idea is not entirely far-fetched, as witnessed by [**?**].

Evolution systems and embedding-projection pairs

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An evolution system is a structure of the form $\mathcal{E} = \langle \mathcal{V}, \mathcal{T}, \Theta \rangle$, where $\mathcal{V}$ is a category (the universe), $\mathcal{T}$ is a class of arrows (called *transitions*) and Θ is an object (called the *origin*). Sequences of transitions starting from the origin are called *evolutions*. For details, see [1].

The theory of universal homogeneous structures (*Fraïssé limits*, see [3]) can be easily interpreted in the language of evolution systems. We shall discuss the case where transitions are embedding-projections pairs, namely, pairs consisting of a monomorphism e, an epimorphism p such that $p \circ e$ is the identity. Such patterns are common in domain theory and also in topology and Banach space theory (see [3, Sections 6,7]), however they have not been much studied in model theory and universal algebra. We show how, in this setting, proper amalgamations lead to new universal and highly homogeneous structures.

A concrete example comes from graph theory, where the origin is a singleton and transitions are one-point extensions together with a projection (we consider reflexive graphs in order to be able to collapse edges). Infinite evolutions lead to structures isomorphic to graphs on the natural numbers $\mathbb{N}$ such that for every $n \in \mathbb{N}$ there is an edge-preserving projection of $[0, n+1]$ onto $[0, n]$ (that is, mapping $n+1$ to a smaller number and being identity on $[0, n]$). Let us call such graphs *point-by-point retractible*.

Theorem 1 *There exists a unique point-by-point retractible graph* $\mathbb{W}$ *with the following extension property:*

(E) *Given a finite induced subgraph* $A \subseteq \mathbb{W}$*, given its one-point extension* $B = A \cup \{v\}$ *and a projection* $p \colon B \to A$*, there exists* $w \in \mathbb{W}$ *with* $w > \max A$*, such that* $A \cup \{w\}$ *is isomorphic to* B *over* A *and* $p(v)$ *is the image of* w *under some projection of* $\mathbb{W}$ *onto* A*.*

In other words, every one-point extension of a finite induced subgraph, in the sense of embedding-projection pairs, is realized in $\mathbb{W}$. Furthermore:

Theorem 2 *Every point-by-point retractible graph can be embedded into* $\mathbb{W}$ *in such a way that there is a projection onto its image.*

Yet another feature of $\mathbb{W}$ is its rich automorphism group; in particular, it acts transitively on $\mathbb{N}$ and the general Fraïssé theory provides automorphisms extending isomorphisms between suitable (and properly embedded) finite graphs.

On the other hand, $\mathbb{W}$ is not isomorphic to the random graph $\mathcal{R}$ and we do not know whether $\operatorname{Aut}\mathbb{W}$ has any nontrivial relation to $\operatorname{Aut}\mathcal{R}$.

One can also consider the topological counterpart of the theory above, looking at the limits of projections instead of colimits of the embeddings. In the case of graphs, we obtain a closed graph structure on the Cantor set with a highly symmetric topological automorphism group.

Finally, let us mention the first work on category-theoretic approach to the theory of universal homogeneous structures, by Droste and Göbel (1989) [4], where many of the applications (e.g. information systems) dealt with left-invertible embeddings. We propose a more detailed analysis, focusing on the simplest possible left-invertible embeddings, namely, adding a single element, building a natural well-ordering of the structure.

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Intmax2: A Secure Blockchain Protocol Enabled by Lattice-Ordered Abelian Groups

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Abstract

Intmax2 is a scaling payment solution for Ethereum. The protocol uses lattice-ordered abelian groups for the balances of accounts. The censorship problem is solved intrinsically on Intmax2 as any user can be an aggregator. The security theorem, mechanically proved in the Lean4 proof assistant, guarantees the key economic safety property of the protocol.

1 Introduction

We present an application of the theory of lattice-ordered abelian groups in the field of blockchain. Intmax2 is a scaling payment solution[1]. The protocol is live on Ethereum as a rollup since 2024 and offers an upper limit of 7500 transactions batches per second where each transaction batch can transfer an unlimited number of tokens to an unlimited number of recipients. The protocol is stateless and have a permissionless block production [2].

The protocol and its safety property has been formally verified in the Lean4 proof assistant. The formalization is open source and mirrors closely the white paper [3, 4].

2 Overview of the Protocol

The main ingredient of Intmax2 is a *rollup contract* deployed on Ethereum. To *deposit* funds on Intmax2, a user sends the funds, together with the rollup address of the recipient, to the rollup contract which then records the deposit in its contract storage. To *transfer* funds on Intmax2, a subset of rollup accounts will first send their transactions to a single *aggregator*, which then inserts the transactions at the leaves of a *merkle tree*. Then the aggregator sends to each sender the merkle root and the merkle proof for that sender's transaction. Each sender then signs the merkle root with their public BLS key and sends this signature back to the aggregator. The aggregator then aggregates the signatures into a single aggregated signature, and sends the merkle root, the aggregated signature and the list of public keys of the senders that was included in the aggregated signature to the rollup contract. The rollup contract then verifies the signature and adds the root, signature and sender list to its storage. Each sender is then responsible for sending the merkle proof of the transaction to each transaction recipient offchain, together with earlier merkle proofs that together prove that the sender had sufficient balance for the transaction. To prove their own balance, each user needs to keep track of all

[1] https://intmax.io/

merkle proofs they have received from aggregators and other users. This collection of merkle proofs, called a *balance proof*, is sent to the rollup contract when a user wants to *withdraw* their funds back to Ethereum.

3 The Security of the Protocol

The rollup contract keeps track of a list $B^* \in \mathcal{B}^*$ of all blocks (i.e. *deposit*, *transfer*, and *withdrawal blocks*) that have been added to the rollup so far. A *state* of the rollup is an assignment of a balance to each account and we denote by $\mathcal{S}$ the set of states. Formally, let $\overline{\mathcal{K}} = \mathcal{K}_1 \coprod \mathcal{K}_2 \coprod \{\text{Source}\}$, where $\mathcal{K}_1$ are accounts on Ethereum, $\mathcal{K}_2$ are accounts on the rollup, and Source is a special account used to represent deposits and withdrawals. The set of states $\mathcal{S}$ is then defined as

$$\mathcal{S} = \{b \in \mathcal{V}^{\overline{\mathcal{K}}} \mid b(k) \geq 0, \forall k \in \overline{\mathcal{K}} \setminus \{\text{Source}\}\},$$

where $\mathcal{V}$ is a lattice-ordered abelian group [1]. The *balance function* takes a balance proof and the current list of added blocks in the rollup contract, and returns the balance of each account in the rollup that can be proven by the balance proof, i.e.

$$Bal : \Pi \times \mathcal{B}^* \to \mathcal{S}.$$

This function is used by users to compute their own balance, as well as by the rollup contract when processing a withdrawal request. Since the amount to be withdrawn is computed by applying the balance function to the balance proof and all previous blocks that have been added to the rollup, a user cannot double-spend by withdrawing the same funds twice.

Given a balance proof $\pi \in \Pi$ and a list of blocks $B^* \in \mathcal{B}^*$ in the rollup contract, the account balances are computed in two steps. First, we extract a list of *partial transactions* from π and B^*, where a partial transaction consists of a sender, a recipient and a (possibly unknown) transaction amount. Then, we compute the balances of every account by applying a *state transition function* on the list of partial transactions.

If we have a sequence of complete (i.e. known) transactions (as in any traditional blockchain protocol), we can compute the balance of each account by applying a transition function for the complete transactions. However, as in the case of the Intmax2 protocol, if some of the transactions are unknown, we cannot know for sure what the balance of each account is. Instead, we can compute a *lower bound* on the balance of each account which is possible if transaction values and account balances take value in a lattice-ordered abelian group.

We say that the rollup contract is *secure* if every withdrawal request succeeds, i.e. the rollup contract has sufficient balance for every withdrawal. This means that if a user has a balance proof which proves the in-rollup balance of one or more of their accounts, they will be able to withdraw these balances to Ethereum. We mechanically proved the security theorem of the Intmax2 protocol in the Lean4 proof assistant.

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Revisiting Completeness of Fixpoint Logics

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We give an algebraic proof strategy for completeness of fixpoint logics, adapting to the modal μ-calculus a topological argument given by Rasiowa and Sikorski for first order logic. This yields a duality theoretic analysis of a representation theorem for flat modal fixpoint logics [1, Theorem 7.1]. We also discuss how to obtain fixpoints which comply with the requirements of this approach.

Context. We study completeness for fragments of the modal μ-calculus obtained by enriching propositional modal logic with least-fixpoint connectives $\sharp_\gamma$ governed by the Kozen-Park axioms. Given a set Γ of basic modal formulas $\gamma(x, \vec{y})$ called *fixpoint schemes*, the Kozen-Park axiomatisation $\mathbf{K}_\sharp(\Gamma)$ extends basic modal logic by adding, for each $\gamma \in \Gamma$, axioms that present $\sharp_\gamma(\vec{y})$ as the least prefixpoint of $\gamma_{\vec{y}} : x \mapsto \gamma(x, \vec{y})$. For the sake of this abstract, we consider fixpoint schemes $\gamma(x, y)$ with a designated recursion variable x and one parameter y.

The algebraic route to completeness proceeds via the *Lindenbaum-Tarski algebra* of the logic under consideration. Jónsson and Tarski showed that Boolean algebras with operators canonically embed into complex algebras of Kripke frames [2, Theorem 3.10]. Instantiating this result to the Lindenbaum algebra of basic modal logic $\mathbf{K}$ allows for an algebraic account of its completeness: the Lindenbaum algebra $\mathcal{L}_\mathbf{K}$ embeds into the complex algebra of its ultrafilter frame – commonly known as the *canonical extension* of the algebra [3, Section 5.3] – in a way which preserves Boolean connectives and $\Diamond$. Expanding this reasoning to fixpoint logics, one further requires the embedding to preserve the designated least fixpoints. A sufficient criterion on the considered fixpoint scheme(s) is *constructivity* in the Lindenbaum algebra of $\mathbf{K}_\sharp$. Here, a fixpoint is constructive if:

$$\forall b \in \mathcal{L}_{\mathbf{K}_\sharp}, \quad \sharp_\gamma(b) \;=\; \sup_{n \geqslant 0} \gamma_b^n(\bot) \qquad \text{in } \mathcal{L}_{\mathbf{K}_\sharp}.$$

As our first contributuion, we construct a dense subframe of the ultrafilter frame on which the canonical embedding preserves both the modality and all designated least fixpoints. This approach expands upon Rasiowa and Sikorski's proof of Gödel's completeness theorem [4], which translates the existence of a model into that of an ultrafilter extending well-chosen filters. This yields a completeness proof for the considered fragments of the modal μ-calculus, and suggests generalisations beyond Kripke frames and constructive fixpoints.

Fixpoint-preservation *à la* Rasiowa-Sikorski. The embedding of a Boolean algebra into its canonical extension need not preserve least fixpoints. We characterise preservation of fixpoints as an instance of general topological constraints on ultrafilters, and argue that the resulting restriction induces a fixpoint-preserving embedding. This construction relies on ultrafilters *respecting* constructive fixpoints in the sense described below.

We view an ultrafilter μ on the Lindenbaum algebra $\mathcal{L}$ as a homomorphism $h_\mu : \mathcal{L} \to 2$. If a constructive fixpoint is presented as a supremum $\sup S$, then μ should preserve this supremum, that is, $h_\mu(\sup S) \;=\; \sup h_\mu(S)$. This condition may be equivalently formulated as '$\sup S \in \mu$ implies that S meets μ'. Thus, for any constructive fixpoint scheme $\gamma \in \Gamma$ and parameter b, we may present fixpoint-preservation of an ultrafilter as a supremum-preservation constraint. We package these conditions as a *sup-constraint*

$$\mathcal{S} \coloneqq \left\{ \{\gamma_b^n(\bot) \mid n \geq 0\} \mid \gamma \in \Gamma, b \in \mathcal{L} \right\}.$$

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We call an ultrafilter $\mathcal{S}$*-complete* if it respects every supremum in $\mathcal{S}$, and denote by $\mathfrak{G}_{\mathcal{S}}$ the set of $\mathcal{S}$-complete ultrafilters. Theorem 1 shows that if there are countably many suprema to respect, then almost every ultrafilter is $\mathcal{S}$-complete.

Theorem 1. *If B is a countable Boolean algebra and $\mathcal{S}$ is a countable* sup*-constraint on B, then $\mathfrak{G}_{\mathcal{S}}$ is dense in the Stone dual of B.*

This density result ensures that restricting the canonical embedding to the complex algebra of the subframe induced by $\mathfrak{G}_{\mathcal{S}}$ still yields an injective homomorphism. Restricting to $\mathcal{S}$-complete ultrafilters ensures that the embedding preserves fixpoints, however it may jeopardise preservation of $\Diamond$. Preservation of the modality is equivalent to the following condition: for every element b of the algebra and transition $\mu \to \nu$ in the full ultrafilter frame such that $b \in \nu$, there must exist a transition $\mu \to \nu'$ in the subframe induced by $\mathfrak{G}_{\mathcal{S}}$ such that $b \in \nu'$. We denote by $\mathrm{T}(\mathfrak{G}_{\mathcal{S}})$ the set of ultrafilters verifying this condition. If the considered modality $\Diamond$ is *residuated*, i.e. it has a right adjoint on the poset underlying the algebra, then Theorem 2 shows that almost every ultrafilter is $\mathcal{S}$-complete *and* verifies the condition specified by $\mathrm{T}(\mathfrak{G}_{\mathcal{S}})$.

Theorem 2. *Suppose B is a countable modal Boolean algebra with a residuated modality, and $\mathcal{S}$ is a countable* sup*-constraint. Then $\mathrm{T}(\mathfrak{G}_{\mathcal{S}}) \cap \mathfrak{G}_{\mathcal{S}}$ is dense in the Stone dual of B.*

In the case of Lindenbaum algebras, the modality $\Diamond$ is residuated [5]. Considering countably many fixpoint schemes Γ, the canonical embedding from the Lindenbaum algebra $\mathcal{L}_{\mathbf{K}_\sharp(\Gamma)}$ to this complex algebra respects $\Diamond$ and fixpoints, thereby yielding completeness of $\mathbf{K}_\sharp(\Gamma)$. A dual template yields embeddings for greatest-fixpoint fragments by imposing inf-constraints. This suggests that generalising to alternating fragments may be possible via simultaneous treatment of sup/inf-constraints. Other directions to be considered are adapting this setting to coalgebraic modal logics, and generalising it to broader classes of fixpoints through general sup-constraints.

A constructivity criterion. Showing that a least fixpoint is constructive may be challenging. A criterion for constructivity was introduced by Santocanale in the form of $\mathcal{O}$*-adjoints* [6]. With **BA** the category of Boolean algebras and **JSL** that of join-semilattices, consider the functor $F : \mathbf{BA} \to \mathbf{JSL}$ obtained by composing the forgetful functor to the category of posets $\mathbf{BA} \to \mathbf{Pos}$ with the free join-semilattice functor $\mathbf{Pos} \to \mathbf{JSL}$. A **BA**-endomorphism $f : B \to B$ is an F*-adjoint* if Ff has a right adjoint g in the **Pos**-enriched category **JSL**, and it is *finitary* if the orbits $\{g^n(D) \mid n \in \mathbb{N}\}$ for D in the join-semilattice FB have an infimum in FB.

Theorem 3 ([6, Lemma 6.6])**.** *Consider a fixpoint scheme $\gamma(x, y)$ such that, for any parameter b in the Lindenbaum algebra, the induced map γ_b is monotone and a finitary left F-adjoint. Then γ is constructive.*

We examine this result through a categorical lens and discuss relations with 2-categorical phenomena, with the aim of generalising this criterion to a broader 2-categorical setting.

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How to build categorical semantics of proofs for the basic modal Lambek logic and beyond

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The identity of proofs problem is a foundational concern that has been actively explored across philosophy, mathematics (such as determining when two proofs represent the same argument) and computer science (such as determining when two algorithms represent the same program). We argue that adopting a semantics of proofs, rather than a semantics of provability, could also reshape how we tackle challenges in modeling multi-agent scenarios and phenomena at the interface of grammatical and semantic competence.

For our semantics of proofs, we opt for *categorical semantics* in the tradition of [1, 2, 4, 5] and [6]. As for the logical framework, we focus on the basic modal Lambek logic, given its fundamental role not only in natural language modelling, but also as the basic substructural logic (see, for instance, [7] and [8]). As for presentation of the logic, we use a *proper display sequent calculus* we call $D^\ell.NL_\diamond$ (see [9, 10] and references therein for the relevant proof-theoretic literature).

We obtained a number of results. First, by lifting the Lindenbaum–Tarski construction [11] to the categorical level, we show how to build the free category F generated by the calculus $D^\ell.NL_\diamond$, where objects are formulas and morphisms are equivalence classes of proofs. The equations defining the free category ensure that the equivalence relation is a *categorical congruence relation*, the *naturality* and *functoriality* of the morphisms corresponding to logical introduction rules, and that the display rules characterizing $D^\ell.NL_\diamond$ are *adjunctions*. We then show that F is a *residuated category* [1].

Second, following the approach inaugurated by Prawitz [12], we define a *normalization procedure* over $D^\ell.NL_\diamond$-proofs. Using techniques from the term-rewriting system literature [17], we show that the procedure terminates and that every proof reduces to exactly one proof in normal form. The relation generated by the normalization procedure generates a category P we call the *Prawitz-Lambek category*. We then show that the free category F and the Prawitz-Lambek category P are equivalent.

Third, we introduce an algorithm, which we call D2D for Display-to-Diagram, transforming a $D^\ell.NL_\diamond$-proof into a 1-cell diagram, namely graphical representations of morphisms in the associated residuated category. To the best of our knowledge, this is the first algorithm of the kind.

We conjecture that this approach can be generalized to any displayable logics of any given signature, namely any logic (possibly with a lattice reduct) endowed with normal modalities of finite arity. Moreover, we conjecture that the techniques and results from algorithmic correspondence theory (see, for example, [13] and references therein) can be lifted to the categorical level, enabling us to capture axiomatic extensions of basic displayable logics that can be presented by analytic-inductive [14, 15] or inductive axioms [16].

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Soberness as idempotent-completeness: towards a formal model theory of virtual ultracategories

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The notion of idempotent-complete (or Cauchy complete) category arises from completeness of metric spaces. In this talk, we honor the topological roots of this notion and reinterpret it in the setting of *virtual ultracategories*, a categorification of topological spaces defined independently in [1, 2, 3] last summer.

Virtual ultracategories (or vu-categories for short) are a multicategorical generalization of ultracategories, abstracting the structure of "categorified convergence of ultrafilters" on the category of points of a topos. The key motivating theorem is that any topos with enough points can be reconstructed as the category of sheaves over its vu-category of points. Hence, vu-categories are strong enough to encode any topos with enough points. This raises the following question: when is a vu-category *sober* (i.e. arises as the vu-category of points of a topos)? In this talk, we focus on the following restricted case:

Question. *When is a full sub-vu-category of a sober vu-category sober? Or from a logical point of view, when is a class of models of a geometric theory, geometrically axiomatizable?*

More precisely, for a subclass $\mathcal{K} \subseteq \mathcal{M}$ of the vu-category $\mathcal{M}$ of points of a topos, the *soberification* of $\mathcal{K}$ yields an extension $\mathcal{K} \subseteq \check{\mathcal{K}} \subseteq \mathcal{M}$ known as *subclosure* in the localic case [4]. The following result shows that this subclosure corresponds exactly to closure under *vu-retractions*, a straightforward analogue of the usual categorical notion of retraction in the setting of vu-categories. This result strongly echoes Lawvere's result [5] relating completeness of metric spaces with idempotent-completeness.

Theorem. *A point belongs to $\check{\mathcal{K}}$ if and only if it is a vu-retract of points in $\mathcal{K}$. In particular, $\mathcal{K}$ is sober (or equivalently, geometrically axiomatizable) if and only if $\mathcal{K}$ is vu-idempotent-complete, and the class of points $\mathcal{K}$ is separating if and only if any point is a vu-retract of points in $\mathcal{K}$.*

In other words, the geometric notion of *soberness*, the logical notion of *geometrically axiomatizable class* of models, and the algebraic notion of *vu-idempotent-completeness* all coincide.

Finally, it is worth mentioning that the notion of vu-retract is powerful enough to recover the standard notion of filtered colimit, which plays a central role in the theory of accessible categories. We thus view the notion of vu-idempotent-completeness as a first step toward a more general understanding of accessibility for vu-categories, opening the way to a formal model theoretical study of these structures.

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The equivalential fragment of $\mathbf{FL_{ew}}$

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Originally, *substructural logics* were introduced as logical systems which, when formulated in Gentzen-style sequent calculi, lack some (or none, as a limit case) of the three fundamental structural rules of contraction, weakening, and exchange. They encompass a wide class of nonclassical logics, which include, for instance, classical and intuitionistic logic, many-valued logics, relevance logics, and linear logic. Semantical methods in general, and particularly algebraic semantics, have provided a powerful tool for analyzing substructural logics from a more uniform perspective. In particular, the algebraic perspective allowed to recognize an important common feature of these logics: the *residuation property*, which is reflected in the fact that the corresponding algebraic models are *residuated structures*.

Nowadays, in the field of algebraic logic one usually refers to substructural logics as the extensions of the Full Lambek Calculus **FL**, an algebraizable logic with equivalent algebraic semantics the variety of FL-algebras, which are residuated lattice-ordered monoids, or residuated lattices for short, in the language $\{\cdot, \backslash, /, \wedge, \vee, 0, 1\}$ [4, 7]. The aforementioned logics can all be seen as axiomatic extensions of **FL**. In particular, many of the most interesting examples of substructural logics, such as classical, intuitionistic, and many-valued logics, fit into the framework of substructural logics with exchange and weakening ($\mathbf{FL_{ew}}$), whose algebraic models are commutative integral and 0-bounded FL-algebras, i.e., FL-algebras where the monoidal operation is commutative, 1 is the top in the lattice order, and 0 is the bottom.

Both the proof-theoretical and the algebraic study of the axiomatic extensions of **FL** has been deeply investigated in the last decades (see again the monographs [4, 7]), in addition to the connections between these two approaches which are analyzed by the research area of *algebraic proof theory* [2, 3]. A large body of research has also been dedicated to investigating interesting *fragments* of substructural logics, particularly the 0-free fragments, or the implicational fragments, both from the algebraic and proof-theoretic perspective (e.g., [1], [6], [9], [10]).

In this contribution we are interested in the *equivalential fragment* of substructural logics, in particular of $\mathbf{FL_{ew}}$. To be precise, let us consider the equivalence connective $\leftrightarrow$ defined by

$$a \leftrightarrow b := (a \to b) \wedge (b \to a).$$

We study the subset of formulas of $\mathbf{FL_{ew}}$ that can be rewritten using only $\leftrightarrow$ and we restrict the semantic to this connective only. It is interesting to observe that the connective $\leftrightarrow$ allows to express equality at the propositional level, in the sense that any two elements x, y in some FL_{ew}-algebra are such that $x = y$ if and only if $x \leftrightarrow y = 1$.

A source of inspiration for our work is given by the study of the equivalential fragment of intuitionistic logic (INT). In [8], Tax provides a Gentzen system that corresponds to the equivalential fragment of intuitionistic logic, and then uses it to give an axiomatization of the $\leftrightarrow$-fragment. His starting idea is to first consider the Gentzen calculus for the implicative-equivalential fragment of INT, and use it to develop the one for $\leftrightarrow$. Later on, Wroński and Kabziński in [5] study the equivalential fragment of intuitionistic logic from an algebraic point of view: they introduce the notion of *equivalential algebra*, proving by means of a purely algebraic analysis that the algebraic $\leftrightarrow$-subreducts of Heyting algebras are a variety axiomatized by equations which are an algebraic translation of Tax's axioms.

Our work takes a first step in the direction of understanding the equivalential fragment of $\mathbf{FL_{ew}}$, $\mathbf{FL^{\leftrightarrow}_{ew}}$, following the initial intuition of Tax. In particular, we provide a sequent calculus for $\mathbf{FL^{\leftrightarrow}_{ew}}$, and also establish that it satisfies cut elimination. This allows us to show that provability is decidable in $\mathbf{FL^{\leftrightarrow}_{ew}}$, and equivalently, the equational theory of the corresponding class of algebraic models is decidable as well.

The work also include new technical results related to this calculus. For example, we prove that the sequent $\Gamma \Longrightarrow \varphi$ is provable in **GEq** if and only if finitely many special sequents $\Gamma_1 \Longrightarrow x_1, \ldots, \Gamma_n \Longrightarrow x_n$ with only a variable on the right are provable. These sequents are effectively obtainable from $\Gamma \Longrightarrow \varphi$.

This is part of a joint work with Tomasz Kowalski and Sara Ugolini.

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Adjoints and the preservation of order

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The base setting for this talk is a category $\mathcal{X}$ equipped with a pseudo-functor $P : \mathcal{X} \to \mathbf{Poset}$ to the category of posets and monotone maps that assigns to every morphism $f : X \to Y$ a covariant Galois connection, or adjunction:

$$(PX, \leq_X) \underset{f_*}{\overset{f^*}{\rightleftarrows}} (PY, \leq_Y) \ .$$

A *topogenous order* on $\mathcal{X}$ is a family of relations $\{\lhd_X \mid X \in \mathcal{X}\}$, each $\lhd_X$ on PX, such that for appropriate elements of PX and PY, and $f : X \to Y$:

- $a \lhd_X b \Rightarrow a \leq_X b$,
- $a \leq_X b \lhd_X c \leq_X d \Rightarrow a \lhd_X d$,
- $a \lhd_Y b \Rightarrow f_*(a) \lhd_X f_*(b)$.

If the adjunction above is given by $a \leq_X f_*(b) \Leftrightarrow f^*(a) \leq_Y b$, then we are interested in *strict morphisms*, those for which it is also true that $a \lhd_X f_*(b) \Leftrightarrow f^*(a) \lhd_Y b$.

The introduction of topogenous orders in [3] was motivated by the original ideas of Császár in [1] where the archetypal example is the order relation on the power set lattice of a topological space X, given for A, $B \in \mathcal{P}(X)$ and $\overline{A}$ the topological closure by:

$$A \lhd_X B \Leftrightarrow \overline{A} \subseteq B.$$

In this context we have two possible Galois adjunctions to consider arising from a continuous map $f : X \to Y$, since pre-image has image as left adjoint and "dual image", given by $f_*(A) = Y \setminus f(X \setminus A)$, as right adjoint.

$$(\mathcal{P}X, \subseteq) \underset{f^{-1}(-)}{\overset{f(-)}{\rightleftarrows}} (\mathcal{P}Y, \subseteq) \qquad\qquad (\mathcal{P}X, \subseteq) \underset{f^{-1}(-)}{\overset{f_*(-)}{\rightleftarrows}} (\mathcal{P}Y, \subseteq)$$

The strict maps with respect to the first and second adjunctions are the closed and open continuous maps respectively.

These results may not be surprising, but they open interesting questions regarding analogous situations. For instance, consider the same relation on topological spaces, but with the adjunction between lattices of open sets:

$$(\Omega X, \subseteq) \underset{f^{-1}(-)}{\overset{f_*(-)}{\rightleftarrows}} (\Omega Y, \subseteq)$$

In this case $f_*(A) = Y \setminus \overline{f(X \setminus A)}$ and strict maps are not necessarily open or closed.

We then turn our attention to pointfree topology and the category of frames, namely complete lattices L which satisfy the infinite distributivity law:

$$(\forall S \subseteq L) \quad a \wedge \bigvee S = \bigvee_{s \in S} (a \wedge s)$$

and frame homomorphisms which preserve $\wedge$ and $\bigvee$. Here we have two very different adjunctions arising from a frame homomorphism $f : L \to M$:

$$L \underset{f^*}{\overset{f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} M \qquad\qquad S(L) \underset{f_{-1}[-]}{\overset{f[-]}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} S(M)$$

The first since f preserves $\bigvee$ and the second is the localic image/pre-image adjunction between the coframes of sublocales of L and M. (See [4].)

Regarding the first adjunction, the completely below relation $\prec\!\!\prec$ on a frame L is a topogenous order and strict maps were studied as "assertive maps" in [2]. Strict maps relative to the rather below relation $\prec$ have not been studied before, while the way below relation $\ll$ brings us to perfect maps.

