STAR-FUNDAMENTAL ALGEBRAS: POLYNOMIAL IDENTITIES AND ASYMPTOTICS

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Abstract. We introduce the notion of star-fundamental algebra over a field of characteristic zero. We prove that in the framework of the theory of polynomial identities, these algebras are the building blocks of a finite dimensional algebra with involution $*$.

To any star-algebra A is attached a numerical sequence $c_n^*(A)$, $n \geq 1$, called the sequence of $*$-codimensions of A. Its asymptotics is an invariant giving a measure of the $*$-polynomial identities satisfied by A. It is well-known that for a PI-algebra such sequence is exponentially bounded and $exp^*(A) = \lim_{n\to\infty} \sqrt[n]{c_n^*(A)}$ can be explicitly computed. Here we prove that if A is a star-fundamental algebra,

$$C_1 n^t exp^*(A)^n \leq c_n^*(A) \leq C_2 n^t exp^*(A)^n, \quad (1)$$

where $C_1 > 0, C_2, t$ are constants and t is explicitly computed as a linear function of the dimension of the skew semisimple part of A and the nilpotency index of the Jacobson radical of A. We also prove that any finite dimensional star-algebra has the same $*$-identities as a finite direct sum of star-fundamental algebras. As a consequence, by the main result in [38] we get that if A is any finitely generated star-algebra satisfying a polynomial identity, then (1) still holds and, so, $\lim_{n\to\infty} \log_n \frac{c_n^*(A)}{exp^*(A)^n}$ exists and is an integer or half an integer.

1. Introduction

Let A be an algebra with involution $*$ over a field F of characteristic zero. This paper is devoted to the computation of an invariant of the ideal of $*$-polynomial identities of A when A is a $*$-fundamental algebra.

Recall that one can attach to any algebra A with involution a numerical sequence $c_n^*(A)$, $n = 1, 2, \ldots$, called the sequence of $*$-codimensions of A. Such sequence is built out of the dimensions of the multilinear $*$-polynomial identities of degree $n \geq 1$ satisfied by the algebra A. Its asymptotics is the invariant we are searching for and it gives a measure of the ideal of the free algebra with involution consisting of the $*$-polynomial identities satisfied by A. Recall that such ideals are precisely the ones invariant under the endomorphisms of the free algebra.

Here we compute such invariant, up to a constant, for any $*$-fundamental algebra. As an outcome of the theory developed here, it turns out that any finite dimensional $*$-algebra has the same $*$-identities as a finite direct sum of $*$-fundamental algebras. It follows that the above invariant can be computed, up to a constant, for any finite dimensional $*$-algebra (actually by [38], for any finitely generated $*$-algebra satisfying a polynomial identity). This motivates the relevance of such algebras.

In order to provide a motivation and a better understanding of the results obtained in this paper we shortly describe the state of the art of the area when dealing with algebras with no additional structure.

In general let $F\langle\mathcal{X}\rangle$ be a free algebra over F on a countable set $\mathcal{X}$. The T-ideals of $F\langle\mathcal{X}\rangle$, i.e., the ideals invariant under all endomorphisms of $F\langle\mathcal{X}\rangle$, are an interesting object of study since they coincide with the sets of polynomial identities satisfied by the algebras over F.

Even though by a famous theorem of Kemer the proper T-ideals are finitely generated [29], they turn out to be quite obscure objects. A way of measuring them is through a numerical sequence called the sequence of codimensions.

Let A be an associative F-algebra and $\mathrm{Id}(A)$ the T-ideal of polynomial identities satisfied by A. In characteristic zero one may restrict oneself to the study of the multilinear polynomials. Then, for every $n \geq 1$, one

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defines $P_n(A)$ to be the space of multilinear polynomials in n fixed variables modulo the identities of A, and the sequence of codimensions of A is $c_n(A) = \dim P_n(A),\ n = 1, 2, \ldots$.

Such sequence has been extensively studied ([11], [12], [13], [30], [31]), but it turns out that it can be explicitly computed only in very few cases. Since Regev in [35] proved that if A satisfies a non-trivial identity (A is a PI-algebra), the corresponding sequence of codimensions is exponentially bounded, the interest focused in the computation of such asymptotics since they represent an invariant of the T-ideal $\mathrm{Id}(A)$. In this perspective, inspired by a conjecture of Amitsur, in [16] and [17] the authors proved that for any PI-algebra A, there exist constants $C_1 > 0, C_2, t, s, d$ such that $C_1 n^t d^n \le c_n(A) \le C_2 n^s d^n$, for all $n \ge 1$, and d is an integer called the PI-exponent $exp(A)$ of A.

Later Berele and Regev in [9] and [6] extended such result by verifying for algebras with 1 a conjecture of Regev stating that the asymptotic equality $c_n(A) \simeq C n^t exp(A)^n$ holds, where C, t are constants and $t \in \frac{1}{2}\mathbb{Z}$. Since the sequence of codimensions is eventually non decreasing ([23]) then by [9, Lemma 39] and [6, Theorem 4.18] it follows that if A is any arbitrary PI-algebra

$$C_1 n^t exp(A)^n \le c_n(A) \le C_2 n^t exp(A)^n \tag{2}$$

holds where $C_1 > 0, C_2, t$ are constants and $t \in \frac{1}{2}\mathbb{Z}$.

This result gives a second invariant of a T-ideal, after the PI-exponent, namely

$$t = \lim_{n\to\infty} \log_n \frac{c_n(A)}{exp(A)^n}.$$

Since the results in [9] and [6] do not provide an interpretation of t, next step is the explicit computation of this second invariant. In case of $k \times k$ matrices Regev in [36] had computed the precise asymptotics. It turns out that if $M_k(F)$ is the algebra of $k \times k$ matrices over F, then $c_n(M_k(F)) \simeq C n^t (k^2)^n$, where C is an explicitly computed constant and $t = -\frac{1}{2}(k^2 - 1)$. Based on this result this asymptotic equality and on further results of [20] and [8] it turns out that one can compute such invariant also for the upper block triangular matrix algebras. These algebras are a special case of the so-called fundamental algebras.

Recall that a fundamental algebra (called also basic algebra) is a finite dimensional algebra over an algebraically closed field F that can be defined in terms of some multialternating polynomials called Kemer polynomials or a pair of integers called the Kemer index or by means of the representation theory of the symmetric group. These algebras were introduced by Kemer as a basic tool of his theory ([27], [28], [29]).

More recently Aljadeff, Janssens and Karasik in [2] were able to compute the invariant t for any fundamental algebra. The result is the following: let A be a fundamental algebra over F. Let q be the number of simple components in the decomposition of a maximal semisimple subalgebra of A and let $s+1$ be the nilpotency index of the Jacobson radical of A. Then $\lim_{n\to\infty} \log_n \frac{c_n(A)}{exp(A)^n} = -\frac{1}{2}(exp(A) - q) + s$. Now, any finite dimensional algebra over F has the same identities as a finite direct sum of fundamental algebras. Hence, since codimensions do not change upon extension of the base field this result rediscovers the result in [9, Lemma 39] given in (2) for finite dimensional algebras. This is the state of the art of the theory.

It is worth mentioning that some aspects of the theory have been generalized by considering an extra structure such as group grading, group action or generalized Hopf algebra action ([24, Section 3], see also [26], [1]).

Next we turn to the theory of algebras with involution.

If A has an involution $*$, e.g. the algebra of $k \times k$ matrices, one can introduce finer invariants defined by the $*$-polynomial identities of A. Recall that by a well-known theorem of Amitsur ([4]) if an algebra A satisfies a $*$-identity, it also satisfies an ordinary identity (no involution), and this gives a close relation between identities and $*$-identities.

As for the general setting, one considers the free algebra with involution $F\langle \mathcal{X}, *\rangle$ on a countable set $\mathcal{X}$. If A is an algebra with involution (or a $*$-algebra), we let $\mathrm{Id}^*(A)$ be the ideal of $*$-polynomial identities satisfied by A. This is a so-called T*-ideal, i.e., an ideal of the free algebra invariant under all endomorphisms commuting with the involution $*$.

As in the ordinary case one constructs the sequence of $*$-codimensions of A by setting $c_n^*(A) = \dim P_n^*(A)$, $n = 1, 2, \ldots$, where $P_n^*(A)$ is the space of multilinear $*$-polynomials in n fixed variables modulo the $*$-identities of A.

Now, in [15] it was shown that for any PI-algebra A, $c_n(A) \le c_n^*(A) \le 2^n c_n(A)$, holds for all $n \ge 1$. On the other hand the explicit computation of the $*$-codimensions has been carried out in very few examples ([32], [33], [34]) and, as in the ordinary case, the attention focused on computing their asymptotics.

An interesting algebra endowed with different involutions is the algebra $M_k(F)$ of $k \times k$ matrices. It turns out that there are only two T*-ideals of identities of $M_k(F)$ and they correspond to the transpose and the symplectic involution (see for instance [21, Section 3.6] or [21, Theorem 3.6.8]). The asymptotics of the $*$-codimensions of $M_k(F)$ were computed in [7] for both types of involution. It turns out that if $*$ is the transpose involution, $c_n^*(M_k(F)) \simeq C_1 n^{-\frac{1}{4}k(k-1)} k^{2n}$, for some constant C_1, and if $*$ is the symplectic involution, k is even and $c_n^*(M_k(F)) \simeq C_2 n^{-\frac{1}{4}k(k+1)} k^{2n}$, for some constant C_2.

As we mentioned above, the $*$-codimensions of a PI-algebra are exponentially bounded, and their exponential growth was computed and shown to be an integer, for any finite dimensional algebra in [18]. For general PI-algebras with involution the result was achieved much later in [14]. It turns out that there exist constants $C_1 > 0, C_2, t_1, t_2, d$ such that

$$C_1 n^{t_1} d^n \le c_n^*(A) \le C_2 n^{t_2} d^n, \tag{3}$$

for all $n \ge 1$, and d is an integer called the $*$-exponent $exp^*(A)$ of A.

The reason for such a delay was due to the lack of a suitable structure theorem for PI-algebras with involution. Such result was proved recently in [1] as a consequence of a close relation between involutions and superinvolutions of an algebra and its Grassmann envelope. It turns out that any PI-algebra with involution has the same $*$-identities as the Grassmann envelope of a finite dimensional superalgebra B with superinvolution. Then when F is algebraically closed, the $*$-exponent can be described as the dimension of a suitable semisimple subalgebra of B.

Next step is to ask if the polynomial factor in (3) is uniquely determined, i.e., $t_1 = t_2$, giving in this way a second invariant of a T*-ideal, after the $*$-exponent. A more concrete question would be the following: can one compute such polynomial factor for a certain class of algebras relating it to the structure of the algebra itself?

In this paper we are able to give a positive answer to this question for the class of $*$-fundamental algebras that here we define. As a consequence we prove that $t_1 = t_2 \in \frac{1}{2}\mathbb{Z}$, for any finitely generated $*$-algebra satisfying a polynomial identity.

First, in accordance to Kemer's theory we introduce the notion of $*$-fundamental algebra. The main feature of these algebras is that any finite dimensional algebra has the same $*$-identities as a finite direct sum of $*$-fundamental algebras. Then we develop the theory of such algebras and as an outcome we are able to compute the polynomial factor of the $*$-codimensions of any $*$-fundamental algebra in terms of some fixed parameters. More precisely we prove the following: let A be a $*$-fundamental algebra over an algebraically closed field and let $A = \bar{A} + J$ be its Wedderburn-Malcev decomposition, as algebra with involution, where $\bar{A}$ is a $*$-semisimple subalgebra and J is the Jacobson radical of A. Let $s+1$ be the nilpotency index of J, i.e, $s \ge 0$ is the smallest integer such that $J^{s+1} = 0$. Let also r be the number of $*$-simple algebras appearing in the decomposition of $\bar{A}$ which are not simple algebras. Then

$$C_1 n^{-\frac{1}{2}(\dim(\bar{A})^- - r)+s} (\dim \bar{A})^n \le c_n^*(A) \le C_2 n^{-\frac{1}{2}(\dim(\bar{A})^- - r)+s} (\dim \bar{A})^n,$$

for some constants $C_1 > 0, C_2$, where $(\bar{A})^- = \{a \in \bar{A} \mid a^* = -a\}$ is the Lie algebra of skew elements of $\bar{A}$.

As a corollary, from [38] we get that if A is any finitely generated PI-algebra with involution over a field of characteristic 0, then

$$C_1 n^t (\exp^* A)^n \le c_n^*(A) \le C_2 n^t (\exp^* A)^n,$$

where $t \in \frac{1}{2}\mathbb{Z}$. Hence $\lim_{n\to\infty} \log_n \frac{c_n^*(A)}{exp^*(A)^n}$ exists and is an integer or half an integer.

2. The general setting

Throughout this paper, we shall denote by F a field of characteristic zero and by A an associative algebra with involution $*$ (or $*$-algebra) over F. We recall that $*$ is an antiautomorphism of order at most two. We refer to [25] for an account of classical results on algebras with involution.

Let $\mathcal{X} = \{x_1, x_2, \ldots\}$ be a countable set and let $F\langle \mathcal{X}, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$ be the free associative algebra with involution on $\mathcal{X}$ over F. Recall that $F\langle \mathcal{X}, *\rangle$ is characterized by the following universal property: if A is any algebra with involution, any set theoretical map $\mathcal{X} \to A$ can be uniquely extended to a homomorphism of algebras with involution. In order to simplify the notation we shall simply write $f(x_1, \ldots, x_n)$, to indicate a $*$-polynomial of $F\langle \mathcal{X}, *\rangle$ in which the variables $x_1, \ldots, x_n$ or their star appear.

Recall that $f(x_1, \ldots, x_n) \in F\langle \mathcal{X}, *\rangle$ is a $*$-polynomial identity (or simply a $*$-identity) of A and we write $f \equiv 0$ if $f(a_1, \ldots, a_n) = 0$, for all $a_1, \ldots, a_n \in A$.

We denote by $\mathrm{Id}^*(A) = \{f \in F\langle \mathcal{X}, *\rangle \mid f \equiv 0 \text{ on } A\}$ the set of $*$-polynomial identities of A. Clearly $\mathrm{Id}^*(A)$ is a T*-ideal of $F\langle \mathcal{X}, *\rangle$, i.e., an ideal invariant under all endomorphisms of the free algebra (commuting with the involution). It is well known that in characteristic zero $\mathrm{Id}^*(A)$ is completely determined by its multilinear polynomials and we denote by

$$P_n^* = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, w_i = x_i \text{ or } w_i = x_i^*, 1 \le i \le n\}$$

the space of multilinear $*$-polynomials of degree n in $x_1, \ldots, x_n$, i.e., for every $i = 1, \ldots, n$, either x_i or x_i^* appears in every monomial of P_n^* at degree 1 (but not both). Notice that $\dim P_n^* = 2^n n!$.

The symmetric group S_n acts on the left on P_n^* : if $\sigma \in S_n$ and $f = f(x_1, \ldots, x_n) \in P_n^*$, then

$$\sigma f = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Since the subspace $P_n^* \cap \mathrm{Id}^*(A)$ is invariant under this action,

$$P_n^*(A) = \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)}$$

has a structure of S_n-module and its dimension, $c_n^*(A)$, is called the nth $*$-codimension of A.