The second adjunction allows the characterisation of closed (respectively open) maps of locales as strict maps. In this case the topogenous order can be described by $S \lhd T$ iff a closed (respectively open) sublocale can be inserted between S and T. (This presents perspectives related to the recent proofs found in [5].)

The talk will outline the properties and characterisation of strict maps in general and then examine the examples mentioned above and how they all fit into the same framework.

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Lifting structure from states to propositions: the various Day extensions available

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Based on joint work with Edmund Robinson in [?].

Abstract. Often one wishes to reason about a system in various states, often equipped with some algebraic structure. Consequently, logicians have amassed a veritable panoply of logical syntaxes with which to manipulate the propositions about such systems.

- In *resource dependent logics*, states are collections of resources which can 'shared' and 'not shared'. Accordingly, there are two conjunctions – one for when resources are allowed to be shared when proving the conjunction, and the other when they cannot.
- For systems evolving through discrete time, there is *linear temporal logic* where the unary 'next state' function is paralleled by the unary 'true in the next state' connective for propositions.
- *Separation logic* was developed to reason about programs interacting with mutable data structures via pointers and heaps. Only disjoint heaps can be combined, so as to avoid a consistency condition on the intersection. The separating conjunction of separation logic expresses that the propositions involved are satisfied on disjoint sections of the heap.

In all these scenarios, structure on the set of states is being lifted to the propositions about these states. While traditionally these syntaxes are considered separately, in order to combine them, e.g. to study resource dependence in a dynamic system, a holistic treatment is desirable.

Let us focus on the first example: resource dependent logics. Collections of resources can be combined, and this yields a *monoidal product* $\otimes$ on the set of states $\mathbb{C}$. Since propositions can be true or false in a state, they can be equated with functions $[\mathbb{C} \to \mathbf{2}]$ (or monotone functions, in the case of intuitionistic semantics), i.e. $\mathbf{2}$-enriched co-presheaves. The classical category-theoretic notion of *Day convolution* from [?] asserts that the monoidal product on $\mathbb{C}$ lifts to a monoidal product $\otimes_D$ on $[\mathbb{C} \to \mathbf{2}]$, which yields precisely the 'non-sharing' conjunction of resource dependent logic (for instance, see [?]).

Our contribution. By generalising Day convolution, we will show that *arbitrary* partial algebraic structure lifts from the set of states to the propositions about those states. Namely, we construct a *morphism of (partial) operads*:

$$(-)_D \colon \mathbf{End}(\mathbb{C})_{\mathrm{par}} \to \mathbf{End}([\mathbb{C} \to \mathbf{2}])^{\mathrm{op}},$$

from the partial endomorphism operad on a poset $\mathbb{C}$ to the (total) endomorphism operad on the co-presheaves. This means that every partial n-ary operation $\theta \colon \mathbb{C}^n \rightharpoonup \mathbb{C}$ yields a canonical *total* n-ary operation on propositions $\theta_D \colon [\mathbb{C} \to \mathbf{2}]^n \to [\mathbb{C} \to \mathbf{2}]$ such that, moreover, those equations that hold between partial operations are also satisfied by their extensions. (We

need not restrict ourselves to **2**-enriched category theory either – a similar construction exists for categories, Lawvere metric spaces, *etc.*) Thus, we recover the definition of many logical connectives found in the literature, including those mentioned above.

Rather than presenting the theorem statement as an unassailable monolith, this talk will focus on the choices inherent to the construction. How is the partial endomorphism operad defined? And why that definition? We will pragmatically survey the possible variations on the above construction, which all centre on the question of *exact squares* discussed in [**?**].

Localic Relations with Open Cones

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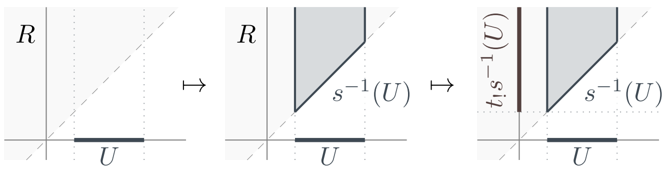
Abstract

We construct an adjunction with identity counit $\mathbf{rocLoc} \underset{\perp}{\rightleftarrows} \boxslash\mathbf{Frm}^{\mathrm{op}}$ between the category **rocLoc** of locales equipped with *open cone localic relations* and monotone maps, and the newly introduced category $\boxslash$**Frm** of *conic frames:* frames L equipped with pairs of *cones* $\vartriangle, \triangledown \colon L \to L$ that serve as localic analogues of point-set upwards and downwards closure operators. Based on arXiv:2605.04038.

Localic relations. For localic preliminaries we refer to [4]. A *localic relation* R on a locale X is a sublocale of the form $r \colon R \rightarrowtail X \times X$. Let **rLoc** be the category containing locales equipped with relations (X, R), where the morphisms $f \colon (X, R) \to (Y, Q)$ are maps of locales $f \colon X \to Y$ that are *monotone*, meaning $(f \times f)[R] \subseteq Q$, which is the suitable internalisation of point-set monotonicity. Thus **rLoc** provides the general backdrop of localic order theory.

To describe R we generally need to combine two notions that may be technically difficult to work with: sublocales and product locales. The point of this work is to provide an intuitive description of R via a localic analogue of up and down closure operators, called *cones.* To make this work, we restrict to a full subcategory $\mathbf{rocLoc} \subseteq \mathbf{rLoc}$ of locales equipped with *open cone relations*: those R whose source and target maps $s, t \colon R \to X$ are open. This gives left adjoints $s_! \dashv s^{-1}$ and $t_! \dashv t^{-1}$, allowing us to define the *induced cones:*

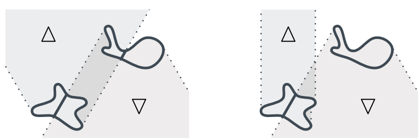
$$\uparrow, \downarrow \colon \mathcal{O}X \longrightarrow \mathcal{O}X; \qquad \uparrow := t_! s^{-1} \quad \text{and} \quad \downarrow := s_! t^{-1}.$$



Any diagonal relation has open cones, and if R, Q have open cones then so does $R \circ Q$. Spatial examples of open cones are discussed in [5], and include the standard ordering $\leqslant$ on $\mathbb{R}$, the interval topology on a distributive lattice, the causal relation of a smooth spacetime [5, Theorem 3.14], and bi-Esakia spaces (Priestley spaces with open cones).

Conic frames. *Conic frames* axiomatise the structure of $\uparrow, \downarrow$. They consist of triples $(L, \vartriangle, \triangledown)$, where L is a frame equipped with abstract *cones* $\vartriangle, \triangledown \colon L \to L$. Of course, an arbitrary pair of functions will not behave like $\uparrow, \downarrow$. The desired behaviour is achieved by imposing *parallelness* and *join-preservation.*

We say $\vartriangle, \triangledown$ are *parallel* if $\vartriangle x \wedge y \sqsubseteq \vartriangle(x \wedge \triangledown y)$ and $x \wedge \triangledown y \sqsubseteq \triangledown(\vartriangle x \wedge y)$ for all $x, y \in L$, a notion introduced in [5]. Intuitively, the illustration on the left describes parallel cones, while the illustration on the right depicts non-parallel cones.



Together with suitably defined morphisms, this gives the category $\boxslash$**Frm** of conic frames. As per motivation, any open cone localic relation R on X defines a conic frame $(\mathcal{O}X, \uparrow, \downarrow)$, giving the functor $\mathsf{Cone} \colon \mathbf{rocLoc} \to \boxslash\mathbf{Frm}^{\mathrm{op}}$.

Conversely, we construct a functor $\mathsf{Rel}\colon \triangle\!\!\!\!\triangledown\mathbf{Frm}^{\mathrm{op}} \to \mathbf{rocLoc}$ as follows. Given $\triangle, \triangledown$, we generate a sublocale $R_{\triangle\!\!\!\triangledown}$ of $X \times X$ using the methods of [4, §III.11] by specifying a binary relation $\sim$ on the coproduct frame $\mathcal{O}X \otimes \mathcal{O}X$ as:

$$A \otimes B \sim C \otimes D \qquad \Longleftrightarrow \qquad \begin{array}{l} A \wedge \triangledown B = C \wedge \triangledown D, \\ \triangle A \wedge B = \triangle C \wedge D. \end{array}$$

The intuition behind this comes from the frame congruence of a point-set relation R on a space, which can be shown to satisfy $A \times B \cap R = C \times D \cap R$ iff $A \cap {\downarrow} B = C \cap {\downarrow} D$ and ${\uparrow} A \cap B = {\uparrow} C \cap D$.

Proposition. *If $(\mathcal{O}X, \triangle, \triangledown)$ is a conic frame, $R_{\triangle\!\!\!\triangledown}$ has open cones with ${\Uparrow}_{R_{\triangle\!\!\!\triangledown}} = \triangle$ and ${\Downarrow}_{R_{\triangle\!\!\!\triangledown}} = \triangledown$.*

Moreover, for any open cone localic relation R we have $R \subseteq R_{\triangle\!\!\!\triangledown}$ iff ${\Uparrow}_R \sqsubseteq \triangle$ and ${\Downarrow}_R \sqsubseteq \triangledown$.

Theorem. *is There is an adjunction* $\mathsf{Cone} \dashv \mathsf{Rel}$ *with identity counit, i.e.* $\mathsf{Cone} \circ \mathsf{Rel} = \mathrm{id}$.

Fixed points and preorders. That the counit is the identity means every conic frame is a fixed point. On the other hand, starting with R we get the induced cones $\Uparrow, \Downarrow$, which in turn induce a localic relation $R_{\Updownarrow}$. An object $(X, R) \in \mathbf{rocLoc}$ is a fixed point precisely when $R \cong R_{\Updownarrow}$ as subobjects of $X \times X$. There are simple spatial counterexamples. However, all kernel pairs of open localic maps are fixed points, and so the following proposition generalises Kock's Godement theorem for effective localic equivalence relations [2].

Proposition. *If R is a weakly closed localic relation with open cones, then $R \cong R_{\Updownarrow}$.*

Localic preorders are localic relations R that satisfy the internal analogue of the reflexivity and transitivity conditions: $\Delta \subseteq R$ and $R \circ R \subseteq R$. Closed localic preorders are studied e.g. in [6], and closed preorders on spaces have historically been important [3].

Proposition. *For any open cone localic relation R and any conic frame $(\mathcal{O}X, \triangle, \triangledown)$:*

- *if R is reflexive then* $\mathrm{id} \sqsubseteq \Uparrow$ *and* $\mathrm{id} \sqsubseteq \Downarrow$*;*
- *if R is transitive then* ${\Uparrow}^2 \sqsubseteq \Uparrow$ *and* ${\Downarrow}^2 \sqsubseteq \Downarrow$*;*
- *$R_{\triangle\!\!\!\triangledown}$ is reflexive iff* $\mathrm{id} \sqsubseteq \triangle$ *and* $\mathrm{id} \sqsubseteq \triangledown$*;*
- *$R_{\triangle\!\!\!\triangledown}$ is transitive iff* $\triangle^2 \sqsubseteq \triangle$ *and* $\triangledown^2 \sqsubseteq \triangledown$.

Future work. It can be shown that the cones $\Uparrow, \Downarrow$ on $\mathcal{O}X$ are really the restriction of more general internal cones $\uparrow, \downarrow\colon \mathrm{S}\ell(X) \to \mathrm{S}\ell(X)$ defined on the coframe of sublocales, obtained via categorical methods. Can a localic relation be (re)constructed from these more general cones?

How do conic frames relate to other frame-theoretic order structures such as biframes and d-frames, and in particular the recently introduced *ad-frames* [1]? Given that closed localic relations with open cones are fixed points, can conic frames be used to study a localic Esakia duality, analogous to [6]?

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A generic construction of free algebras in varieties of Hilbert algebras and Brouwerian semilattices

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Brouwerian semilattices (Br) are among the most investigated classes of *varieties of logic*. In the seminal Köhler [4] in particular, a construction of the free Brouwerian semilattice is presented, based in part on an earlier de Bruijn [3]. It leads to a recursive formula for the number of meet-irreducible elements, which was used to compute the exact number of elements of the 3-generated free Browerian semilattice. Of the more recent works, Bezhanishvili et al. [1] use a colouring technique to construct free algebras in *nuclear Brouwerian semilattices*, which expand Brouwerian semilattices with a strong form of a nucleus (an S4-like modality).

Hilbert algebras (Hi), implicative subreducts of Brouwerian semilattices, are less studied. A number of observations can be found in de Bruijn [3] and Urqhardt [7]. More recently, finite Hilbert algebras (but not the free algebra) were studied in Celani and Cabrer [2]. All these works use techniques based either on a syntactic analysis of terms, or on dualities and Kripke frames.

We take a different route: a purely algebraic one, motivated by Słomczyńska [6]. Beginning from an n-generated free algebra $\mathbf{F}_{\mathcal{V}}(n)$—which obviously exists and is finite—in some subvariety $\mathcal{V}$ of Hi (or of its variant Ho expanded by zero), we analyse the structure of $\mathbf{F}_{\mathcal{V}}(n)$ given by its subdirectly irreducible factors, that is, its completely meet-irreducible congruences. The poset $\mathbf{Cm}\,\mathbf{F}_{\mathcal{V}}(n)$ of these congruences is isomorphic to the poset $\mathbf{Cm}\,\mathbf{F}_{\widehat{\mathcal{V}}}(n)$ of completely meet-irreducible congruences of the free algebra $\mathbf{F}_{\widehat{\mathcal{V}}}(n)$, where $\widehat{\mathcal{V}}$ is a natural counterpart of $\mathcal{V}$ in Br (or Bo, the zero-expanded variant). Since the lattices of subvarieties of Hi and Br, and respectively of Ho and Bo, are isomorphic, that gives us skeletons (Kripke frames) for all free algebras in a uniform way. The frames have a multilayer structure, corresponding to the higher and higher subdirectly irreducibles (with height measured by chains of meet-irreducible congruences). The crucial difference between varieties with zero and the varieties without zero occurs only at the first layer, higher up the construction of the frame is the same.

The next step, where the distinction between Brouwerian semilattices and Hilbert algebras comes into play, is the choice of some antichains in the frames. If we take all antichains, we obtain a Brouwerian semilattice, if we omit some, we lose some meets, so we obtain a *partial* Brouwerian semilattice. Carefully selecting the right antichains we arrive at our description of $\mathbf{F}_{\mathcal{V}}(n)$. Algorithm 1 below shows how to build $\mathbf{Cm}\,\mathbf{F}_{\mathcal{V}}(n) \cong \mathbb{P}(n) = \bigcup_{i=1}^{n} P_i(n)$, generically, for $\mathcal{V} = \mathsf{Br}, \mathsf{Hi}, \mathsf{Bo}, \mathsf{Ho}$. The last ingredient we need to build the free Hilbert algebra (without or with zero) is a function selecting the right antichains. For any generator x, we put

$$S(x) := \big\{(L,i) \in \mathbb{P}(n)\big\} : x \notin L \text{ and } x \in L' \text{ for all } (L',j) > (L,i)\big\},$$

with the notation explained in Algorithm 1. Finally, we can state

Theorem 1. *For any $n > 0$, we have*

$$\mathbf{F}_{\mathsf{Hi}}(n) \cong \left(\bigcup_{i=1}^{n} \mathcal{P}(S(x_i)); \to, \emptyset\right) \qquad \mathbf{F}_{\mathsf{Ho}} \cong \left(\bigcup_{i=1}^{n+1} \mathcal{P}(S(x_i)) \cup \mathcal{P}(S(0)); \to, \emptyset, P_1(n)\right)$$

where $\mathcal{P}$ *is the power set operation,* 1, 0 *are interpreted as* $\emptyset$, $P_1(n)$ *and* $\to$ *is defined suitably.*

We deal with subvarieties generically by appropriately restricting G in Algorithm 1.

Algorithm 1 $\mathbb{P}(n)$ construction

```
Require: J ⊆ X with |X| = n, 1 ≤ i ≤ n        ▷ X is the set of generators. Points are pairs (J, i)
  Let Proj(U) := {J : (J, i) ∈ U for some i}
  P_1(n) ← {(J, 1) : J ⊂ X}; ≤ ← =             ▷ If V = Ho, Bo, then P_1(n) ← {(J, 1) : J ⊆ X}
  function BuildNextLayer(⋃_{i=1}^{k-1} P_i(n))
      P_k(n) ← ∅
      for G ∈ Up(⋃_{r=1}^{k-1} P_r(n)), G ∩ P_{k-1}(n) ≠ ∅, L ⊂ ⋂ Proj(G) do
          P_k(n) ← P_k(n) ∪ {(L, k)}
          for (J, j) ∈ G do
              ≤ ← ≤ ∪ ⟨(L, k), (J, j)⟩
          end for
      end for
  end function
  for k ∈ {2, ..., n} do                        ▷ If V = Ho, Bo, then k ∈ {2, ..., n + 1}
      BuildNextLayer(⋃_{i=1}^{k-1} P_i(n))
  end for
```

Combining our description of $\mathbf{F}_{\mathcal{V}}(n)$ with results in Słomczyńska [5] yields simple proofs of well known results on hereditary structural completeness of Hi, Br, and Bo, and structural incompleteness of Ho. As a byproduct we also get

Theorem 2. *Let* $\mathcal{V} \subseteq \mathsf{Ho}$*, and let* $t_1, \ldots, t_k, r$ *be terms. If* $t_i(1, \ldots, 1) = 1$ *for all* $i \in \{1, \ldots, k\}$*, then*

$$\mathbf{F}_{\mathcal{V}}(n) \models \bigwedge_{i=1}^{k} t_i = 1 \implies r = 1 \iff \mathbf{F}_{\mathcal{V}}(n) \models t_1 \to (\ldots \to (t_k \to r) \ldots) = 1.$$

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Rings and Boolean Algebras as Algebraic Theories

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In universal algebra, an equational class of algebras is specified by a signature of operations and a set of equational identities. An algebraic theory (also called a clone [14] or Lawvere theory [9]) encodes the same information in different terms: it is a collection of abstract operations of finite arities together with a rule for their composition. A model or algebra of the theory is then a set equipped with actual operations corresponding to the elements of the theory, in such a way that composition is respected.

In this talk, based on [5], we focus on two classes of algebraic theories: affine and hyperaffine theories. Affine theories are commutative and idempotent theories equipped with a Mal'cev operation [12, 2], and hyperaffine theories are commutative theories in which every operation is a decomposition operator. In the first part, we build upon ideas from Johnstone [8] and Garner [6] to show that hyperaffine theories are entirely determined by a set of binary operations called *coefficients*, which constitutes a Boolean ring. Our analysis provides a novel proof of the fact that the category of non-degenerate hyperaffine theories is equivalent to the category of non-degenerate Boolean algebras, originally proved in [6]. We then apply a similar framework to the case of affine theories: the coefficients of an affine theory form a commutative ring, which completely determines the theory. Consequently, we establish the result that the category of non-degenerate affine theories is equivalent to the category of non-degenerate commutative rings [5, Theorem 3.14].

The models of a hyperaffine theory with coefficient ring B are Bergman's B-sets [1], sets equipped with a binary action of the Boolean algebra B. B-sets naturally model the behavior of the if-then-else operator in programming languages [10, 7], and Bergman provided a representation of them in terms of sheaves over Boolean spaces. On the other hand, the models of an affine theory with coefficient ring R are affine R-modules [4, 13], also called Mal'cev modes [11]. These algebras can also be described through a binary action of the ring [11, Theorem 6.3.4] on a set. Given a Boolean ring B, as B is commutative, we have two ways of encoding B as an algebraic theory: either as a hyperaffine theory or as an affine theory. We are thus led to the question: what are the models of an affine theory whose ring of coefficients is Boolean? In what way do they differ from B-sets? We answer these questions by providing two characterisations for the models of an affine theory over a Boolean ring B. The first one is purely algebraic: the models of an affine theory over a Boolean ring are Boolean vector spaces equipped with a compatible action of B which is idempotent and commutative [5, Theorem 5.7]. The second one shows that these structures admit a sheaf representation analogous to B-sets [5, Theorem 5.9].

As a final contribution, we highlight a connection between hyperaffine algebraic theories and nBAs, a recently introduced generalisation of Boolean algebras [3]. We prove that every hyperaffine theory naturally induces an nBA structure on the set of its n-ary operations; conversely, a coherent sequence of nBAs uniquely determines a hyperaffine theory [5, Theorem 4.2]. This link originates from the dual interpretation of the $(n+1)$-ary operator in the signature of nBAs, viewed on the one hand as a composition operator, and on the other as a generalised if-then-else.

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Esakia duality for temporal Heyting algebras

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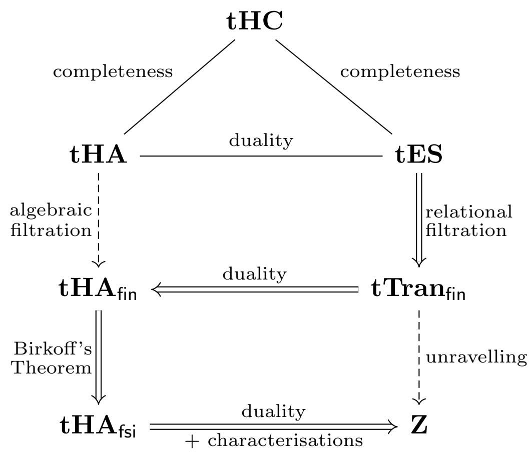
The temporal Heyting calculus **tHC**, first presented in [?], is the natural temporal augmentation of the modalized Heyting calculus **mHC**, also first studied in [?]. It has as its algebraic models the category of temporal Heyting algebras **tHA**, a class of Heyting algebras with a "forward-looking" $\Box$ that has a left-adjoint, "backward-looking" $\blacklozenge$. Being intuitionistic, however, it lacks the modalities $\Diamond$ and $\blacksquare$ typically defined in terms of negation.

	Past	Future
$\exists$	$\blacklozenge$	$\Diamond$
$\forall$	$\blacksquare$	$\Box$

We present a suitable category of topological models for the logic (temporal Esakia spaces **tES**) and develop an Esakia duality between the categories of algebraic and topological models, extending the duality given in [?]. This includes defining a class of filters on **tHA** and closed upsets on **tES** such that we have a poset-isomorphism between **tHA** congruences, our class of filters, and our class of closed upsets. Having achieved this, we classify simple and subdirectly-irreducible (s.i.) temporal Heyting algebras via their dual spaces as was done for "BAOs" in [?] and "distributive modal algebras" in [?]. Finally, we use this characterization to achieve a relational completeness result combining finiteness (achieved via the finite model property) and a notion of "rootedness" dual to subdirect-irreducibility (analogous to the result that **IPC** is complete with respect to the class of finite trees).

This work serves as a strong endorsement for the use of duality in the study of logic, as two central steps toward achieving the final completeness result were significantly more tractable on their respective sides of the algebra-topology duality. We take the following methodological path to our final completeness result, where $\mathbf{tTran}_{\mathsf{fin}}$ is the class of finite relational models of **tHC**, $\mathbf{tHA}_{\mathsf{fin}}$ the class of finite algebraic models, $\mathbf{tHA}_{\mathsf{fsi}}$ the class of finite s.i. algebraic models, and **Z** the class of finite Z-rooted relational models of **tHC** (where Z-rootedness is the above-described notion of rootedness dual to subdirect-irreducibility). The dashed arrows represent paths not taken.



The method of 'algebraic filtration' posed a considerable challenge as algebraic filtrations of Heyting algebras with modal operators have yet to be treated anywhere in the literature. Furthermore, this algebraic filtration would have to encode the transitive closure of the smallest filtration on all three relations, something that would likely involve several nontrivial adaptations of the standard method of intuitionistic algebraic filtration [?]. In contrast, proving the finite model property relationally is completely painless, simply requiring the aforementioned filtration technique.

Analogously, the method of "unraveling" [?, §4.5] does not seem to have been studied in any sufficiently similar context. This would amount to giving an arbitrary finite relational model the shape corresponding to subdirect-irreducibility (Z-rootedness) while preserving the satisfaction of modal formulas. Given the complex nature of the Z relation (zigzagging until it meets non-reflexive points, etc.), this would likely have been correspondingly complex. In contrast, reducing $\mathbf{tHA}_{\mathsf{fin}}$ to $\mathbf{tHA}_{\mathsf{fsi}}$ without interrupting logical truth comes for free via Birkoff's Theorem. A final application of our dual characterisation of finite s.i. tHAs brings us to the desired completeness result.

Stone Duality for Monads

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This talk is based on our recent joint work [**Garner2026-ja**].

Summary. In computer science, a monad T represents a *notion of computation*, with the Kleisli maps $f : A \to TB$ interpreted as a program with input type A and output type B. If we interpret *computation as interaction* with some environment, then the environment can be represented by a *state transition system.* In this work, we construct the universal transition system $\mathbb{B}T$ interacting with a given monad $T\colon \mathsf{Set} \to \mathsf{Set}$. The transition system $\mathbb{B}T$ is universal in the same sense of the classical Stone duality for Boolean algebras. Recall that the Stone space $\mathsf{Spec}(\mathfrak{B})$ is the universal space of states in which to interpret the propositions of a Boolean algebra $\mathfrak{B}$ in the sense that $\mathsf{Spec}(-)$ is the left adjoint to a functor $X \mapsto \mathsf{Clop}(X)$ constructing a Boolean algebra from any space X. The classical stone duality is then the equivalence of categories we get by restricting this adjunction to its fixpoints. Similarly, our construction of this universal transition system is one direction of a dual adjunction, which we cut down to obtain the Stone duality of the title.

We begin with the case for finitary monads on Set, whereby we consider a transition system to be a category $\mathbb{C}$ internal to the category of topological spaces (or *topological categories* for brevity, though be warned that this terminology is highly overloaded in the literature). What this means is that $\mathbb{C}$ has a *space* of objects/states $\mathbb{C}_0$ and space of morphisms/transitions $\mathbb{C}_1$, such that all the structure maps of $\mathbb{C}$ are continuous functions. Computations of output type B interacting with such a system induce, for each starting state $c \in \mathbb{C}_0$, some transition $s(c) : c \to c'$, along with some output value $o(b) \in B$. In this way we obtain from $\mathbb{C}$ the monad $\Gamma\mathbb{C}$ of such pairs of *continuous* functions (s, o), where $s : \mathbb{C}_0 \to \mathbb{C}_1$ is a section of the source map $\mathbb{C}_1 \to \mathbb{C}_0$, and $o : \mathbb{C}_0 \to A$. In other words, the topologies on $\mathbb{C}_0$ and $\mathbb{C}_1$ are used to control what we consider computable. If we further restrict to those image-finite o (i.e. landing in some finite subset of A), we then recover a finitary monad $\Gamma_\omega(\mathbb{C})$.

Now, the universal transition system associated to a finitary monad T, which we call the *topological behaviour category* $\mathbb{B}T$, is characterized universally as the left adjoint to the assignment $\Gamma_\omega : \mathbb{C} \mapsto \Gamma_\omega\mathbb{C}$ (we will address functoriality later). The objects of $\mathbb{B}T$ are certain natural transformations $\beta\colon T \to \mathsf{id}_{\mathsf{Set}}$ specifying how to run each computation term $t \in TA$ down to its return value $\beta(t) \in A$; while morphisms with domain β are equivalence classes $[t]_\beta$ of computations $t \in T1$ considered up to a notion of "trace equivalence" $\sim_\beta$. Intuitively, we have $t_1 \sim_\beta t_2$ if β runs through the same sequence of operations in both t_1 and t_2 to get the final output. In fact this category is not new: it is the behaviour category of [**Garner2022-rj**]. What *is* new are the topologies, which are generated by the elements of T to ensure that all the computations from T, and as little else, are considered computable.

As a concrete example, consider when T is generated by computations $\mathsf{get}_n \in TA$ for each $n \in \mathbb{N}$ and for some finite set A, satisfying equations expressing that get_n reads from a memory cell of the computer at address n. Here, $\mathbb{B}_0T$ is simply the set of possible memory configurations, while a transition in $\mathbb{B}_1T$ is an assignment of new values to finitely many memory cells. Now, the computations $(s, o) : \Gamma_\omega(\mathbb{B}T)(A)$, may, without further constraint, refer to the contents of *infinitely* many cells of the current memory configuration. Yet, by the finitary nature of syntax, computations in TA may query only *finitely* many cells. So $\Gamma_\omega(\mathbb{B}T)$ admits many more computations than T, most of which are computationally unreasonable. This gap is closed by introducing the topology on $\mathbb{B}T$, and restricting $\Gamma_\omega(\mathbb{B}T)$ to involve only *continuous* functions.

The construction $T \mapsto \mathbb{B}T$ and $\mathbb{C} \mapsto \Gamma_\omega\mathbb{C}$ functorially extends to the promised adjunction

$$\mathbb{B} : \mathsf{Mnd}_\omega(\mathsf{Set}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{TopRetro}^{\mathsf{op}} : \Gamma_\omega.$$

Here, $\mathsf{Mnd}_\omega(\mathsf{Set})$ is the usual category of finitary monads and monad morphisms. The twist is that the category $\mathsf{TopRetro}$ whose objects are topological categories have for morphisms, not the usual functors, but rather *retrofunctors* [**NiuSpivak2025**], which I will introduce and motivate in the talk. We also have a similar adjunction accounting for ranked monads by the changing the base of internalization from topological spaces to locales. The reason for this change is the existence of (necessarily infinitary) monads whose topological space of states $\mathbb{B}_0T$ "ought to be" a non-trivial point-less space, which I will briefly touch upon in the talk.

$$\mathsf{LB} : \mathsf{Mnd}_r(\mathsf{Set}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{LocRetro}^{\mathsf{op}} : \Gamma.$$

The fixed points. Going back and forth along the adjunction, the monad $\Gamma_\omega(\mathbb{B}T)$ amounts to completing a monad T with *prescience.* To each $t \in TA$, there is a new operation $\bar{t} \in \Gamma_\omega(\mathbb{B}T)A$ which intuitively, performs t, keeps track of the result, and then undoes the performance of t. Such monads were characterised by the first-named author [**Garner2024-yc**] as those monads with a cartesian-closed category of Eilenberg-Moore algebras.

In the other direction, the topological behaviour category $\mathbb{B}T$ is always (what we call) *ample*: the source map is a local homeomorphism, and the space of objects is a Stone space. The cartesian closed monads and the ample topological categories are precisely the fixpoints of the adjunction $\Gamma_\omega \dashv \mathbb{B}$. As alluded to in the introduction above, the resulting equivalence between the fixpoints is the *Stone Duality for Monads* of the title. In fact, this subsumes the classical Stone duality: any Boolean algebra $\mathfrak{B}$ induces a cartesian-closed monad $T_\mathfrak{B}$ of distributions, whose $\mathbb{B}T_B$ is the classical Stone dual $\mathsf{Spec}(\mathfrak{B})$ equipped with only identity transitions.

From a logical perspective, we can think of the ample topological category corresponding to a cartesian closed monad T as a topologized Kripke frame for a dynamic modal logic whose propositions are interpreted by clopen sets in the space of objects, while the space of morphisms describes the modalities $[t]\varphi$ for each $t \in T1$.

Regular epimorphisms in the regular category of rational polyhedra and $\mathbb{Z}$-maps

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The relation between Łukasiewicz logic and polyhedral geometry has long been a source of explicit bridges between algebra, logic, piecewise-linear topology, and the geometry of numbers [2]. An important piece of this relation is the duality between finitely presented MV-algebras and rational polyhedra [3]. Let $\mathsf{MV}_{\mathrm{fp}}$ be the category of finitely presented MV-algebras and let $\mathsf{PL}_{\mathbb{Z}}$ denote the category of rational polyhedra (finite unions of rational polytopes in some $\mathbb{R}^n$) whose arrows are $\mathbb{Z}$-maps, i.e., continuous piecewise-affine maps with integer coefficients (see [5] and references therein for more background on PL-geometry). Thus,

$$\mathsf{MV}_{\mathrm{fp}} \;\simeq\; \mathsf{PL}_{\mathbb{Z}}^{\mathrm{op}}.$$

Informally, an MV-algebra presented by finitely many generators and relations corresponds to a rational polyhedron cut out by the $\mathbb{Z}$-maps associated to the relators, and homomorphisms correspond contravariantly to $\mathbb{Z}$-maps.

We recall a few standard definitions [1]. Recall that an arrow e is called a *cover* (or *extremal epimorphism*) if whenever $e = m \circ g$ with m mono, then m is an iso.

Definition 1. A category $\mathcal{C}$ is *regular* if

1. $\mathcal{C}$ has finite limits;
2. every arrow f admits an *image factorization* $f = m \circ e$ with m mono and e a cover;
3. covers are stable under pullback.

Regularity is a basic property of categories of algebras: it ensures that images exist and behave well under pullback, so that quotients can be treated via kernel pairs. In particular, it is the standard categorical setting for interpreting coherent logic and for building sites/toposes from algebraic–geometric data. Since finitely presented MV-algebras inherits regularity from the whole category of MV-algebras, one immediately gets that $\mathsf{PL}_{\mathbb{Z}}$ is *co-regular.* The main result in this contribution is that the category $\mathsf{PL}_{\mathbb{Z}}$ is also regular, together with a neat characterization of regular epimorphisms in this category[1].

It is folklore that the category PL of polyhedra and continuous piecewise-affine maps between them is regular. The main steps in the proof are two: first notice that the category of polytopes and affine maps between them inherits regularity from the *ambient* category Set; then prove that regularity is preserved by *patching* together polytopes and affine maps to obtain the category PL. To export this technique to the category $\mathsf{PL}_{\mathbb{Z}}$ we devise an abstract version of the above proof that generalises to any (regular) ambient category, and then we choose appropriately another category to replace Set. The central technical role is played by "denominators", as we briefly explain below.

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*Presenting author
[1]Notice that in any regular category "regular epimorphism" = "cover".

Definition 2. For any poset Ω, define the category Set_Ω as follows: objects are pairs (X, d) where X is a set and $d\colon X \to \Omega$ is an arbitrary function called *denominator map*; arrows $f\colon (X, d_X) \to (Y, d_Y)$ are functions $f\colon X \to Y$ such that for all $x \in X$:

$$d_X(x) \leq d_Y(f(x)) \tag{1}$$

Lemma 3 (see e.g., [4])**.** *If Ω is a frame then Set_Ω is a (pre)topos. Thus, in particular it is a regular category.*

It is folklore that the set of natural numbers ordered by divisibility is a co-frame. Thus, in our application we take Ω to be $\mathbb{N}$ ordered by *reverse divisibility*:

$$m \leq n \iff \exists k \in \mathbb{N} \ \ (m = nk),$$

From Lemma 3 we immediately obtain that $\mathsf{Set}_\mathbb{N}$ is a regular category.

We can think of the Euclidean spaces $\mathbb{R}^n$ as living in the ambient category $\mathsf{Set}_\mathbb{N}$ by endowing them with the canonical denominator map $\delta\colon \mathbb{R}^n \to \mathbb{N}$ defined by:

$$\delta_n(x) = \begin{cases} \mathrm{den}(x) & \text{if } x \in \mathbb{Q}^n, \\ 0 & \text{if } x \notin \mathbb{Q}^n, \end{cases}$$

where $\mathrm{den}(x)$ is the least common multiple of the denominators of the coordinates of x. Affine maps with integer coefficients are morphisms in Set_Ω as they satisfy (1), and so are $\mathbb{Z}$-maps. Crucially, also the reverse holds, i.e., a PL-map between rational polyhedra that satisfies (1) is a $\mathbb{Z}$-map. This bridges the *topological* PL character with the *arithmetic* character of $\mathbb{Z}$-maps and provides the crucial step in proving that $\mathsf{PL}_\mathbb{Z}$ is regular.

Finally, our abstract framework also enables a characterization of regular epic $\mathbb{Z}$-maps, by lifting the one of regular epic epimorphisms in $\mathsf{Set}_\mathbb{N}$. It is an exercise to show that regular epimorphisms in Set_Ω are the surjective functions $f\colon (X, d_X) \to (Y, d_Y)$ with the following property:

$$d_Y(y) = \bigvee \{d_X(x) \mid f(x) = y\}. \tag{2}$$

This leads to the following characterization of regular epic $\mathbb{Z}$-maps.

Theorem 4. *A $\mathbb{Z}$-map is regular epic if and only if it is surjective and satisfies* (2)*, i.e. taking into account the reversed order, $d_Y(y) = \gcd\{d_X(x) \mid f(x) = y\}$.*

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Arboreal Categories from Shapes

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Several categorical approaches to notions of equivalence or bisimilarity that are relevant in logic and theoretical computer science have been investigated. Among these are arboreal categories [**?**], whose objects have an intrinsic process structure giving rise to a well-behaved notion of bisimilarity. This relation can then be transported along an adjunction into a category $\mathcal{E}$ of "extensional" objects. The main examples of these so-called **arboreal adjunctions** involve an extensional category of relational structures and recover logical equivalence for various fragments of infinitary first-order logic, hence providing an abstract framework for game comonads (see [**?**] for a recent survey, including extensions and variations of substantial resource-sensitive model-theoretic results such as Rossman's equirank preservation theorem obtained in this framework).

In this talk, we present work in progress on the construction of a class of arboreal adjunctions, which can be seen as a vast generalisation of the arboreal adjunction for Basic Modal Logic. The construction of the corresponding arboreal categories leads naturally to a certain modal-like syntax, which in this sense is obtained *from* the arboreal category and not viceversa. The resulting modalities express the presence of "shapes" inside a relational structure, which can also be understood as conjunctive queries with a source and target as in [**?**].

First we describe the general abstract setting. Given a distinguished object $*$ in $\mathcal{E}$, let $\mathcal{E}_*$ denote the slice $*/\,\mathcal{E}$ and $\mathcal{E}_{**}$ denote the category of cospans $* \to A \leftarrow *$. We fix a subcategory $\mathcal{S}$ of $\mathcal{E}_{**}$, whose objects we call **shapes**, and define the category $\mathcal{S}$-Tree of **$\mathcal{S}$-trees** as follows. Its objects are the pairs (T, t) where T is a tree and $t\colon T \to \mathcal{S}$ is a **labelling**, that is, a function from the non-root elements of T to the objects of $\mathcal{S}$. Morphisms in $\mathcal{S}$-Tree are the pairs $(f, \gamma)\colon (T, t) \to (S, s)$ where $f\colon T \to S$ is a morphism of trees (i.e., a monotone function that preserves the root and the height of elements) and γ is a collection of morphisms $\gamma_x\colon t(x) \to s(f(x))$ in $\mathcal{S}$, for every non-root x.

Theorem. *If all morphisms in $\mathcal{S}$ are epimorphisms, then $\mathcal{S}$-Tree is an arboreal category.*

In order to relate $\mathcal{S}$-Tree to the extensional objects, we define a "realisation" functor $M_{\mathcal{S}}$: $\mathcal{S}$-$\mathsf{Tree} \to \mathcal{E}_*$ by looking at an S-tree as building instructions for an object of $\mathcal{E}_*$, glueing together the shapes at each node of the tree according to the pattern indicated by the tree structure. Formally, it is enough to specify $M_{\mathcal{S}}$ on the dense subcategory of paths $\mathcal{S}$-$\mathsf{Path} \hookrightarrow \mathcal{S}$-$\mathsf{Tree}$ and then extend cocontinously: given a path (P, p) represented as a list of shapes $[(S_1, s_0^1, s_f^1), \ldots, (S_n, s_0^n, s_f^n)]$, we define its image along $M_{\mathcal{S}}$ as the colimit of the diagram

and this definition extends naturally to morphisms between paths. The question arises whether $M_{\mathcal{S}}$ is a left adjoint, which would allow us to define a bisimilarity relation on $\mathcal{E}_*$ from the bisimilarity relation of the arboreal category $\mathcal{S}$-Tree (see [**?**, Sections 4 and 6]).

In order to move forward and relate the general setting to logic, we specialise to the case $\mathcal{E} = \mathsf{Str}(\sigma)$ where $\mathsf{Str}(\sigma)$ is the category of relational structures for some signature σ and where $*$ is a discrete singleton structure. We interpret an object $(S, s_0, s_f) \in \mathcal{S} \hookrightarrow \mathsf{Str}_{**}(\sigma)$ as a defining a modality $\Diamond_{(S,s_0,s_f)}$, or $\Diamond_S$ for short, in the following sense: $\Diamond_S\varphi$ is true at a point a_0 of structure A iff there is a morphism $f : S \to A$ sending $s_0 \mapsto a_0$ such that φ is true at $f(s_f) \in A$. Let $\mathsf{Gra} = \mathsf{Struct}_*(\{E\})$ denote the category of pointed directed graphs. Under suitable conditions on $\mathcal{S}$, the following holds.

Fact. *The functor* $M_{\mathcal{S}} : \mathcal{S}\text{-}\mathsf{Tree} \to \mathsf{Gra}_*$ *is comonadic.*

The proof is based on analysing the density comonad of $M_{\mathcal{S}}$ and showing that its category of coalgebras is isomorphic to $\mathcal{S}\text{-}\mathsf{Tree}$.

From this starting point, we wish to generalise from Gra to $\mathsf{Struct}(\sigma)$ for arbitrary choices of σ and, perhaps more importantly, to allow for *decorations*. In the abstract setting, a **choice of decorations** consists of a fibration $D : \mathcal{D} \to \mathcal{E}*$; intuitively, objects in the fiber over $(X, x) \in \mathcal{E}$ correspond to different ways of decorating (X, x) with additional structure. At the concrete level of relational structures, we take D to be the reduct functor $\mathsf{Struct}_*(\sigma') \to \mathsf{Struct}_*(\sigma)$, where $\sigma' = \sigma \cup \tau$ is an extension of σ, that forgets the tuples from relations in τ. By taking the pullback of D and $M_{\mathcal{S}}$ in the category of large categories we obtain a fibration $\overline{D} : \mathcal{S}\text{-}\mathsf{Tree}_D \to \mathcal{S}\text{-}\mathsf{Tree}$ where the total category $\mathcal{S}\text{-}\mathsf{Tree}_D$ is a category of $\mathcal{S}$-trees decorated with tuples in an appropriate sense.

Conjecture. *For a reduct functor* $D : \mathsf{Struct}_*(\sigma') \to \mathsf{Struct}_*(\sigma)$*, the category* $\mathcal{S}\text{-}\mathsf{Tree}_D$ *is arboreal and comonadic over* $\mathsf{Struct}_*(\sigma')$*.*

We believe the arboreal adjunctions obtained in this way will allow us to capture a wide range of logics including ones that have received a comonadic treatment such as Basic Modal Logic, Path Predicate Modal Logic and First Order Logic [**?**, **?**], as well as others which have not received such treatment such as Unary Negation First Order Logic (UNFO) [**?**] and extensions of PDL with conjunctive queries in the spirit of the logic CPDL+ [**?**].

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Delimited control, the Casari schema, and KM

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Let IPC denote the intuitionistic propositional calculus. We will add a unary modal operator $\Box$ to IPC and consider the following axioms:

$$(\mathsf{r})\ p \to \Box p \qquad (\mathsf{derv})\ \Box p \to (q \vee (q \to p)) \qquad (\mathsf{sl\ddot{o}b})\ (\Box p \to p) \to p \qquad (\mathsf{nnv})\ \neg\neg\Box\bot$$

The system mHC (termed the *modalized Heyting calculus* by Esakia in [3]) results from adding (r) and (derv) to IPC. Adding the *strong Löb axiom* (slöb) to mHC results in the system KM (the *Kuznetsov-Muravitsky proof-intuitionistic calculus*; [8]). We will also consider the system obtained from mHC by adding the axiom (nnv) (*not-not-verum*), which is seen to be an instance of the strong Löb axiom with $p = \bot$. We refer to this system as $\mathsf{KM}_\bot$.

Let IQC denote the predicate intuitionistic calculus, and let QL denote the standard predicate extension of L extending IQC, where L is one of the propositional systems introduced above. It is well-known that all of the above propositional systems are conservative extensions of IPC, but that their predicate extensions are *not* conservative over IQC. Indeed, [4] observes that both the *Kuroda principle* and the *relativized Kuroda principle*, also known as the *Casari schema* (see, e.g., [2, Sec. 7])

$$(\mathsf{KP})\ \forall x\, \neg\neg p(x) \to \neg\neg\forall x\, p(x) \qquad (\mathsf{rKP})\ \forall x\, ((p(x) \to \forall y\, p(y)) \to \forall y\, p(y)) \to \forall x\, p(x)$$

are provable in QKM. In fact, these axioms characterize the non-modal fragments of $\mathsf{QKM}_\bot$ and QKM respectively. Write int(L) for the intuitionistic/non-modal fragment of L, where L is one of the predicate systems introduced above; then the results of [4] can be interpreted as:

Theorem 1 ([4])**.** $\mathsf{int}(\mathsf{QKM}_\bot) = \mathsf{IQC} + (\mathsf{KP})$ and $\mathsf{int}(\mathsf{QKM}) = \mathsf{IQC} + (\mathsf{rKP})$

As it turns out, the system IQC + (KP) has a deep relationship to the concept of *delimited control operators* from theoretical computer science. It is well-known that control flow operators, such as Scheme's callcc, which extend lambda calculus with the ability to manipulate control flow by accessing the current continuation, provide a Curry–Howard interpretation of full classical logic. What is lesser known is that *restricted* forms of control operators can provide a similar interpretation for intermediate logics. Indeed, [7] observes that the *delimited control operators* shift and reset are enough to prove (KP), while retaining the characteristically intuitionistic disjunction and existence properties.

[7] essentially extends the standard natural deduction system NJ for IQC with two new rules

$$\frac{\Gamma, \mathsf{k} : \neg A \vdash_\bullet \mathsf{p} : \bot}{\Gamma \vdash_\bullet \&\mathsf{k}.\mathsf{p} : \bot}(\mathsf{shift}) \qquad \frac{\Gamma \vdash_\bullet \mathsf{p} : \bot}{\Gamma \vdash \#\mathsf{p} : \bot}(\mathsf{reset})$$

and we refer to this system as $\mathsf{NJ}_\bullet$ (a slight modification of the system presented in [7]). In this system, the turnstile is allowed to be decorated with an annotation $\bullet$, which is allowed on all axiom instances and ignored in the intuitionistic inference rules. In a decorated context (only), the rule (shift) can be used to perform double-negation elimination. The annotation can only

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be removed by use of the (reset) rule. For the proof-term interpretation of these rules, the term &k.p can be understood as binding the current continuation, *up to the nearest delimiting* #, as k in the term p (as opposed to the "top-level" continuation of the whole program). A derivation $\Gamma \vdash \varphi$ is *proper* if the turnstile is undecorated. It follows from [7] that $\mathsf{NJ}_\bullet$ provides a natural deduction calculus for the intuitionistic fragment of $\mathsf{QKM}_\perp$:

Theorem 2. Let φ be an intuitionistic (□-free) formula. Then

$$\vdash \varphi \text{ is derivable in } \mathsf{NJ}_\bullet \quad \text{iff} \quad \mathsf{QKM}_\perp \vdash \varphi \quad \text{iff} \quad \mathsf{IQC} + (\mathsf{KP}) \vdash \varphi$$

Multi-prompt delimited control operators ([5], [1]) are similar to (shift) and (reset), but with the ability to *tag* delimiters # and binders & with a label, such that the continuation is bound up to the nearest *matching* delimiter. This gives rise to an analogous extension $\mathsf{NJ_d}$ (briefly alluded to at the end of [6]) with the addition of the rules:

$$\frac{\Gamma, \mathsf{k} : A \to T \vdash_\Delta \mathsf{p} : T \qquad \alpha : T \in \Delta}{\Gamma \vdash_\Delta \&_\alpha \mathsf{k.p} : T}(\mathsf{shift}_\alpha) \qquad \frac{\Gamma \vdash_{\alpha:T,\Delta} \mathsf{p} : T}{\Gamma \vdash_\Delta \#_\alpha \mathsf{p} : T}(\mathsf{reset}_\alpha)$$

Here, the turnstile is decorated with a whole list of *prompt variables*, which are a distinct stock of variables also typed with formulas. Again, axiom instances are available with any list of prompt-variables desired. The (shift_α) rule allows double negation elimination "relative to T", as long as some $\alpha : T$ is in the prompt context, while (reset_α) allows removal of a prompt variable $\alpha : T$ from the context, when the conclusion is T.

We extend the results of [7] and [6] by demonstrating that NJ_d plays an analogous role to $\mathsf{NJ}_\bullet$ for the systems QKM and IQC + (rKP) – that is, it proves the Casari schema and, moreover, provides a natural deduction calculus for the intuitionistic fragment of QKM. Finally, we will show that this fragment remains essentially intuitionistic, by showing that IQC + (rKP) has the disjunction and existence properties. Unlike [7], which shows this property of $\mathsf{NJ}_\bullet$ by strong normalization of proof terms, we accomplish it through a direct consideration of topos semantics and the Freyd cover.

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Extended Abstract : Stone Duality Proofs for Colorless Distributed Computability Theorems

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In a distributed system, several processes attempt to coordinate through some form of communication in order to solve a task. Characterizing task-solvability for general models of communication has been a central question in distributed computing since its foundation. In 2004, Herlihy and Shavit [4], as well as Saks and Zaharoglou [5], building on the work of Borowsky and Gafni [1, 2], were awarded the Gödel prize for showing that central problems in distributed computability can be understood and solved using topology.

Indeed, the possible epistemic states of a distributed system fit nicely into finite combinatorial objects called simplicial complexes, which have a clear topological interpretation. Concretely, such an object consists of all possible global states of the system, glued along shared subsets. Each global state in a distributed system is a collection of local states. This not only represents the combinatorics of possible configurations, but also encodes the lack of global information available to each process: a process in a certain local state cannot distinguish between two global states if its local state lies in their intersection. Simplicial complexes, which can also be thought of as spaces built by gluing together points, lines, triangles and their higher dimensional analogues, thus precisely encode the epistemic ambiguities of the system.

A typical distributed problem, a *task*, is for example presented as a triple $(\mathcal{I}, \mathcal{O}, \Delta)$ where $\mathcal{I}$ and $\mathcal{O}$ are simplicial complexes, and Δ a relation between them. Furthermore, at least for some tasks, it is known that continuous maps between the geometric realizations of associated simplicial complexes classify which tasks can be solved by the protocol. While many generalizations and applications of this topological approach have been developed, still to this day, the direct connection between distributed computing and continuous maps between geometric realizations was unclear. Here, we identify *spectral spaces*, arising as limits of the finite combinatorial topology models of distributed computing, as the underlying phenomenon behind this remarkable connection. Our work shows that this topological characterization is in fact a result of Stone duality, which is a well established tool for such correspondences in computer science. This additionally allows us to obtain a version of the original result which applies to a much wider class of protocols.

In short, our main contribution is the identification of spectral topology as a natural and fruitful extension of the simplicial semantics for distributed computing. This perspective allows us to generalize known colorless topological computability results to any round-based, full-information model of computation. We achieve this by encoding such a protocol as an endofunctor Π on the category of simplicial complexes. Using this, we associate a spectral space $\Pi^\infty(\mathcal{I})$ to any input complex $\mathcal{I}$, which can be described abstractly in terms of a projective limit, but also concretely as a space of sequences. These spaces characterize computability, as is shown in our main theorem, which states that a protocol Π solves a task $(\mathcal{I}, \mathcal{O}, \Delta)$ if, and only if, there exists a spectral map from $\Pi^\infty(\mathcal{I})$ to the output complex $\mathcal{O}$ which respects the specification Δ. For a more developed description of our techniques and results, see the pre-print [3].

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TOWARDS A 2-CATEGORICAL LOGIC OF DEPENDENT TYPE THEORY

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We introduce KZ-sketches, a 2-categorical formalism for axiomatizing 2-dimensional theories within 2-functorial semantics. This closes a gap between the theory of bipresentable 2-categories and the corresponding form of Gabriel–Ulmer duality [DLO24], on the one hand, and the technology of Kan injectivity [DLLS22], on the other.

As a main application, we use KZ-sketches to axiomatize 2-categories of pseudomodels of categories with representable maps (CwRs), introduced in [Uem23]. These capture, essentially, 2-categories of natural models of Dependent Type Theory ([Awo18]) equipped with specified classes of type constructors.

Conceptually, this provides new evidence for the guiding idea ([Pow95]) that logics, in a broad and robust sense, should be understood as 2-dimensional theories. Technically, our approach yields an immediate and useful consequence: 2-categories of pseudomodels of CwRs are finitely bipresentable. Hence they enjoy strong formal properties: they are bicomplete and bicocomplete, their bilimits admit a well-behaved description, and they provide a natural setting for transfinite constructions such as the 2-small object argument and biadjoint functor theorems.

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Machine Space, Exponential Spaces and Compactness

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The interpretation of topology in terms of *verifiable properties* provides insight into many aspects of the field. However, if this perspective is taken completely seriously it prompts some tricky questions concerning the failure of exponentials of topological spaces. In this talk, we will address these questions by introducing the notion of *machine space* and demonstrate its utility by extending Escardó's algorithm for universal quantification over compact spaces (see [3]) from data types to general spaces.

A property P is said to be *verifiable* if whenever an element satisfies P, this can be established by finite means. Notably absent are any constraints on elements which do not satisfy P — that is, you need not be able to verify $\neg P$.

The logic of verifiability admits finite conjunctions and arbitrary disjunctions. Identifying each proposition U with the set of points which satisfy it recovers the usual notion of a *topological space* with verifiable properties corresponding to open sets. This interpretation of topology is due to Smyth in [4]. One interesting aspect of this interpretation is that compact spaces are precisely those that admit universal quantification of open predicates [2].

Philosophical problems The verifiability interpretation raises the following problem of a philosophical nature: do the verifiable properties on a space themselves have verifiable properties so that they can be arranged into a topological space? Naively the answer would appear to be 'yes'. For instance, let $x \in X$ be a point in a space $(X, \mathcal{O}X)$. It appears that we should be able to verify whether a verifiable property in $\mathcal{O}X$ contains x or not. While this is a compelling argument, the mathematics tells a different story. What we are asking for is equivalent to asking for a certain exponential object to exist in the category of topological spaces, but it is well known that these exponentials do not always exist. How can we reconcile this failure with our philosophical intuition?

A seemingly unrelated problem: if compact spaces allow for universal quantification of open predicates, is there a *uniform* procedure for universal quantification that works for all compact spaces?

Mathematical Solutions In what follows we make use of the point-free approach to topology.

It can be instructive to imagine that for each open in a space, there exists some machine or program which carries out the verification procedure. Each machine takes points of the space as input and will either halt (in finite time) or run forever, depending on whether the point belongs to the associated open or not.

We might be tempted to identify machines with opens, which can in turn be seen to correspond to continuous maps from X into the Sierpiński space Σ. Since we are thinking of these machines as 'real things' we can interact with, we can expect there to be natural verifiable properties concerning the machines themselves (for example, does the machine halt on some given point?) yielding a *space of machines* for each space X. However, if X is not locally compact, the exponential object Σ^X, which we would think of as the space of opens, does not exist and so we are forced to distinguish between machines and opens — unlike functions from X to Σ, machines are not extensional.

In order to formalise these ideas, we describe an explicit construction of a space of machines. We replace the (potentially nonexistent) space of opens Σ^X with a certain *weak exponential*.

Our starting point is to fix a presentation for X with generators G. Since G is a set, the space Σ^G exists. The space X embeds canonically into Σ^G and so the opens of X are restrictions of opens in Σ^G. We will view distinct opens of Σ^G which restrict to the same open U in X as distinct machines which accept (i.e. halt on) precisely the elements of U. A crucial point is that the space Σ^G is locally compact and we can take Σ^{Σ^G} to be the space of machines of X. This space may be thought of as a reasonably canonical weak exponential associated to X with base Σ.

We then relate machine space to Σ^X, when the latter exists. In particular, there is a canonical quotient map $\Sigma^{\Sigma^G} \twoheadrightarrow \Sigma^X$ sending each machine to its associated open. Given an open and a point in a general space X there is *not* an obvious way to verify that the point lies in the open, as the open alone does not give any such procedure. This helps explain the fact that for general spaces the collection of opens equipped with the Scott topology does not have a continuous evaluation map. We could however expect the evaluation map to be continuous if there were some way to associate a machine to each open. Indeed, we show that when X is locally compact the canonical quotient map of machine space onto Σ^X always has a section, allowing a continuous assignment of opens to machines which represent them.

Another way in which machine space is useful is in understanding compactness. If K is a compact, locally compact space, then we can check if a verifiable property holds on all of K and so we should be able to verify whether the corresponding open in Σ^K is equal to the largest element of Σ^K, namely K itself. In other words, K is compact if and only if the singleton $\{K\}$ is open in Σ^K (see [2]). Of course, this viewpoint is inapplicable outside of the locally compact case, since the exponential Σ^K will not exist.