In order to capture the exponential rate of growth of the sequence of $*$-codimensions, in [14] the authors proved that for any associative $*$-algebra A, satisfying an ordinary identity, the limit

$$\exp^*(A) = \lim_{n\to\infty} \sqrt[n]{c_n^*(A)}$$

exists and is an integer. It is called the $*$-exponent of A. Moreover $\exp^*(A)$ can be explicitly computed; it turns out to be the dimension of a suitable finite dimensional semisimple $*$-algebra when the base field F is algebraically closed.

An important example of algebra with involution is $M_k(F)$, the algebra of $k \times k$ matrices over F. The significant involutions on $M_k(F)$ are the transpose involution t and the symplectic involution s. Recall that s is defined only when $k = 2m$ is even as follows: let $C \in M_{2m}(F)$ be written as $C = \begin{pmatrix} B & D \\ E & G \end{pmatrix}$ where B, D, E, G are $m \times m$ matrices; then $C^s = \begin{pmatrix} G^t & -D^t \\ -E^t & B^t \end{pmatrix}$. The relevance of t and s in PI-theory is given by the fact that if $*$ is any involution on $M_k(F)$, then $\mathrm{Id}^*(M_k(F)) = \mathrm{Id}^t(M_k(F))$ or $\mathrm{Id}^s(M_k(F))$ (see [21, Theorem 3.6.8]).

Now assume that A is a finite dimensional algebra with involution $*$ over an algebraically closed field F of characteristic zero.

By the Wedderburn-Malcev theorem [21, Theorem 3.4.4] for algebras with involution we can write

$$A = \bar{A} \oplus J$$

where $\bar{A}$ is a semisimple subalgebra of A, $J = J(A)$ is the Jacobson radical and both $\bar{A}$ and J are stable under the involution. Moreover

$$\bar{A} = A_1 \oplus \cdots \oplus A_q, \tag{4}$$

where $A_1, \ldots, A_q$ are $*$-simple algebras.

Recall that a $*$-simple algebra is either simple or a direct sum of a simple algebra and its opposite with exchange involution ([37, Proposition 2.13.24]). Also, as mentioned above, a simple algebra with involution has the same $*$-identities as the algebra $M_k(F)$ with transpose or symplectic involution. Hence in (4) we shall assume that there is $r \ge 0$ such that $A_i \cong M_{d_i}(F) \oplus M_{d_i}(F)^{op}$ with exchange involution if $i \le r$ and $A_i \cong M_{d_i}(F)$ with transpose or symplectic involution if $r+1 \le i \le q$. In other words among the $*$-simple algebras $A_1, \ldots, A_q$ only the last $q - r$ are simple. In order to simplify the notation we shall identify $A_i \equiv M_{d_i}(F)$, when $r+1 \le i \le q$, and $A_i \equiv M_{d_i}(F) \oplus M_{d_i}(F)^{op}$, when $i \le r$.

If we denote by e_i the unit element of the $*$-simple component A_i, then $1_{\bar{A}} = \sum_{i=1}^{q} e_i$ is the unit element of $\bar{A}$. In case A has a unit element $1 = 1_{\bar{A}}$, since for $i \neq k$, $e_i \bar{A} e_k = 0$, we have the decomposition

$$A = \oplus_{i,k=1}^{q} e_i A e_k = (\oplus_{i=1}^{q} e_i A_i e_i) \oplus (\oplus_{i,k=1}^{q} e_i J e_k).$$

When A does not have a unit element, we consider the algebra $A' = F \oplus A$ obtained from A by adjoining 1. Recall that the multiplication in A' is defined as follows: $(\alpha + a)(\beta + b) = \alpha\beta + \alpha b + \beta a + ab$, for every $\alpha + a, \beta + b \in A'$. Clearly $*$ on A extends to A' by defining $(\alpha + a)^* = \alpha + a^*$. Let $1_{A'}$ be the unit element of A'. Then if we define $e_0 = 1_{A'} - \sum_{i=1}^{q} e_i \in A'$, since $e_0 e_i = e_i e_0$ for $i \neq 0$, we have the decomposition

$$A' = F \oplus (\oplus_{i=1}^{q} e_i A_i e_i) \oplus (\oplus_{i,k=0}^{q} e_i J e_k).$$

Clearly the relation between the $*$-codimensions of A and A' is $c_n^*(A) \leq c_n^*(A')$. We remark that their exponential growth can also be different (see [22, Lemma 1]).

In what follows we shall be dealing with multilinear $*$-polynomials. Hence in order to check that any such polynomial is a $*$-identity of an algebra A, it will be enough to evaluate the variables on elements of a basis of A. To this end we choose a basis of our finite dimensional algebra $A = \bar{A} + J$ as the union of a basis of J and a basis of $\bar{A}$, which is the union of bases of the $*$-simple components.

Now, since $A_i = M_{d_i}(F)$ or $A_i = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$, we can decompose $e_i = \sum_{j=1}^{d_i} e_{j,j}^{(i)}$ or $e_i = \sum_{j=1}^{d_i} (e_{j,j}^{(i)}, 0) + \sum_{j=1}^{d_i} (0, e_{j,j}^{(i)})$, where the $e_{j,j}^{(i)}$'s are the matrix units of $M_{d_i}(F)$. By abuse of notation, in case $A_i = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$, $e_{j,j}^{(i)}$ will denote also $(e_{j,j}^{(i)}, 0)$ or $(0, e_{j,j}^{(i)})$. Hence we can write the elements of A as a linear combination of elements of the spaces

$$e_{j,j}^{(i)} A_i e_{k,k}^{(i)}, \quad e_{j,j}^{(i)} J e_{m,m}^{(l)}, \quad 1 \leq j,k \leq d_i,\ 1 \leq m \leq d_l,\ 1 \leq i,l \leq q$$

and, when A does not have a unit element, we have to add the spaces

$$e_{1,1}^{(0)} J e_{k,k}^{(i)}, \quad e_{j,j}^{(i)} J e_{1,1}^{(0)}, \quad e_{1,1}^{(0)} J e_{1,1}^{(0)},$$

where $e_{1,1}^{(0)} = e_0$.

Definition 2.1. *Let f be a $*$-polynomial. A substitution of all the variables of f with elements of one of the spaces $e_{j,j}^{(i)} A_i e_{k,k}^{(i)}$, $e_{j,j}^{(i)} J e_{k,k}^{(l)}$ is called an elementary substitution. A variable in an elementary substitution will be called semisimple or radical if it is evaluated in an element of $\bar{A}$ or J, respectively.*

3. Alternating polynomials

Let $f(x_1, \ldots, x_n, Y)$ be a $*$-polynomial depending on the variables $x_1, \ldots, x_n$ and on a finite set of variables $Y \subseteq \mathcal{X}$. We assume that f is linear in the variables $x_1, \ldots, x_n$, i.e., for every $i = 1, 2, \ldots, n$, either x_i or x_i^* appears in every monomial of f at degree 1. We say that f is alternating in $x_1, \ldots, x_n$ if f vanishes whenever we identify any two of these variables. Notice that we identify only the indices of the two variables leaving the exponents ($*$ or no $*$) unchanged. Since the characteristic of the base field is different from 2 this is equivalent to say that

$$f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, Y) = -f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n, Y), \text{ for all } 1 \leq i < j \leq n.$$

For instance $x_1^* x_2 - x_2 x_1^*$ is not alternating in x_1 and x_2 whereas $x_1 x_2^* - x_2 x_1^*$ is alternating in x_1 and x_2.

A basic example of an alternating polynomial is the nth Capelli polynomial defined as follows: if $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_{n+1}\}$ then

$$Cap_n(X, Y) = Cap_n(x_1, \ldots, x_n, y_1, \ldots, y_{n+1}) = \sum_{\sigma \in S_n} (\mathrm{sgn}\sigma) y_1 x_{\sigma(1)} y_2 x_{\sigma(2)} \cdots y_n x_{\sigma(n)} y_{n+1},$$

where S_n is the symmetric group. Such polynomial is alternating in the variables $x_1, \ldots, x_n$. It is well known that $Cap_{d^2}(X, Y)$ is not an identity of $d \times d$ matrices over F (see for instance [21, Prop. 1.7.1]). Hence, since any ordinary polynomial can be viewed as a $*$-polynomial, we get that $Cap_{d^2}(X, Y)$ is not a $*$-identity of A, where either $A = M_d(F) \oplus M_d(F)^{op}$ with exchange involution or $A = M_d(F)$ with transpose or symplectic involution. Moreover, any element $e_{h,k}$ of A can be obtained as an evaluation of $Cap_{d^2}(X,Y)$, where in case $A = M_d(F) \oplus M_d(F)^{op}$, $e_{h,k}$ denotes $(e_{h,k}, 0)$ or $(0, e_{h,k})$. For instance

$$Cap_{d^2}(e_{1,1}, e_{1,2}, \dots, e_{1,d}, e_{2,1}, \dots, e_{d,d-1}, e_{d,d},\ e_{h,1}, e_{1,1}, e_{2,1}, e_{d-1,d}, e_{d,k}) = e_{h,k}, \tag{5}$$

where for $x_1, \dots, x_{d^2}$, we substituted all the $e_{i,j}$'s ordered according to the left lexicographic order of the indices, the indeterminates y_1, y_{d^2+1} were replaced by $e_{h,1}, e_{d,k}$, respectively and for all other y_i's we made the unique substitution making $y_1x_1y_2x_2 \cdots x_{d^2}y_{d^2+1}$ the only monomial with non-zero evaluation. Clearly, in case $A = M_d(F) \oplus M_d(F)^{op}$, the $e_{i,j}$'s in the previous evaluation must have all the same component different from zero.

Proposition 3.1. *Let A be a $*$-simple algebra. For every $\mu \geq 1$ there exists a multilinear $*$-polynomial*

$$f(X_1, \dots, X_\mu, Y) \notin Id^*(A) \tag{6}$$

alternating on each of the disjoint sets $X_1, \dots, X_\mu$, where $|X_1| = \cdots = |X_\mu| = \dim(A)$ and $|Y| < \infty$.

Such a polynomial has the property that it can take any value of the type $e_{i,i}$, $1 \leq i \leq d$, when evaluated in A.

The proof of the Proposition follows from the following considerations.

Definition 3.1. *For every $\mu \geq 1$, define*

$$Cap_{\mu,n}(X_1, \dots, X_\mu, Y) := \prod_{i=1}^{\mu} Cap_n(X_i, Y_i),$$

where $X_i = \{x_{i,1}, \dots, x_{i,n}\}$, $Y_i = \{y_{i,1}, \dots, y_{i,n+1}\}$, $i = 1, \dots, \mu$, are distinct sets of variables and $Y = \cup Y_i$.

Notice that if $A = M_d(F)$ with transpose or symplectic involution, Cap_{μ,d^2} is the required polynomial (6) that we are searching for.

Recall that given any $*$-polynomial $f(x_1, \dots, x_n, Y)$ linear in each of the variables in $X = \{x_1, \dots, x_n\}$, the operator of alternation Alt_X is defined as

$$Alt_X f(x_1, \dots, x_n, Y) = \sum_{\sigma \in S_n} (\mathrm{sgn}\sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(n)}, Y).$$

The new polynomial $Alt_X f(x_1, \dots, x_n, Y)$ is multilinear and alternating in $x_1, \dots, x_n$.

Now given $X = \{x_1, \dots, x_{2n}\}$ and $Y = \{y_1, \dots, y_{2n+2}\}$ define

$$Cap_n(X, Y, *) := Cap_n(x_1, \dots, x_n, y_1, \dots, y_{n+1}) Cap_n(x^*_{n+1}, \dots, x^*_{2n}, y^*_{n+2}, \dots, y^*_{2n+2}).$$

Notice that $Cap_{d^2}(X, Y, *) \notin \mathrm{Id}^*(A)$, where $A = M_d(F) \oplus M_d(F)^{op}$, and any value of the type $e_{i,i} \in A$ can be obtained by evaluating the variables x_i, y_i, x^*_i and y^*_i as in (5).

Now by applying the operator of alternation to $Cap_n(X, Y, *)$ we get a $*$-polynomial:

$$G_{2n}(X, Y, *) = Alt_X Cap_n(X, Y, *)$$

multilinear and alternating in $x_1, \dots, x_{2n}$.

Definition 3.2. *For every $\mu \geq 1$, define*

$$G_{\mu,2n}(X_1, \dots, X_\mu, Y, *) := \prod_{i=1}^{\mu} G_{2n}(X_i, Y_i, *),$$

where $X_i = \{x_{i,1}, \dots, x_{i,2n}\}$, $Y_i = \{y_{i,1}, \dots, y_{i,2n+2}\}$, $i = 1, \dots, \mu$, are distinct sets of variables and $Y = \cup Y_i$.

Notice that if $A = M_d(F) \oplus M_d(F)^{op}$ with exchange involution, $G_{\mu,2d^2}$ is the polynomial (6) we are looking for. Moreover, by evaluating $G_{2d^2}(X, Y, *)$ as in (5), we can get, up to a scalar, any value of the type $e_{i,i}$, $1 \leq i \leq d$.

4. Star-Reduced algebras

In this section A will be a finite dimensional algebra with involution $*$ over an algebraically closed field F of characteristic zero. We write $A = \bar{A} \oplus J$, where $\bar{A} = A_1 \oplus \cdots \oplus A_q$ with $A_1, \ldots, A_q$ $*$-simple algebras and $s \geq 0$ is the smallest integer such that $J^{s+1} = 0$.

We make the following.

Definition 4.1. *The algebra A is $*$-reduced if up to a rearrangement of the $*$-simple components $A_1 J A_2 J \cdots J A_q \neq 0$.*

We remark that in the non-involution setting this property is called reduced or full. In the next lemma we use the symbol $\hat{}$ to indicate omission. For instance $A_1 \oplus \hat{A}_2 \oplus A_3 = A_1 \oplus A_3$.

Lemma 4.1. *A is $*$-reduced if and only if either A is $*$-simple or $Id^*(A) \subsetneq \cap_{i=1}^q Id^*(B_i)$, where $B_i = A_1 \oplus \cdots \oplus \hat{A}_i \oplus \cdots \oplus A_q + J$, $1 \leq i \leq q$.*

Proof. Let A be not $*$-simple. Suppose that there exists a multilinear $*$-polynomial f such that $f \notin \mathrm{Id}^*(A)$ and $f \in \bigcap_{i=1}^q \mathrm{Id}^*(B_i)$, and take a non-zero evaluation φ. Since f is a $*$-identity for each B_i, in order to get a non-zero evaluation of f, we must substitute at least one element from each $*$-simple component A_i. Let $\mathcal{M}$ be a monomial of f such that $\varphi(\mathcal{M}) \neq 0$. Since $A_i A_j = 0$ for $i \neq j$, between two variables of $\mathcal{M}$ that are evaluated in A_i and A_j, respectively, we must have a variable evaluated in J. Thus $A_{\sigma(1)} J A_{\sigma(2)} \cdots J A_{\sigma(q)} \neq 0$, for some rearrangement of the $*$-simple components.