As might be expected, this can be extended to the general setting of compact (but not necessarily locally compact) locales by replacing Σ^K with machine space. This perspective on the universal quantification is essentially due to Escardó [1]. Instead of asking if an open of K equals K, we ask if an open in Σ^G *contains* K — that is, if the machine accepts all points of K. By the Hofmann–Mislove theorem, there is an associated open corresponding to all of the machines which cover K, which plays the same role as the singleton $\{K\}$ in the previous approach.

Our contribution is obtaining this open via an explicit algorithm for universally quantifying over a compact space. This is a very general and purely topological/localic version of Escardó's algorithm [1, 3] for universal quantification over Cantor space. This talk is based on [2].

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Lax comma categories: descent and exponentiability

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Janelidze's categorical Galois theory [7, 9, 1] provides a unifying perspective on various Galois-type theorems, which include, for instance, Magid's Galois theory for commutative rings, and Grothendieck's Galois theory for étale coverings of schemes. In fact, such results were generalized and found applications in other settings [8, 10, 11].

Aiming to develop a higher dimensional counterpart of Janelidze's Galois theory, Clementino and Lucatelli Nunes initiated their research by establishing the core concepts of categorical Galois for lax comma 2-categories [4], which are the 2-dimensional analogue of comma categories. As these 2-categories play an important role in this development, it then became apparent that their systematic study would provide some guiding principles towards this end.

We will give an overview our results from [2, 5, 6] as well as ongoing work on various sorts of lax comma categories, of both a categorical and topological flavor. Among the lax comma categories that concern us, we have the following in particular:

- $\mathsf{Set} \Downarrow \mathcal{X} \simeq \mathsf{Fam}(\mathcal{X})$ for a category $\mathcal{X}$ – the category of *families* of objects in $\mathcal{X}$.
- $\mathsf{Ord} \Downarrow X$ for X a (pre)ordered set – the category of *ordered families* of elements of X.
- $\mathsf{Cat} \Downarrow \mathcal{X}$ for a category $\mathcal{X}$.
- $\mathsf{Top} \Downarrow X$ for a topological space X; we consider every topological space X as a preorder by considering the *dual* of the specialisation preorder. This allows us to view Top as a preorder-enriched category, by considering the pointwise order on $\mathsf{Top}(X, Y)$. Explicitly, for continuous maps $f, g\colon X \to Y$, we have $f \leq g$ if and only if $f(x) \leq g(x)$ (in the dual of the specialisation preorder in Y).

One useful guideline for studying lax comma categories is the following: any properties of the lax comma 2-category $\mathbb{A} \Downarrow X$ are often inherited from X. As an example, we have the following result:

Theorem 1. *If $\mathcal{X}$ is a (co)complete category, then $\mathsf{Cat} \Downarrow \mathcal{X}$ has the respective property.*

Analogous results are confirmed to hold in the ordered [3] and in the topological setting as well. Regarding exponentiable objects, we have the following:

Theorem 2. *If X is topological $\wedge$-semilattice with bottom element, then the following are equivalent:*

(i) The forgetful functor $\mathsf{Top} \Downarrow X \to \mathsf{Top}$ reflects exponentiable objects.
(ii) X is a topological Heyting $\bigwedge$-semilattice.

Regarding the study of (effective) descent morphisms, we make use of the following general result:

Theorem 3. *The functor $\mathbb{A} \Downarrow X \to \mathbb{A}$ preserves descent and effective descent morphisms.*

Of particular interest for (higher) categorical Galois theory, we shall discuss effective descent morphisms in the categorical and topological setting, as these are the fundametal tools for the “categorical” Galois theorems; see [9, Theorem 4.2], [1, Chapter 7].

Namely, we (1) obtained a characterisation of effective descent morphisms in $\mathsf{Top} \Downarrow X$ for suitable topological $\bigwedge$-semilattices, and (2) we are able to obtain sufficient conditions for effective descent morphisms in $\mathsf{Cat} \Downarrow \mathcal{X}$ for suitable categories $\mathcal{X}$. Such conditions are naturally expressed in terms of effective descent morphisms in $\mathsf{Fam}(X)$ and $\mathsf{Fam}(\mathcal{X})$ respectively, plus some distributivity conditions.

This is based on joint work with Maria Manuel Clementino, Dirk Hofmann and Fernando Lucatelli Nunes.

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There are only denumerably many locally tabular bi-intermediate logics of trees and of co-trees

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Much like the intuitionistic propositional calculus IPC is algebraized by the variety of Heyting algebras, the bi-intuitionistic propositional calculus bi-IPC is algebraized by the variety bi-HA of *bi-Heyting algebras* (Heyting algebras whose order duals are also Heyting algebras). This logic is the natural symmetric extension of IPC and can be obtained by enlarging the language with the *co-implication* $\leftarrow$, a binary connective that behaves dually to the intuitionistic implication. Moreover, the celebrated Esakia duality [3] restricts to one between the category of bi-Heyting algebras and their homomorphisms, and the category of *bi-Esakia spaces* (Esakia spaces whose order duals are also Esakia spaces) and *bi-Esakia-morphisms* (continuous p-morphisms that also satisfy the *down* condition: $y \leq f(x) \implies \exists z \leq x\,(f(z) = y)$).

A notable extension of bi-IPC is $\mathsf{bi\text{-}GD} := \mathsf{bi\text{-}IPC} + (p \to q) \vee (q \to p)$, the *bi-intuitionistic Gödel-Dummett* logic, which is algebraized by the variety bi-GA of *bi-Gödel algebras*. The duals of these algebras are the *bi-Esakia co-forests* (disjoint unions of co-trees equipped with a bi-Esakia topology), and *bi-Esakia co-trees* (co-trees equipped with a bi-Esakia topology) are duals to subdirectly irreducible (SI for short) bi-Gödel algebras (see, e.g., [2]). In particular, finite SI bi-Gödel algebras dualize to finite co-trees.

The lattice $\Lambda(\mathsf{bi\text{-}GA})$ of varieties of bi-Gödel algebras has been extensively studied in [2, 4]. For example, it is shown in [2, Thm. 4.16] that $\Lambda(\mathsf{bi\text{-}GA})$ has the size of the continuum, and it follows from [2, Cor. 5.31] that, for a variety $\mathsf{V} \in \Lambda(\mathsf{bi\text{-}GA})$,

$$\mathsf{V} \text{ is locally finite } \iff \mathsf{V} \subseteq \mathsf{V}_n := \{\mathbf{A} \in \mathsf{bi\text{-}GA} \colon \mathfrak{C}_n \not\hookrightarrow \mathbf{A}_*\}, \text{ for some } n \in \mathbb{Z}^+, \tag{1}$$

where $\mathfrak{C}_n \not\hookrightarrow \mathbf{A}_*$ denotes the nonexistence of an order embedding from the n-comb $\mathfrak{C}_n$ (see Figure 1) into the bi-Esakia dual of $\mathbf{A}$.

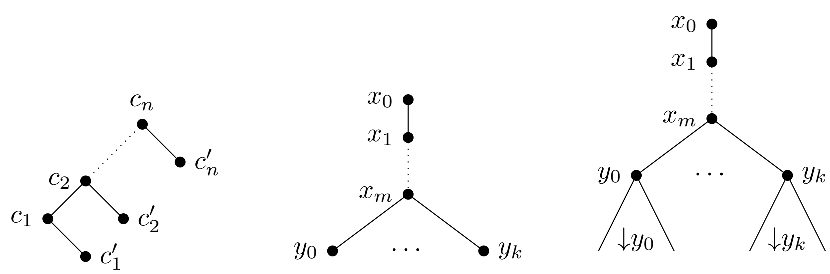

Figure 1: From left to right: the n-comb $\mathfrak{C}_n$; the co-tree $\tau(m,k) \in \mathcal{T}_2$; the co-tree $(\tau(m,k), [\downarrow y_0, \ldots, \downarrow y_k]) \in \mathcal{T}_{n+1}$.

In this talk, I will present a proof (detailed in [5]) that, while $\Lambda(\mathsf{bi\text{-}GA})$ has the size of the continuum, it contains only countably many locally finite varieties of bi-Gödel algebras, all of which are finitely axiomatizable. Using the duality between finite co-trees and SI bi-Gödel algebras, [1, Thm. 6.21] shows that the aforementioned result is equivalent to proving that every $\mathcal{T}_n \coloneqq (\{\mathcal{X} \colon \mathcal{X}$ is a finite co-tree s.t. $\mathfrak{C}_n \not\twoheadrightarrow \mathcal{X}\}/\cong)$ does not contain infinite antichains w.r.t. $\leq_p$, the *bi-p-morphic image relation* ($\mathcal{X} \leq_p \mathcal{Y}$ iff $\exists$ surjective bi-Esakia-morphism $f \colon \mathcal{Y} \twoheadrightarrow \mathcal{X}$).

In fact, we established the stronger claim that every $(\mathcal{T}_n, \leq_p)$ is a *better partial order* (BPO for short), a particular type of well partial orders (recall that well partial orders lack infinite antichains) whose robust definition ensures the "better-behaviour" of infinite constructions. To do so, we identified a partial order that strengthens the Higman's multiset embeddability relation: given a poset $(P, \leq)$, the *multiset projectivity relation* relation $<<$ on the set $P^{\#}$ of finite multisets of P is defined by

$$N << M \iff \exists \text{ surjective } f \colon M \twoheadrightarrow N \text{ s.t. } f(p) \leq p \text{ for all } p \in M.$$

We then show that if the starting poset $(P, \leq)$ is a BPO, then so is $(P^{\#}, <<)$ (this transfer property does not hold under the weaker hypothesis that $(P, \leq)$ is only a well partial order).

The remainder of the proof uses an inductive argument. $(\mathcal{T}_2, \leq_p)$ is a BPO because it is order isomorphic to $(\omega, \leq) \times (\omega, \leq)$ (the standard order on the natural numbers is a BPO, and products of BPOs are always BPOs). Figure 1 shows the structure of the co-trees in $\mathcal{T}_2$. Then, by assuming that $\mathcal{T}_n$ is a BPO, it follows by above that so is $(\mathcal{T}_n^{\#}, <<)$, hence also $(\mathcal{T}_2, \leq_p) \times (\mathcal{T}_n^{\#}, <<)$ is a BPO. Finally, by showing that every co-tree in $\mathcal{T}_{n+1}$ (see Figure 1) can be identified with an element of $(\mathcal{T}_2, \leq_p) \times (\mathcal{T}_n^{\#}, <<)$ in a unique order reflecting manner, a well known result for BPOs ensures that $(\mathcal{T}_{n+1}^{\#}, <<)$ is in fact a BPO, as desired.

By combining this with previous results on $\Lambda(\mathsf{bi\text{-}GA})$, we obtained an informative depiction of this lattice, which will also be presented during the talk.

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On the Decidability of Comparing Many-Valued Logics: A WSkS Approach

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The advantages of moving beyond truth-functional semantics for propositional logics have been increasingly noted in recent years. Partial non-deterministic logical matrices (PNmatrices) [1], where connectives are interpreted as multifunctions, greatly extend the expressive power of many-valued semantics and support modular, algorithmic constructions of non-classical logics. Their systematic study [2] has shown that PNmatrix semantics can be automatically generated for certain families of sequent calculi and composed in a modular fashion to yield semantics for combined logics [4]. A basic question is then: given two finite PNmatrices, can we effectively compare the logics they determine? That is, can we algorithmically check inclusion between their associated consequence relations or sets of valid formulas?

For ordinary (deterministic) finite matrices, the answer is positive [5]. Since deterministic valuations are homomorphisms, we can collapse variables sharing a value without changing the truth values of formulas, reducing the comparison to a finite number of variables n, which we fix as the maximum of the sizes of the sets of truth values of the two matrices. Furthermore, the logical equivalence relation induced by a finite matrix partitions the set of formulas over n variables into a finite number of equivalence classes, i.e., the logics are locally tabular. Representatives for these equivalence classes can be automatically generated, meaning the comparison of their consequence relations reduces to checking them on the finite union of the sets of representatives for each matrix. For PNmatrices, this reduction fails because valuations are not uniquely determined by their values on variables, causing local tabularity to fail even under finite variable restrictions. The comparison problem's complexity thus depends on the underlying notion of logic: it is undecidable for Fmla- and $\mathsf{Set}\times\mathsf{Fmla}$-logics [6, 5], but remains open for Scottian $\mathsf{Set}\times\mathsf{Set}$-logics. This aligns with fundamental differences: deterministic $\mathsf{Set}\times\mathsf{Set}$-logics are always finitely axiomatizable [11], but their $\mathsf{Set}\times\mathsf{Fmla}$-counterparts are not [12], which allows for consequence relations that cannot be characterized by any finite set of rules, opening the door to undecidability.

Despite these challenges, progress has been made in characterizing the Nmatrix models of such logics through generalizations of standard class operators and closure properties under ultraproducts [3]. To formalize our contribution, we distinguish three common notions of logic over a signature Σ:

- a Fmla-*logic* is a set $\mathsf{Thm}\subseteq\mathbf{Fm}$ (theorems) closed under substitutions;
- a $\mathsf{Set}\times\mathsf{Fmla}$-*logic* is a Tarskian consequence relation $\vdash\subseteq\mathsf{Set}\times\mathsf{Fmla}$;
- a $\mathsf{Set}\times\mathsf{Set}$-*logic* is a Scottian consequence relation $\rhd\subseteq\mathsf{Set}\times\mathsf{Set}$.

These notions are related: every $\mathsf{Set}\times\mathsf{Set}$-*logic* $\rhd$ induces a $\mathsf{Set}\times\mathsf{Fmla}$-logic $\vdash_\rhd$ and a Fmla-logic $\mathsf{Thm}_\rhd$. We denote the n-variable fragment of a logic by $\rhd^{|n}{=}\rhd\cap(\wp(\mathbf{Fm}_n)\times\wp(\mathbf{Fm}_n))$.

Our main contribution is the classification of the decidability status for the comparison problem across these settings for finite PNmatrices, as summarized in Table 1.

Logic type	Full fragment	n-variable fragment
Fmla-logic	Undecidable [6]	Undecidable [7]
Set $\times$ Fmla-logic	Undecidable [6]	Undecidable [7]
Set $\times$ Set-logic	Open	**Decidable** [7] (via WSkS)

Table 1: Decidability of comparing finite PNmatrices.

We focus on the logic characterized by PNmatrices. A Σ-*PNmatrix* is a tuple $\mathbb{M} = \langle A, \cdot_{\mathbb{M}}, D\rangle$, where A is a set of truth values, $\cdot_{\mathbb{M}}$ is a multifunction into $\wp(A)$, and $D \subseteq A$ is the set of designated values. A valuation v is a function such that $v(f(\varphi_1, \ldots, \varphi_k)) \in f_{\mathbb{M}}(v(\varphi_1), \ldots, v(\varphi_k))$. The resulting Scottian logic $\rhd_{\mathbb{M}}$ is defined by: $\Gamma \rhd_{\mathbb{M}} \Delta$ iff for every $v \in \mathsf{Val}(\mathbb{M})$, $v(\Gamma) \subseteq D$ implies $v(\Delta) \cap D \neq \emptyset$.

The most recent results on this problem, presented at ReacTS [7], show that while restricting to the n-variable fragment $\rhd^{|n}$ keeps the comparison problem undecidable for Fmla and Set $\times$ Fmla logics, it establishes the decidability of the Scottian (Set $\times$ Set) case by reducing it to the validity of formulas in Weak Second-order Logic of k Successors (WSkS).

Theorem 1. *For every $n \in \mathbb{N}$, the problem of determining whether $\rhd^{|n}_{\mathbb{M}_1} \subseteq \rhd^{|n}_{\mathbb{M}_2}$ for finite PN-matrices $\mathbb{M}_1, \mathbb{M}_2$ is decidable.*

This reduction involves encoding the set of partial bivaluations over a finite prefix subtree of the infinite k-ary tree. Each node in the tree represents a potential subformula, and WSkS predicates label these nodes to satisfy the constraints of a valuation. Since the validity problem for WSkS is decidable [10], this provides an effective method for comparing Scottian logics under finite variable restrictions, reinforcing the importance of the multiple-conclusion context in non-classical logic. Crucially, this reduction allows the comparison problem to be fully automated. In the presentation, we will showcase concrete examples solved using our own implementation, which compiles the matrices into WSkS and uses the MONA solver [8] to verify inclusion. This implementation allows us to automatically compare n-variable fragments of non-monadic non-deterministic matrices – highlighting cases where no other existing automated tool could be applied – and even extract separating sequents when the inclusion fails.

In ongoing work, we explore the possibility of lifting these results to the comparison of equational consequences in multialgebras. Specifically, we focus on the multiple-conclusion equational consequence $\rhd_{\mathcal{A}}$ over a class $\mathcal{A}$ of multialgebras defined by: $\{\varphi_i \approx \psi_i : i \in I\} \rhd_{\mathcal{A}} \{\varphi_j \approx \psi_j : j \in J\}$ whenever for every $\mathbf{A} \in \mathcal{A}$ and every valuation $v : \mathbf{Fm} \to \mathbf{A}$, $v(\varphi_i) = v(\psi_i)$ for all $i \in I$ implies $v(\varphi_j) = v(\psi_j)$ for some $j \in J$. In future work, we are interested in studying whether both the positive decidability results and the negative undecidability results described here lift to this equational setting, and if the WSkS-based reduction can be effectively adapted.

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Quasi-equational bases for oriented paths

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For a finite relational structure P let $Q(P)$ be the class of all relational structures isomorphic to a substructure of a power of P, and let $Q_I(P)$ be the class of structures isomorphic to an induced substructure of a power of P. The class $Q_I(P)$ is axiomatizable by universal sentences of the form

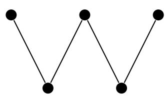
$$(\forall \bar{x})\ [\varphi_1(\bar{x}) \wedge \cdots \wedge \varphi_n(\bar{x}) \Rightarrow \varphi(\bar{x})],$$

where φ_i, φ are atomic formulas. In digraphs these are of the form $x \to y$ (there is an edge from x to y) and $x = y$. We call such sentences *quasi-identities*, for historical reasons. The class $Q(P)$ is axiomatizable by quasi-identites in which the conclusion φ is an equation. We call such sentences *equality-forcing quasi-identities*. We say that P is *finitely q-based* (or *finitely efq-based*) if $Q_I(P)$ (or $Q(P)$) is axiomatizable by a finite number of (equality-forcing) quasi-identities. We study the following question.

Problem 1. Which finite relational structures are finitely (ef)q-based?

While the algebraic analogue of this problem has stimulated research in universal algebra in recent decades, see e.g. [2, 3], the results for relational structures are rare. As an example we recall Pultr's and Nešetřil's theorem stating that the only finitely q-based finite simple graphs are disjoint unions of complete bipartite graphs [1].

In the hope of discovering a general pattern, we decided to look closer at a class of digraphs that, on one side, behave essentially different than simple graphs and, on the other, are not too complicated. We focus on *oriented paths*, that is, digraphs obtained from a path in a simple graph by choosing a direction for each of its edges. One may visualize oriented paths by Hasse-like diagrams where arrows are edges whose direction is always upwards, see below for examples.



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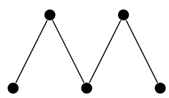
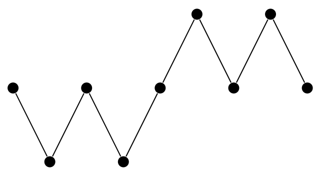


m



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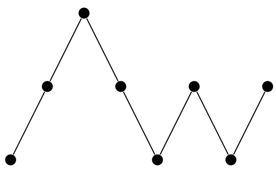
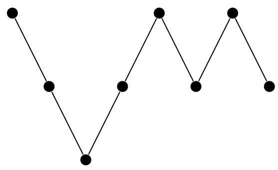


Λw



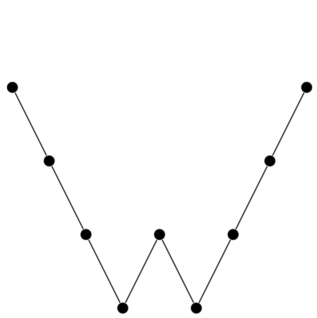
Vm

Recall that a digraph is *(ef-)critical relative to* P if it does not belong to $Q_I(P)$ $(Q(P))$ but each its proper induced subdigraph does. We obtained the following result.

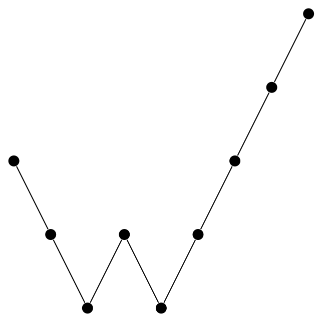
Theorem 2. *There is a linear function* $c\colon \mathbb{N} \to \mathbb{N}$ *such that for every finite oriented path* P *containing* w *or* m *as a subdigraph and with* $|P| = k$*, the following holds:*

P is finitely efq-based iff all critical digraphs relative to P have at most $c(k)$ vertices.

It is not difficult to show that if P is an oriented path containing w or m and $Q_I(P)$ contains one of wIm, Λw or Vm, then P is not finitely q-based. Similarly, if $Q(P)$ contains one of wIm, Λw or Vm, then P is not finitely efq-based. It would seem that we are very close to a complete characterisation, alas, it still evades us. For consider oriented paths that contain w or m but $\{wIm, \Lambda w, Vm\} \cap Q(P) = \emptyset$. We call them valleys and hills. Critical digraphs relative to valleys and hills may be very complex. As examples, consider the following valleys

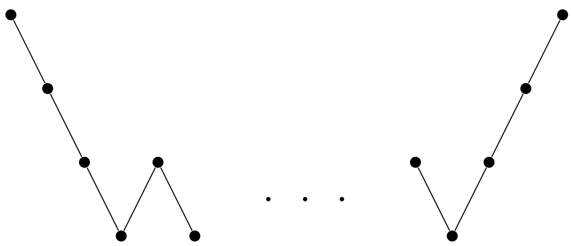


W_{33}



W_{24}

Then W_{33} is finitely (ef)q-based while W_{24} is not, which is witnessed by the following infinite sequence of critical digraphs relative to W_{24}:



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A Graphical Calculus for Categorical Linear Dependency

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I introduce *graded linear categories* with a *universe* (or small codomain fibration), a categorical model of quantitative linear dependent types, and prove that a *graphical* calculus [15] for intuitionistic linear logic (ILL) [8] yields an *initial* graded linear category with a universe. This is the first graphical initiality result for a blend of ILL and dependent types.

Linear maps are fundamental in logic, mathematics and computer science. They typically arise in the linear part of *linear-nonlinear adjunctions* [3]; *e.g.*, *linear categories* [3, 4], a categorical model of ILL, have such an adjunction. Linear categories appear not only in logic but also in various other fields; they are ubiquitous when nonlinear maps are refined through linear ones. *Dependent types* [9] are indispensable part of the foundations of mathematics, providing an apparatus for predicates, families and substitution. Most categorical models of dependent types [10, 5, 7, 2] are based upon the *nonlinear* structure of cartesian categories.

Can linearity and dependency *interact* categorically? This problem has been a challenge since the categorical models of dependent types rely on cartesian (or nonlinear) structures. For this reason, most attempts [6, 11, 14] treat linearity and dependency *separately*. Nevertheless, a syntactic advance was made by McBride [12], later refined by Atkey [1] and extended by Moon *et al.* [13]. That line of work, known as *quantitative* or *graded dependent types*, achieves a genuine blend of ILL and dependent types, *e.g.*, types depending on linear types. However, a categorical study of the blend has been largely open, and I address this problem as follows.

First, based on the idea of graded dependent types, I define *graded linear categories*, linear categories equipped with *grading* for quantitative resource usage, and show that, when they have a universe, they form a quantitative linear analogue of *comprehension categories* [10], a categorical model of dependent types, supporting major type formers such as dependent sums, dependent products and exponential (or of-course modality). Through exponential, graded linear categories with a universe have a dependent generalisation of a linear-nonlinear adjunction, whose nonlinear part forms comprehension categories. I next prove that the graphical calculus for ILL, which forms an initial linear category, *intrinsically* has quantitative information and yields an *initial* graded linear category with a universe. Finally, I outline a quantitative variant of fibrations and an equivalence between them and graded linear categories with a universe. From an indexed categorical perspective, my structure has fibrewise graded linear categories that are stable under reindexing functors. From a fibrational point of view, the total categories also carry the graded linear structure. While a graded linear category with a universe has the dependent linear-nonlinear adjunction, it may not be locally cartesian closed.

These results provide novel graphical and categorical frameworks for the study of the interaction between linearity, dependency and quantitative resource usage.

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Partially-ordered Płonka sums of po-algebras

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Płonka sums have played a significant role in universal algebra and logic, e.g., with the construction of Clifford semigroups from groups, involutive bisemilattices from Boolean algebras, regularization of varieties, and its connection to logics of variable inclusion [2]. More recently they have been applied to the structure theory of commutative idempotent involutive residuated lattices [4], locally integral involutive po-semigroups [3], and residuated po-monoids [1]. This work establishes a comprehensive framework for Płonka sums of residuated structures within the setting of a certain kind of po-algebras (see [5] for details on po-algebras), introducing the construction of *po-Płonka sum*. Moreover, we obtain some syntactic restrictions on inequalities that ensure such inequalities will be preserved by po-Płonka sums. Finally, we define Płonka partitions for posets and explain how they are related to poset po-Płonka sums. The main application of our results is to residuated lattices, but the framework presented here applies to n-ary residuated partially ordered algebras in general. Even in the setting with no fundamental operations or only unary operations, po-Płonka sums provide a new tool for constructing or decomposing posets and lattices with pairs of operations that are residuals, Galois connections or dual Galois connections.

Let Alg_τ be the category of algebras of type $\tau : \mathcal{F} \to \mathbb{N}$ where $\mathcal{F}$ is the set of function symbols. A join-semilattice $\mathbf{I} = (I, \vee)$ can be considered as a category where a (unique) morphism exists from i to j if and only if $i \leq j$. From any functor $\varphi : \mathbf{I} \to \mathsf{Alg}_\tau$—i.e., any indexed family of homomorphisms $\varphi_{ij} : \mathbf{A}_i \to \mathbf{A}_j$ for $i \leq j$ in $\mathbf{I}$ such that $\varphi_{ii} = \mathrm{id}_{\mathbf{A}_i}$ and $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$—its Płonka sum $\mathbf{A} \in \mathsf{Alg}_\tau$ is constructed by endowing the disjoint union $A = \uplus_I A_i$ with a structure of τ-algebra such that for every n-ary $f \in \mathcal{F}$, all $k = i_1 \vee \cdots \vee i_n \in I$, and $a_j \in A_{i_j}$ the fundamental operation $f^{\mathbf{A}}$ is defined by

$$f^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{A}_k}(\varphi_{i_1 k}(a_1), \ldots, \varphi_{i_n k}(a_n)).$$

A *po-algebra signature* is a map $\tau : \mathcal{F} \to \{\vee, \wedge\} \times \bigcup_{n=0}^{\infty} \{1, \partial\}^n$, where $\tau(f)_0$ indicates whether f is potentially a left or a right adjoint and $\tau(f)_i$ denotes the tonicity of the ith argument of the operation symbol f, for $i = 1, \ldots, n$. This means that in a po-algebra $\mathbf{A}$, each $f^{\mathbf{A}}$ is monotone in the ith variable if $\tau(f)_i = 1$ and antitone if $\tau(f)_i = \partial$.

A *po-metamorphism* is a pair of monotone maps $\psi, \varphi : \mathbf{A} \to \mathbf{B}$ such that $\psi(x) \leq^{\mathbf{B}} \varphi(x)$ and for all $f, g \in \mathcal{F}$ such that $\tau(f)_0 = \vee$ and $\tau(g)_0 = \wedge$ we have

$$\varphi(f^{\mathbf{A}}(x_1, \ldots, x_n)) = f^{\mathbf{B}}(\varphi^{\tau_1}(x_1), \ldots, \varphi^{\tau_n}(x_n)), \qquad \text{where } \tau_i = \tau(f)_i$$
$$\psi(g^{\mathbf{A}}(x_1, \ldots, x_n)) = g^{\mathbf{B}}(\psi^{\tau_1}(x_1), \ldots, \psi^{\tau_n}(x_n)), \qquad \text{where } \tau_i = \tau(g)_i,$$

$\varphi^\partial = \psi = \psi^1$ and $\psi^\partial = \varphi = \varphi^1$. Note that if a metamorphism ψ, φ satisfies $\varphi = \psi$ then it is a po-algebra homomorphism. Po-metamorphisms form a category denoted by $\mathsf{Alg}_\tau^{\leq}$ where $id_{\mathbf{A}}, id_{\mathbf{A}}$ is the identity po-metamorphism on $\mathbf{A}$.