Suppose now that A is $*$-reduced and $A_1 J A_2 J \cdots J A_q \neq 0$. Then there exist elements $a_1 \in A_1, \ldots, a_q \in A_q$ and $u_1, \ldots, u_{q-1} \in J$ such that

$$a_1 u_1 a_2 u_2 \cdots u_{q-1} a_q \neq 0.$$

If e_i denotes the unit element of A_i, set $v_i = a_i u_i$, $1 \leq i \leq q-1$, and from the above inequality we get

$$e_1 v_1 e_2 v_2 \cdots v_{q-1} e_q \neq 0.$$

This says that

$$e^{(1)}_{i_1,i_1} v_1 e^{(2)}_{i_2,i_2} \cdots v_{q-1} e^{(q)}_{i_q,i_q} \neq 0,$$

for some matrix units $e^{(j)}_{i_j,i_j} \in A_j$, and we may assume that $v_j \in e_j J e_{j+1}$. Recall that in case $A_j = M_{d_j}(F) \oplus M_{d_j}(F)^{op}$ then $e^{(j)}_{i_j,i_j}$ means either $(e^{(j)}_{i_j,i_j}, 0)$ or $(0, e^{(j)}_{i_j,i_j})$.

Now take any integer $\mu \geq q-1$ and write $\mu = t + q$. For each $j = 1, \ldots, q$ let

$$f_j = f_j(X^{(j)}_1, \ldots, X^{(j)}_\mu, Y^{(j)}) = Cap_{\mu, d_j^2}(X^{(j)}_1, \ldots, X^{(j)}_\mu, Y^{(j)})$$

or

$$f_j = f_j(X^{(j)}_1, \ldots, X^{(j)}_\mu, Y^{(j)}, *) = G_{\mu, 2d_j^2}(X^{(j)}_1, \ldots, X^{(j)}_\mu, Y^{(j)}, *)$$

according as $A_j = M_{d_j}(F)$ or $A_j = M_{d_j}(F) \oplus M_{d_j}(F)^{op}$ and let

$$f = f_1 z_1 f_2 z_2 \cdots z_{q-1} f_q.$$

Notice that the polynomial f depends on the integer t. If we evaluate f in A by evaluating f_j in A_j so that its value is $e^{(j)}_{i_j,i_j}$ and z_j in v_j, we get the value $e^{(1)}_{i_1,i_1} v_1 e^{(2)}_{i_2,i_2} \cdots v_{q-1} e^{(q)}_{i_q,i_q} \neq 0$. We call φ such an evaluation.

Set $X_l = X^{(1)}_l \cup \cdots \cup X^{(q)}_l$, $1 \leq l \leq \mu$, $Z_l = X_l \cup \{z_l\}$, $1 \leq l \leq q-1$, $Y = Y^{(1)} \cup \cdots \cup Y^{(q)}$ and let

$$\tilde{f} = \tilde{f}(Z_1, \ldots, Z_{q-1}, X_q, \ldots, X_\mu, Y) = Alt_{Z_1} \cdots Alt_{Z_{q-1}} Alt_{X_q} \cdots Alt_{X_\mu} f(Z_1, \ldots, Z_{q-1}, X_q, \ldots, X_\mu, Y)$$

be the polynomial obtained from f by alternating each set Z_l, $1 \leq l \leq q-1$, and each set X_i, $q \leq i \leq \mu$. Notice that $|Z_l| = d+1$ and $|X_i| = d$ where $d = \dim \bar{A}$.

We claim that for the above evaluation φ we have $\varphi(\tilde{f}) = \alpha \varphi(f) \neq 0$, for some integer α.

In fact, when we exchange two variables of an alternating set Z_l or X_i that are semisimple, if they are evaluated, say, in A_k and A_l, with $k \neq l$, the corresponding evaluation gives zero since $A_k A_l = A_l A_k = 0$. On the other hand suppose one of the two variables, say z_j, is a radical variable and let $\varphi(z_j) \in e_j J e_{j+1}$. Since e_j and e_{j+1} belong to distinct simple components, we still get zero when we exchange z_j with a semisimple variable. Hence $\varphi(\tilde{f})$ coincides with $\varphi(f)$ up to an integer α counting, for any $*$-simple component, the number of permutations of each alternating set. This proves the claim and $\tilde{f} \notin \mathrm{Id}^*(A)$.

Next we show that if the integer μ is taken such that $J^\mu = 0$, then $\tilde{f} \in \mathrm{Id}^*(B_i)$, $1 \leq i \leq q$. Recall that the polynomial $\tilde{f}$ is alternating on each set Z_j, $1 \leq j \leq q-1$, and on each set X_k, $q \leq k \leq \mu$. Moreover $|Z_j| = d+1$ and $|X_k| = d$. Since $\dim B_i/J < d$, in order to get a non-zero evaluation of $\tilde{f}$, we must evaluate at least one variable of each alternating set into a radical element. Since $J^\mu = 0$ we get that $\tilde{f} \in \mathrm{Id}^*(B_i)$, as wished. ☐

Definition 4.2. *A multilinear $*$-polynomial f such that $f \in \bigcap_{i=1}^q Id^*(B_i)$ and $f \notin Id^*(A)$ will be called $*$-reduced.*

5. The Kemer index for algebras with involution

As in the previous section A will be a finite dimensional algebra with involution $*$ over an algebraically closed field F of characteristic zero. We have $A = \bar{A} \oplus J$ where $\bar{A} = A_1 \oplus \cdots \oplus A_q$ with $A_1, \ldots, A_q$ $*$-simple algebras and $J = J(A)$.

Definition 5.1. *The (t,s)-index of A is $Ind_{t,s}(A) = (\dim \bar{A}, s_A)$ where $s_A \geq 0$ is the smallest integer such that $J^{s_A+1} = 0$.*

Notice that this is the same as the (t,s)-index of A as an algebra without involution.

Definition 5.2. *Let $\Gamma \subseteq F\langle \mathcal{X}, *\rangle$ be a T^*-ideal. We define $\beta(\Gamma)$ to be the greatest integer t such that for every $\mu \geq 1$, there exists a multilinear $*$-polynomial $f(X_1, \ldots, X_\mu, Y) \notin \Gamma$ alternating in the μ sets X_i with $|X_i| = t$. Then we define $\gamma(\Gamma)$ to be the greatest integer s for which there exists for all $\mu \geq 1$, a multilinear $*$-polynomial $f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_s, Y) \notin \Gamma$ alternating in the μ sets X_i with $|X_i| = \beta(\Gamma)$ and in the s sets Z_j with $|Z_j| = \beta(\Gamma) + 1$.*

Definition 5.3. *$Ind_K^*(\Gamma) = (\beta(\Gamma), \gamma(\Gamma))$ is called the Kemer $*$-index of Γ.*

In case $\Gamma = \mathrm{Id}^*(A)$, we also say that $(\beta(\Gamma), \gamma(\Gamma)) = (\beta(A), \gamma(A)) = Ind_K^*(A)$ is the Kemer $*$-index of A. Even if Definition 5.2 and 5.3 make sense for a wider class of $*$-algebras, i.e., finitely generated PI-algebras, we shall make use of them only for finite dimensional algebras.

As an example we consider $A = M_d(F) \oplus M_d(F)^{op}$ with exchange involution. For every $\mu \geq 1$ the polynomial $G_{\mu,2d^2}$ given in Definition 3.2 is alternating in the μ sets X_i with $|X_i| = 2d^2$ and is not a $*$-identity of A. Moreover since $\dim A = 2d^2$ any $*$-polynomial alternating in $2d^2 + 1$ elements is a $*$-identity of A. Hence $\beta(A) = 2d^2$ and $\gamma(A) = 0$. Notice that in case $A = M_d(F)$ with transpose or symplectic involution, $Ind_K^*(A) = (d^2, 0)$. In fact $Cap_{\mu,d^2} \notin \mathrm{Id}^*(A)$ and it has the prescribed properties.

We remark that by the definition of $\gamma(\Gamma)$ there exists a smallest integer μ_0 such that every multilinear $*$-polynomial $f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_{\gamma(\Gamma)+1}, Y)$, alternating in $\mu \geq \mu_0$ sets X_i with $\beta(\Gamma)$ elements and in $\gamma(\Gamma)+1$ sets Z_j with $\beta(\Gamma) + 1$ elements lies in Γ.

Definition 5.4. *A multilinear $*$-polynomial $f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_{\gamma(\Gamma)}, Y) \notin \Gamma$ which is alternating in $\mu > \mu_0$ sets X_i with $|X_i| = \beta(\Gamma)$ and in $\gamma(\Gamma)$ sets Z_i with $|Z_i| = \beta(\Gamma) + 1$ is called a Kemer $*$-polynomial related to Γ.*

Remark 5.1. *If A is a finite dimensional $*$-algebra, then $Ind_K^*(A) \leq Ind_{t,s}(A)$ in the left lexicographic order.*

Proof. Let $f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_{\gamma(A)}, Y) \notin \mathrm{Id}^*(A)$ be a Kemer $*$-polynomial with $|X_i| = \beta(A)$, $|Z_i| = \beta(A)+1$. If $\beta(A) > \dim \bar{A}$, then in order to have a non-zero evaluation of f in A we have to evaluate at least one variable of each set X_i into J. Since μ can be taken arbitrarily large and J is nilpotent we get a contradiction. Hence $\beta(A) \leq \dim \bar{A}$. Now if $\beta(A) = \dim \bar{A}$ then $\gamma(A) \leq s_A$, where $s_A \geq 0$ is such that $J^{s_A+1} = 0$. If not, in order to have a non-zero evaluation we must evaluate at least one variable of each Z_i into J. Since the number of Z_i's is greater than s_A we get a contradiction. ☐

In what follows, by abuse of notation we shall write $s_A = s$.

6. Star-fundamental algebras

We start with the following construction. Let $A = \bar{A} + J$ be a finite dimensional algebra with involution over an algebraically closed field F, $J^s \neq 0$, $J^{s+1} = 0$ and let $n = \dim J$. Then define

$$A' = \bar{A} * F\langle x_1, \dots, x_n, * \rangle,$$

the free product of $\bar{A}$ and the free algebra $F\langle x_1, \dots, x_n, * \rangle$. Clearly

$$A' = \bar{A} \oplus I,$$

where I is the $*$-ideal of A' generated by $x_1, \dots, x_n$. If I_1 is the $*$-ideal generated by $\{f(A') \mid f \in \mathrm{Id}^*(A)\}$, then since $f(\bar{A}) = 0$, for $f \in \mathrm{Id}^*(A)$, we have that $I_1 \subseteq I$.

We define

$$\mathcal{A}_s = A'/(I^{s+1} + I_1).$$

Since $\mathrm{Id}^*(A)$ is generated by its multilinear $*$-polynomials, I_1 is also generated by the evaluations of the multilinear $*$-polynomials in $\mathrm{Id}^*(A)$. In order to get a close connection between A and $\mathcal{A}_s$, we notice that by the universal property of the free product, given any elements $a_1, \dots, a_n \in A$ there is a unique $*$-homomorphism $\varphi_{a_1,\dots,a_n} : A' \to A$ such that $\varphi_{a_1,\dots,a_n}$ is the identity on $\bar{A}$ and $\varphi_{a_1,\dots,a_n}(x_i) = a_i$, $1 \leq i \leq n$. Now take $a_1, \dots, a_n \in J$. Then in this case $I^{s+1} + I_1 \subseteq Ker\varphi_{a_1,\dots,a_n}$ and $\varphi_{a_1,\dots,a_n}$ induces a $*$-homomorphism which we still call $\varphi_{a_1,\dots,a_n} : \mathcal{A}_s \to A$.

In particular if we choose $a_1, \dots, a_n$ as generators of J, then $\varphi_{a_1,\dots,a_n} : \mathcal{A}_s \to A$ becomes surjective and, so, A is isomorphic to a quotient of $\mathcal{A}_s$. It follows that $\mathrm{Id}^*(\mathcal{A}_s) \subseteq \mathrm{Id}^*(A)$.

The basic properties of the algebra $\mathcal{A}_s$ are the following.

Lemma 6.1.

1) *$\mathcal{A}_s$ is a finite dimensional algebra and $Id^*(\mathcal{A}_s) = Id^*(A)$.*
2) *$Ind_{t,s}(\mathcal{A}_s) = Ind_{t,s}(A)$.*
3) *Any evaluation of a multilinear $*$-polynomial f in A factorizes through an evaluation of f in $\mathcal{A}_s$ in the following sense: given a multilinear $*$-polynomial $f(y_1, \dots, y_m)$ and an evaluation $f(a_1, \dots, a_m) \in A$ where $a_1, \dots, a_k \in \bar{A}$, $a_{k+1}, \dots, a_m \in J$, there is an evaluation $f(a_1, \dots, a_k, x_1, \dots, x_{m-k}) \in \mathcal{A}_s$ such that*

$$f(a_1, \dots, a_m) = \varphi_{a_{k+1},\dots,a_m} f(a_1, \dots, a_k, x_1, \dots, x_{m-k}),$$

where $\varphi_{a_{k+1},\dots,a_m} : \mathcal{A}_s \to A$ is the homomorphism which is the identity map on $\bar{A}$ and $\varphi_{a_{k+1},\dots,a_m}(x_i) = a_{k+i}$, $1 \leq i \leq m-k$.

Proof.

1) The algebra A'/I^{s+1} is finite dimensional. In fact, its coset representatives are a linear combination of words of the type $a_0 w_1 a_1 w_2 a_2 \cdots w_t a_t$ where the a_i are elements of $\bar{A}$, the w_i are words in the x_i and x_i^*, $1 \leq i \leq n$, and the degree of $w_1 w_2 \cdots w_t$ is at most s. Since $\mathcal{A}_s = A'/(I^{s+1} + I_1) \cong \frac{A'/I^{s+1}}{(I^{s+1}+I_1)/I^{s+1}}$ is isomorphic to a quotient of A'/I^{s+1}, then also $\mathcal{A}_s$ is finite dimensional.

Since I_1 is the $*$-ideal of A' generated by all valuations of the $*$-identities of A in A', $\mathrm{Id}^*(A) \subseteq \mathrm{Id}^*(\mathcal{A}_s)$. The other inclusion was proved earlier.

2) As we remarked above, $I_1 \subseteq I$. Then consider the ideal $I' = I/(I^{s+1} + I_1)$ of $\mathcal{A}_s$. We have that $\mathcal{A}_s/I' \cong A'/I \cong \bar{A}$ and since $(I')^{s+1} = 0$, $\bar{A}$ is a maximal semisimple subalgebra of $\mathcal{A}_s$. Thus $Ind_{t,s}(\mathcal{A}_s) = (\dim \bar{A}, -)$.

As we remarked above, if $a_1, \dots, a_n$ are generators of J, then $\varphi_{a_1,\dots,a_n} : \mathcal{A}_s \to A$ is surjective and I' is mapped onto J. Since $J^s \neq 0$ then also $(I')^s \neq 0$. Thus $Ind_{t,s}(\mathcal{A}_s) = (\dim \bar{A}, s) = Ind_{t,s}(A)$.

3) This follows from the construction of $\mathcal{A}_s$. □

From the above discussion we have that $\mathcal{A}_s \cong \bar{A} + I'$, $(I')^s \neq 0$, $(I')^{s+1} = 0$ and $Ind_{t,s}(\mathcal{A}_s) = (\dim \bar{A}, s)$.

Definition 6.1. *We define*

$$\mathcal{B}_0 = \mathcal{A}_s/(I')^s.$$

Hence $Id^*(A) = Id^*(\mathcal{A}_s) \subseteq Id^*(\mathcal{B}_0)$, and $Ind_{t,s}(\mathcal{B}_0) = (\dim \bar{A}, s-1)$.

We fix again the notation. Let $A = \bar{A} + J$, $\bar{A} = A_1 \oplus \cdots \oplus A_q$, where the A_i's are $*$-simple algebras and let $s \geq 0$ be the smallest integer such that $J^{s+1} = 0$.

Now, for any $1 \leq i \leq q$ we denote

$$B_i = A_1 \oplus \cdots \hat{A}_i \cdots \oplus A_q + J,$$

where the symbol $\hat{A}_i$ means that the algebra A_i is omitted in the direct sum.

Definition 6.2. *The algebra A is $*$-fundamental if either A is $*$-simple or $s > 0$ and*

$$Id^*(A) \subsetneq \cap_{i=1}^q Id^*(B_i) \cap Id^*(\mathcal{B}_0).$$

In this case a multilinear $*$-polynomial f is $*$-fundamental if $f \in \cap_{i=1}^q Id^*(B_i) \cap Id^*(\mathcal{B}_0)$ and $f \notin Id^*(A)$.

We remark that in case $s > 0$, the algebras B_i have a lower (t, s)-index. In fact the first index is lower. Also the algebra $\mathcal{B}_0$ has a lower (t, s)-index, since $Ind_{t,s}(\mathcal{B}_0) = (\dim \bar{A}, s - 1)$.

It is clear that any $*$-fundamental algebra is $*$-reduced by Lemma 4.1. We should mention that in the non-involution setting fundamental algebras are also called basic algebras (see for instance [1], [2]).

Proposition 6.1. *Every finite dimensional algebra with involution has the same $*$-identities as a finite direct sum of $*$-fundamental algebras.*

Proof. If A is semisimple the conclusion of the proposition is clear since each A_i is $*$-fundamental. Let $J \neq 0$ and suppose A not $*$-fundamental so that $\mathrm{Id}^*(A) = \cap_{i=1}^q \mathrm{Id}^*(B_i) \cap \mathrm{Id}^*(\mathcal{B}_0)$. Hence A has the same $*$-identities as $B_1 \oplus \cdots \oplus B_q \oplus \mathcal{B}_0$ and each summand has a lower (t, s)-index. The proof is completed by induction on the (t, s)-index. □

Next we want to characterize $*$-fundamental algebras in terms of the Kemer $*$-index. To this end we make the following.

Definition 6.3. *A multilinear $*$-polynomial f has property K with respect to A if every non-zero elementary evaluation in A has precisely s radical substitutions.*

Lemma 6.2. *Let f be a multilinear $*$-polynomial.*

1) *f is $*$-reduced if and only if in every non-zero elementary evaluation $f(a_1, \ldots, a_m)$, for every i, $1 \leq i \leq q$, there is at least one variable x_j evaluated in $a_j \in A_i$.*
2) *Let f be $*$-reduced and $s > 0$. Then f is $*$-fundamental if and only if f has property K.*

Proof. 1) Suppose $f(a_1, \ldots, a_m) \neq 0$. If no variable of f is evaluated in a $*$-simple component, say A_k, then f is actually evaluated in B_k. Since f is $*$-reduced, $f \in \bigcap_{i=1}^q \mathrm{Id}^*(B_i)$; hence in particular $f \in \mathrm{Id}^*(B_k)$ and $f(a_1, \ldots, a_m) = 0$, a contradiction. A similar argument proves also the converse.

2) Let f be $*$-fundamental. We shall prove that f has property K.

Consider an elementary evaluation $f(a_1, \ldots, a_m) \neq 0$ in A. Since $J^{s+1} = 0$, at most s among the a_i's belong to J and let $a_1, \ldots, a_k \in \bar{A}$, $a_{k+1}, \ldots, a_m \in J$.

By the universal property of $\mathcal{A}_s$ given in Lemma 6.1, $f(a_1, \ldots, a_m) = \varphi_{a_{k+1},\ldots,,a_m} f(a_1, \ldots, a_k, x_1, \ldots, x_{m-k})$, where $\varphi_{a_{k+1},\ldots,,a_m} : \mathcal{A}_s \to A$ is the $*$-homomorphism which is the identity map on $\bar{A}$ and maps $x_i \to a_{k+i}$, $1 \leq i \leq m - k$.

Now, if $m - k \leq s - 1$, i.e., at most $s - 1$ of the a_i's belong to J, $f(a_1, \ldots, a_k, x_1, \ldots, x_{m-k})$ is still non-zero in the projection $\mathcal{A}_s \to \mathcal{A}_s/(I')^s = \mathcal{B}_0$. Since by hypothesis f is a $*$-identity of $\mathcal{B}_0$, we get a contradiction.

Conversely, suppose that a multilinear $*$-polynomial f has property K. Being f $*$-reduced we already know that $f \in \cap_{i=1}^q \mathrm{Id}^*(B_i)$. Moreover if every non-zero elementary evaluation has s radical substitutions, the corresponding $*$-polynomial through which f factorizes is a $*$-identity of $\mathcal{B}_0$ and, so, $f \in \mathrm{Id}^*(\mathcal{B}_0)$. □

The proof of the following theorem follows the main lines of the original proof of Kemer and the approach given by the authors in [3]. The reader should keep in mind that the ingredients of the proof are $*$-polynomials versus ordinary polynomials and $*$-simple algebras versus simple algebras. In the upcoming construction of Kemer $*$-polynomials the multialternating polynomials corresponding to $*$-simple components which are not simple are those given in Definition 3.2 and they differ from the ordinary ones.

Theorem 6.1. *Let A be a finite dimensional algebra with involution. Then A is $*$-fundamental if and only if $Ind^*_K(A) = Ind_{t,s}(A)$.*

Proof. Suppose $Ind^*_K(A) = Ind_{t,s}(A)$. If A is not $*$-fundamental by Proposition 6.1 and its proof, A satisfies the same $*$-identities as a finite direct sum of $*$-fundamental algebras $B_1 \oplus \cdots \oplus B_r$ having lower (t,s)-index. Since the Kemer $*$-index of a direct sum is the largest of the Kemer $*$-indices of the components, and $Ind^*_K(B_i) \leq Ind_{t,s}(B_i)$, we get $Ind^*_K(A) = \max_i Ind^*_K(B_i) \leq \max_i Ind_{t,s}(B_i) < Ind_{t,s}(A)$, a contradiction.

Now we assume that A is a $*$-fundamental algebra and we distinguish several cases. Recall that $A = \bar{A} + J$, $\bar{A} = A_1 \oplus \cdots \oplus A_q$, with $A_i = M_{d_i}(F)$ or $A_i = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$ and $J^{s+1} = 0$.

CASE 1. $s = 0$. In this case A is $*$-simple. Hence the Kemer $*$-index of A is $(\dim A, 0)$, and so, it coincides with the (t,s)-index.

CASE 2. $\dim \bar{A} = 0$. We have $A = J$, a nilpotent algebra. Then $x_1 \cdots x_s$ is a Kemer $*$-polynomial and $Ind^*_K(A) = Ind_{t,s}(A) = (0, s)$.

CASE 3. $\dim \bar{A} > 0$ and $s > 0$. Let $f(z_1, \dots, z_s, y_1, \dots, y_m)$ be a $*$-fundamental polynomial for A. Recall that f has property K and, so, any non-zero evaluation has precisely s radical substitutions and also, since f is $*$-reduced, all the $*$-simple components must appear among the semisimple evaluations. So, denote by $\eta : F\langle \mathcal{X}, * \rangle \to A$ a non-zero elementary evaluation, i.e, $\eta(f) = f(r_1, \dots, r_s, b_1, \dots, b_m) \neq 0$, where $r_1, \dots, r_s \in J$ and $b_1, \dots, b_m \in \bar{A}$.

Out of the polynomial f we shall construct a Kemer $*$-polynomial for A alternating on μ sets each of size $t = \dim \bar{A}$ and on s sets each of size $\dim \bar{A} + 1$. It will follow that $Ind^*_K(A) = Ind_{t,s}(A)$.

We need to distinguish the cases $q > 1$ and $q = 1$.

CASE 3.1. Let $q > 1$ and let $\mathcal{M}$ be a monomial in the elements $r_1, \dots, r_s, b_1, \dots, b_m$ appearing in the evaluation η of f which is non-zero. We may clearly assume that all variables appearing in $\mathcal{M}$ are without $*$. Given a $*$-simple component A_i there is an $a_i \in A_i$, $a_i \in \{b_1, \dots, b_m\}$ such that either $\mathcal{M} = w a_i r_{t_i} a_j w'$ or $\mathcal{M} = w a_j r_{t_i} a_i w'$, for some $a_j \in A_j$, $a_j \in \{b_1, \dots, b_m\}$, $j \neq i$, and for some $r_{t_i} \in J$. Here w, w' are eventually empty monomials in the remaining elements. In this way we associate to every $*$-simple component A_i a radical element r_{t_i} and, so, a radical variable z_{t_i} that we call *selected*.

Notice that it can happen that the same radical variable is associated to two distinct $*$-simple components one to the left and one to the right of the variable. In any case the number of selected radical variables is at most q. Let $z_{t_1}, \dots, z_{t_u}$ be such radical variables, $u \leq q$.

Let $\nu = \mu + s > q + s$ be any integer. For every $i \in \{1, \dots, q\}$ let $f_{\nu,c_i}(X^{(i)}) = f_{\nu,c_i}(X^{(i)}_1, \dots, X^{(i)}_\nu, Y^{(i)})$ be the $*$-polynomial constructed in Definition 3.1 or 3.2 alternating on each of the sets $X^{(i)}_j$, $|X^{(i)}_j| = c_i = \dim A_i$ and taking a non-zero value in A_i.

Next we introduce new variables $u_1, v_1, \dots, u_q, v_q$ distinct from the variables of f and from those of $f_{\nu,c_i}(X^{(i)})$, for $1 \leq i \leq q$. We construct a map $\psi : F\langle \mathcal{X}, * \rangle \to F\langle \mathcal{X}, * \rangle$ which is the identity on all the non selected variables and if a selected variable $z_l \in \{z_{t_1}, \dots, z_{t_u}\}$ is associated to a $*$-simple component A_i which lies to the left, we set $\psi(z_l) = u_i f_{\nu,c_i}(X^{(i)}) v_i z_l$. On the other hand if z_l is associated to a $*$-simple component A_i which lies to the right, we set $\psi(z_l) = z_l u_i f_{\nu,c_i}(X^{(i)}) v_i$. Finally if z_l is associated to two $*$-simple components A_i to the left and A_j to the right, we set $\psi(z_l) = u_i f_{\nu,c_i}(X^{(i)}) v_i z_l u_j f_{\nu,c_j}(X^{(j)}) v_j$. Define $\hat{f} = \psi(f)$. Recall that if a selected variable z_l is evaluated in r_l, we may assume that $r_l = e^{(i)}_{h,h} r_l e^{(j)}_{k,k}$, for some idempotents $e^{(i)}_{h,h}, e^{(j)}_{k,k}$ with $h \in \{1, \dots, d_i\}$, $k \in \{1, \dots, d_j\}$ and $i \neq j$. Notice that in case $A_i = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$, by $e^{(i)}_{h,h}$ we mean either $(e^{(i)}_{h,h}, 0)$ or $(0, e^{(i)}_{h,h})$.

Next we extend the evaluation η to the new variables as follows. If a selected variable z_l is associated to a $*$-simple component A_i and z_l is evaluated in $r_l = e^{(i)}_{h,h} r_l e^{(j)}_{k,k}$, the variables u_i, v_i and those appearing in $f_{\nu,c_i}(X^{(i)})$ take values in A_i in such a way that $\eta(f_{\nu,c_i}(X^{(i)})) = \eta(u_i) = \eta(v_i) = e^{(i)}_{h,h}$. A similar evaluation is performed in case z_l is associated to the right or to both sides of $*$-simple components.

Notice that $\eta(\psi(z_l)) = \eta(z_l)$ so that $\eta(\hat{f}) = \eta(f) \neq 0$.

Set

$$X_j = X^{(1)}_j \cup \cdots \cup X^{(q)}_j, \quad 1 \leq j \leq \nu = \mu + s$$

and

$$Z_i = X_{\mu+i} \cup \{z_i\}, \quad 1 \le i \le s.$$

Hence $|X_i| = \dim \bar{A}$ and $|Z_i| = \dim \bar{A} + 1$.

Next we alternate in the previous polynomial $\hat{f}$ independently each set X_i, $1 \le i \le \mu$, and each set Z_i, $1 \le i \le s$, and we set

$$g = Alt_{X_1} \cdots Alt_{X_\mu} Alt_{Z_1} \cdots Alt_{Z_s} \hat{f}$$

the $*$-polynomial so obtained. Hence g is alternating on μ sets each of size $t = \dim \bar{A}$ and on s sets each of size $\dim \bar{A} + 1$. Since $Ind^*_K(A) \le Ind_{t,s}(A)$, if we prove that g has a non-zero evaluation in A, it will follow that g is a Kemer $*$-polynomial and $Ind^*_K(A) = Ind_{t,s}(A)$ will follow, as wished.

Let S_U be the symmetric group acting on the set U. Then we can write g as

$$g = \sum_{\sigma \in G} (sgn\sigma)\sigma\hat{f}, \tag{7}$$

where $G := \prod_{i=1}^{\mu} S_{X_i} \times \prod_{j=1}^{s} S_{Z_j}$. Consider the subgroup $H = \prod_{j=1}^{\nu} \prod_{i=1}^{q} S_{X_j^{(i)}}$. Clearly if $\sigma \in H$, $(sgn\sigma)\sigma\hat{f} = \hat{f}$ and, so,

$$\eta(g) = \eta(\sum_{\sigma \in G} (sgn\sigma)\sigma\hat{f}) = \eta(\sum_{\sigma \in G \setminus H} (sgn\sigma)\sigma\hat{f}) + |H|\eta(\hat{f}).$$

Since $\eta(\hat{f}) \neq 0$, in order to show that $\eta(g) \neq 0$ it is enough to prove that $\eta(\sigma\hat{f}) = 0$ for any $\sigma \in G \setminus H$.

Now, if $\sigma(z_t) = z_t$ for all radical variables there is at least one variable in some $X_k^{(i)}$ which is exchanged with some variable in $X_k^{(j)}$ with $i \neq j$. Then in the evaluation $\eta(\sigma(u_i f_{\nu,c_i}(X^{(i)})v_i))$ one variable in $X_k^{(i)}$ is evaluated in an element of A_j, $j \neq i$. But since u_i and v_i are evaluated in A_i and $A_i A_k = A_k A_i = 0$ for all $k \neq i$, it follows that $\eta(\sigma\hat{f}) = 0$.

Hence we may assume that $\sigma \in G \setminus H$ is such that $\sigma(z_t) \neq z_t$ for some t.

Let η' be the evaluation of $f = f(z_1, \ldots, z_s, y_1, \ldots, y_m)$ such that $\eta'(y_i) = \eta(y_i)$ and $\eta'(z_t) = \eta(\sigma\psi(z_t))$. It follows that $\eta'(f) = \eta(\sigma\hat{f})$. Notice that if z_t is not a selected variable, $\eta'(z_t) = \eta(\sigma(z_t))$.

A basic remark is the following.