A *semilattice directed system* of po-algebras and metamorphisms is a functor $\Phi : \mathbf{I} \to \mathsf{Alg}_\tau^{\leq}$. Let $\psi_{ij}, \varphi_{ij} = \Phi(i \leq j) : \mathbf{A}_i \to \mathbf{A}_j$. The *po-Płonka sum* $\mathbf{A} = \int_{\mathbf{I}} \Phi$ is defined by endowing the disjoint union $A = \uplus_I A_i$ with the relation

$$a \leq^{\mathbf{A}} b \iff \varphi_{i_1 j}(a) \leq^{\mathbf{A}_j} \psi_{i_2 j}(b), \qquad \text{for } j = i_1 \vee i_2 \text{ in } \mathbf{I} \text{ and } a \in A_{i_1}, b \in A_{i_2},$$

and for $k = i_1 \vee \cdots \vee i_n \in I$ and $a_j \in A_{i_j}$, the fundamental operations for all $f, g \in \mathcal{F}$ such that $\tau(f)_0 = \vee$ and $\tau(g)_0 = \wedge$ are defined by

$$f^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{A}_k}(\varphi^{\tau_1}_{i_1 k}(a_1), \ldots, \varphi^{\tau_n}_{i_n k}(a_n)), \qquad \text{where } \tau_i = \tau(f)_i$$
$$g^{\mathbf{A}}(a_1, \ldots, a_n) = g^{\mathbf{A}_k}(\psi^{\tau_1}_{i_1 k}(a_1), \ldots, \psi^{\tau_n}_{i_n k}(a_n)), \qquad \text{where } \tau_i = \tau(g)_i.$$

We also need the following properties for all $i < j, k \in I$ and $l = j \vee k$:

(S1) $\varphi_{jl} \circ \psi_{ij} \le \psi_{kl} \circ \varphi_{ik}$,

(S2) $\varphi_{ij}(x) \le \psi_{ij}(y) \implies x < y$, for all $x, y \in A_i$.

Theorem 1. *Let Φ be a directed system of po-metamorphisms with po-Płonka sum $\mathbf{A}$.*

1. *The binary relation $\le^{\mathbf{A}}$ is a partial order if and only if (S_1) and (S_2) hold.*
2. *If for all $i \in I$, an operation $f^{\mathbf{A}_i}$ has a residual $g_j^{\mathbf{A}_i}$ with respect to each argument x_j then $f^{\mathbf{A}}$ has residual $g_j^{\mathbf{A}}$ in the po-Płonka sum, i.e., the following equivalence is preserved*

$$f(x_1, \ldots, x_j, \ldots, x_n) \le y \iff x_j \le g_j(x_1, \ldots, y, \ldots, x_n)$$

where $\tau(f)_0 = \vee$, $\tau(g_j)_0 = \wedge$.

For the preservation of lattice operations we first generalize [6, Theorem 3.2] to po-Płonka sums.

Theorem 2 (Associativity for po-Płonka sums). *Let $\Phi : \mathbf{I} \to \mathsf{Alg}^{\le}_{\tau}$ be a directed system of po-Algebras satisfying $(S_1), (S_2)$, where $\Phi(i) = \mathbf{A}_i$ for every $i \in I$, and $\theta : \mathbf{J} \to \mathsf{Jslat}$ a directed system of join-semilattices, where $\theta(j) = \mathbf{I}_j$ for every $j \in J$, whose Płonka sum is $\mathbf{I}$. Define the directed system $\Phi' : \mathbf{J} \to \mathsf{Alg}^{\le}_{\tau}$, where $\Phi'(j) = \int_{\mathbf{I}_j} \Phi$, while for every $j, k \in J$ with $j \le k$ let*

$$\varphi'_{jk} := \bigcup_{i \in I_j} \varphi_{i\theta_{jk}(i)} \quad \text{and} \quad \psi'_{jk} := \bigcup_{i \in I_j} \psi_{i\theta_{jk}(i)},$$

then Φ' satisfies $(S_1), (S_2)$ and $\int_{\mathbf{I}} \Phi = \int_{\mathbf{J}} \Phi'$.

Then we identify conditions under which the po-Płonka sum of two lattices is again a lattice, and finally apply Theorem 2 to show that po-Płonka sums over certain join-semilattices can be constructed in a step-by-step manner by using only two lattices at each step. With this approach, well-behaved po-Płonka sums of lattice-ordered algebras are again lattice ordered.

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Product Łukasiewicz Unbound Logic

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Recently, the Łukasiewicz unbound logic has been introduced in [5]. This logic very closely follows the original Łukasiewicz logic; however, instead of (bounded) MV-algebras [4], it is based on specific unbounded residuated lattices, Abelian ℓ-groups [7, 8]. As Abelian ℓ-groups are models of Abelian logic [3, 11], the Łukasiewicz unbound logic can be seen as an expansion of Abelian logic obtained by adding an additional constant `f` for falsum and adding a new rule, which enforces `f` to be a negative element and ensures the resulting logic is semilinear. This logic can be defined semantically, as the Łukasiewicz unbound logic is strongly finitely complete w.r.t. the pointed Abelian ℓ-group of reals with the designated element -1. In this talk, our goal is to use similar methods to construct Product Łukasiewicz unbound logic, the logic which is strongly finitely complete w.r.t. the ordered unital field $\boldsymbol{R}$.

Recall that Product Łukasiewicz logic [6] is defined as the expansion of Łukasiewicz logic with the additional product operation $\odot$ satisfying the following axioms:

$$(\chi \odot \varphi) \ominus (\chi \odot \psi) \leftrightarrow \chi \odot (\varphi \ominus \psi) \qquad \text{(Distributivity)} \qquad \text{(P1)}$$

$$\varphi \odot (\psi \odot \chi) \leftrightarrow (\varphi \odot \psi) \odot \chi \qquad \text{(Associativity)} \qquad \text{(P2)}$$

$$\varphi \to \varphi \odot \overline{1} \qquad \text{(Unit element)} \qquad \text{(P3)}$$

$$\varphi \odot \psi \to \varphi \qquad \text{(Monotonicity)} \qquad \text{(P4)}$$

$$\varphi \odot \psi \to \psi \odot \varphi \qquad \text{(Commutativity)} \qquad \text{(P5)}$$

Also, let us recall the extension of Łukasiewicz product logic PŁ$'$ [10] with the *zero divisors rule*

$$\frac{\neg(\varphi \odot \varphi)}{\neg\varphi}. \qquad \text{(ZD)}$$

This rule ensures that the models of this logic are exactly the algebras without non-trivial zero divisors. To introduce Product Łukasiewicz unbound logic, we need to start with the logic of unital commutative ℓ-rings [2]. We can axiomatize this logic as the expansion of Abelian logic by the following axioms:

$$\varphi \cdot (\psi + \chi) \leftrightarrow (\varphi \cdot \psi) + (\varphi \cdot \chi) \qquad \text{(Distributivity)} \qquad \text{(R1)}$$

$$(\varphi \cdot \psi) \cdot \chi \leftrightarrow \varphi \cdot (\psi \cdot \chi) \qquad \text{(Associativity)} \qquad \text{(R2)}$$

$$\varphi \cdot 1 \leftrightarrow \varphi \qquad \text{(Unit element)} \qquad \text{(R3)}$$

$$\frac{\varphi, \psi}{\varphi \cdot \psi} \qquad \text{(Compatibility with order)} \qquad \text{(R4)}$$

$$\varphi \cdot \psi \leftrightarrow \psi \cdot \varphi \qquad \text{(Commutativity)} \qquad \text{(R5)}$$

$$\frac{\varphi \leftrightarrow \psi}{\varphi \cdot \chi \leftrightarrow \psi \cdot \chi} \qquad \text{(Congruence)} \qquad \text{(R6)}$$

To obtain our desired completeness theorem, we must focus on those commutative ℓ-rings which are semilinear. These ℓ-rings are called function rings (f-rings) [9, 7]. We introduce the logic of unital f-rings as the extension of the logic of unital commutative ℓ-rings by the following axiom [1]:

$$(\chi \vee 0) \wedge ((-\chi \vee 0) \cdot (\psi \vee 0)) \leftrightarrow 0 \qquad \text{(Disjointness Preservation)} \tag{F1}$$

To obtain finitary strong completeness w.r.t. $\boldsymbol{R}$, one has to add the following zero divisor rule:

$$\frac{\varphi \cdot \varphi \leftrightarrow 0}{\varphi \leftrightarrow 0}. \tag{Unbound ZD}$$

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Spectral spaces from closure operators on semilattice-ordered semigroups

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Spectral spaces play an important role in the interaction between algebra and topology. They arise naturally in many classical settings, most prominently in spectrum constructions equipped with hull-kernel topologies, and they provide a useful framework for studying algebraic structures by topological methods [7, 3, 6]. In this talk, we study how closure operators on semilattice-ordered semigroups give rise to spectral spaces and to naturally induced topological operations. Our approach is also related to earlier work on topological aspects of semilattice-ordered semigroups and on spectral constructions arising from closure operators on monoids and additively idempotent semirings [2, 9, 8].

Starting from a semilattice-ordered semigroup together with a closure $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$, we consider the family

$$X := \{A \subseteq S \mid \mathrm{cl}(A) = A\}$$

of all subsets fixed by the closure operator, endowed with the topology induced from the hull-kernel topology on $\mathcal{P}(S)$. In related settings, earlier work showed that algebraic closure operators yield spectral fixed-point spaces [1, 4, 5].

Recall that a subset Y of a spectral space X is said to be retrocompact in X if for every quasi-compact open subset U of X, $U \cap Y$ is quasi-compact. Our first result shows that algebraicity of cl admits the following topological characterization:

Theorem 1. *The set X, endowed with the subspace topology induced from the hull-kernel topology on $\mathcal{P}(S)$, is a spectral space and a retrocompact subset in $\mathcal{P}(S)$ if and only if* cl *is algebraic.*

We then investigate the multiplication $\star : X \times X \to X$ defined by

$$A \star B := \mathrm{cl}(AB),$$

where $AB = \{ab \mid a \in A, b \in B\}$. If the closure operator cl is algebraic, then $\star$ is continuous with respect to the patch topology, the standard compact Hausdorff refinement of the spectral topology. If, in addition, cl satisfies a natural multiplicativity condition, then $\star$ is associative and order-preserving in each variable. In this way, closure operators provide a systematic method for constructing topological magmas and, under multiplicativity assumptions, topological semigroups from semilattice-ordered semigroups. Thus, the space X carries not only a spectral topology but also a naturally induced algebraic structure compatible with the patch topology.

A central part of the talk concerns the question of when the map $\star : X \times X \to X$ is spectral. We establish a criterion for this property and use it to compare the global behavior of the multiplication with the behavior of the corresponding right and left translation maps, namely

$$R_B : X \to X, \qquad R_B(A) = A \star B$$

and

$$L_A : X \to X, \qquad L_A(B) = A \star B.$$

Let

$$X_{\mathrm{fin}} := \{F^{\mathrm{cl}} \mid F \in \mathcal{P}_{\mathrm{fin}}(S)\}$$

denote the family of all finitely generated cl-closed subsets of S. Our second main result is the following:

Theorem 2. *Let S be a semilattice-ordered semigroup and let* cl *be an algebraic closure operator on S. Then the following conditions are equivalent:*

1. *the map $\star : X \times X \to X$ is spectral,*
2. *for every $B \in X$ the right translation $R_B : X \to X$ is spectral,*
3. *for every $A \in X$ the left translation $L_A : X \to X$ is spectral,*
4. *for every finite subset $F \subseteq S$ there exists a finite (possibly empty) family of pairs $(G_i, H_i) \in \mathcal{P}_{\mathrm{fin}}(S) \times \mathcal{P}_{\mathrm{fin}}(S)$, $i \in I$, such that for all $A, B \in X$,*
$$F \subseteq A \star B \iff \exists i \in I \text{ such that } G_i \subseteq A \text{ and } H_i \subseteq B,$$
5. *for every finite subset $F \subseteq S$ there exists a finite (possibly empty) family of pairs $(G_i, H_i) \in X_{\mathrm{fin}} \times X_{\mathrm{fin}}$, $i \in I$, such that for all $A, B \in X_{\mathrm{fin}}$,*
$$F \subseteq A \star B \iff \exists i \in I \text{ such that } G_i \subseteq A \text{ and } H_i \subseteq B.$$

The next result shows that spectrality of $\star$ follows from a well-quasi-order condition.

Proposition 3. *Let S be a semilattice-ordered semigroup and let* cl *be an algebraic closure operator on S. If the poset $(X_{\mathrm{fin}}, \subseteq)$ is well-quasi-ordered, then the map $\star : X \times X \to X$ is spectral.*

This result makes the criterion from Theorem 2 applicable in practice. In the talk, I will present examples showing how this theory applies to closure operators naturally associated with semilattice-ordered semigroups.

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Conjunctive Concept Algebras – Unnamed Perspective

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Named Perspective and Unnamed Perspective In database theory, we distinguish two perspectives on relations, called the *named perspective* and the *unnamed perspective* [1]. Under the named perspective, a relation on G is a set of *named tuples* $t : I \to G$, where I is the *schema* of the relation. Under the unnamed perspective, a relation on G is a set of ordered n-tuples, where n is the *arity* of the relation. When visualizing a relation as a table, the tuples are the rows, and the columns are either named (named perspective) or ordered (unnamed perspective). The perspective determines the algebraic operations available on the relations; two different variants of Codd's relational algebra are distinguished, cf. [1, Sect. 5.1].

Conjunctive Queries We distinguish between conjunctive calculus queries and tableau queries, and their formalization depends on the perspective, cf. [1, Sect. 4]. Under the unnamed perspective, a *conjunctive calculus query* can be represented by a pair (φ, t), where φ is a conjunction of atomic formulas (over a relational signature M), and $t = (x_1, \ldots, x_n)$ is an ordered tuple of variables occurring in φ. The *result* of the query (φ, t) in a structure $\mathfrak{G}$ (over the signature M) is the relation $\text{res}_{\mathfrak{G}}(\varphi, t) := \{(\alpha(x_1), \ldots, \alpha(x_n)) \mid \mathfrak{G} \models \varphi[\alpha]\}$. Under the named perspective, the only difference is that t is a named tuple, and the relation $\text{res}_{\mathfrak{G}}(\varphi, t)$ is a set of named tuples. Alternatively, under either perspective, we may represent the formula φ by a structure $\mathfrak{A}$ (over the signature M). The corresponding query $(\mathfrak{A}, t)$ is a *tableau query*. Tableau queries offer a natural generalization to infinite queries (i.e. we allow $\mathfrak{A}$ to be infinite). The *result* $\text{res}_{\mathfrak{G}}(\mathfrak{A}, t)$ can be defined accordingly.

Concept Lattice of a Relational Structure We focus here on the unnamed perspective using tableau queries. The result operation $\text{res}_{\mathfrak{G}}$ has a dual adjoint, if the orders on relations and queries are chosen in a particular way. If R and S are finitary relations on G, say $R \subseteq A^m$ and $S \subseteq A^n$, then $R \leq S$ iff $m \geq n$ and $\{(g_1, \ldots, g_n) \mid (g_1, \ldots, g_m) \in R\} \subseteq S$. Correspondingly, for conjunctive queries, we have $(\mathfrak{A}, (x_1, \ldots, x_n)) \lesssim (\mathfrak{B}, (y_1, \ldots, y_m))$ iff $n \leq m$ and a homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ with $f(x_1) = y_1, \ldots, f(x_n) = y_n$ exists. We refer to the dual adjoint as $\text{info}_{\mathfrak{G}}$, and being a dual adjoint means that the pair $(\text{info}_{\mathfrak{G}}, \text{res}_{\mathfrak{G}})$ forms a Galois connection. As in Formal Concept Analysis [2], a *concept* of $\mathfrak{G}$ is defined as a stable pair $(R, (\mathfrak{A}, t))$ under the Galois connection, i.e. $\text{res}_{\mathfrak{G}}(\mathfrak{A}, t) = R$ and $\text{info}_{\mathfrak{G}}(R) = (\mathfrak{A}, t)$. Likewise, the relation R is the *extent* and the query $(\mathfrak{A}, t)$ is the *intent* of the concept $(R, (\mathfrak{A}, t))$, and the concepts form a complete lattice, called the *concept lattice* of $\mathfrak{G}$. Note that the concept extents are precisely the definable relations of $\mathfrak{G}$ (in terms of tableau queries). We refer to [4] for a description of the concept lattice under the named perspective.

Conjunctive Concept Algebras Again, we focus here on the unnamed perspective. The concept lattice of $\mathfrak{G}$ extends to a concept algebra $\mathfrak{C}(\mathfrak{G}) = (\mathfrak{B}(\mathfrak{G}), \vee, \wedge, \bot, \top, c_i, \eta_i, d_{ij}, \#)_{i,j \in \omega}$, where the underlying set $\mathfrak{B}(\mathfrak{G})$ is the set of all concepts of $\mathfrak{G}$, and $\vee$, $\wedge$, $\bot$ and $\top$ are the supremum, infimum, bottom element and top element of the concept lattice. The remaining operations are natively defined on relations, and generalize to concepts: the unary *deletion operation* c_i, defined for each $i \in \omega$, deletes the i-th column of a relation; the unary *insertion operation* η_i inserts a new column at position i into a relation, which ranges over all possible elements (as in a Cartesian product); the extent of the *equality concept* d_{ij}, for $i \leq j$, consists of all tuples $(g_0, \ldots, g_j)$ satisfying $g_i = g_j$; moreover, we have $d_{ji} = d_{ij}$ for all $i, j \in \omega$; finally, the *arity* $\#(R, (\mathfrak{A}, t))$ of a concept $(R, (\mathfrak{A}, t))$ can be defined as the arity of the relation R. We call $\mathfrak{C}(\mathfrak{G})$ the *conjunctive concept algebra* of $\mathfrak{G}$. For comparison, under the named perspective, we define $\mathfrak{C}(\mathfrak{G}) = (\mathfrak{B}(\mathfrak{G}), \vee, \wedge, \bot, \top, c_i, d_{ij}, \text{schema})$. In this case, no insertion operations are necessary, and the deletion operations are much simpler, because no index shifts are involved.

Main Objective With the additional algebraic operations, introduced by the conjunctive concept algebras, it seems possible to obtain an axiomatic characterization of the concept algebras $\mathfrak{C}(\mathfrak{G})$.

Previous Work In previous work [5], covering the named perspective, we have obtained partial results. First, an axiomatic characterization of the $\wedge$-subalgebras of the concept algebras $\mathfrak{C}(\mathfrak{G})$, by *projectional semilattices.* The projectional semilattice axioms compare surprisingly well to the axioms for cylindric algebras. Second, an axiomatic characterization of the $\bigwedge$-subalgebras of the concept algebras $\mathfrak{C}(\mathfrak{G})$, as the complete projectional semilattices.

Main Results In this work, we present analogous characterizations for the unnamed perspective. The results differ significantly, because of the different nature of algebraic operations in the unnamed perspective.

Related Work The infimum in the relation order, which we have described above, is the natural join. This is more apparent in the named perspective; the complete lattice of tables (i.e. relations in the named perspective), which we originally used [3], turns out to be the Tropashko lattice $(R(G, \text{Var}), \bowtie, \oplus)$, cf. [6]. Its operations feature in the Galois connection, e.g.

$$\text{res}_{\mathfrak{G}}(Q_1 + Q_2) = \text{res}_{\mathfrak{G}}(Q_1) \bowtie \text{res}_{\mathfrak{G}}(Q_2)\ , \tag{1}$$

$$\text{info}_{\mathfrak{G}}(T_1 \oplus T_2) = \text{info}_{\mathfrak{G}}(T_1) \times \text{info}_{\mathfrak{G}}(T_2) \tag{2}$$

holds for all queries Q_1, Q_2 and tables T_1, T_2 (and likewise for arbitrary families), where " + " and " × " refer to the sum and product of tableau queries. In later work [4], we have made some changes: (a) table headers are finite (but G may still be infinite), (b) only one empty table exists (not one per header type) and (c) an unsatisfiable query is added (cf. [1, p. 48]). The assumptions reflect in our axioms in [5] (named perspective) and in the present work (unnamed perspective). We have axiomatized closure systems, so no table suprema ($\oplus$) are involved.

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Refutational Display Calculi

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Refutation calculi are formal systems which derive the invalid formulas of a given logic [**?**]. Starting with Łukasiewicz's work [**?**], refutation calculi for various logics have been developed using different proof-theoretic formats, including Hilbert-style calculi [**?**, **?**], Gentzen-style sequent calculi [**?**, **?**] and hypersequent calculi [**?**]. For example, in [**?**] it is shown that the refutation system of Łukasiewicz for propositional classical logic augmented with the following rule is complete with respect to the classical modal logic K, where $\phi \dashv \psi$ is interpreted as "the sequent $\phi \vdash \psi$ is not valid".

$$\frac{\dashv \lambda \quad \dashv \psi \vee \theta_1 \quad \cdots \quad \dashv \psi \vee \theta_n}{\dashv \lambda \vee \Box\theta_1 \vee \cdots \vee \Box\theta_n \vee \Diamond\psi}$$

In this work, we investigate refutation calculi based on display calculi. Display calculi, introduced by Belnap [**?**], are deductive systems dealing with sequents $\Pi \vdash \Sigma$ where Π and Σ are *structures*, i.e. syntactic objects inductively defined from formulas using so-called *structural* connectives. The rules that govern the interaction between these connectives ensure that any substructure of any sequent $\Pi \vdash \Sigma$ can always be displayed, either only in precedent position or only in succedent position.

Display calculi have been applied to give cut-free presentations of many non-classical logics, including modal and substructural ones [**?**, **?**]. In this work, we focus on proper display calculi for the family of basic LE-logics, i.e. the logics canonically associated with basic normal lattice expansions of any signature [**?**, **?**, **?**, **?**]. Rather than employing display calculi in the usual role of proof construction, we investigate how they can be adapted for generating refutations – i.e., derivations of invalid sequents, or *antisequents*. Here is an example in a LE-signature containing a unary normal box operator, where $\check{\Box}$ and $\hat{\top}$ denote the structural counterparts of $\Box$ and $\top$, respectively.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\hat{\top} \dashv p}\,Ax_3 \qquad \vee_{L_2}\dfrac{\dfrac{}{q \dashv p}\,Ax_4}{p \vee q \dashv p}}{\Box(p \vee q) \dashv \check{\Box} p}\,\Box_L}{\Box(p \vee q) \dashv \Box p}\,\Box_R \qquad \dfrac{\dfrac{\dfrac{}{\hat{\top} \dashv q}\,Ax_3 \qquad \vee_{L_1}\dfrac{\dfrac{}{p \dashv q}\,Ax_4}{p \vee q \dashv q}}{\Box(p \vee q) \dashv \check{\Box} q}\,\Box_L}{\Box(p \vee q) \dashv \Box q}\,\Box_R}{\Box(p \vee q) \dashv \Box p \vee \Box q}\,\vee_R$$

The refutational framework allows certain interesting properties of lattices to be expressed through rules, such as the following result due to Whitman [**?**], which asserts that if $a \wedge b \leq c \vee d$ is valid in a lattice, then one of

$$b \leq c \vee d, \quad a \leq c \vee d, \quad a \wedge b \leq d, \quad a \wedge b \leq c$$

must also be valid. The corresponding refutational rule is the following.

$$\frac{B \dashv C \vee D \quad A \dashv C \vee D \quad A \wedge B \dashv D \quad A \wedge B \dashv C}{A \wedge B \dashv C \vee D}$$

Through proof analysis, in our calculi we generalized Whitman's theorem in the presence of normal n-ary modal operators, proving the soundness of the following 'deep nested' refutational rules, where Σ (resp. Π) is built from structural $\Box$-like (resp. $\Diamond$-like) connectives.[1]

$$\wedge_L \ \frac{\varphi \dashv \Sigma \quad \psi \dashv \Sigma \quad (\varphi \wedge \psi \dashv \Sigma[A_i]^{pre})_{i \in I} \quad (\varphi \wedge \psi \dashv \Sigma[B_i]^{pre})_{i \in I} \quad (\varphi \wedge \psi \dashv \Sigma[C_j]^{suc})_{j \in J} \quad (\varphi \wedge \psi \dashv \Sigma[D_j]^{suc})_{j \in J}}{\varphi \wedge \psi \dashv \Sigma[A_i \wedge B_i]^{pre}_{i \in I}[C_j \vee D_j]^{suc}_{j \in J}}$$

$$\frac{\Pi \dashv \varphi \quad \Pi \dashv \psi \quad (\Pi[A_i]^{pre} \dashv \varphi \vee \psi)_{i \in I} \quad (\Pi[B_i]^{pre} \dashv \varphi \vee \psi)_{i \in I} \quad (\Pi[C_j]^{suc} \dashv \varphi \vee \psi)_{j \in J} \quad (\Pi[D_j]^{suc} \dashv \varphi \vee \psi)_{j \in J}}{\Pi[A_i \wedge B_i]^{pre}_{i \in I}[C_j \vee D_j]^{suc}_{j \in J} \dashv \varphi \vee \psi} \ \vee_R$$

We introduce refutation display calculi D.LE^{r} for basic normal LE-logics. Starting from initial antisequents, these calculi derive new antisequents through the application of display rules, structural rules, and logical introduction rules. We establish soundness and completeness of the basic D.LE^{r} calculus (in any signature) exploiting proof-analysis results on derivable sequents. The resulting systems are terminating tableaux calculi for basic LE-logics: if a sequent is valid, then some branch exists in its tableau tree such that all the sequents at its terminal node are valid.

[1] We write $[A_i \wedge B_i]^{pre}$ (resp. $[C_j \vee D_j]^{suc}$) to denote that the formula $A_i \wedge B_i$ (resp. $C_j \vee D_j$) appears in the antisequent in precedent (resp. succedent) position.

On 1-variable fragments of modal predicate logics

Valentin Shehtman*

We consider normal 1-modal predicate logics without equality as they are defined in [2]. So a logic is a set of formulas containing classical validities, the axioms of **K** and closed under Modus Ponens, $\Box$- and $\forall$-introduction, and predicate substitution.

$\widehat{L}$ denotes the *Kripke completion* of a predicate logic L, i.e., the set of all formulas valid on all predicate Kripke frames (with expanding domains) validating L.

In *pure 1-variable formulas* there is a single variable x, and predicate letters are monadic. Every pure 1-variable formula A translates into a 2-modal propositional formula A_*: every atom $P_i(x)$ is replaced with the proposition letter p_i and every quantifier $\forall x$ with $\blacksquare$.

The *1-variable fragment* of a 1-modal predicate logic L is

$$L{-}1 := \{A_* \mid A \in L,\ A \text{ is pure 1-variable}\}.$$

For a 1-modal logic $\mathbf{\Lambda}$ (in the language with $\Box$), $\mathbf{\Lambda}{*}\mathbf{S5}$ denotes its fusion with **S5** (in the language with $\blacksquare$), and

$$\mathbf{\Lambda} \lrcorner\, \mathbf{S5} := \mathbf{\Lambda} * \mathbf{S5} + \Box\blacksquare p \to \blacksquare\Box p,\ [\mathbf{\Lambda}, \mathbf{S5}] := \mathbf{\Lambda} * \mathbf{S5} + \Box\blacksquare p \leftrightarrow \blacksquare\Box p.$$

$\mathbf{Q\Lambda}$ denotes the minimal predicate extension of a propositional modal logic $\mathbf{\Lambda}$, and
$\mathbf{Q\Lambda C} := \mathbf{Q\Lambda} + Ba$ where $Ba := \forall x\, \Box P(x) \to \Box\forall x\, P(x)$.