Remark 6.1. *In the evaluation η' all the variables y_i remain semisimple while some of the radical variables z_t may become semisimple. If this happens, $\eta'(f) = 0$ by property K.*

Suppose first that $\sigma(z_t) \neq z_t$ for a non-selected variable z_t. Then $\sigma(z_t) \in X_{\mu+t}$ and, so, $\eta'(z_t) = \eta(\sigma(z_t))$ is a semisimple element. Hence the evaluation $\eta'(f) = \eta(\sigma\hat{f})$ vanishes by Remark 6.1 and we are done.

Therefore we may assume that all the non-selected variables are fixed by σ.

Let $z_1, \ldots, z_k$ be the selected variables exchanged with elements $x_i \in X_{\mu+i}$ by σ. If for one of these variables z_l we have $\eta'(z_l) = \eta(\sigma(\psi(z_l)) = 0$ we are done since $\eta'(f) = 0$ in this case. The same conclusion holds if one of the elements $\eta'(z_l) = \eta(\sigma(\psi(z_l))$ is a semisimple element by Remark 6.1.

Consider now z_l, $1 \le l \le k$, and suppose for instance that $\psi(z_l) = u_i f_{\nu,c_i}(X^{(i)})v_i z_l$ so that $\sigma(\psi(z_l)) = u_i\sigma(f_{\nu,c_i}(X^{(i)}))v_i\sigma(z_l)$. We consider the case when the polynomial $f_{\nu,c_i}(X^{(i)})$ is associated to the left of the variable z_l. The other cases are treated similarly.

Since $\sigma(z_l) \neq z_l$, $\sigma(z_l) \in X_{\mu+l}$ is a semisimple element. The evaluation of $\eta'(z_l)$ is

$$\eta'(z_l) = \eta(\sigma(\psi(z_l))) = \eta(u_i\sigma(f_{\nu,c_i}(X^{(i)}))v_i)\eta(\sigma(z_l))$$

$$= e_{h,h}^{(i)}\eta(\sigma(f_{\nu,c_i}(X^{(i)}))e_{h,h}^{(i)}\eta(\sigma(z_l))$$

and if it is 0 or semisimple we are done by property K. Hence we may assume that $\eta'(z_l) \neq 0$ is radical and it happens if $\eta(\sigma(f_{\nu,c_i}(X^{(i)})))$ is a non-zero radical element of $e_i J e_i$ and also $\eta(\sigma(z_l)) \in A_i$.

But if $\eta(\sigma(f_{\nu,c_i}(X^{(i)})))$ is a radical element this means that we have substituted through σ some of the variables of $X_l^{(i)}$ with some selected radical variables. Recall that the selected variables are evaluated in some $e_a J e_b$ with $a \neq b$.

Hence if only one say z_t is exchanged in $f_{\nu,c_i}(X^{(i)})$ and $\eta(z_t) \in e_aJe_b$, since all the other elements of $\sigma(f_{\nu,c_i}(X^{(i)}))$ are in $\cup X_j, j = 1,\dots,\nu$, we obtain that $\eta(\sigma(f_{\nu,c_i}(X^{(i)})) \in e_aJe_b$ or e_bJe_a, $a \neq b$, contrary to our assumption.

It follows that at least two selected variables must be exchanged in $f_{\nu,c_i}(X^{(i)})$. This happens for each of the selected variables z_l, $1 \le l \le k$. Hence we have substituted in each polynomial $f_{\nu,c_i}(X^{(i)})$ associated to a variable z_l, $1 \le l \le k$, at least two selected variables z_t, $1 \le t \le k$. Since such polynomials associated to distinct selected variables involve distinct variables in $\cup X_j$, we need at least $2k$ selected variables z_l. But this is impossible since we have only k such variables at our disposal.

CASE 3.2. Let $q = 1$. We distinguish two subcases. Assume first that the algebra A has a unit e_1. Recall that f is a $*$-fundamental polynomial for A with a non-zero evaluation $\eta(f) \neq 0$ as above.

Hence $e_1\eta(f) = \eta(f)$ and there exists an index h such that $e^{(1)}_{h,h}\eta(f) \neq 0$.

Consider the polynomial $f_{\nu,c_1}(X^{(1)})f$ where $f_{\nu,c_1}(X^{(1)}) = f_{\nu,c_1}(X^{(1)}_1,\dots,X^{(1)}_\nu, Y^{(1)})$ is, as above, the $*$-polynomial constructed in Definition 3.1 or 3.2 with $\nu = \mu + s$. Clearly η can be extended to an evaluation of $f_{\nu,c_1}(X^{(1)})f$ such that $\eta(f_{\nu,c_1}(X^{(1)})f) = e^{(1)}_{h,h}\eta(f) \neq 0$. Next we perform the alternation of the s sets $Z_i = X^{(1)}_{\mu+i} \cup \{z_i\}$, $1 \le i \le s$, obtaining the polynomial $\tilde{f}$ and we show that the evaluation $\eta(\tilde{f})$ is non-zero. In fact, when in the alternation of the s sets $Z_i = X^{(1)}_{\mu+i} \cup \{z_i\}$, we export a radical variable outside the polynomial f and we import in f a semisimple variable as above, by property K the value of $\tilde{f}$ is zero. Hence $\tilde{f}$ is a Kemer $*$-polynomial and we are done.

Finally assume that $q = 1$ and the algebra $A = A_1 + J$ does not have a unit. We regard J as a (A_1, A_1)-bimodule by considering the left and right multiplication by the unit element of A_1. Then J decomposes into the direct sum of bimodules (see [19, Lemma 2])

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11},$$

where for $i \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. Similarly, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively. Moreover, for $i, k, l, m \in \{0, 1\}$, $J_{ik}J_{lm} \subseteq \delta_{kl}J_{im}$ where δ_{kl} is the Kronecker delta.

If all the variables appearing in f are evaluated in $A_1 + J_{11}$ we can replace A with $A_1 + J_{11}$ and we are in the previous case of an algebra with 1. Otherwise, since f is $*$-fundamental and $s > 0$ at least one variable is evaluated under η into $J_{10} \cup J_{01} \cup J_{00}$. Since η is a non-zero evaluation and f is $*$-reduced at least one variable must be evaluated in J_{10} or J_{01}. Let z_1 be such variable evaluated for instance in J_{10} and assume that $\eta(z_1) = e^{(1)}_{h,h}r_1$, with $r_1 \in J$.

Then we replace the variable z_1 with $uf_{\nu,c_1}(X^{(1)})vz_1$, where $f_{\nu,c_1}(X^{(1)}) = f_{\nu,c_1}(X^{(1)}_1,\dots,X^{(1)}_\nu, Y^{(1)})$ with $\nu = \mu + s$, and let $\hat{f}$ be the resulting $*$-polynomial. We extend η to an evaluation of $\hat{f}$ as above so that $\eta(f_{\nu,c_1}(X^{(1)})) = \eta(u) = \eta(v) = e^{(1)}_{h,h}$ and $\eta(\hat{f}) = \eta(f) \neq 0$. Next we perform the alternation of the s sets $Z_i = X_{\mu+i} \cup \{z_i\}$, $1 \le i \le s$, and we show that the same evaluation η is non-zero.

As before, when in the alternation a variable $z_i \neq z_1$ is substituted in $X^{(1)}_{\mu+i}$ we get zero by property K. Finally if z_1 is substituted in $X^{(1)}_{\mu+i}$, the $*$-polynomial $uf_{\nu,c_1}(X^{(1)})v$ vanishes since z_1 is evaluated in J_{10}. In this way we obtain a Kemer $*$-polynomial and we are done. □

7. Some tools

Recall that two functions $f(x), g(x)$ of a real variable are asymptotically equal and we write $f(x) \simeq g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. Also if A is an algebra with involution $*$, we denote by $A^- = \{a \in A \mid a = -a^*\}$ the set of skew elements of A. Recall that A^- is a Lie algebra under the bracket $[x, y] = xy - yx$.

We start by recalling the following result proved in [7]

Theorem 7.1. *Let $M_d(F)$ be the algebra of $d \times d$ matrices over F with transpose or symplectic involution. Then*

1) *If $* = t$ is the transpose involution, $c^*_n(M_d(F)) \simeq C_1 n^{-\frac{1}{4}d(d-1)} d^{2n}$, for some constant C_1.*

2) *If* $* = s$ *is the symplectic involution,* $c_n^*(M_d(F)) \simeq C_2 n^{-\frac{1}{4}d(d+1)} d^{2n}$*, for some constant* C_2*.*

Recalling the dimension of the space of skew elements, the above asymptotic equalities can be rewritten as follows.

If $*$ *is either the transpose or the symplectic involution, then*

$$c_n^*(M_d(F)) \simeq C n^{-\frac{1}{2}\dim M_d(F)^-} (\dim M_d(F))^n,$$

for some constant C*.*

Notice that if A is an algebra with involution $*$ then the map $\varphi : A \to A^{op}$ such that $\varphi(a) = a^*$, is an isomorphism of algebras. Hence $\mathrm{Id}(A) = \mathrm{Id}(A^{op})$.

Theorem 7.2. *If* $A = M_d(F) \oplus M_d(F)^{op}$ *with exchange involution, then*

$$c_n^*(A) \simeq C n^{-\frac{1}{2}(\dim A^- - 1)} (\dim A)^n,$$

for some constant C*.*

Proof. Let us write $y_i = x_i + x_i^*$ and $z_i = x_i - x_i^*$, for symmetric and skew variables of the free algebra with involution, respectively. It is clear that any multilinear $*$-polynomial identity of an algebra A can be written as a linear combination of multilinear polynomial identities in those symmetric and skew variables. We point out that multilinear in this case means that in every monomial either y_i or z_i appears.

Now for any k, $0 \le k \le n$, define

$$P_{k,n-k} = span\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, w_i = y_i \text{ for } 1 \le i \le k \text{ and } w_i = z_i, \text{ for } k+1 \le i \le n\}.$$

If $f(x_1, \ldots, x_n) \in P_n$ let $\tilde{f} = f(y_1, \ldots, y_k, z_{k+1}, \ldots, z_n) \in P_{k,n-k}$ be the $*$-polynomial obtained from f by replacing $x_1, \ldots, x_n$ with $y_1, \ldots, y_k, z_{k+1}, \ldots, z_n$, respectively. Clearly $f \to \tilde{f}$ is a linear isomorphism of P_n onto $P_{k,n-k}$. We claim that such isomorphism extends to a linear isomorphism of $\dfrac{P_n}{P_n \cap \mathrm{Id}(M_d(F))}$ onto $\dfrac{P_{k,n-k}}{P_{k,n-k} \cap \mathrm{Id}^*(A)}$.

In fact, if $f \in P_n, f \notin \mathrm{Id}(M_d(F))$ let $a_1, \ldots, a_n \in M_d(F)$ be such that $f(a_1, \ldots, a_n) \ne 0$. Then

$$\tilde{f}((a_1, a_1), \ldots, (a_k, a_k), (a_{k+1}, -a_{k+1}), \ldots, (a_n, -a_n)) = (f(a_1, \ldots, a_n), f(a_1, \ldots, a_k, -a_{k+1}, \ldots, -a_n) \ne (0, 0).$$

Viceversa, if $\tilde{f} \notin \mathrm{Id}^*(A)$ and a non-zero evaluation is given by the above elements, then we deduce that either $f(a_1, \ldots, a_n) \ne 0$ or $f(a_1, \ldots, a_k, -a_{k+1}, \ldots, -a_n) \ne 0$. As we remarked before the theorem $\mathrm{Id}(M_d(F)) = \mathrm{Id}(M_d(F)^{op})$, hence we deduce that $f(x_1, \ldots, x_n) \notin \mathrm{Id}(M_d(F))$.

As a consequence we have that

$$c_n(M_d(F)) = \dim \frac{P_n}{P_n \cap \mathrm{Id}(M_d(F))} = \dim \frac{P_{k,n-k}}{P_{k,n-k} \cap \mathrm{Id}^*(A)} = c_{k,n-k}(A).$$

Now, by [36] $c_n(M_d(F)) \simeq C n^{-\frac{1}{2}(d^2-1)} d^{2n}$ and by [10] $c_n^*(A) = \sum_{k=0}^n \binom{n}{k} c_{k,n-k}(A)$. Hence we get that

$$c_n^*(A) = \sum_{k=0}^n \binom{n}{k} c_n(M_d(F)) = 2^n c_n(M_d(F)) \simeq C n^{-\frac{1}{2}(d^2-1)} (2d^2)^n.$$

□

In the following sections we shall need the following result on the asymptotics of product of $*$-codimensions.

Lemma 7.1. *Let* $\bar{A} = A_1 \oplus \cdots \oplus A_q$ *be a finite dimensional* $*$*-semisimple algebra over a field of characteristic zero, where the* A_i*'s are* $*$*-simple algebras. Then we have*

$$\sum_{m_1 + \cdots + m_q = m} \binom{m}{m_1, \ldots, m_q} c_{m_1}^*(A_1) \cdots c_{m_q}^*(A_q) \simeq C m^{-\frac{1}{2}(\dim(\bar{A})^- - r)} (\dim \bar{A})^m,$$

where C *is a constant,* r *is the number of* $*$*-simple algebras* A_i *which are not simple.*

Proof. Recall that by a theorem of Beckner and Regev [5], if $(p_1,\dots,p_q)\in\mathbb{Q}^q$ is such that $\sum_i p_i = 1$ and $F(x_1,\dots,x_q)$ is a continuous function homogeneous of degree r with $0<F(p_1,\dots,p_q)<\infty$, then

$$\sum_{m_1+\cdots+m_q=m}\binom{m}{m_1,\dots,m_q}p_1^{m_1}\cdots p_q^{m_q}F(m_1,\dots,m_q)\simeq m^rF(p_1,\dots,p_q). \tag{8}$$

We apply this formula to the following setting: suppose that the algebras $A_1,\dots,A_r$ are not simple, i.e, $A_i = M_{d_i}(F)\oplus M_{d_i}(F)^{op}$ with exchange involution, $1\le i\le r$, and $A_{r+1},\dots,A_q$ are simple algebras, i.e., $A_i = M_{d_i}(F)$ with transpose or symplectic involution, $r+1\le i\le q$. If $d=\dim\bar A$, then $d=\sum_{j=1}^r 2d_j^2+\sum_{j=r+1}^q d_j^2$ and we set $p_j=\frac{2d_j^2}{d}$ if $1\le j\le r$ and $p_j=\frac{d_j^2}{d}$ if $r+1\le j\le q$.

Consider the function

$$F(x_1,\dots,x_q)=\prod_{j=1}^r x_j^{-\frac12(\dim A_j^- -1)}\prod_{j=r+1}^q x_j^{-\frac12\dim A_j^-}$$

which is a homogeneous polynomial of degree

$$-\frac12\sum_{j=1}^r(\dim A_j^- -1)-\frac12\sum_{j=r+1}^q\dim A_j^- = -\frac12(\dim(\bar A)^- - r).$$

Then applying the formula (8) we get

$$\sum_{m_1+\cdots+m_q=m}\binom{m}{m_1,\dots,m_q}\prod_{j=1}^r(\frac{2d_j^2}{d})^{m_j}\prod_{j=r+1}^q(\frac{d_j^2}{d})^{m_j}\prod_{j=1}^r m_j^{-\frac12(dimA_j^- -1)}\prod_{j=r+1}^q m_j^{-\frac12\dim A_j^-}\simeq Cm^{-\frac12(\dim(\bar A)^- - r)}, \tag{9}$$

where $C=F(\frac{2d_1^2}{d},\dots,\frac{d_q^2}{d})$ is a constant. Thus

$$\sum_{m_1+\cdots+m_q=m}\binom{m}{m_1,\dots,m_q}c^*_{m_1}(A_1)\cdots c^*_{m_q}(A_q)$$

$$\simeq C_1\sum_{m_1+\cdots+m_q=m}\binom{m}{m_1,\dots,m_q}\prod_{j=1}^r m_j^{-\frac12(\dim A_j^- -1)}(2d_j^2)^{m_j}\prod_{j=r+1}^q m_j^{-\frac12\dim A_j^-}(d_j^2)^{m_j}$$

$$\simeq C_2 m^{-\frac12(\dim(\bar A)^- - r)}(\dim\bar A)^m.$$

□

8. An upper bound for finite dimensional $*$-algebras

In this section $A=\bar A+J$ is a finite dimensional algebra with involution over an algebraically closed field F of characteristic zero. We have $\bar A=A_1\oplus\cdots\oplus A_q$, where $J^s\ne0$, $J^{s+1}=0$ and the A_i's are $*$-simple algebras with $A_i=M_{d_i}(F)\oplus M_{d_i}(F)^{op}$ with exchange involution, $1\le i\le r$ and $A_i=M_{d_i}(F)$ with transpose or symplectic involution, $r+1\le i\le q$. Fix a basis $\{u_1,\dots,u_m\}$ of A which is the union of the standard bases of the $*$-simple components and of a basis $\{w_1,\dots,w_p\}$ of J. If $1\in A$, $w_t\in e^{(i)}_{j,j}Je^{(l)}_{k,k}$, for some $1\le j\le d_i$, $1\le k\le d_l$, $1\le i,l\le q$, where in case $A_i=M_{d_i}(F)\oplus M_{d_i}(F)^{op}$, $e^{(i)}_{j,l}$ stands for $(e^{(i)}_{j,l},0)$ or $(0,e^{(i)}_{l,j})$. When A does not have a unit element, w_t can belong also to the spaces $e^{(0)}_{1,1}Je^{(l)}_{k,k}$, $e^{(i)}_{j,j}Je^{(0)}_{1,1}$, $e^{(0)}_{1,1}Je^{(0)}_{1,1}$, where $e^{(0)}_{1,1}=e_0$.

We define the generic elements of A and their star as

$$U_j=\sum_{i=1}^m\xi_{i,j}u_i,\quad U_j^*=\sum_{i=1}^m\xi_{i,j}u_i^*,\quad \text{for } j\ge1, \tag{10}$$

where the elements $\xi_{i,j}$ are commutative variables. Hence $U_j,U_j^*\in A\otimes F[\xi_{i,j}\mid 1\le i\le m, j\ge1]$.

Let

$$V_n=\operatorname{span}\{U^{\varepsilon_1}_{\sigma(1)}\cdots U^{\varepsilon_n}_{\sigma(n)}\mid\sigma\in S_n,\ \varepsilon_i=1\text{ or }*\}.$$

Since a multilinear $*$-polynomial is a $*$-identity of A if and only if it vanishes on the generic elements U_j (see [21, Theorem 1.4.4]), it follows that

$$c_n^*(A) = \dim V_n. \tag{11}$$

The aim of this section is to compute an upper bound of $c_n^*(A)$. We follow the approach of the proof given in [2]. Nevertheless in the involution case we face the difficulties coming from a more subtle decomposition of the Wedderburn-Malcev structure theorem of finite dimensional algebras.

We start by fixing a basis of the $*$-simple components. Let $\{a_1^{(l)}, \dots, a_{m_l}^{(l)}\}$ be a basis of the $*$-simple algebra A_l, $1 \le l \le q$. Recall that either $A_l = M_{d_l}(F)$ with transpose or symplectic involution or $A_l = M_{d_l}(F) \oplus M_{d_l}(F)^{op}$ with exchange involution. Thus in the first case $m_l = d_l^2$ and a basis consists of the matrix units $e_{i,j}^{(l)}$, and in the second case $m_l = 2d_l^2$ and a basis consists of the elements $(e_{i,j}^{(l)}, 0)$ and $(0, e_{i,j}^{(l)})$.

Next we rewrite the generic elements in the following form

$$U_j = \sum_{l=1}^{q} U_j^{(l)} + W_j, \quad U_j^* = \sum_{l=1}^{q} (U_j^{(l)})^* + W_j^*, \tag{12}$$

where

$$U_j^{(l)} = \sum_{i=1}^{m_l} \xi_{i,j}^{(l)} a_i^{(l)}, \quad (U_j^{(l)})^* = \sum_{i=1}^{m_l} \xi_{i,j}^{(l)} (a_i^{(l)})^*, \quad W_j = \sum_{i=1}^{p} \eta_{i,j} w_i, \quad W_j^* = \sum_{i=1}^{p} \eta_{i,j} w_i^*.$$

Thus $U_1, U_1^*, \dots, U_n, U_n^* \in A \otimes F[\xi_{i,j}^{(l)}, \eta_{k,j} \mid 1 \le j \le n, 1 \le i \le m_l, 1 \le k \le p, 1 \le l \le q]$, where the $\eta_{i,j}$ are also commutative indeterminates.

Notice that if $A_l = M_{d_l}(F)$, then $U_j^{(l)}$ is a generic matrix and $(U_j^{(l)})^*$ is its star. When $A_l = M_{d_l}(F) \oplus M_{d_l}(F)^{op}$, then $U_j^{(l)}$ is actually a pair of generic matrices. Anyway by abuse of notation we shall call $U_j^{(l)}$ a generic matrix also in this last case. The following remark is in order.

Remark 8.1. *Consider k_l generic matrices $U_{j_1}^{(l)}, \dots, U_{j_{k_l}}^{(l)}$ and let $\tilde{U}_{j_1}^{(l)}, \dots, \tilde{U}_{j_{k_l}}^{(l)}$ be the same generic matrices partially evaluated (we have specialized some of the coefficients $\xi_{i,j_r}^{(l)}$ to scalars in some way). Then*

$$\dim span\{(\tilde{U}_{j_{\sigma(1)}}^{(l)})^{\varepsilon_1} \cdots (\tilde{U}_{j_{\sigma(k_l)}}^{(l)})^{\varepsilon_{k_l}} \mid \sigma \in S_{k_l}, \varepsilon_i = 1 \text{ or } *\}$$

$$\le \dim span\{(U_{j_{\sigma(1)}}^{(l)})^{\varepsilon_1} \cdots (U_{j_{\sigma(k_l)}}^{(l)})^{\varepsilon_{k_l}} \mid \sigma \in S_{k_l}, \varepsilon_i = 1 \text{ or } *\} = c_{k_l}^*(A_l).$$

In order to compute an upper bound of $c_n^*(A)$ we define

$$Y_n = \text{span}\{T_{\sigma(1)} \cdots T_{\sigma(n)} \mid T_j = U_j^{(l)} \text{ or } (U_j^{(l)})^* \text{ or } \eta_{i,j} w_i \text{ or } \eta_{i,j} w_i^*, \text{ for some } 1 \le l \le q, 1 \le i \le p\}.$$

By linearity we have that $V_n \subseteq Y_n$; so $c_n^*(A) \le \dim Y_n$ and our aim is to find a suitable upper bound of $\dim Y_n$.

Notice that, since $J^{s+1} = 0$, in every non-zero monomial of Y_n at most s elements w_i or w_i^* can appear.

For any $u \le s$, let $(w_{t_1}^{\varepsilon_1}, \dots, w_{t_u}^{\varepsilon_u}) = \vec{w}$ be a fixed sequence of elements of the basis of J and their star, and consider the subspace $Y_n^{\vec{w}}$ of Y_n defined as

$$Y_n^{\vec{w}} = \text{span}\{M_0 \eta_{t_1,j_1} w_{t_1}^{\varepsilon_1} M_1 \eta_{t_2,j_2} w_{t_2}^{\varepsilon_2} \cdots M_{u-1} \eta_{t_u,j_u} w_{t_u}^{\varepsilon_u} M_u \mid \{j_1, \dots j_u\} \subseteq \{1, \dots, n\},$$

$$M_0, M_1, \dots, M_u \text{ are multilinear monomials in } U_k^{(l)} \text{ or } (U_k^{(l)})^*,\ k \in \{1, \dots, n\} \setminus \{j_1, \dots, j_u\},\ 1 \le l \le q\}.$$

In particular when $u = 0$, $\vec{w} = \emptyset$ and

$$Y_n^{\vec{w}} = \text{span}\{M_0 \mid M_0 \text{ multilinear monomial in } U_k^{(l)} \text{ or } (U_k^{(l)})^*,\ 1 \le k \le n,\ 1 \le l \le q\}.$$

Thus $\dim Y_n \le \sum_{\vec{w}} \dim Y_n^{\vec{w}}$ where the sum runs over all sequences $\vec{w} = (w_{t_1}^{\varepsilon_1}, \dots, w_{t_u}^{\varepsilon_u})$ with $0 \le u \le s$.

Since

$$M_0 \eta_{t_1,j_1} w_{t_1}^{\varepsilon_1} M_1 \eta_{t_2,j_2} w_{t_2}^{\varepsilon_2} \cdots M_{u-1} \eta_{t_u,j_u} w_{t_u}^{\varepsilon_u} M_u = \prod_{i=1}^{u} \eta_{t_i,j_i} M_0 w_{t_1}^{\varepsilon_1} M_1 w_{t_2}^{\varepsilon_2} \cdots M_{u-1} w_{t_u}^{\varepsilon_u} M_u,$$

we define a new space

$$\bar{Y}_n^{\vec{w}} = \text{span}\{M_0 w_{t_1}^{\varepsilon_1} M_1 w_{t_2}^{\varepsilon_2} \cdots M_{u-1} w_{t_u}^{\varepsilon_u} M_u \mid M_0, M_1, \ldots, M_u \text{ are monomials in } U_k^{(l)} \text{ or } (U_k^{(l)})^*, \; k \in \{1,\ldots,n\} \setminus \{j_1,\ldots,j_u\},\ 1 \le l \le q\}. \tag{13}$$

It follows that

$$\dim Y_n^{\vec{w}} \le u! \binom{n}{u} \dim \bar{Y}_n^{\vec{w}},$$

where $\binom{n}{u}$ counts the distinct subsets $\{j_1,\ldots,j_u\}$ of $\{1,\ldots,,n\}$ and $u!$ counts the number of permutations of $j_1,\ldots,j_u$.

Thus since $n >> s$,

$$c_n^*(A) \le \dim Y_n \le \sum_{\vec{w}} \dim Y_n^{\vec{w}} \le Cs! \binom{n}{s} \dim \bar{Y}_n^{\vec{w}}, \tag{14}$$

where C counts the number of sequences (of length $\le s$) of the basis of J and their star, and $\vec{w}$ is a sequence such that $\dim \bar{Y}_n^{\vec{w}}$ is maximal.

In the next step we shall compute an upper bound of $\dim \bar{Y}_n^{\vec{w}}$, for any $\vec{w} = (w_{t_1}^{\varepsilon_1},\ldots,w_{t_u}^{\varepsilon_u})$. In order to do so, for fixed $n_1,\ldots,n_q$ such that $n_1+\cdots+n_q = n-u$, we define

$$Z_{n_1,\ldots,n_q} = \text{span}\{M = M_0 w_{t_1}^{\varepsilon_1} M_1 w_{t_2}^{\varepsilon_2} \cdots M_{u-1} w_{t_u}^{\varepsilon_u} M_u \in \bar{Y}_n^{\vec{w}} \mid \ n_i = \text{number of } \ U_j^{(i)} \text{ or } (U_j^{(i)})^* \text{ appearing in } \ M,\ 1 \le i \le q\}.$$

Clearly

$$\dim \bar{Y}_n^{\vec{w}} \le \sum_{n_1+\cdots+n_q = n-u} \binom{n-u}{n_1,\ldots,n_q} \dim Z_{n_1,\ldots,n_q}, \tag{15}$$

and our aim is to find an upper bound of $\dim Z_{n_1,\ldots,n_q}$. In order to simplify the notation, from now on we assume that A has 1 since the case when A does not have 1 follows the same pattern of proof.

Lemma 8.1. *There are constants $b_1,\ldots,b_q$, $\sum_{i=1}^q b_i \le s+1$, such that*

$$\dim Z_{n_1,\ldots,n_q} \le C c_{k_1}^*(A_1) \cdots c_{k_q}^*(A_q),$$

for some constant C, where $k_i = n_i + b_i + 1$, for $i = 1,\ldots,q$.

Proof. Let us consider a non-zero monomial

$$\mathcal{M} = M_0 w_{t_1}^{\varepsilon_1} M_1 w_{t_2}^{\varepsilon_2} \cdots M_{u-1} w_{t_u}^{\varepsilon_u} M_u \in Z_{n_1,\ldots,n_q}.$$

Since $(U_j^{(l)})^\varepsilon (U_m^{(r)})^\delta = 0$ for $l \ne r$, where $\varepsilon, \delta \in \{1,*\}$, in order for $\mathcal{M}$ to be non-zero, each M_i must be computed in generic elements $U_j^{(a_i)}$ (and their star) of a $*$-simple algebra A_{a_i} for $0 \le i \le u$. We shall write

$$M_i = M_i(U_j^{(a_i)}, *).$$

Recall that for every $1 \le i \le u$, we have that $w_{t_i}^{\varepsilon_i} = e_{h_i,h_i}^{(a_{i-1})} w_{t_i}^{\varepsilon_i} e_{k_i,k_i}^{(a_i)}$. Here $e_{j,l}^{(i)}$ is the usual matrix unit in case $A_i = M_{d_i}(F)$ and $e_{j,l}^{(i)} = (e_{j,l}^{(i)}, 0)$ or $(0, e_{l,j}^{(i)})$ in case $A_i = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$. In the computation of an upper bound for $\dim Z_{n_1,\ldots,n_q}$ we have to keep in mind that the sequence $(w_{t_1}^{\varepsilon_1},\ldots,w_{t_u}^{\varepsilon_u})$ is fixed and, so, also the indices $h_1, k_1, \ldots, h_u, k_u$. Also, in order for $\mathcal{M}$ to be non-zero, in case $A_{a_i} = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$, we must have that $e_{k_i,k_i}^{(a_i)} = (e_{k_i,k_i}^{(a_i)}, 0)$ and $e_{h_{i+1},h_{i+1}}^{(a_i)} = (e_{h_{i+1},h_{i+1}}^{(a_i)}, 0)$ or $e_{k_i,k_i}^{(a_i)} = (0, e_{k_i,k_i}^{(a_i)})$ and $e_{h_{i+1},h_{i+1}}^{(a_i)} = (0, e_{h_{i+1},h_{i+1}}^{(a_i)})$. Notice that $a_0, a_1, \ldots, a_u \in \{1,\ldots,,q\}$ are not necessarily distinct and the number of generic elements $U_j^{(b_j)}$ or their star appearing in this monomial is $n-u$.