Definition 1 *The* product *of 1-frames $F_1 = (W_1, R_1)$ and $F_2 = (W_2, R_2)$ is the 2-frame $F_1 \times F_2 = (W_1 \times W_2, R_h, R_v)$, where*

$$(x, y)R_h(x', y') \iff xR_1x' \,\&\, y = y';\ (x, y)R_v(x', y') \iff x = x' \,\&\, yR_2y'.$$

A semiproduct *of F_1 and F_2 is a subframe of their product $(F_1 \times F_2) \restriction W$ such that $R_h(W) \subseteq W$.*

Definition 2 *The* product $\mathbf{\Lambda}_1 \times \mathbf{\Lambda}_2$ *of 1-modal propositional logics $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ is the 2-modal logic of the class $\{F_1 \times F_2 \mid F_1 \models \mathbf{\Lambda}_1,\ F_2 \models \mathbf{\Lambda}_2\}$.*

The semiproduct $\mathbf{\Lambda}_1 \leftthreetimes \mathbf{\Lambda}_2$ *of $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ is the 2-modal logic of the class of semiproducts of frames $F_1 \models \mathbf{\Lambda}_1$ and $F_2 \models \mathbf{\Lambda}_2$*

Proposition 3 *[9] Let $\mathbf{\Lambda}$ be a 1-modal propositional logic. Then*

$$\mathbf{\Lambda} \lrcorner\, \mathbf{S5} \subseteq \mathbf{Q\Lambda}{-}1 \subseteq \widehat{\mathbf{Q\Lambda}}{-}1 = \mathbf{\Lambda} \leftthreetimes \mathbf{S5};\ [\mathbf{\Lambda}, \mathbf{S5}] \subseteq \mathbf{Q\Lambda C}{-}1 \subseteq \widehat{\mathbf{Q\Lambda C}}{-}1 = \mathbf{\Lambda} \times \mathbf{S5}.$$

Definition 4 *A modal propositional logic $\mathbf{\Lambda}$ is called* quantifier-friendly *if $\mathbf{\Lambda} \lrcorner\, \mathbf{S5} = \mathbf{Q\Lambda}{-}1$ and* Barcan-friendly *if $\mathbf{\Lambda} \times \mathbf{S5} = \mathbf{Q\Lambda C}{-}1$*

Corollary 5 *If $\mathbf{\Lambda}$ is quantifier-friendly and $\mathbf{Q\Lambda}$ is Kripke complete, then $\mathbf{\Lambda} \lrcorner\, \mathbf{S5} = \mathbf{\Lambda} \leftthreetimes \mathbf{S5}$ ($\mathbf{\Lambda}$ is* semiproduct-matching with **S5***).*

If $\mathbf{\Lambda}$ is Barcan-friendly and $\mathbf{Q\Lambda C}$ is Kripke complete, then $[\mathbf{\Lambda}, \mathbf{S5}] = \mathbf{\Lambda} \times \mathbf{S5}$ ($\mathbf{\Lambda}$ is product-matching with **S5***).*

A propositional modal logic is *Horn axiomatizable* if it is axiomatized by modal axioms corresponding to first-order Horn formulas and (maybe) variable-free modal axioms, cf. [1].

We also consider specific modal logics:
$\mathbf{S4.1} = \mathbf{S4} + \Box\Diamond p \to \Diamond\Box p$, $\mathbf{D4.1} = \mathbf{S4} + \Box\Diamond p \to \Diamond\Box p$, $\mathbf{S4.2} = \mathbf{S4} + \Diamond\Box p \to \Box\Diamond p$,
$\mathbf{D4.2} = \mathbf{S4} + \Diamond\Box p \to \Box\Diamond p$, $\mathbf{S4.2.1} = \mathbf{S4.2} + \mathbf{S4.1}$, $\mathbf{D4.2.1} = \mathbf{D4.2} + \mathbf{D4.1}$.

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Theorem 6 *If* $\mathbf{\Lambda}$ *is Horn axiomatizable and Kripke complete, then*

(1) $\mathbf{\Lambda}$ *is product-matching with* $\mathbf{S5}$*;*

(2) $\mathbf{\Lambda}$ *is quantifier-friendly;*

(3) $\mathbf{\Lambda}$ *is semiproduct-matching with* $\mathbf{S5}$ *whenever* $\mathbf{Q\Lambda}$ *is Kripke complete.*

(1) is the well-known fact, cf. [1]. (2) is stated in [8]. (3) follows from (2) and Proposition 3.

The proof of (2) essentially uses the simplicial semantics of modal predicate logics (cf. [6], where it was called 'metaframe semantics'). Given $\mathbf{\Lambda} \rightthreetimes \mathbf{S5} \not\vdash A_*$, we first construct a 2-modal frame F separating A_* from $\mathbf{\Lambda} \rightthreetimes \mathbf{S5}$, and next a simplicial frame $\mathbb{F}$ separating A and from $\mathbf{Q\Lambda}$. The frame $\mathbb{F}$ as a simplicial set is the nerve of a groupoid on the points of F with self-isomorphism group $\mathbf{Z}$. It is important that nerves of groupoids are Kan complexes [5]; this guarantees the 'solidity' (or Beck - Chevalley) property for $\mathbb{F}$ needed for soundness.

The same method allows us to prove the following

Theorem 7 *The logics* $\mathbf{S4.1}$, $\mathbf{S4.2}$, $\mathbf{S4.2.1}$, $\mathbf{D4.1}$, $\mathbf{D4.2}$, $\mathbf{D4.2.1}$ *are quantifier-friendly.*
The logics $\mathbf{S4.1}$, $\mathbf{D4.1}$ *are Barcan-friendly.*

Hence using completeness results from [4], [7] and Proposition 3 we obtain

Corollary 8 *The logics* $\mathbf{S4.2}$, $\mathbf{D4.2}$ *are semiproduct-matching with* $\mathbf{S5}$.

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On lifting maps to the Wallman compactification

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Abstract

Given a bounded distributive lattice L, its Wallman space is the compact T_1 space $(\mathsf{W}(L), \tau_L)$ whose points are the minimal prime filters of L, with topology generated by

$$\mathcal{N}_p^{\mathsf{W}(L)} = \{F \in \mathsf{W}(L) \mid p \in F\},$$

for $p \in L$. As described in [2], this construction yields a duality between compact T_1 spaces with closed continuous maps and a suitable category of lattices.

This duality is closely connected with both Stone duality and the Omega-point duality. In Stone duality, points of spectral spaces are represented by prime filters on distributive lattices of compact-open sets, whereas in the Omega-point duality points of sober spaces are represented by completely prime filters on frames of opens. These notions do not coincide in general. The Wallman construction provides a third perspective: compact T_1 spaces are represented by minimal prime filters on their frames of open sets. Thus the Wallman perspective lies between the Stone and Omega-point ones, extending their common intuition to compact T_1 spaces, which are not fully captured by either duality.

Moreover, this construction leads to the definition of a canonical compactification of a T_1 space. Indeed, if (X, P) is a T_1 space, then applying the Wallman construction to P gives a compact T_1 space, its Wallman compactification $(\mathsf{W}(X), \tau_P)$, into which X embeds densely. In this sense, the Wallman compactification can be regarded as a T_1-analogue of the Stone–Čech compactification. Categorically, however, the analogy is not immediate. The Stone–Čech compactification shows that compact Hausdorff spaces form a reflective subcategory of Top, and an epireflective subcategory of Tych. For compact T_1 spaces, the corresponding statement is more delicate: the Wallman compactification is not functorial for arbitrary continuous maps. Thus one is led to ask under which restrictions on morphisms the Wallman compactification becomes functorial and whether, with these morphisms, compact T_1 spaces form an epireflective subcategory. This problem was investigated by Harris and Hájek; see [3, 4, 1].

From this perspective the problem can be formulated in terms of lifting continuous maps. Given a continuous map between T_1 spaces $f : (X, P) \to (Y, Q)$, one asks whether there exists a unique continuous map $\mathsf{W}(f) : (\mathsf{W}(X), \tau_P) \to (\mathsf{W}(Y), \tau_Q)$ extending f.

In general the situation is quite delicate. A continuous map between T_1 spaces need not admit any lifting to the corresponding Wallman compactifications, and even when such a lifting exists it may fail to be unique. Moreover, restricting attention to maps that admit a unique lifting does not resolve the difficulty: these maps do not form a class closed under composition, and no maximal subclass with this property exists.

For this reason the problem has been approached by identifying subclasses of continuous maps that ensure good lifting behaviour. The first such class was introduced by Harris in [5] under the name of $\mathcal{WC}$-maps. A continuous map between T_1 spaces is called a $\mathcal{WC}$-map if it admits a unique closed lifting to the Wallman compactifications. This class contains several familiar types of maps, including the canonical embeddings of a T_1 space into its Wallman compactification, closed surjections, and continuous maps whose codomain is a normal Hausdorff space.

Harris later showed in [1] that a larger class of maps can be considered, namely the class of $\mathcal{WO}$-maps, which properly contains the class of $\mathcal{WC}$-maps and for which the Wallman compactification becomes an epireflective functor.

We say that $f : (X, Q) \to (Y, P)$ is a $\mathcal{WO}$*-map* if for every finite open cover $\mathcal{V}$ of Y, there exists a finite open cover $\mathcal{U}$ of X such that for every $U \in \mathcal{U}$, there exists some $V \in \mathcal{V}$ such that

$$U \prec_f V ,$$

that is

$$\overline{f[C]} \subseteq V$$

for every closed subset $C \subseteq U$.

A notable feature of $\mathcal{WO}$-maps is that they admit a relatively simple internal characterization. No comparable description is known for $\mathcal{WC}$-maps. The problem of finding such an internal characterization was posed by Harris in [1] and, to the best of our knowledge, has remained open since the 1970s.

The results presented in the talk are the following:

- We introduce a contravariant adjunction between the category $\mathsf{T}_{1\mathcal{WO}}$ of T_1 spaces with $\mathcal{WO}$-maps and the category $\mathsf{DL}_{\mathcal{WO}}$ of distributive bounded lattices equipped with morphisms corresponding to $\mathcal{WO}$-maps. Under this adjunction, a T_1 space is associated with the lattice of its open sets, while a lattice is sent to its Wallman compactification, that is, the space of its minimal prime filters equipped with a Stone-type topology. On morphisms, a $\mathcal{WO}$-map is sent to the corresponding morphism between the lattices of open sets given by taking preimages. Conversely, a morphism $i : P \to Q$ in $\mathsf{DL}_{\mathcal{WO}}$ is sent to the continuous map that assigns to each minimal prime filter F of Q the unique minimal prime filter of P contained in its preimage $i^{-1}[F]$.
- We solve the problem of providing an internal characterization of $\mathcal{WC}$-maps posed by Harris. To this end we prove that a continuous map

  $$f : (X, P) \to (Y, Q)$$

  is a $\mathcal{WC}$-map if and only if the following conditions hold:
  - for every pair of open sets $V_1, V_2 \subseteq Y$ covering Y, there exist open sets $U_1, U_2 \subseteq X$ covering X such that $U_1 \prec_f V_1$ and $U_2 \prec_f V_2$;
  - for every open set $V \subseteq Y$, if U_1 and U_2 are open subsets of X such that $U_i \prec_f V$ for $i = 1, 2$, then $U_1 \cup U_2 \prec_f V$.

This is joint work with Matteo Viale.

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The compactness theorem for topos-valued models

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Models of coherent theories internal to Grothendieck toposes capture many classical notions. For example,

- if T is the theory of local rings, and X is a topological space, then a $Sh(X)$-valued model is a locally ringed space with underlying space X, that is, a family of local rings, "continuously parametrized by X".
- If T is the theory of fields, then a $\mathbf{Set}^{\circlearrowleft}$-valued model is a difference field.
- If T is any theory, and B is a complete Boolean algebra, then a $Sh(B, \tau_{can})$-valued model (where τ_{can} is the canonical topology) is the same as a B-valued model.

These examples, as well as applications like [**?**], motivate topos-valued model theory.

It is convenient to identify coherent theories with small coherent categories, $\mathcal{E}$-valued models with $\mathcal{E}$-valued coherent functors, and homomorphisms with natural transformations (see [**?**]).

A possible question is whether the compactness theorem applies to $\mathcal{E}$-valued models. The answer depends on what we mean by the compactness theorem:

1. Every (finitely) consistent theory has an $\mathcal{E}$-model. This is trivial: given a **Set**-model M, we can simply take the sheafification of the "constant M" presheaf. In categorical terms, it is the composite $\mathcal{C} \xrightarrow{M} \mathbf{Set} \xrightarrow{!^*} \mathcal{E}$.
2. The Łoś lemma holds, i.e. $\mathbf{Coh}(\mathcal{C}, \mathcal{E})$ is closed under ultraproducts in the functor category $[\mathcal{C}, \mathcal{E}]$. This is (usually) false, I will give several counterexamples. (However, it is true for presheaf toposes, where ultraproducts are computed pointwise.)
3. Every type can be realized in a model. In positive model theory, an x-type is a prime filter on the Lindenbaum-Tarski algebra of positive existential formulas in context x. In the language of categorical logic, it is a prime filter $Sub_{\mathcal{C}}(x) \to 2 = Sub_{\mathbf{Set}}(1)$. In the setting of $\mathcal{E}$-valued model theory, it is a lattice homomorphism $Sub_{\mathcal{C}}(x) \to Sub_{\mathcal{E}}(1)$. By considering slices, it suffices to treat homomorphisms $Sub_{\mathcal{C}}(1) \xrightarrow{p} Sub_{\mathcal{E}}(1) \subseteq \mathcal{E}$. We say that p is realized by the model (coherent functor) $M : \mathcal{C} \to \mathcal{E}$, if $p \leq M|_{Sub_{\mathcal{C}}(1)}$, that is, if we have a (necessarily unique) natural transformation in

$$\begin{array}{ccc} Sub_{\mathcal{C}}(1) & \xrightarrow{\quad p \quad} & \mathcal{E} \\ & \Downarrow & \nearrow_{M} \\ \downarrow & & \\ \mathcal{C} & & \end{array}$$

A topos $\mathcal{E}$ *can realize types*, if the above lifting property holds wrt. any $\mathcal{C} \hookleftarrow Sub_{\mathcal{C}}(1) \xrightarrow{p} \mathcal{E}$.

I will sketch the proof of the following claims:

- Every presheaf topos can realize types.

- If we knew that $Sh(L, \tau_{coh})$ can realize types for any complete distributive lattice L (where τ_{coh} is the finite union topology), then it would follow that every Grothendieck topos can realize types.
- If B is a complete Boolean algebra, then $Sh(B, \tau_{coh})$ can realize types. And more generally, if Q is a well-founded forest, and the lattice L is a retract of $Up(Q)$, then $Sh(L, \tau_{coh})$ can realize types. (The first claim is in [**?**], the second is a generalization, using prime products.)

A first-order extension of multi-lingual sequent calculus

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Abstract

In [1] Abramsky introduced the program of 'domain theory in logical form', whose aim is to provide logical systems that capture the structural behaviour of domains used in denotational semantics. In this approach, topological and order-theoretic structures are represented syntactically by logical calculi, allowing domain-theoretic constructions to be studied at the level of proof systems.

Kegelmann [3] showed that a finitary proof-theoretic framework for representing stably compact spaces and their associated semantic structures is provided by the multi-lingual sequent calculus (**MLS**).

The syntax of **MLS** is based on token algebras, that is, distributive lattices generated from a set of tokens using the connectives $\wedge$, $\vee$, together with the constants $\top$ and $\bot$. Intuitively, tokens represent semantic relations between opens and Scott-open filters of the underlying space.

A sequent in **MLS** has the form

$$\Gamma \vdash \Delta$$

where Γ and Δ are finite sets of formulas generated from tokens. The intended reading is that the conjunction of the formulas in Γ entails the disjunction of the formulas in Δ. The consequence relation is defined by the following inference rules:

$$\frac{}{\bot \vdash}\,(L\bot) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot}\,(R\bot)$$

$$\frac{\Gamma \vdash \Delta}{\top, \Gamma \vdash \Delta}\,(L\top) \qquad \frac{}{\vdash \top}\,(R\top)$$

$$\frac{\phi, \psi, \Gamma \vdash \Delta}{\phi \wedge \psi, \Gamma \vdash \Delta}\,(L\wedge) \qquad \frac{\Gamma \vdash \Delta, \phi \qquad \Gamma \vdash \Delta, \psi}{\Gamma \vdash \Delta, \phi \wedge \psi}\,(R\wedge)$$

$$\frac{\phi, \Gamma \vdash \Delta \qquad \psi, \Gamma \vdash \Delta}{\phi \vee \psi, \Gamma \vdash \Delta}\,(L\vee) \qquad \frac{\Gamma \vdash \Delta, \phi, \psi}{\Gamma \vdash \Delta, \phi \vee \psi}\,(R\vee)$$

In [5], Moshier further shows that the patch construction for stably compact spaces admits a natural proof-theoretic interpretation within **MLS**. From the logical point of view, the patch construction amounts to freely adding opposites to a given logic, that is, introducing tokens that satisfy Gentzen-style rules for negation. In this sense, the patch construction can be understood as the universal way of extending the positive fragment represented by **MLS** with a form of negation.

In this work we investigate how this propositional framework can be extended to a first-order setting. Our approach is based on the theory of hyperdoctrines from categorical logic. See [2], [4], [6] for a comprehensive discussion. Following the standard methodology, we identify the fibres of the doctrine, define the appropriate reindexing operations, and study the interpretation of quantifiers as adjoint functors between fibres. In particular, we analyze the conditions under which the Beck–Chevalley property holds in this setting. This perspective clarifies the role of logical structure in **MLS** and opens the way toward a first-order version of domain theory in logical form.

Keywords— non-classical logics, hyperdoctrines, sequent calculus

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On varieties of modular lattices with complementation*

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A *lattice with complementation* is a complemented lattice equipped with a fixed complementation $x \mapsto x'$. In this talk we focus on modular lattices with complementation and in particular on those which satisfy De Morgan's laws. A simple example is $\mathbf{M}_n = (M_n, \vee, \wedge, ', 0, 1)$, the lattice $M_n = (M_n, \vee, \wedge)$ with complementation defined by the cycle $(a_0\ a_1\ \dots\ a_{n-1})$; see Fig. 1.

Among other results, we axiomatize the varieties $\mathrm{V}(\mathbf{M}_n)$ for every integer $n \geq 3$. It is no surprise that, to some extent, we rely on Jónsson's axiomatization of the varieties of lattices generated by the M_n's (see [3] or [2, Sect. 5.4]).

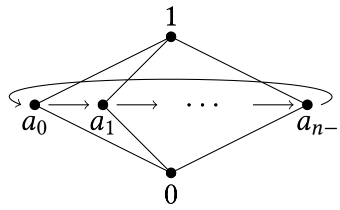

Figure 1: $\mathbf{M}_n$

We first show that the variety $\mathrm{V}(\mathbf{M}_3)$ is axiomatized, relative to the variety $\mathcal{M}_{DM}$ of modular lattices with complementation satisfying De Morgan's laws, exactly by the identity that axiomatizes the lattice variety $\mathrm{V}(M_3)$ relative to modular lattices, i.e.

$$u \wedge (x \vee y) \wedge (x \vee z) \wedge (y \vee z) \leq (u \wedge x) \vee (u \wedge y) \vee (u \wedge z).$$

However, for $n > 3$, Jónsson's identities axiomatizing the lattice variety $\mathrm{V}(M_n)$ are satisfied also by (subdirectly irreducible) lattices with complementation that do not belong to $\mathrm{V}(\mathbf{M}_n)$. We therefore need another identity that singles out $\mathbf{M}_n$. To this end, for every integer $n \geq 3$, we consider the term

$$\tau_n(x) = \bigwedge_{2 \leq k < n} \left(x \vee x^{(k)}\right),$$

where the derivative-like notation $x^{(k)}$ is shorthand for $x''^{\cdots\prime}$. We prove that $\mathrm{V}(\mathbf{M}_n)$ is axiomatized, relative to $\mathcal{M}_{DM}$, by the identities

$$x_0 \wedge \bigwedge_{1 \leq i < j \leq n} (x_i \vee x_j) \leq \bigvee_{1 \leq i \leq n} (x_0 \wedge x_i)$$

and

$$x \wedge (\tau_n(y) \vee z) \approx (x \wedge \tau_n(y)) \vee (x \wedge z).$$

Using the $\mathbf{M}_n$'s, we also show that the variety $\mathcal{M}_{DM}$ has uncountably many subvarieties.

*Part of joint work with Václav Cenker, Ivan Chajda and Helmut Länger

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Makkai's lost proof of projectivity of N in the free topos

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Like many constructive systems, *intuitionistic higher-order logic* (IHOL), the logic of elementary toposes, enjoys various meta-theorems saying that existence proofs can be distilled into constructions. In particular, a strong "Rule of Countable Choice" is admissible: if IHOL proves a statement "$\forall x{:}N,\ \exists y{:}A,\ \varphi(x,y)$", then there is some IHOL-definable function $N \to A$ witnessing this. Viewed categorically, this says that the natural numbers object in the free topos is projective: every cover $\{\, x{:}N,\ y{:}A \mid \varphi(x,y) \,\} \twoheadrightarrow N$ admits some splitting.

This result was announced around 1980 by several authors — at least by Friedman and Scedrov and Makkai, as noted in [LS86, §II.20], based on Troelstra and Hayashi's similar result for second-order arithmetic [Tro73; Hay77] — but their proofs never made it to print, and the result has remained in purgatory ever since, on the record but without proof [Oos06].

We present a full exposition of this result — based loosely on the unpublished notes of Makkai [Mak80] (with permission of the author), but with an even more thoroughly categoricalised approach. The development turns out to give a scenic tour through a range of topics in categorical logic:

1. an algebraic stratification of elementary ses into *r-ranked partial toposes*, analogous to the finite-rank fragments of IHOL;
2. a dip into the theory of *arithmetic universes* [Mai10] to provide and analyse *internal* free ranked toposes, in particular giving a slightly novel AU-based analysis of Gödel-numbering to conclude the enumerability of free internal structures;
3. construction of *internal universes* in toposes, to internalise the 'standard model' of the finite-rank fragments (and using the language of fibred/indexed categories to communicate between external and internal ranked toposes);
4. finally, putting these together into a reflection theorem, yielding the overall result: N is projective in the free topos, and hence the rule of countable choice is admissible for IHOL.

Both the talk and the associated article (which will be available as a preprint by the time of the conference) aim to give a self-contained and elementary exposition, accessible to essentially anyone acquainted with elementary toposes and their logic.

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Algebraic proof theory for distributive sequent calculi and GBI-algebras

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We relativize the substructural hierarchy to the setting of the distributive sequent calculus **GBI** and GBI-algebras. Further, we show that an extension of the sequent calculus **GBI** by structural rules is analytic if and only if it satisfies a stronger form of analyticity, namely strong atomic analyticity, if and only if the corresponding variety of GBI-algebras GBI is closed under MacNeille completions.

A *GBI-algebra* (generalized bunched implication algebra) is a structure $(A, \wedge, \vee, \cdot, \backslash, /, \to, 1, \top)$ with a residuated lattice reduct and a Brouwerian algebra reduct. As a result, GBI-algebras have residuals for multiplication and meet and are the equivalent algebraic semantics for the distributive sequent calculus **GBI**. Commutative GBI-algebras (or BI-algebras) have been used in modeling dynamic memory allocation and pointer arithmetic in computer science.

In [1] each equation, which can serve as an axiom for an extension, is placed at a specific level of the *substructural hierarchy* based on the iterations of *positive* and *negative* connectives in it. We relativize this hierarchy to the distributive setting of **GBI**, and we denote the n-th positive (resp. negative) level of the hierarchy by $\mathcal{P}_n$ (resp. $\mathcal{N}_n$).

A *distributive residuated frame* [2] is a structure $\mathbf{W} = (W, W', N, \varepsilon, \circ, \curlywedge, \backslash\!\!\backslash, /\!\!/, \leftthreetimes, \rightthreetimes)$ where $N \subseteq W \times W'$ is a relation satisfying the conditions

$$x \circ y N z \text{ iff } y N x \backslash\!\!\backslash z \text{ iff } x N z /\!\!/ y \quad \text{and} \quad x \curlywedge y N z \text{ iff } y N x \leftthreetimes z \text{ iff } x N z \rightthreetimes y.$$

The *dual algebra* $\mathbf{W}^{+}$ of such a frame is a complete GBI-algebra. Also, for every GBI algebra $\mathbf{A}$ there is a natural frame $\mathbf{W_A}$ associated to it and its dual algebra $\mathbf{W_A^{+}}$ is the Dedekind-MacNeille completion of $\mathbf{A}$. For a set R of structural rules we denote by $\mathbf{GBI}_R$ the extension of **GBI** by R, and for a cardinal κ we denote by $\mathbf{GBI}_R^\kappa$ the extension of $\mathbf{GBI}_R$ by κ-infinitary versions of the connectives for $\wedge$ and $\vee$ and infinitary versions of the associated $\wedge, \vee$-rules.

Given a set R of structural rules and a cardinal κ, we say that the system $\mathbf{GBI}_R^\kappa$ is a *conservative extension* (respectively *atomic conservative extension*) of **GBI** provided that if S is a set of sequents (resp. *atomic sequents* i.e., sequents consisting of atomic formulas), and s is a sequent in the language of **GBI**, $S \vdash_{\mathbf{GBI}_R^\kappa} s$ implies that $S \vdash_{\mathbf{GBI}_R} s$.

A sequent calculus is *strongly analytic* if, for every proof of a sequent from a set of atomic sequents closed under cut, there is a cut-free proof with the subformula property.

The calculus **GBI** can be extended with sequent rules to provide extensions of the logic. Rules that do not mention logical connectives (and only pertain to how the metalogic structure behaves) are known as structural rules. The latter naturally correspond to structural quasiequations in the sense that the extensions of the logic are algebraizable with respect to the subquasivarieties axiomatized by the quasiequations. We further identify a combinatorial condition on a structural quasiequation (based on the lack of cycles in an associated graph) that we refer to as *acyclicity.*

Theorem 1. *Every equation in the $\mathcal{N}_2$ level of the distributive substructural hierarchy is equivalent to a finite set of structural quasiequations in the language of GBI-algebras. Further every acyclic $\mathcal{N}_2$-equation is equivalent to a finite set of analytic structural rules.*

As in [1] this equivalence allows us to easily identify, by the (combinatorial) condition of acyclicity, a class of equations in the $\mathcal{N}_2$ level which correspond to analytic structural rules.

Theorem 2. *Analytic quasiequations are preserved under MacNeille completions. In other words, if a GBI-algebra* $\mathbf{A}$ *satisfies a quasiequation, then so does* $\mathbf{W}_{\mathbf{A}}^{+\!\!\!\!+}$.

This result holds for the variety of GBI-algebras but not for distributive residuated lattices in general. The Dedekind-MacNeille completion is known to not preserve distributivity of lattices; however, as stated in [2] a consequence of the existence of the meet residual is that distributivity in GBI-algebras is preserved by the MacNeille completion.

Theorem 3. *If* R *is a set of analytic structural rules, then for every cardinal* κ *the sequent calculus* $\mathbf{GBI}_R^\kappa$ *is strongly analytic.*

This establishes a connection between analytic structural rules and strong analyticity of the infinitary sequent calculus. The following theorem completes the equivalence of analytic extensions and closure under MacNeille completions.

Theorem 4. *Let* R *be a set of structural rules. If* $\mathbf{GBI}_R^\omega$ *is an atomic conservative extension of* $\mathbf{GBI}_R$*, then* R *is equivalent to a set of analytic structural rules.*

As a result, we have the following theorem in the same manner as [1], connecting $\mathcal{N}_2$-equations, structural rules, and extensions of the sequent calculus **GBI**.

Theorem 5.

1. *Every* $\mathcal{N}_2$ *equation is equivalent to a set of structural rules (and quasiequations).*
2. *For any set* R *of structural rules, the following are equivalent:*
   - *(i)* R *is equivalent to a set of acyclic structural rules.*
   - *(ii)* R *is equivalent to a set of analytic structural rules.*
   - *(iii)* $R^\bullet$ *(the set of quasiequations interpreting* R*) is preserved by MacNeille completions.*
   - *(iv)* $\mathbf{GBI}_R^\kappa$ *is a conservative extension of* $\mathbf{GBI}_R$ *for every cardinal* κ.
   - *(v)* R *is equivalent to a set of rules to* R' *such that* $\mathbf{GBI}_{R'}^\kappa$ *is strongly analytic for every cardinal* κ.

   If R *implies left weakening (integrality), then all the above hold.*
3. *For any set* E *of* $\mathcal{N}_2$*-equations, the following are equivalent:*
   - *(i)* E *is equivalent to a set of acyclic quasiequations.*
   - *(ii)* E *is equivalent to a set of analytic quasiequations.*
   - *(iii) The variety* GBI_E *(the subvariety of* GBI *satisfying* E*) admits MacNeille completions.*
   - *(iv) The variety* GBI_E *admits completions.*

   If E *implies integrality (left weakening), then all the above hold.*

Additionally, the results can be extended to the setting of nGBI-algebras, which are the non-necessarily associative generalizations of GBI-algebras, and to the calculus with an arbitrary constant 0 included in the language. In fact without 0 in the language the analytic quasiequations are equivalent to simple equations.