Then $\mathcal{M}$ is a linear combination of monomials of the type

$$\begin{aligned}\mathcal{N}_{k_0,h_{u+1}} &= e_{k_0,k_0}^{(a_0)} M_0 e_{h_1,h_1}^{(a_0)} w_{t_1}^{\varepsilon_1} e_{k_1,k_1}^{(a_1)} M_1 e_{h_2,h_2}^{(a_1)} w_{t_2}^{\varepsilon_2} e_{k_2,k_2}^{(a_2)} \cdots e_{k_{u-1},k_{u-1}}^{(a_{u-1})} M_{u-1} e_{h_u,h_u}^{(a_{u-1})} w_{t_u}^{\varepsilon_u} e_{k_u,k_u}^{(a_u)} M_u e_{h_{u+1},h_{u+1}}^{(a_u)} \\ &= (M_0)_{k_0,h_1} e_{k_0,h_1}^{(a_0)} w_{t_1}^{\varepsilon_1} (M_1)_{k_1,h_2} e_{k_1,h_2}^{(a_1)} w_{t_2}^{\varepsilon_2} \cdots (M_{u-1})_{k_{u-1},h_u} e_{k_{u-1},h_u}^{(a_{u-1})} w_{t_u}^{\varepsilon_u} (M_u)_{k_u,h_{u+1}} e_{k_u,h_{u+1}}^{(a_u)}\end{aligned}$$

$$= (M_0)_{k_0,h_1}(M_1)_{k_1,h_2}\cdots(M_{u-1})_{k_{u-1},h_u}(M_u)_{k_u,h_{u+1}}e^{(a_0)}_{k_0,h_1}w^{\varepsilon_1}_{t_1}e^{(a_1)}_{k_1,h_2}w^{\varepsilon_2}_{t_2}\cdots e^{(a_{u-1})}_{k_{u-1},h_u}w^{\varepsilon_u}_{t_u}e^{(a_u)}_{k_u,h_{u+1}},$$

where $(M_i)_{k_i,h_{i+1}}$ is the (k_i,h_{i+1})-entry of $M_i = M_i(U_j^{(a_i)},*)$ in case $A_{a_i} = M_{d_{a_i}}(F)$ and, in case $A_{a_i} = M_{d_{a_i}}(F)\oplus M_{d_{a_i}}(F)^{op}$, $(M_i)_{k_i,h_{i+1}} = ((M_i^0)_{k_i,h_{i+1}},0)$ or $(M_i)_{k_i,h_{i+1}} = (0,(M_i^1)_{h_{i+1},k_i})$, with $M_i = (M_i^0, M_i^1)$, according as $e^{(a_i)}_{h_{i+1},h_{i+1}} = (e^{(a_i)}_{h_{i+1},h_{i+1}},0)$ or $e^{(a_i)}_{h_{i+1},h_{i+1}} = (0,e^{(a_i)}_{h_{i+1},h_{i+1}})$; here $k_0\in\{1,\ldots,d_{a_0}\}$ and $h_{u+1}\{1,\ldots,d_{a_u}\}$.

Thus we can write

$$\mathcal{N}_{k_0,h_{u+1}} = F(U,*)e^{(a_0)}_{k_0,h_1}w^{\varepsilon_1}_{t_1}e^{(a_1)}_{k_1,h_2}w^{\varepsilon_2}_{t_2}\cdots e^{(a_{u-1})}_{k_{u-1},h_u}w^{\varepsilon_u}_{t_u}e^{(a_u)}_{k_u,h_{u+1}}, \tag{16}$$

where

$$F(U,*) = \prod_{i=0}^{u}(M_i)_{k_i,h_{i+1}}$$

is a multilinear function of the generic elements $U_j^{(m)}$ and their star which depends on the monomials $M_0,\ldots,M_u$ and the two indices k_0,h_{u+1}.

Next we group together all indices a_i which are equal among themselves. Let $s_1,\ldots,s_{b_l}$ be all the distinct indices i such that $a_i = l$, i.e.,

$$a_{s_1} = \cdots = a_{s_{b_l}} = l.$$

Hence their number is b_l. Let us also write $\bar{k}_i = k_{s_i}$, $\bar{h}_{i+1} = h_{s_i+1}$, for $1\le i\le b_l$. Notice that $b_1+\cdots+b_q = u+1\le s+1$. Then set

$$F_l(U_j^{(l)},*) = \prod_{i=1}^{b_l}(M_{s_i})_{\bar{k}_i,\bar{h}_{i+1}}.$$

and consequently

$$F(U,*) = \prod_{l=1}^{q}F_l(U_j^{(l)},*), \tag{17}$$

where we set $F_l(U_j^{(l)},*) = 1$ if $U_j^{(l)}$ or its $*$ does not appear in $\mathcal{N}_{k_0,h_{u+1}}$, for any j.

Let $\mathcal{S}_l = \mathrm{span}\{F_l(U_j^{(l)},*)\}$ be the F-vector space spanned by all the expressions $F_l(U_j^{(l)},*)$, for fixed k_0,h_{u+1}. In order to compute $\dim\mathcal{S}_l = \dim\mathcal{S}_l e^{(l)}_{1,1}$, in case $A_l = M_{d_l}(F)$ we write

$$F_l(U_j^{(l)},*)e^{(l)}_{1,1} = e^{(l)}_{1,\bar{k}_1}M_{s_1}e^{(l)}_{\bar{h}_2,\bar{k}_2}M_{s_2}e^{(l)}_{\bar{h}_3,\bar{k}_3}\cdots e^{(l)}_{\bar{h}_{b_l},\bar{k}_{b_l}}M_{s_{b_l}}e^{(l)}_{\bar{h}_{b_{l+1}},1}, \tag{18}$$

and, in case $A_l = M_{d_l}(F)\oplus M_{d_l}(F)^{op}$, we obtain an expression similar to (18) where the computation is performed for instance on the first component. The above elements are just the evaluations of a monomial in n_l generic matrices of A_l and their star, and in b_l+1 matrix units $e^{(l)}_{i,j}$, where n_l = number of generic matrices in $M_{s_1}M_{s_2}\cdots M_{s_{b_l}}$.

Define $k_l = n_l+b_l+1$. Then

$$k = \sum_{l=1}^{q}k_l = n-u+\sum_{l=1}^{q}b_l+q = n+K, \tag{19}$$

where $K = -u+\sum_{l=1}^{q}b_l+q\le -u+s+q+1$ is a constant independent of n.

Now we can apply Remark 8.1 to the space $\mathcal{S}_l e^{(l)}_{1,1}$ by noticing that we have specialized b_l+1 generic matrices to the elements $e^{(l)}_{1,\bar{k}_1},\ldots,e^{(l)}_{\bar{h}_{b_{l+1}},1}$. Hence we get that

$$\dim\mathcal{S}_l\le c^*_{k_l}(A_l).$$

Recalling the definition of $Z_{n_1,\ldots,n_q}$, we get that

$$\dim Z_{n_1,\ldots,n_q}\le C\prod_{l=1}^{q}\dim\mathcal{S}_l\le Cc^*_{k_1}(A_1)\cdots c^*_{k_q}(A_q),$$

where $C\le\dim\bar{A}$ counts the number of indices k_0,h_{u+1}. □

Now we can collect the results so far obtained and prove the following

Theorem 8.1. *Let $A = \bar{A} + J$ be a finite dimensional algebra with involution $*$ over an algebraically closed field F of characteristic zero. Let $\bar{A} = A_1 \oplus \cdots A_r \oplus A_{r+1} \oplus \cdots \oplus A_q$ be a direct sum of $*$-simple algebras with $A_1, \ldots, A_r$ not simple algebras, $J^s \neq 0, J^{s+1} = 0$. Then*

$$c_n^*(A) \leq Cn^{-\frac{1}{2}(\dim(\bar{A})^- - r)+s}(\dim \bar{A})^n.$$

for some constant C.

Proof. By (14) and (15) we have that

$$c_n^*(A) \leq C' \binom{n}{s} \sum_{n_1+\cdots+n_q=n-u} \binom{n-u}{n_1, \ldots, n_q} \dim Z_{n_1,\ldots,n_q},$$

for some constant C'. Then by Lemma 8.1 and Lemma 7.1 we get that

$$c_n^*(A) \leq C' \binom{n}{s} \sum_{n_1+\cdots+n_q=n-u} \binom{n-u}{n_1, \ldots, n_q} c_{k_1}^*(A_1) \cdots c_{k_q}^*(A_q),$$

where $k_i = n_i + b_i + 1$, $\sum_{i=1}^q b_i = u + 1$. Hence

$$c_n^*(A) \leq C' \binom{n}{s} \sum_{k_1+\cdots+k_q=k} \binom{k}{k_1, \ldots, k_q} c_{k_1}^*(A_1) \cdots c_{k_q}^*(A_q) \leq C' \binom{n}{s} k^{-\frac{1}{2}(\dim(\bar{A})^- - r)}(\dim \bar{A})^k$$

$$\leq C' \binom{n}{s} (n+K)^{-\frac{1}{2}(\dim(\bar{A})^- - r)}(\dim \bar{A})^{n+K} \leq C'' n^{-\frac{1}{2}(\dim(\bar{A})^- - r)+s}(\dim \bar{A})^n,$$

for some constant C'', where K is a constant independent of n (see (19)). □

9. A lower bound for $*$-fundamental algebras

The aim of this section is to find a suitable lower bound of $c_n^*(A)$ for any $*$-fundamental algebra. We start by remarking a result about generic elements. Let $A = M_k(F)$ be a $*$-simple algebra and let $U_1, \ldots, U_t$ be generic elements of A defined as in (10). Clearly if $A = M_k(F)$, then U_j is a generic matrix and we write

$$U_j = \sum_{i,l=1}^{k} \xi_{i,l}^j e_{i,l}.$$

Notice that U_j^* is also a generic matrix.

When $A = M_k(F) \oplus M_k(F)^{op}$, then U_j is actually a pair of generic matrices:

$$U_j = (U_j^0, U_j^1) = (\sum_{i,l=1}^{k} \xi_{i,l}^j e_{i,l}, \sum_{i,l=k+1}^{2k} \xi_{i,l}^j e_{i,l}).$$

Anyway by abuse of notation we shall call U_j a generic matrix also in this last case. Even in this case U_j^* is a generic matrix.

Let $g(x_1, \ldots, x_t)$ be a multilinear $*$-monomial. Then $g(U_1, \ldots, U_t)$ is a non-zero element in $A \otimes F[\xi]$, where $F[\xi] = F[\xi_{i,l}^j]$, $1 \leq j \leq t$, with $1 \leq i, l \leq k$ or $1 \leq i, l \leq 2k$, according as $A = M_k(F)$ or $A = M_k(F) \oplus M_k(F)^{op}$.

If $A = M_k(F) \oplus M_k(F)^{op}$, then $g(U_1, \ldots, U_t) = (g^0, g^1)$ with $g^0, g^1 \neq 0$. We have the following.

Lemma 9.1. *Under the above conditions, all the diagonal entries of $G = g(U_1, \ldots, U_t)$ are non-zero polynomials.*

Proof. In case $A = M_k(F) \oplus M_k(F)^{op}$, we prove the result only for one component, say g^0 and we shall write $g = g^0$.

Suppose by contradiction that some entry on the diagonal, say G_{11}, vanishes. Now for any invertible matrix $h \in GL(k, F)$ we have that

$$hGh^{-1} = g(hU_1h^{-1}, \ldots, hU_th^{-1}).$$

Let $\bar{h}: F[\xi] \longrightarrow F[\xi]$ be the automorphism induced by h on the polynomial ring $F[\xi]$, that is:

$$\bar{h}(\xi^j_{i,l}) = \bar{h}((U_j)_{i,l}) := (hU_jh^{-1})_{i,l} \text{ and, so, } (hGh^{-1})_{i,l} = \bar{h}(G_{i,l}).$$

Since $G_{11} = 0$ then, for all $h \in GL(k, F)$, the entry $(hGh^{-1})_{1,1} = 0$. Now suppose that there exist indices $i \neq j$ with $G_{i,j} = 0$. Then by conjugating by a permutation matrix we get

$$G_{\sigma(i),\sigma(j)} = (\sigma G \sigma^{-1})_{i,j} = 0,$$

and, so, all off diagonal entries are zero. Hence, since $G_{11} = 0$ the element G has determinant zero, a contradiction since all non-zero elements of the algebra of generic matrices are invertible. This proves that all off diagonal entries are non-zero. But if we conjugate G with the matrix $h = 1 - e_{2,1}$ we have:

$$0 = \bar{h}(G_{1,1}) = (hGh^{-1})_{1,1} = G_{1,2} \neq 0,$$

a contradiction. □

The main result of this section is the following.

Theorem 9.1. *Let $A = \bar{A} + J$ be a $*$-fundamental algebra over an algebraically closed field F of characteristic zero and let $\bar{A} = A_1 \oplus \cdots A_r \oplus A_{r+1} \oplus \cdots \oplus A_q$ be a direct sum of $*$-simple algebras with $A_1, \ldots, A_r$ not simple algebras, $J^{s+1} = 0, s \geq 0$. Then*

$$c^*_n(A) \geq Cn^{-\frac{1}{2}(\dim(\bar{A})^- - r) + s}(\dim \bar{A})^n,$$

for some constant $C > 0$.

Proof. If A is $*$-simple the result follows from Theorems 7.1 and 7.2. Hence we may assume that $s > 0$ and let (d, s) be the (t, s)-index of A.

Since A is $*$-fundamental, there exists a $*$-fundamental polynomial $f(Z_1, \ldots, Z_s, y_1, \ldots, y_q, X) \notin \mathrm{Id}^*(A)$ multilinear and alternating in s sets of variables $Z_i = \{x_{i,1}, \ldots, x_{i,d+1}\}$ each with $d + 1$ variables, linear in the variables $y_1, \ldots, y_q$ and eventually depending on some extra variables X. In order to get a non-zero evaluation, for all $i = 1, \ldots q$, the variable y_i (up to a permutation of the indices) must be evaluated in A_i and precisely one variable of each Z_i must be evaluated in J. Let us assume that the first variable $z_i := x_{i,1}$, $1 \leq i \leq s$, is evaluated in J.

Now we are going to consider a non-zero elementary evaluation η of f.

In such an evaluation y_i is evaluated in $e^{(i)}_{h_i,k_i}$, $1 \leq i \leq q$. Recall that if $A_i = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$ the element $e^{(i)}_{h_i,k_i}$ stands for $(e^{(i)}_{h_i,k_i}, 0)$ or $(0, e^{(i)}_{k_i,h_i})$.

Let $\tilde{f}$ be the polynomial obtained from f by replacing y_i by $u_iv_iw_iy_i$, where u_i, v_i, w_i are new variables different from the ones appearing in f. Clearly the evaluation η of f can be extended to a non-zero evaluation η' of $\tilde{f}$ by evaluating the new variables u_i, v_i, w_i in $e^{(i)}_{h_i,h_i}$, $1 \leq i \leq q$.

Next we performe a partial evaluation of $\tilde{f}$ by evaluating only the variables $u_i, w_i, y_i, x_{j,k}$, $k > 1$, $1 \leq j \leq s$, as in η'. In this way we obtain an expression $g(v_1, \ldots, v_q, z_1, \ldots, z_s)$ which is a linear combination of monomials made out of elements of A and the variables v_i, z_j and their $*$. We call such g a generalized $*$-polynomial.

Recall that if we evaluate some z_j in $\bar{A}$ or some v_i in J then g vanishes.