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Concerning Uniform Menger Quotients

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Abstract

The Menger property is a classical covering property in topology and plays an important role in the theory of selection principles. In point–free topology, covering properties may be studied in the setting of frames and locales, where topological notions are expressed algebraically. Bayih, Dube, and Ighedo [3] introduced and studied Menger and weakly Menger locales in point-free topology. Nearness frames provide a framework for studying proximity-like and uniform structures in this setting. In particular, morphisms between nearness frames allow one to investigate how covering properties behave under frame-theoretic constructions such as quotients.

A subset A of a space X is said to be *relatively Menger* [2] if for every sequence $\{C_n : n \in \mathrm{N}\}$ of open covers of X, there exists, for each n, a finite set $J_n \subseteq C_n$ such that

$$A \subseteq \bigcup_{n\in\mathrm{N}} \bigcup J_n.$$

Motivated by the above notion of relatively Menger subsets, in this work we introduce and study in nearness frames a uniform analogue of relatively Menger subsets. This is not an outrageous idea, for instance, in [4] bounded frame quotients were introduced in frames using bounded subsets in classical topology. Also, in [5], a notion of remote subsets was adapted to locales.

We define the notion of a *uniformly Menger quotient* and establish several characterizations and structural properties.

Let (L, μ) and (M, ν) be nearness frames and let

$$h : (L, \mu) \to (M, \nu)$$

be a uniform frame homomorphism. We refer a reader to [1] for a theory of nearness frames.

Definition 0.1. *Let $h : (L, \mu) \to (M, \nu)$ be a uniform frame quotient. We say that h is* uniformly Menger *if for every sequence*

$$\{C_n : n \in \mathrm{N}\}$$

of uniform covers of L, there exist finite subsets $D_n \subseteq C_n$ such that

$$\bigvee_{n\in\mathrm{N}} \bigvee h[D_n] = 1_M.$$

Thus finitely many elements can be selected from each cover so that the union of their images under h forms a cover of the codomain frame M. This definition extends the classical Menger covering principle to the setting of uniform frame quotients.

We outline our first result showing equivalent descriptions of uniformly Menger frame quotients.

Theorem 0.2. *Let $h : (L, \mu) \to (M, \nu)$ be a uniform frame quotient. The following conditions are equivalent.*

1. *h is uniformly Menger.*

2. *For every sequence* $\{C_n : n \in \mathrm{N}\}$ *of directed uniform covers of* L*, there exist finite subsets* $D_n \subseteq C_n$ *such that*
$$\bigvee_{n\in\mathrm{N}} \bigvee h[D_n] = 1_M.$$

3. *For every sequence of uniform covers of* L*, there exists a sequence of finite selections whose images under* h *cover* M*.*

Another characterization is in terms of σ-filters.

Proposition 0.3. *Let* (L, μ) *be a nearness frame and* $h : (L, \mu) \to (M, \nu)$ *be a uniform quotient of* (L, μ)*. The following statements are equivalent:*

1. h *is uniformly Menger.*

2. *For every* σ*-ultrafilter* F *in* M*,* $h^{-1}(F) \cap \bigcup_{n\in\mathrm{N}} C_n \neq \emptyset$ *for every countable collection* $\{C_n : n \in \mathrm{N}\} \subseteq \mu$*.*

3. *For each* σ*-filter* K *in* M*,* $h^{-1}(K) \cap \bigcup_{n\in\mathrm{N}} D_n \neq \emptyset$ *for every countable collection* $\{D_n : n \in \mathrm{N}\} \subseteq \mu$*.*

4. *For every* σ*-filter* R *in* M *and for each countable collection* $\{C_n : n \in \mathrm{N}\}$ *of subsets of* L*,*
$$\left\{\bigvee\{c \in C_n : c \wedge x = 0\} : x \in h^{-1}(R)\right\} \notin \mu$$
for every n*.*

5. *Every* σ*-filter* T *in* M *is contained in some* σ*-filter* K *in* M *satisfying that* $h^{-1}(K) \cap \bigcup_{n\in\mathrm{N}} C_n \neq \emptyset$ *for every countable collection* $\{C_n : n \in \mathrm{N}\} \subseteq \mu$*.*

We now compare the uniformly Menger property with some other uniform quotients.

Definition 0.4. *A uniform frame quotient* $h : (L, \mu) \to (M, \nu)$ *is called* uniformly Rothberger *if for every sequence of uniform covers* $\{C_n : n \in \mathrm{N}\}$ *of* L *there exist elements* $d_n \in C_n$ *such that*
$$\bigvee_{n\in\mathrm{N}} h(d_n) = 1_M.$$

Definition 0.5. *A uniform frame quotient* $h : (L, \mu) \to (M, \nu)$ *is* pre-Lindelöf *if every uniform cover of* L *admits a countable subfamily whose images under* h *cover* M*.*

Proposition 0.6. *Uniformly Menger quotients are between uniformly Rothberger quotients and pre-Lindelöf uniform quotients.*

It is of interest to extend this work by exploring how uniformly Menger quotients relate with pseudocompactness and also consider coproducts.

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The Deducibility Problem for Certain Extensions of the Non-Associative Full Lambek Calculus is Undecidable

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Abstract

We show that every logic in the interval between **nFL** and **FL** has an undecidable deducibility problem thus extending the main result of [3]. This is accomplished by showing that the word problem for every variety between the varieties of RL and nRL is undecidable and exploiting the corresponding algebraization results.

The nonassociative full Lambek calculus **nFL** is a generalization of the full Lambek calculus **FL** where the sequent comma lacks associativity. While provability of **nFL** is decidable in polynomial space, in [3] it was shown that **nFL** has an undecidable deducibility problem. This was done by encoding the halting problem for 2-tag systems (which is undecidable [4]) into the $\{\cdot, \vee\}-$fragment of **nFL**.
A *not necessarily associative residuated lattice* (nRL) is an algebraic structure $\mathbf{R} = (R, \vee, \wedge, \cdot, \backslash, /, 1)$, where $(R, \vee, \wedge)$ is a lattice, $(R, \cdot, 1)$ is a unital groupoid and the *law of residuation* holds: For all $x, y, z \in R$,

$$x \cdot y \leq z \iff y \leq x \backslash z \iff x \leq z/y$$

where $\leq$ is the induced lattice order. Furthermore, a *residuated lattice* (RL) is an algebraic structure as above where $\cdot$ is associative. Note that nonassociative residuated lattices form a variety, nRL, and residuated lattices also form a variety, RL. It is well known that pointed residuated lattices[1] (FL-algebras) are the equivalent algebraic semantics for **FL** in the sense of Blok and Pigozzi [2], pointed nonassociative residuated lattices (nFL-algebras) are the equivalent algebraic semantics for **nFL**, and that each form a variety. This equivalence allows us to exploit the fact that the deducibility problem of an axiomatic extension of **nFL** is undecidable iff the corresponding variety of nFL-algebras has an undecidable quasiequational theory.

A 2-tag system consists of a finite alphabet $\mathcal{T} := \{a_1, ..., a_n\}$ and a production function $\tau : \mathcal{T} \to \mathcal{T}^*$. We define the 1-step computation relation $\rightsquigarrow_1$ by $w \rightsquigarrow_1 w'$ if $|w| \geq 2$, $w = a_i a_j u$, and $w' = u\tau(i)$; i.e., we delete the first two letters $a_i a_j$ of w and we append the word $\tau(i)$ in the end. Furthermore, we let $\rightsquigarrow$ be the reflexive and transitive closure of $\rightsquigarrow_1$. We say that w is *accepted* if there is some $a \in \mathcal{T} \cup \{\varepsilon\}$ for which $w \rightsquigarrow a$ and we let $\mathrm{Acc}(\tau)$ denote the set of accepted words.

We amend the simulation of 2-tag systems provided in [3] by defining a rewrite system R_τ with ordering $\leq_{R_\tau}$ on a special set of $\{\cdot, \vee, 1\}$-terms which simulates computations in $\tau : \mathcal{T} \to \mathcal{T}^*$ by means of a set of instructions (as inequations) P. In particular, our special set consists of finite joins of right associated terms with respect to $\cdot$ over a chosen alphabet Σ. We represent the word $w \in \mathcal{T}^*$ in R_τ by $\overrightarrow{ewX}$, where $^\rightarrow$ takes words in Σ^* and right associates them with respect to $\cdot$. The system verifies that appending and deletion were applied correctly by splitting, via $\vee$, into parallel auxiliary computations from the main computation. We say that a term is *accepted* if it reaches $eX \vee \bigvee_{i=0}^{n} eA$ via $\leq_{R_\tau}$ for some $n \in \mathbb{N}$ where only successful auxiliary computations reach eA. We denote the set of accepted terms by $\mathrm{Acc}(R_\tau)$.

[1]Pointed residuated lattices are residuated lattices with a distinguished element 0. However, the constant 0 is not germane to our results.

Theorem 1. *Let $\tau : \mathcal{T} \to \mathcal{T}^*$ be a 2-tag system. Then for each $w \in \mathcal{T}^*$, $w \in \mathrm{Acc}(\tau)$ iff $\overrightarrow{ewX} \in \mathrm{Acc}(R_\tau)$.*

For a given 2-tag system $\tau : \mathcal{T} \to \mathcal{T}^*$ and a $\{\cdot, \vee, 1\}$-term u over Σ, we define the following quasiequation, which allows us to encode $\mathrm{Acc}(R_\tau)$ in the theory of nRL :

$$\mathrm{acc}_{R_\tau}(u) := (\&\mathrm{P} \implies u \leq eX \vee eA)$$

where &P is the conjunction of the instructions (inequations) in P.

We define the structure $\mathbf{W}_{R_\tau} := (\Sigma^*, \Sigma^* \times \Sigma^*, N_{R_\tau}, \circ, \varepsilon)$ where $(\Sigma^*, \circ, \varepsilon)$ is the free monoid over Σ and $xN_{R_\tau}(y, z) \iff \overrightarrow{yxz} \in \mathrm{Acc}(R_\tau)$ for $x, y, z \in \Sigma^*$. We note that

$$xsN_{R_\tau}(y, z) \iff sN_{R_\tau}(yx, z) \iff xN_{R_\tau}(y, sz)$$

for each $x, y, z, s \in \Sigma^*$. This verifies that $\mathbf{W}_{R_\tau}$ is a residuated frame in the sense of [1]. In general from a residuated frame $\mathbf{W}$, we get a residuated lattice $\mathbf{W}^+$ as shown in [1]. For a non-accepted term u, $\mathbf{W}^+_{R_\tau}$ witnesses the failure of the quasiequation $\mathrm{acc}_{R_\tau}(u)$.

Theorem 2. *Let $\mathcal{V}$ be a subvariety of* nRL *containing $\mathbf{W}^+_{R_\tau}$ for some 2-tag system $\tau : \mathcal{T} \to \mathcal{T}^*$. Then $u \in \mathrm{Acc}(R_\tau)$ if and only if $\mathcal{V} \models \mathrm{acc}_{R_\tau}(u)$.*

In the proof of the forward direction of Theorem 2, the fact that our rewrite system does not build in associativity of $\cdot$ is essential since we show that computations of accepted terms are sound in **nRL**. However, for the backwards direction we construct $\mathbf{W}^+_{R_\tau}$ which is a residuated lattice and in particular associative. We get the following result.

Theorem 3. *If $\mathcal{V}$ is a subvariety of* nRL *containing $\mathbf{W}^+_{R_\tau}$ where $\tau : \mathcal{T} \to \mathcal{T}^*$ is a 2-tag system so that membership in $\mathrm{Acc}(\tau)$ (and therefore membership in $\mathrm{Acc}(R_\tau)$) is undecidable, then $\mathcal{V}$ has an undecidable word problem/quasiequational theory. In particular, any variety in the interval from* RL *to* nRL *has undecidable word problem/quasiequational theory.*

Consequently, every logic in the interval between **nFL** and **FL** has an undecidable deducibility problem.

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Amalgamation for finitely-generated varieties of idempotent semilinear residuated lattices.

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The amalgamation property plays an important role in the areas of model theory and algebra. Amalgamation for a variety of residuated lattices—the equivalent algebraic semantics of the Full Lambek calculus—corresponds to a version of interpolation for the corresponding logic (see [1]). We note that [4] provides a criterion for establishing amalgamation; if a variety $\mathcal{V}$ has the congruence extension property and the class $\mathcal{V}_{\mathrm{FSI}}$ of its finitely subdirectly irreducibles is closed under subalgebras, then $\mathcal{V}$ has the amalgamation property if and only if $\mathcal{V}_{\mathrm{FSI}}$ has the one-sided amalgamation property. In the context of idempotent semilinear varieties, the class of finitely subdirectly irreducibles of $\mathcal{V}$ is precisely the class of all chains in $\mathcal{V}$.

Given a class $\mathcal{K}$ of similar algebras, a *span in* $\mathcal{K}$ is an ordered pair $(\varphi_1 : \mathbf{A} \to \mathbf{B}, \varphi_2 : \mathbf{A} \to \mathbf{C})$ of embeddings $\varphi_1 : \mathbf{A} \to \mathbf{B}$ and $\varphi_2 : \mathbf{A} \to \mathbf{C}$ where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$. An *amalgam of* (φ_1, φ_2) *in* $\mathcal{K}$ is an ordered pair $(\psi_1 : \mathbf{B} \to \mathbf{D}, \psi_2 : \mathbf{C} \to \mathbf{D})$ of embeddings where $\mathbf{D} \in \mathcal{K}$ and $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$. The class $\mathcal{K}$ is said to have the *amalgamation property* if every span in $\mathcal{K}$ has an amalgam in $\mathcal{K}$. The *one-sided amalgamation property* is defined similarly except we ask that ψ_2 is merely a homomorphism (as opposed to an embedding). If $\mathcal{V}$ is a variety, its *amalgamation closure* is the smallest variety containing $\mathcal{V}$ with the amalgamation property, whenever it exists.

To investigate the amalgamation property of a class, it is key to understand prominent features of the structure of the algebras in the class. A correspondence theorem for all idempotent residuated chains is provided in [2] in terms of enhanced monoidal preorders. For a residuated lattice $\mathbf{A}$, we set $x^\ell = 1/x$ and $x^r = x\backslash 1$ and say $\mathbf{A}$ is *quasi-involutive* if $\mathbf{A}$ satisfies the equation $x = x^{\ell r} \wedge x^{r\ell}$. In [3] it is shown that the study of idempotent residuated chains can be reduced to that of their quasi-involutive skeleton. In this work, we extend the results of [5] to the quasi-involutive setting and further to the general setting of idempotent semilinear algebras.

For brevity, let qiS_2 denote the variety of all semilinear quasi-involutive idempotent residuated lattices and note the the class qiRL_2^c of FSI's of qiS_2 consists of the quasi-involutive idempotent residuated chains. We define a relation on a quasi-involutive idempotent residuated chain $\mathbf{A}$ as follows: for $x, y \in A$, we write $x \equiv_1 y$ if $\langle x \rangle = \langle y \rangle$ where $\langle x \rangle$ denotes the 1-free subalgebra of $\mathbf{A}$ generated by x; also, by $\langle x \rangle_1$ we denote the subalgebra of $\mathbf{A}$ generated by x. We refer to equivalence classes of the form $[x]_{\equiv_1}$ as *blocks*.

Theorem 1. *If* $\mathbf{A}$ *is a quasi-involutive idempotent residuated chain, then its corresponding enhanced monoidal preorder* $\mathbf{P}_A$ *is an ordinal sum of its blocks.*

It turns out, there are four distinct types of blocks $(\mathbf{N}, \mathbf{B}, \mathbf{U}, \mathbf{D})$ depending on the freedom of x in the generation of $\langle x \rangle$. In general, the representation of quasi-involutive idempotent residuated chains via blocks is too coarse to understand when the amalgamation closure exists. In turn, we identify a necessary condition on how blocks are ordered for an 1-amalgam to exist.

Definition 2. A class $\mathcal{K}$ of quasi-involutive idempotent residuated chains is called AP-*consistent* if the following two conditions hold:

1. for every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ with proper nontrivial isomorphic subalgebras $\mathbf{A}' \leq \mathbf{A}$, $\mathbf{B}' \leq \mathbf{B}$ in $\mathcal{K}$ (i.e. $\mathbf{A}' \cong \mathbf{B}'$) and for every $x \in A \setminus A'$, $y \in B \setminus B'$ with the property that $x^\star \in A'$ and $y^\star \in B'$ are identifiable ($z^\star = z^\ell \wedge z^r$) under the isomorphism $\mathbf{A}' \cong \mathbf{B}'$, then $\langle x \rangle_1 \cong \langle y \rangle_1$,

2. if C is a common block among both $\mathbf{A}$ and $\mathbf{B}$ and for each block C_x, C_y with $x \in C_x$, $y \in C_y$, C_x directly below C in $\mathbf{P}_A$, C_y directly below C in $\mathbf{P}_B$, and $x^\star = y^\star$ in C, then C_x and C_y are the same block.

We say a class $\mathcal{K}$ is AP-*inconsistent* whenever it is not AP-consistent.

Due to the behavior of homomorphic images of algebras in qiRL_2^c, the second condition guarantees that all homomorphic images of $\mathcal{K}$ also satisfy the first condition. In fact, AP-consistency is characterized through having all homomorphic images of $\mathcal{K}$ satisfying condition (1). This serves as an illustrative way of checking failure of the amalgamation property.

Theorem 3. *If $\mathcal{V} \in \Lambda(\mathsf{qiSL}_2)$ is generated by $\mathcal{K} \subseteq \mathsf{qiRL}_2^c$ and $\mathcal{K}$ is AP-inconsistent, then $\mathcal{V}$ lacks an amalgamation closure. In particular, $\mathcal{V}$ does not have the amalgamation property.*

A *forest* is a poset where all principal downsets are chains. We denote by $MC(\mathbf{F})$ the set of all maximal chains of a forest $\mathbf{F}$. We summarize the characterization of finitely generated subvarieties of qiS_2 with the amalgamation property in the following theorem.

Theorem 4. *Let $\mathcal{V} = \mathsf{V}(\mathcal{K})$ be a finitely generated subvariety of qiSL_2. Then, $\mathcal{V}$ has the amalgamation property if and only if $\mathcal{K}$ is AP-consistent and there exists a forest $\mathbf{F}$ (dependent on the blocks of the algebras in $\mathsf{HS}(\mathcal{K})$) such that $\mathcal{V} = \mathsf{V}(MC(\mathbf{F}))$.*

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Conditionals in Constructive Quantum Logics

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Holliday [4] has recently introduced a logic called *fundamental logic*, which generalizes both intuitionistic logic (the minimal logic of constructive reasoning) and orthologic (the minimal logic of quantum reasoning). From the algebraic perspective, fundamental logic is the logic of the variety of fundamental lattices:

Definition 0.1. A *fundamental lattice* is a tuple $(L, \leq, \wedge, \vee, \neg, 0, 1)$ such that $(L, \leq, \wedge, \vee, 0, 1)$ is a bounded lattice and $\neg : L \to L$ is an antitone map with the following properties:

- $a \wedge \neg a = 0$;
- $a \leq \neg\neg a$.

Since fundamental logic is weaker than both intuitionistic logic and orthologic, fundamental lattices generalize both Heyting lattices (i.e., the $\to$-free reducts of Heyting algebras) and ortholattices. Fundamental logic, however, is strictly weaker than the intersection of intuitionistic logic and orthologic. Moreover, the logic lacks an implication connective, even though some extensions of orthologic, such as orthomodular logic, can be equipped with a reasonable implication connective for which *modus ponens* holds [3].

In this talk based on a joint project with Juan P. Aguilera, I will present some recent results which shed some new light on constructive quantum logics, i.e., logics generalizing both constructive and quantum reasoning, both in a signature with and without an implication connective. I will start from results in the signature $\{\wedge, \vee, \neg\}$.

In [2], we introduce the following restriction of the class of fundamental lattices:

Definition 0.2. An Ex lattice is a fundamental lattice $(L, \leq, \wedge, \vee, \neg, 0, 1)$ satisfying the following inequalities:

(Nu) $\neg\neg a \wedge \neg\neg b \leq \neg\neg(a \wedge b)$;

(Vi) $\neg\neg a \wedge b \wedge (c \vee d) \leq a \vee (b \wedge c) \vee (b \wedge d)$;

(Cl) $\neg((a \wedge b) \vee (a \wedge c)) \leq (a \wedge (b \vee c)) \vee \neg(a \wedge (b \vee c))$.

Ex lattices generalize both ortholattices and Heyting lattices. Their key property is given by the following result.

Theorem 0.3 ([2])**.** *Any* Ex *lattice L subdirectly embeds into the product of an ortholattice O_L and a Heyting lattice A_L.*

I will discuss two straightforward consequences of this theorem, listed below.

Corollary 0.4. *The intersection of orthologic and (the $\to$-free reduct of) intuitionistic logic is* Ex *logic, i.e., the logic of* Ex *lattices.*

Corollary 0.5. *The lattice of extensions of* Ex *logic is isomorphic to the product of the lattice of extensions of intuitionistic logic in the signature* $\{\wedge, \vee, \neg\}$ *and the lattice of extensions of orthologic.*

I will also discuss how to extend these results to a signature with an implication connective. In this context, Heyting lattices are replaced with Heyting algebras, and ortholattices are replaced with orthomodular algebras (i.e., orthomodular lattices equipped with the Sasaki implication $\rightarrow$, defined as $a \rightarrow b := \neg a \vee (a \wedge b)$). We start with the following definition.

Definition 0.6. An Ex algebra is a structure $(L, \leq, \wedge, \vee, \rightarrow, 0, 1)$ such that:

- Letting $\neg a := a \rightarrow 0$, $(L, \leq, \wedge, \vee, \neg, 0, 1)$ is an Ex lattice;
- The following inequalities hold:

(nS) $\neg(a \rightarrow b) = \neg(\neg a \vee (a \wedge b))$.

(MP) $a \wedge (a \rightarrow b) = a \wedge b$.

(M) $(a \wedge b) \rightarrow b = 1$.

(W4) $a \rightarrow (b \wedge c) \leq a \rightarrow b$.

(iCl$_1$) $a \wedge (a \wedge b) \rightarrow c \leq b \vee b \rightarrow c$.

(iCl$_2$) $((b \wedge c) \vee (b \wedge d)) \rightarrow a \leq (b \wedge (c \vee d)) \vee (b \wedge (c \vee d)) \rightarrow a$.

Ex algebras generalize both orthomodular algebras and Heyting algebras, and are a special case of lattices equipped with one of Holliday's preconditionals [5]. The key property of Ex algebras is established by the following result.

Theorem 0.7 ([1])**.** *Any* Ex *algebra L subdirectly embeds into the product of an orthomodular algebra O_L and a Heyting algebra I_L.*

I will conclude with two consequences of this theorem, listed below.

Corollary 0.8. *The intersection of intuitionistic logic and orthomodular logic is precisely* iEx *logic, i.e., the logic of* Ex *algebras.*

Corollary 0.9. *The lattice of extensions of* iEx *logic is isomorphic to the product of the lattice of superintuitionistic logics and the lattice of superorthomodular logics.*

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Maximal sublattices of finite semidistributive lattices

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In this talk we report on our findings in [2], where we studied maximal sublattices of finite semidistributive lattices via their complements.

A proper sublattice $\mathcal{S}$ of a lattice $\mathcal{L}$ is said to be *maximal* if for every $x \in \mathcal{L} \setminus \mathcal{S}$, the sublattice generated by $\mathcal{S} \cup \{x\}$ is all of $\mathcal{L}$. Maximal sublattices of finite distributive lattices were fully described in [3, Theorem 4] and [5, Theorem 3]: the complements of maximal sublattices of distributive lattices are always intervals of the form $[a, b] = \{x \in L \ : \ a \leq x \leq b\}$ with a being a unique join-irreducible element, and b being a unique meet-irreducible one in the interval.

The class of distributive lattices is contained in the class of bounded lattices; by which we mean the bounded homomorphic images of free lattices [4, Chapter II]. The complements of maximal sublattices of bounded lattices were also shown to be intervals in [1, Theorem 7].

This suggests the following question.

Question 1. *For which classes of lattices are the complements of maximal sublattices intervals?*

The class of semidistributive (SD) lattices is the extension of the class of bounded lattices, and in [1, page 116] the same question was asked of SD lattices.

In [2] we started a systematic study of the following hypothesis:

Hypothesis 1. *The complements of maximal sublattices of (finite) SD lattices are also intervals.*

Since the class of semidistributive lattices is the intersection of classes of join- and meet-semidistributive lattices, we study also complements for these classes.

A join-semidistributive ($\mathrm{SD}_\vee$) lattice is a lattice $\langle \mathcal{L}, \vee, \wedge \rangle$ in which the following quasi-identity holds:

$$x \vee y = x \vee z \implies x \vee (y \wedge z) = x \vee y$$

for all $x, y, z \in \mathcal{L}$. Meet-semidistributive lattices ($\mathrm{SD}_\wedge$) are dually defined.

Hypothesis 2. *The complement of a maximal sublattice of a (finite) lattice satisfying $\mathrm{SD}_\vee$, is the union of intervals with a common minimal element.*

Immediately, one has also the dual formulation that the complement of a maximal sublattice of a (finite) $\mathrm{SD}_\wedge$ lattice is the union of intervals with a common maximal element. If the Hypothesis 2 is true, then the hypothesis for SD follows.

If we do not restrict ourselves to $\mathrm{SD}_\vee$ or $\mathrm{SD}_\wedge$ lattices, we can have both more than one minimal and more than one maximal element in the complement of a maximal sublattice.

Here we summarize our results towards these conjectures obtained in [2].

*Speaker

Theorem 1. *Let $\mathcal{L}$ be an SD$_\vee$ lattice and $\mathcal{M}$ be a maximal sublattice of $\mathcal{L}$. If there is a greatest element in the complement $\mathcal{C} = \mathcal{L} \setminus \mathcal{M}$, then $\mathcal{C}$ is an interval. The dual statement holds for SD$_\wedge$ lattices.*

Corollary 1. *Let $\mathcal{L}$ be an SD$_\vee$ lattice, $\mathcal{M} \leq \mathcal{L}$ be a maximal sublattice and a be a coatom in $\mathcal{L}$. If $a \in \mathcal{C} = \mathcal{L} \setminus \mathcal{M}$, then $\mathcal{C}$ is an interval.*

As an immediate consequence of Theorem 1 and Corollary 1 and their dual statements for SD$_\wedge$ lattices we obtain the following.

Theorem 2. *Suppose $\mathcal{L}$ is an SD lattice and that $\mathcal{M}$ is a maximal sublattice of $\mathcal{L}$ with complement $\mathcal{C}$. If $\mathcal{C}$ contains either a greatest or least element or if it contains either an atom or a coatom of $\mathcal{L}$, then $\mathcal{C}$ is an interval.*

And now the main result follows.

Theorem 3. *Let $\mathcal{L}$ be an SD lattice and $\mathcal{M}$ be a maximal sublattice of $\mathcal{L}$. If there is an element $a \in \mathcal{C} = \mathcal{L} \setminus \mathcal{M}$ that is comparable to every element of $\mathcal{C}$, then $\mathcal{C}$ is an interval.*

We discuss these results and some other properties of the complements of maximal sublattices of finite lattices.

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Modal Measurable Logics via a Modal Loomis-Sikorski Representation Theorem

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Countable modal logic The aim of our work is to develop a modal formalism for reasoning about relational structures based on measurable spaces. Alternatively, such structures can be viewed as Kripke frames equipped with a measurable space structure. Specifically, we work with a modal language with countable meets and joins. Ideally, these should be interpreted as intersections and unions; we investigate to what extent this is feasible when aiming for a completeness theorem.

As a starting point, we consider the countably infinite analogue K_σ of the classical modal logic K, obtained by extending the classical propositional logic with countable joins and meets and with a unary modal operator $\diamond$ that distributes over countable joins. We call the algebras corresponding to K_σ *modal σ-algebras.* These arise from extending σ-Boolean algebras with a countable-join-preserving operator.

The language of countable modal logic can be interpreted in *modal measurable spaces.* These are structures of the form (X, R, Σ) where (X, Σ) is a measurable space, R is a relation on X, and $a \in \Sigma$ implies $\diamond a := \{x \in X \mid R[x] \cap a \neq \emptyset\} \in \Sigma$. In such spaces, a valuation $V : \text{Prop} \to \Sigma$ of the proposition letters can be extended to an interpretation for all formulas via

$$V(\bigvee_{n\in\omega} \varphi_n) = \bigcup_{n\in\omega} V(\varphi_n), \qquad V(\neg\varphi) = X \setminus V(\varphi), \qquad V(\diamond\varphi) = \diamond V(\varphi),$$

so all formulas are interpreted as measurable sets. Then $(\Sigma, \diamond)$ forms a modal σ-algebra. We call modal σ-algebras arising from a modal measurable space *concrete.* In order to obtain completeness, it suffices to prove that every modal σ-algebra is isomorphic to, or can be embedded in, a concrete one.

A representation theorem In case of σ-Boolean algebras, such a representation already exists. The Loomis-Sikorski theorem, proven independently by Loomis [5] and Sikorski [8] in the 1940s, states that every "abstract" σ-Boolean algebra (that is, every Boolean algebra in which countable joins and meets exist) is a quotient of a "concrete" σ-algebra of sets (where the meets and joins are intersections and unions) modulo a σ-ideal.

We extend this theorem to the modal case. This requires a few changes. First, we replace σ-ideals by *modal σ-ideals*, which are simply σ-ideals that are also closed under $\diamond$. Second, it turns out that we need the *infinite descending chain rule*: for every chain $(b_n)_{n\in\omega}$ such that $b_0 \geq b_1 \geq b_2 \geq \cdots$,

$$\bigwedge_{n\in\omega} b_n = \bot \quad \text{implies} \quad \bigwedge_{n\in\omega} \diamond b_n = \bot. \tag{1}$$

This rule resembles one used in a duality for Markov processes [2]. We call modal σ-algebras satisfying this rule *measurable.* With this terminology, we obtain the following theorem.

Modal Loomis-Sikorski theorem: Every measurable modal σ-algebra is the quotient of a concrete modal σ-algebra with a modal σ-ideal.

Modal Loomis-Sikorski Logic Based on this result, we define *Modal Loomis-Sikorski Logic* as the extension of K_σ with an analogue of the infinite descending chain rule. We can equip this with a "measurable" semantics based on *modal measurable spaces* with a modal σ-ideal N of marked sets that we think of as "designated null-sets". Together, these constitute a *marked measurable space*. Modal formulas are then interpreted in the modal σ-algebra $(\Sigma, \diamondsuit)$ quotiented by N.

The idea of interpreting formulas in a space modulo an ideal originates with Scott [7], and was also used by Lando [4], G. Bezhanishvili and Fernández-Duque [1], and by Pavlov [6]. Indeed, [6] states that "One can argue that an object of the right category of spaces in measure theory is not a set equipped with a σ-algebra of measurable sets, but rather a set S equipped with a σ-algebra M of measurable sets and a σ-ideal N of negligible sets, i.e. sets of measure 0." Moreover, Jamneshan and Tao write in [3] that the quotient algebra should be viewed "as a 'point-free' or 'pointless' abstraction of [a measurable space] X in which the null sets have been 'deleted'."

Using our modal version of the Loomis–Sikorski theorem, we can prove:

Completeness theorem: Every axiomatic extension of Modal Loomis-Sikorski Logic is sound and complete with respect to a class of marked modal measurable spaces.

Future work We view this work as a first step in developing a modal logical framework for reasoning about measure-based dynamical systems and (point-free) ergodic theory. This naturally gives rise to many avenues for future research. First, we can ask whether it is possible to axiomatise the logic of modal measurable spaces. A second natural question is whether Modal Loomis-Sikorski Logic is complete with respect to a suitable class of (unmarked) modal measurable spaces. Finally, a natural next step would be to make the relations probabilistic and, more generally, to incorporate probability explicitly into the framework.

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Biquasiintuitionistic logic and related structures

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Consider a lattice $(K, \leq, \wedge, \vee)$ and a map $^\dagger : K \to K$ satisfying n_0): if $x \leq y$, then $y^\dagger \leq x^\dagger$ $\forall x, y \in K$. Vakarelov [16] developed a framework allowing to study a vast family of logics with negations that are sound and complete with respect to various subclasses of algebras $(K, \leq, \wedge, \vee, ^\dagger)$ satisfying n_0). We assume n_0) everywhere below. In particular, consider $\forall x, y, z \in K$: n_1) $x \leq x^{\dagger\dagger}$; n_2) $x^{\dagger\dagger} \leq x$; n_3) $x \wedge x^\dagger \leq y$; n_4) $y \leq x \vee x^\dagger$; n_5) if $x \wedge y \leq z$, then $x \wedge z^\dagger \leq y^\dagger$; n_6) if $x \leq y \vee z$, then $z^\dagger \leq x^\dagger \vee y$. Boundedness and infinite meet (resp., join) distributivity of $(K, \leq, \wedge, \vee)$ defines intuitionistic (resp., cointuitionistic) logic and the variety of Heyting (resp., Brouwer) algebras, with negation satisfying n_1), n_3), and n_5) (resp., n_2), n_4), and n_6)) [2]. We introduce [3] quasiintuitionistic (resp., coquasiintuitionistic) algebras and logic, defined as $(K, \leq, \wedge, \vee, ^\dagger)$ with the bounded and distributive $(K, \leq, \wedge, \vee, 0, 1)$ together with n_1) and n_3) (resp., n_2) and n_4)). $(K, \leq, \wedge, \vee, ^\circ, ^\bullet, 0, 1)$ with a bounded distributive $(K, \leq, \wedge, \vee, 0, 1)$ and $^\circ$ (resp., $^\bullet$) satisfying n_1) and n_3) (resp., n_2) and n_4)) will be called a biquasiintuitionistic algebra. The class of these algebras corresponds to the biquasiintuitionistic logic. They turn into Skolem (i.e. 'Heyting–Brouwer') algebras and biintuitionistic logic of [12, 13] under the assumptions of infinite distributivity, n_5) for $^\circ$, and n_6) for $^\bullet$. We prove [4] that the coquasiintuitionistic logic is the natural setting for Lawvere's border operator ∂, introduced in [10] for cointuitionistic logic. Defined by $\partial x := x \wedge x^\bullet$ $\forall x \in K$, it satisfies $\partial\partial = \partial$, $\partial(x \wedge y) = (\partial x \wedge y) \vee (x \wedge \partial y)$ $\forall x, y \in K$ (i.e. Leibniz's rule), and $x^{\bullet\bullet} \vee \partial x = x$ $\forall x \in K$, generalising the properties of the topological border operator of [17]. 'Naturality' means that it is the minimal bounded logic such that the two first properties above (or the third one) hold(s). ($\mathrm{d}x := x \vee x^\circ$ has order dual properties.) We also show [4] that the countably additively complete biquasiintuitionistic algebra is the minimal bounded lattice allowing for the adjoint pair of Reyes–Zolfaghari modal operators, $\Diamond_{\mathbb{N}} \dashv \Box_{\mathbb{N}}$, introduced in [14] for the Skolem algebra: $\Box_0 := \mathrm{id}_K =: \Diamond_0$, $\Box_n(\cdot) := (\Box_{n-1}(\cdot))^{\bullet\circ}$ and $\Diamond_n(\cdot) := (\Diamond_{n-1}(\cdot))^{\circ\bullet}$ $\forall n \in \mathbb{N}$, $\Box_{\mathbb{N}} := \bigwedge_{n\in\mathbb{N}} \Box_n$, $\Diamond_{\mathbb{N}} := \bigvee_{n\in\mathbb{N}} \Diamond_n$. A biquasiintuitionistic algebra $(K, \leq, \wedge, \vee, ^\circ, ^\bullet, 0, 1)$ will be called an Akchurin algebra iff the reduct $(K, \leq, \wedge, \vee, 0, 1)$ is equipped with a Skolem algebra structure. So, the logic corresponding to the class of Akchurin algebras is a product of biquasiintuitionistic and biintuitionistic logics over $(K, \leq, \wedge, \vee, 0, 1)$. (See [9] for a biquasiintuitionistic extension of the full nonassociative Lambek–Grishin calculus.)

For any quasiintuitionistic (resp., coquasiintuitionistic) algebra $(K, \leq, \wedge, \vee, 0, 1, ^\circ)$ (resp., $(K, \leq, \wedge, \vee, 0, 1, ^\bullet)$), $(\cdot)^{\circ\circ}$ is a closure (resp., $(\cdot)^{\bullet\bullet}$ is an interior) operator, i.e. it is a monad (resp., comonad). So, it defines a full reflective (resp., coreflective) subcategory $(K_{\circ\circ} := \{x \in K \mid x = x^{\circ\circ}\}, \leq)$ (resp., $(K_{\bullet\bullet} := \{x \in K \mid x = x^{\bullet\bullet}\}, \leq)$). In general, $(\cdot)^{\circ\circ}$ (resp., $(\cdot)^{\bullet\bullet}$) does not preserve finite meets (resp., finite joins), while it preserves 0, 1, and finite joins (resp., finite meets), with the join (resp., meet) operation on $(K_{\circ\circ}, \leq)$ (resp., $(K_{\bullet\bullet}, \leq)$) defined by $\cdot \curlyvee \cdot := (\cdot \vee \cdot)^{\circ\circ}$ (resp., $\cdot \curlywedge \cdot := (\cdot \wedge \cdot)^{\bullet\bullet}$). So, the image of the action of $(\cdot)^{\circ\circ}$ (resp., $(\cdot)^{\bullet\bullet}$) on a quasiintuitionistic (resp., coquasiintuitionistic) algebra $(K, \leq, \wedge, \vee, 0, 1, ^\circ)$ (resp., $(K, \leq, \wedge, \vee, 0, 1, ^\bullet)$) defines an internal algebra $(K_{\circ\circ}, \leq, \wedge, \curlyvee, 0, 1, ^\circ)$ (resp., $(K_{\bullet\bullet}, \leq, \curlywedge, \vee, 0, 1, ^\bullet)$). We show [3] that these internal algebras are orthocomplemented lattices. The Kolmogorov–Glivenko theorem [8, 6] identifies the double negated formulas of the intuitionistic logic with the formulas of the classical logic. The dual Kolmogorov–Glivenko theorem [15] is equivalent to: the internal algebra of a Brouwer algebra is boolean [11]. So, our result provides a generalised (resp., generalised dual) Kolmogorov–Glivenko theorem, that relates (resp., co)quasiintuitionistic logic with its internal orthocomplemented logic, at the expense of considering formulas not containing meets (resp., joins). Two internal boolean lattices of a Skolem algebra are isomorphic, so an Akchurin algebra admits: a) three different internal lattices: two orthocomplemented and one boolean; b) two pairs of the Reyes–Zolfaghari modal operators; c) two operator pairs (∂, d).

Spectral presheaves, introduced in [7] as the main object of the topos theoretic approach to quantum mechanics, were generalised in [1] to provide a generalisation of the Stone duality for complete orthomodular lattices $(L, ^\perp)$. For the poset $\mathcal{V}_\mathrm{c}(L)$ of the complete boolean subalgebras of $(L, ^\perp)$ ordered by inclusion, the spectral presheaf is defined as a functor $\Sigma^L \in \mathrm{Ob}(\mathtt{Set}^{(\mathcal{V}_\mathrm{c}(L))^{\mathrm{op}}})$ assigning the Stone spectra sp_S to objects and the restrictions of these spectra to morphisms. The set $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ consists of all subpresheaves of Σ^L such that their images are closed-and-open in the Stone topology. Equipped with a lattice structure given by subpresheaves as $\leq$ and topological closures (resp., interiors) of sums (resp., intersections) as $\vee$ (resp., $\wedge$), $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ becomes a complete Skolem algebra [1]. We generalise this setting to complete orthocomplemented lattices $(L, ^\perp)$, and show that $\mathrm{Sub}_{\mathrm{clop}}(\Sigma^L)$ is an Akchurin algebra, with $(\cdot)^\bullet := \delta^\wedge \circ {}^\perp \circ \varepsilon^\wedge$, $(\delta^\wedge(x))(B) := \{s \in \mathrm{sp}_\mathrm{S}(B) \,:\, s(\bigwedge\{y \in B \,:\, y \geq x\}) = 1\}\ \forall B \in \mathrm{Ob}(\mathcal{V}_\mathrm{c}(L))\ \forall x \in L$, $\varepsilon^\wedge$ defined as the right adjoint of $\delta^\wedge$, and $^\circ$ defined dually to $^\bullet$ [3]. The coquasiintuitionistic (resp., cointuitionistic) reduct of this algebra is fully paraconsistent (i.e. $\partial x \neq 0\ \forall x \in \mathrm{Sub}_{\mathrm{clop}}(\Sigma^L) \setminus \{\varnothing, \Sigma^L\}$) iff L has more than 4 elements (resp., has more than 4 elements and $\mathcal{V}_\mathrm{c}(L)$ is connected) [5]. We also prove that the internal orthocomplemented lattices of this Akchurin algebra are isomorphic to the underlying lattice $(L, ^\perp)$ [3]. So, the generalised Kolmogorov–Glivenko theorem for the internal Akchurin logic of the spectral presheaf reconstructs its underlying lattice and $\mathcal{V}_\mathrm{c}(L)$.

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Exactly Five Subquasivarieties of Sugihara Algebras have the Amalgamation Property

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The amalgamation property[1] is of fundamental importance in algebraic logic, as it is strongly related to various types of interpolation properties for corresponding logical systems. Rising to prominence due in large part of Maksimova's pioneering work on amalgamation and interpolation for superintuitionistic logics [8], the link between amalgamation and interpolation is today very well known and represents one of the standard methodologies for studying interpolation.

In this work, we study amalgamation in quasiequational classes of *Sugihara algebras*, i.e., the constant-free subreducts of Sugihara monoids. Sugihara algebras and Sugihara monoids originated in the algebraic study of relevant logics several decades ago, and in modern research have received considerable attention because of their utility as instructive case studies and their central role in structural decompositions of other classes of residuated algebraic structures; see, e.g., [2, 3, 6]. Notably, Marchioni and Metcalfe have obtained an exhaustive description of varieties of Sugihara monoids with the amalgamation property [9] and used this description to give a complete list of axiomatic extensions of the logic **R**-mingle with the Craig interpolation property.

While [8, 9] and a range of other studies have succeeded in classifying all subvarieties of a given variety V of residuated structures that have the amalgamation property, such classifications are dramatically more difficult to obtain for the *subquasivarieties* of a variety of residuated structures. The complications in passing from subvarieties to subquasivarieties are numerous, but among the most notable difficulties is that a subquasivariety of a variety V may fail to have the relative congruence extension property even when V has the congruence extension property. Because of the intimate link between the (relative) congruence extension property and the amalgamation property (see, e.g., [7]), this is a serious challenge to applying the usual tools for studying amalgamation in classes of residuated structures.

The present study succeeds in overcoming the aforementioned challenges by exploiting some important features—most notably, the existence of certain almost-minimal quasivarieties—of the lattice of subquasivarieties of Sugihara algebras. This allows us to adhere closely to proofs that avoid dealing with the (relative) congruence extension property. Following the general method of studying amalgamation via so-classed closure properties [5], we are able to give an exhaustive classification of subquasivarieties of Sugihara algebras with the amalgamation property.

Theorem 1. *There are exactly five quasivarieties of Sugihara algebras with the amalgamation property.*

In light of the fact that the lattice of subquasivarieties of Sugihara algebras is relatively poorly understood (e.g., we do not even know its cardinality), the preceding result is rather

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[1] A class K of similar algebras is said to have the *amalgamation property* (or **AP**) is whenever $\varphi_{\mathbf{B}}\colon \mathbf{A} \hookrightarrow \mathbf{B}$ and $\varphi_{\mathbf{C}}\colon \mathbf{A} \hookrightarrow \mathbf{C}$ are embeddings in K, there exist embeddings $\psi_{\mathbf{B}}\colon \mathbf{B} \hookrightarrow \mathbf{D}$ and $\varphi_{\mathbf{C}}\colon \mathbf{C} \hookrightarrow \mathbf{D}$ in K such that $\psi_{\mathbf{C}} \circ \varphi_{\mathbf{C}} = \psi_{\mathbf{B}} \circ \varphi_{\mathbf{B}}$

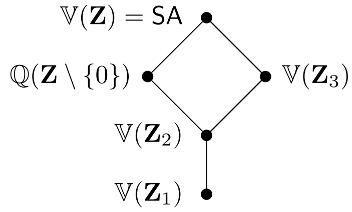

Figure 1: Hasse diagram of the subquasivarieties of the variety SA of Sugihara algebras with the amalgamation property.

striking. Moreover, because we are able to execute our proofs without application of the congruence extension property, we are able to combine the previous theorem with the known classification of quasivarieties of Sugihara algebras with the relative congruence extension property [1] to obtain the following surprising result.

Theorem 2. *For an arbitrary subquasivariety* Q *of Sugihara algebras,* Q *has the amalgamation property if and only if* Q *has the transferable injections property. Consequently, by application of well-known bridge theorems of abstract algebraic logic, an arbitrary (not necessarily axiomatic) extension of the logic* **R***-mingle has the Maehara interpolation property if and only if it has the Robinson property.*

Aside from their importance to relevant logic, our hope is that—as has often been the case with studies of Sugihara algebras and Sugihara monoids—these results provide a useful case study from which the study of more complicated kinds of residuated structures may be based. With this in mind, we highlight in this work a few important lessons from the preceding theorems that we believe may be useful in studying amalgamation in other quasivarieties of residuated structures. More information on this work may be found in our preprint [4].

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A Double-Categorical Perspective (or two) on Dual Fibrations

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Abstract

We investigate two senses in which the dual fibration construction can be applied to double categories, and explore the logical significance of each.

The significance of fibrations to categorical logic is well-understood [2]. Elementary existential fibrations (EEFs), for example, model specifications of regular logic, the fragment of first-order logic whose logical operations consist only of verum, conjunction and existential quantification.

A general goal of ours is to lift logically significant aspects of fibrations into double categories to study them there, in order to leverage the additional expressive power of double categories. This has been started in Nasu's thesis where the regular logic case is considered [3]. In this talk, we will discuss work in progress on the significance of dual fibrations. Given any fibration $p : \mathcal{E} \to \mathcal{B}$, one may define the dual fibration $p^\bullet : \mathcal{E}^\bullet \to \mathcal{B}$ over the same base category, where the new total category $\mathcal{E}^\bullet$ is obtained by taking fibrewise opposites of the original.

Given a double category, there are at least two (related) ways in which a fibration may arise, and thus at least two reasonable places to consider taking dual fibrations. The first comes from a well-known result of Shulman that a double category $\mathbf{D}$ is an equipment if and only if its underlying source-target pairing $\langle s, t\rangle : D_1 \to D_0 \times D_0$ is a fibration (equivalently an opfibration) [5]. Equipments are double categories for which the data of the tight structure is effectively encoded as part of the loose structure, in the form of so-called companions f_* and conjoints f^* associated to each tight morphism f.

Shulman's result can also be broken down in a more granular way (which, to date, seems only to have been written down on the nLab [1]): a double category has companions (resp. conjoints) iff its underlying source-target span is a two-sided fibration (resp. opfibration). We conjecture that taking duals of such two-sided (op)fibrations induces Paré's construction of the double category of (co)retrocells on a double category with (chosen) companions (resp. conjoints) [4]. Given a double category $\mathbf{D}$ with chosen companions, a retrocell α in $\mathbf{D}$ (left) is a double cell in $\mathbf{D}$ of the form (right):

$$
\begin{array}{ccc}
A & \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & B \\
{\scriptstyle f}\downarrow & \overset{\alpha}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle g} \\
C & \underset{S}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & D
\end{array}
\qquad\qquad
\begin{array}{ccccc}
A & \overset{f_*}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & C & \overset{S}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & D \\
\| & & \alpha & & \| \\
A & \underset{R}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & B & \underset{g_*}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} & D
\end{array}
$$

Paré also showed that retrocells of $\mathbf{D}$ can be assembled into a double category $\mathbf{D}^{ret}$ with the same objects, tight morphims and loose morphisms as $\mathbf{D}$, and that $\mathbf{D}^{ret}$ has a canonical choice of companions.

Alternatively, given a cartesian equipment $\mathbf{D}$ (an equipment with a suitable notion of product) with category of objects D_0, one can take the underlying source-target pairing, which is a fibration (in fact a bifibration) by Shulman's theorem, and then pull back along the functor that sends objects I to pairs $(I, 1)$, where 1 is the terminal object in D_0, to obtain a new so-called unilateral fibration denoted $uni(\mathbf{D})$. Nasu showed in [3] that when $\mathbf{D}$ additionally satisfies a Frobenius axiom, $uni(\mathbf{D})$ retains all the information of the original equipment. For example, when $\mathbf{D}$ is the Frobenius cartesian equipment $\mathbf{Rel}$, whose objects are sets, tight morphisms are functions, loose morphisms from A to B are relations $R \subseteq A \times B$ (composed in the usual way), and cells are inclusions of relations along pairs of functions, $uni(\mathbf{Rel})$ is the subobject fibration on the category of sets.

It is also possible, given an EEF p, to construct a double category (in fact a Frobenius cartesian equipment), which we denote $\mathbf{Bil}(p)$. A version of this construction, denoted $\mathbf{Fr}(-)$, first appeared in [5], and Nasu showed in [3] that $\mathbf{Bil}(-)$ and $uni(-)$ restrict to an equivalence between EEFs and Frobenius cartesian equipments.

Our main motivating example for exploring dual fibrations in this second context is the double category $\mathbf{Rel}^\bullet$, which has the same objects, tight and loose morphisms as $\mathbf{Rel}$, but where the composition of two relations $R \subseteq A \times B$ and $S \subseteq B \times C$ is given by what Peirce called the relative product, i.e. those pairs (a, c) such that for all b in B, (a, b) is in R or (b, c) is in S, and cells in $\mathbf{Rel}^\bullet$ are given by taking opposite inclusions of cells from $\mathbf{Rel}$. $\mathbf{Rel}^\bullet$ is also a Frobenius cartesian equipment, and $uni(\mathbf{Rel})$ and $uni(\mathbf{Rel}^\bullet)$ are dual fibrations of one another.

Generalizing this observation raises three questions: what properties must a fibration satisfy in order that its dual fibration be an EEF, what are the double-categorical analogues of these properties, and what is their logical significance? Answering the first question amounts to an easy exercise in arrow-flipping. Towards answering the second and third questions, we note that these properties point toward a distinctly non-intuitionistic type of negation (we conjecture that it amounts to linear negation). For although binary relative products are definable in any Heyting category, they do not become an associative operation unless one requires, in particular, a "dual Frobenius reciprocity" condition, whose type-theoretic interpretation is

$$\forall x. \, (\varphi \vee \psi) \equiv \big(\forall x. \, \varphi\big) \vee \psi$$

for x not free in ψ. We will elaborate on this, and further explore answers to the second and third questions, in this talk.

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Examples of Arrow Algebras

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Abstract

Arrow Algebras appear all throughout Categorical Logic. We look at certain structures that admit an Arrow Algebra and what properties they satisfy. We found that the Arrow Tripos coincides with the given Tripos construction in all cases. The Diller-Nahm Dialectica and modified Dialectica Arrow Algebras are well-behaved: they are quasi-implicative and strong, but not implicative. The Herbrand Arrow Algebra gives a minimal example of an Arrow Algebra and satisfies no additional properties. All three are not Implicative Algebras, justifying the comparably weak axioms of an Arrow Algebra.

1 Introduction

Arrow Algebras were introduced as a generalization of Locales and Implicative Algebras because many structures throughout Categorical Logic naturally carry this structure [1, 7]. An Arrow Algebra is defined as a quadruple $(A, \preccurlyeq, \to, S)$ such that

- $(A, \preccurlyeq)$ is a complete lattice,
- $\to : A \times A \to A$ is an operation that is monotone in its second argument and antitone in its first argument, i.e. for all $a' \preccurlyeq a,\ b \preccurlyeq b'$, we have $a \to b \preccurlyeq a' \to b'$ and
- $S \subseteq A$ is a *separator*, i.e. a designated set of values we regard to be true. It is closed upwards and under Modus Ponens, so if $a \in S$ and $a \preccurlyeq b$, then $b \in S$ as well as if $a, a \to b \in S$, then $b \in S$. Furthermore, it contains the combinators

$$\mathbf{k} := \bigcurlywedge_{a,b\in A} a \to b \to a,$$

$$\mathbf{s} := \bigcurlywedge_{a,b,c\in A} (a \to b \to c) \to (a \to b) \to (a \to c),$$

$$\mathbf{a} := \bigcurlywedge_{I,x,(y_i),(z_i)} (\bigcurlywedge_{i\in I} x \to y_i \to z_i) \to x \to (\bigcurlywedge_{i\in I} y_i \to z_i).$$

The combinators allow us to do logic inside an Arrow Algebra: **k** and **s** axiomatize intuitionistic logic and **a** is a weakening of the identity $\bigcurlywedge_{b\in B} a \to b = a \to \bigcurlywedge_{b\in B} b$, which is valid among all Locales. Clearly, any Locale is an Arrow Algebra with $S = \{\top\}$. With this definition, we have

$$\bigcurlywedge_{p_1,\ldots,p_n\in A} \varphi \in S$$

for any implicative intuitionistic tautology φ with free variables $p_1, \ldots, p_n$ [1, Prop 3.5].

Given an Arrow Algebra $\mathcal{A} = (A, \preccurlyeq, \to, S)$, we can define a functor $\mathsf{P}_{\mathcal{A}} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{preHeyt}$ by sending a set X to $(A^X, \vdash_{S^X})$, where we set $\varphi \vdash_{S^X} \psi$ iff

$$\bigcurlywedge_{x\in X} \varphi(x) \to \psi(x) \in S.$$

By **preHeyt**, we denote the category of Heyting prealgebras, i.e. preorders whose posetal reflection is a Heyting algebra. On morphisms, $\mathsf{P}_{\mathcal{A}}$ acts by precomposition. This construction makes $\mathsf{P}_{\mathcal{A}}$ a Tripos, which we call the Arrow Tripos [1, Thm 5.10].

2 Main Contributions

We showed that the Herbrand Topos, the Diller-Nahm variant of Gödel's Dialectica Interpretation and the modified Dialectica Tripos respectively admit an Arrow Algebra [2, 8, 9]. We named these the Herbrand, Diller-Nahm Dialectica and modified Dialectica Arrow Algebra respectively. Besides the similarity in construction, their corresponding Arrow Triposes coincide exactly with the Tripos construction in all three cases [9, Sec 4].

We also looked at which additional properties they satisfy. Having very similar definitions, the Diller-Nahm Dialectica and modified Dialectica Arrow Algebra satisfy the same properties and are well-behaved. Most notably, they are strong, i.e. satisfy the following weakening of the axiom for Implicative Algebras [7, Def 2.1]

$$\mathbf{a'} := \bigcurlywedge_{a,B} (\bigcurlywedge_{b \in B} a \to b) \to a \to \bigcurlywedge_{b \in B} b \in S.$$

We cannot use Kleene's first algebra K_1 solving the Halting Problem to refute strongness (c.f. [1, Prop 3.18] and [9, Sec. 3]). Furthermore, they are quasi-implicative, but not implicative. The Herbrand Arrow Algebra is particularly interesting, because it gives us a certain "minimal" example of an Arrow Algebra. It satisfies none of the additional properties an Arrow Algebra can have, not even binary implicativity (i.e. $a \to (b \curlywedge b') = (a \to b) \curlywedge (a \to b')$ is not valid) and thus is neither quasi-implicative nor implicative. It is also not strong, because otherwise we could instantiate it with K_1 and solve the Halting Problem. Since none of the examples are Implicative Algebras, this justifies to the comparably weak axioms of Arrow Algebras.

Next, we want to use these results to find a Gleason cover for Arrow Algebras. For topological spaces, this the minimal cover of a compact Hausdorff space by an extremally disconnected one [3] and has since been generalized to Topoi, in which case the covering Topoi are DeMorgan-Topoi [5, 6]. Via the internal Locales of a Topoi, which of course are Arrow Algebras, we want to lift this to Arrow Algebras. Johnstone also showed that the properties of the Herbrand Topos arise because it can be obtained as the Gleason cover of an ordinary realizability Topos [2, 4]. We would like to see whether we can obtain a similar result for Arrow Algebras.

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