Let $\mathbf{n} = (n_1, \ldots, n_q)$ be a composition of n into q parts, i.e., $\sum_{i=1}^q n_i = n$, and set $m = n + n_{q+1}$, where $n_{q+1} = s$.

The set $\{1, \ldots, m\}$ is partitioned in $q + 1$ subsets $B_1, \ldots, B_{q+1}$, with

$$B_i = \{b_{i-1} + 1, b_{i-1} + 2, \ldots, b_i\},\ |B_i| = n_i,$$

where $b_0 = 0, b_i := n_1 + \ldots + n_i$, $1 \leq i \leq q + 1$, and we set $\mathcal{P} = \{B_1, \ldots, B_{q+1}\}$. $\mathcal{P}$ is called a multipartition of m.

Fix a sequence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, where $\varepsilon_i = 1$ or $*$, and consider the set of variables $T = \{x'_1, \ldots, x'_n, x_{n+1}, \ldots, x_m\}$, where $x'_i = x_i^{\varepsilon_i}$, $1 \leq i \leq n$. Let $M^\varepsilon_1, \ldots, M^\varepsilon_q$ be the monomials in some of the variables of T defined by

$$M^\varepsilon_i = M^\varepsilon_i(x'_{b_{i-1}+1}, x'_{b_{i-1}+2}, \ldots, x'_{b_i}) = x'_{b_{i-1}+1}x'_{b_{i-1}+2} \cdots x'_{b_i}.$$

In other words M^ε_i is a multilinear product of n_i consecutive variables x'_h indexed by the elements of B_i.

Now if in g we replace v_i by the monomial M_i^ε, $1 \le i \le q$, and z_j by the variable $x_{n+j}, 1 \le j \le s$, then we get a generalized $*$-polynomial:

$$g_{\mathcal{P}}^{\varepsilon}(x'_1, \dots, x'_n, x_{n+1}, \dots, x_m) := g(M_1^\varepsilon, \dots, M_q^\varepsilon, x_{n+1}, \dots, x_m).$$

Consider m generic elements of A

$$U_j = \sum_{i=1}^{q} U_j^{(i)} + W_j, \; j = 1, \dots, m,$$

with $U_j^{(i)}$ generic elements of A_i and W_j generic elements of J as in (12).

For $\sigma \in S_m$, set

$$M_i^{\varepsilon,\sigma} = M_i^\varepsilon(U^{(i)}_{\sigma(b_{i-1}+1)}, U^{(i)}_{\sigma(b_{i-1}+2)}, \dots, U^{(i)}_{\sigma(b_i)}) = (U^{(i)}_{\sigma(b_{i-1}+1)})^{\varepsilon_{b_{i-1}+1}} (U^{(i)}_{\sigma(b_{i-1}+2)})^{\varepsilon_{b_{i-1}+2}} \cdots (U^{(i)}_{\sigma(b_i)})^{\varepsilon_{b_i}}.$$

In case $A_i = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$ then

$$\begin{aligned} M_i^{\varepsilon,\sigma} &= ((M_i^{\varepsilon,\sigma})^0, (M_i^{\varepsilon,\sigma})^1) \\ &= ((U^{(i)}_{\sigma(b_{i-1}+1)})^{a_{i_1}} (U^{(i)}_{\sigma(b_{i-1}+2)})^{a_{i_2}} \cdots (U^{(i)}_{\sigma(b_i)})^{a_{i n_i}}, \; (U^{(i)}_{\sigma(b_{i-1}+1)})^{b_{i_1}} (U^{(i)}_{\sigma(b_{i-1}+2)})^{b_{i_2}} \cdots (U^{(i)}_{\sigma(b_i)})^{b_{i n_i}}), \end{aligned}$$

where $a_{i_j} = 0$ if $\varepsilon_{b_{i-1}+j} = 1$ and $a_{i_j} = 1$ if $\varepsilon_{b_{i-1}+j} = *$, whereas $b_{i_j} = 1$ if $\varepsilon_{b_{i-1}+j} = 1$ and $b_{i_j} = 0$ if $\varepsilon_{b_{i-1}+j} = *$. We set $(M_i^{\varepsilon,\sigma})_{h_i,h_i} = (((M_i^{\varepsilon,\sigma})^0)_{h_i,h_i}, 0)$ or $(0, ((M_i^{\varepsilon,\sigma})^1)_{h_i,h_i})$ according as y_i is evaluated in $(e^{(i)}_{h_i,k_i}, 0)$ or $(0, e^{(i)}_{k_i,h_i})$, respectively.

We have

$$g_{\mathcal{P}}^{\varepsilon}(U_{\sigma(1)}, \dots, U_{\sigma(m)}) = \prod_{i=1}^{q} (M_i^{\varepsilon,\sigma})_{h_i,h_i} G(W_{\sigma(n+1)}, \dots, W_{\sigma(m)}), \tag{20}$$

where

$$G(W_{\sigma(n+1)}, \dots, W_{\sigma(m)}) = g(e^{(1)}_{h_1,h_1}, \dots, e^{(q)}_{h_q,h_q}, W_{\sigma(n+1)} \dots, W_{\sigma(m)}),$$

and, in case $A_i = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$, $e^{(i)}_{h_i,h_i} = (e^{(i)}_{h_i,h_i}, 0)$ if y_i is evaluated in $(e^{(i)}_{h_i,k_i}, 0)$ and $e^{(i)}_{h_i,h_i} = (0, e^{(i)}_{h_i,h_i})$ if y_i is evaluated in $(0, e^{(i)}_{h_i,k_i})$.

In order to complete the proof we need the following.

Lemma 9.2. *Let $\mathcal{P} = \{B_1, \dots, B_{q+1}\}$ be as above and let $\Sigma = \{\sigma \in S_m \mid \sigma(B_i) = B_i, 1 \le i \le q$ and $\sigma(n+j) = n+j, 1 \le j \le s\}$. Then if $G_{\mathcal{P}} = span\{g_{\mathcal{P}}^{\varepsilon}(U_{\sigma(1)}, \dots, U_{\sigma(m)}), \; \sigma \in \Sigma, \; \varepsilon \in \{1, *\}^n\}$ we have that*

$$\dim G_{\mathcal{P}} = c^*_{n_1}(A_1) \cdots c^*_{n_q}(A_q).$$

Proof. Since $\sigma(n+i) = n+i, \; i = 1, \dots, s$, then $G(W_{\sigma(n+1)}, \dots, W_{\sigma(m)}) = G(W_{n+1}, \dots, W_m) \neq 0$. Hence if we write $\sigma \in \Sigma$ as $\sigma = \tau_1 \cdots \tau_q$, where $\tau_i = \sigma|_{S_{n_i}} = \sigma|_{S_{B_i}}$ we have that

$$G_{\mathcal{P}} \cong \mathcal{M} = span\{\prod_{i=1}^{q} (M_i^{\varepsilon,\tau_i})_{h_i,h_i}, | \; \tau_i \in S_{n_i}, \; \varepsilon \in \{1, *\}^n\}.$$

For each i the polynomial functions $(M_i^{\varepsilon,\tau_i})_{h_i,h_i}$ depend only upon the generic elements $U_j^{(i)}$, where $j \in B_i$. Moreover, if $i \neq l$, M_i^{ε,τ_i} and $M_l^{\varepsilon',\tau_l}$ are in disjoint sets of variables for any $\tau_i \in S_{n_i}, \tau_l \in S_{n_l}$ and $\varepsilon, \varepsilon' \in \{1, *\}^n$.

Therefore the space $\mathcal{M}$ is the tensor product of the spaces spanned by the polynomials $(M_i^{\varepsilon,\tau_i})_{h_i,h_i}, 1 \le i \le q, \; \tau_i \in S_{n_i}, \varepsilon \in \{1, *\}^n$.

Since M_i^{ε,τ_i} is a non-zero element in $A_i \otimes F[\xi^{(i)}_{j,l}]$ (the variables which appear are all distinct) then, by Lemma 9.1, all the diagonal entries are non-zero polynomials. Hence the linear map $M_i^{\varepsilon,\tau_i} \mapsto (M_i^{\varepsilon,\tau_i})_{h_i,h_i}$ from the space generated by the multilinear monomials in $U^{(i)}_{\tau_i(b_{i-1}+1)}, U^{(i)}_{\tau_i(b_{i-1}+2)}, \dots U^{(i)}_{\tau_i(b_i)}$ and their $*$ to the polynomial ring $F[\xi^{(i)}_{j,l}]$ is injective.

Hence the space spanned by the monomials $(M_i^{\varepsilon,\tau_i})_{h_i,h_i}$ is isomorphic to the span of multilinear products of n_i generic elements of A_i which by (11) has dimension $c^*_{n_i}(A_i)$. □

Now we denote by $\mathcal{C}_{\mathbf{n}}$ the set of all multipartitions $\{C_1, \ldots, C_{q+1}\}$ of m such that each C_j has n_j elements. It is clear that $\mathcal{C}_{\mathbf{n}}$ has $\frac{m!}{\prod_{i=1}^{q+1} n_i!} = \binom{m}{n_1,,\ldots,n_{q+1}}$ elements. Let $\mathcal{C} = \{C_1, \ldots, C_{q+1}\} \in \mathcal{C}_{\mathbf{n}}$ be a fixed multipartition of m and let

$$\mathcal{M}_{\mathcal{C}} = span\{g_{\mathcal{P}}^{\varepsilon}(U_{\sigma(1)}, \ldots, U_{\sigma(m)}) \mid \varepsilon \in \{1, *\}^n,\ \sigma \in S_m \text{ and } \sigma(B_i) = C_i, 1 \le i \le q\}.$$

Notice that the spaces $\{\mathcal{M}_{\mathcal{C}}\}_{\mathcal{C}\in\mathcal{C}_{\mathbf{n}}}$ form a direct sum since they are made of homogeneous elements of different multidegree in the variables. Such a sum is contained in the space obtained by specializations of the span of multilinear products of $n+t$ generic elements of A, where t is independent of n and is equal to s plus the number of variables of $\tilde{f}$ which are evaluated in A in the partial evaluation. Hence

$$\sum_{\substack{\mathbf{n}=(n_1,\ldots,n_q)\\ \sum_i n_i = n}} \sum_{\mathcal{C}\in\mathcal{C}_{\mathbf{n}}} \dim(\mathcal{M}_{\mathcal{C}}) \le c^*_{n+t}(A) \tag{21}$$

with t some fixed number. By symmetry we have $\dim \mathcal{M}_{\mathcal{C}} = \dim \mathcal{M}_{\mathcal{P}}$ for all $\mathcal{C} \in \mathcal{C}_{\mathbf{n}}$. Now, since

$$\dim \mathcal{M}_{\mathcal{P}} \ge \dim G_{\mathcal{P}} = c^*_{n_1}(A_1)\cdots c^*_{n_q}(A_q)$$

and $n_{q+1} = s$, we have

$$\sum_{\mathcal{C}\in\mathcal{C}_{\mathbf{n}}} \dim \mathcal{M}_{\mathcal{C}} = \binom{m}{n_1, \ldots, n_{q+1}} \dim \mathcal{M}_{\mathcal{P}} = \binom{m}{s}\binom{n}{n_1, \ldots, n_q} \dim \mathcal{M}_{\mathcal{P}}$$

$$\ge C_1 n^s \binom{n}{n_1, \ldots, n_q} c^*_{n_1}(A_1)\cdots c^*_{n_q}(A_q),$$

for some constant $C_1 > 0$. Hence, by (21) and Lemma 7.1 we get

$$\begin{aligned} c^*_{n+t}(A) &\ge C_1 n^s \sum_{\substack{\mathbf{n}=(n_1,\ldots,n_q)\\ \sum_i n_i = n}} \binom{n}{n_1, \ldots, n_q} c^*_{n_1}(A_1)\cdots c^*_{n_q}(A_q) \\ &\ge C_2 n^{-\frac{1}{2}(\dim(\bar{A})^- - r)+s} (\dim \bar{A})^n, \end{aligned} \tag{22}$$

for some constant $C_2 > 0$. Hence formula (22) yields the conclusion

$$c^*_l(A) \ge C_2(l-t)^{-\frac{1}{2}(\dim(\bar{A})^- - r)+s}(\dim \bar{A})^{l-t} \ge C_3 l^{-\frac{1}{2}(\dim(\bar{A})^- - r)+s}(\dim \bar{A})^l, \tag{23}$$

for some constant C_3.

□

10. The main results

Let A be a $*$-fundamental algebra. Recall that when A is $*$-simple the asymptotics of $c^*_n(A)$ are given in Theorem 7.1 and Theorem 7.2. Then putting together these results and the bounds of $c^*_n(A)$ obtained in Theorem 8.1 and Theorem 9.1 we get the following.

Theorem 10.1. *Let $A = \bar{A} + J$ be a $*$-fundamental algebra over an algebraically closed field F of characteristic zero and let $s \ge 0$ be the least integer such that $J^{s+1} = 0$. Write $\bar{A} = A_1 \oplus \cdots A_r \oplus A_{r+1} \oplus \cdots \oplus A_q$, a direct sum of $*$-simple algebras with $A_1, \ldots, A_r$ not simple algebras, then*

$$C_1 n^{-\frac{1}{2}(\dim(\bar{A})^- - r)+s}(\dim \bar{A})^n \le c^*_n(A) \le C_2 n^{-\frac{1}{2}(\dim(\bar{A})^- - r)+s}(\dim \bar{A})^n,$$

for some constants $C_1 > 0, C_2$. Hence

$$\lim_{n\to\infty} \log_n \frac{c^*_n(A)}{exp^*(A)^n} = -\frac{1}{2}(\dim(\bar{A})^- - r) + s.$$

Now let A be a finitely generated $*$-algebra satisfying a polynomial identity. By [38] A has the same $*$-identities as a finite dimensional $*$-algebra B. By Proposition 6.1, B has the same $*$-identities as a finite direct sum of $*$-fundamental algebras $D_1 \oplus \cdots \oplus D_m$. Since $c^*_n(B) \simeq c^*_n(D_l)$ for a suitable D_l, we get the following.

Theorem 10.2. *Let A be a finitely generated $*$-algebra over a field F of characteristic zero. If A satisfies a polynomial identity then*

$$C_1 n^t exp^*(A)^n \le c_n^*(A) \le C_2 n^t exp^*(A)^n,$$

where $t \in \frac{1}{2}\mathbb{Z}$, for some constants $C_1 > 0, C_2$. Hence $\lim_{n\to\infty} \log_n \frac{c_n^*(A)}{exp^*(A)^n}$ exists and is a half integer.

It is worth mentioning that by Theorem 8.1 and Theorem 10.1 the above upper bound can be specialized for any finite dimensional $*$-algebra which is $*$-simple or non-semisimple. We have.

Theorem 10.3. *Let $A = \bar{A} + J$ be a finite dimensional algebra with involution $*$ over an algebraically closed field F of characteristic zero. Let $\bar{A} = A_1 \oplus \cdots A_r \oplus A_{r+1} \oplus \cdots \oplus A_q$ be a direct sum of $*$-simple algebras with $A_1, \ldots, A_r$ not simple algebras. Then if A is $*$-simple or non-semisimple*

$$c_n^*(A) \le C n^{-\frac{1}{2}(\dim(\bar{A})^- - r) + s} (\dim \bar{A})^n.$$

for some constant C, where $s + 1$ is the nilpotency index of J.

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