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ORIGINAL PAPER

Cardinal inequalities involving the Hausdorff pseudocharacter

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Abstract

We establish several bounds on the cardinality of a topological space involving the Hausdorff pseudocharacter $H\psi(X)$. This invariant has the property $\psi_c(X) \leq H\psi(X) \leq \chi(X)$ for a Hausdorff space X. We show the cardinality of a Hausdorff space X is bounded by $2^{pwL_c(X)H\psi(X)}$, where $pwL_c(X) \leq L(X)$ and $pwL_c(X) \leq c(X)$. This generalizes results of Bella and Spadaro, as well as Hodel. We show additionally that if X is a Hausdorff linearly Lindelöf space such that $H\psi(X) = \omega$, then $|X| \leq 2^\omega$, under the assumption that either $2^{<\mathfrak{c}} = \mathfrak{c}$ or $\mathfrak{c} < \aleph_\omega$. The following game-theoretic result is shown: if X is a regular space such that player two has a winning strategy in the game $G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$, $H\psi(X) < \kappa$ and $\chi(X) \leq 2^{<\kappa}$, then $|X| \leq 2^{<\kappa}$. This improves a result of Aurichi, Bella, and Spadaro. Generalizing a result for first-countable spaces, we demonstrate that if X is a Hausdorff almost discretely Lindelöf space satisfying $H\psi(X) = \omega$, then $|X| \leq 2^\omega$ under the assumption $2^{<\mathfrak{c}} = \mathfrak{c}$. Finally, we show $|X| \leq 2^{wL(X)H\psi(X)}$ if X is a Hausdorff space with a π-base with elements with compact closures. This is a variation of a result of Bella, Carlson, and Gotchev.

Keywords Cardinality bounds · Hausdorff pseudocharacter · Topological games

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1 Introduction

In his paper [19], besides popularizing a new method for proving cardinal function inequalities, Richard Hodel introduced a new cardinal function which captures exactly how much

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of the character is needed in two well-known cardinal inequalities for Hausdorff spaces: Arhangel'skii's theorem stating that $|X| \leq 2^{\chi(X) \cdot L(X)}$ and the Hajnal–Juhász inequality stating that $|X| \leq 2^{\chi(X) \cdot c(X)}$ (where $\chi(X)$, $L(X)$ and $c(X)$ denote, respectively, the character, the Lindelöf number and the cellularity of X).

Let X be a Hausdorff space. The Hausdorff pseudocharacter of X, denoted by $H\psi(X)$, is defined as the minimum cardinal κ such that for every $x \in X$ there is a family $\mathcal{V}_x$ of open neighbourhoods of x such that $|\mathcal{V}_x| \leq \kappa$ and, for every pair of distinct points $x, y \in X$ there are $V_x \in \mathcal{V}_x$ and $V_y \in \mathcal{V}_y$ such that $V_x \cap V_y = \emptyset$.

Let $\psi(X)$ denote the pseudocharacter of X. It is clear that $\psi(X) \leq H\psi(X) \leq \chi(X)$, for every Hausdorff space X, and Hodel showed that $\chi(X)$ can be replaced with $H\psi(X)$ in both Arhangel'skii's Theorem and the Hajnal–Juhász inequality.

Various authors have later attempted to refine cardinal inequalities involving the character by means of the Hausdorff pseudocharacter and similarly defined cardinal functions, including Kortezov [21]. Fedeli [17], Stavrova [23], Bella, Bonanzinga and Matveev [7] and Bonanzinga, Cuzzupé and Pansera [14]. In this paper we implement this approach systematically on a series of cardinal function inequalities that have appeared in the literature in the last few years.

In [12] Bella and Spadaro gave a common proof of Arhangel'skii's Theorem and the Hajnal–Juhász inequality by establishing that $|X| \leq 2^{pwL_c(X)\chi(X)}$ for any Hausdorff space X. The cardinal invariant $pwL_c(X)$, defined in 2.8, has the properties $pwL_c(X) \leq L(X)$ and $pwL_c(X) \leq c(X)$. In Sect. 2 we show that the character can be replaced with the Hausdorff pseudocharacter $H\psi(X)$ in this inequality. As a corollary follow the results of Hodel mentioned above. We also construct an example of a Hausdorff space X such that $\psi(X) < H\psi(X) < \chi(X)$.

In Sect. 3 we give an analogous improvement to a cardinal inequality for linearly Lindelöf spaces that was proved by the first author in [6], which is in turn related to Arhangel'skii and Buzyakova's upper bound on the cardinality of linearly Lindelöf first-countable Tychonoff spaces [1].

In Sect. 4 we present a refinement of a game-theoretic cardinal bound proved by Aurichi, Bella, and Spadaro in [2]. We show that if X is a regular space such that player two has a winning strategy in the game $G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$, $H\psi(X) < \kappa$ and $\chi(X) \leq 2^{<\kappa}$, then $|X| \leq 2^{<\kappa}$.

In Sect. 5 we show that the consistent upper bound on the cardinality of almost discretely Lindelöf first-countable Hausdorff spaces proved by the first and the third author in [11] continues to hold if "first-countable" is replaced with "Hausdorff pseudocharacter".

In the final section of the paper we show that $|X| \leq 2^{wL(X)H\psi(X)}$ if X is Hausdorff space with a compact π-base; that is, a π-base with elements with compact closures. This is a variation of a result in [8] due to Bella, Carlson, and Gotchev. We ask if this bound is a bound for the cardinality of any normal space. This question is related to a question asked by Bell, Ginsburg, and Woods in [5].

All spaces are defined to be at least Hausdorff. We refer to [16, 20] and [22] for undefined notions.

2 Hausdorff spaces

Let X be a Hausdorff space and let the Hausdorff pseudocharacter $H\psi(X)$ be bounded above by a cardinal κ. For all $x \in X$ fix families $\mathcal{V}_x$ of open sets containing x such that $|\mathcal{V}_x| \leq \kappa$ and for all $x \neq y$ there exist $U \in \mathcal{V}_x$ and $V \in \mathcal{V}_y$ such that $U \cap V = \varnothing$.

Furthermore, assume the families $\mathcal{V}_x$ are closed under finite intersections. For $A \subseteq X$, define $c(A) = \{x \in X : V \cap A \neq \varnothing \text{ for all } V \in \mathcal{V}_x\}$. The following is a straightforward modification of Proposition 2.4 in [15].

Proposition 2.1 *Let X be a Hausdorff space, $D \subseteq X$, and suppose there exists a cardinal κ such that for all $x \in X$ there exists $\mathcal{B}_x \in \left[[D]^{\leq\kappa}\right]^{\leq\kappa}$ such that $\{x\} = \bigcap_{B\in\mathcal{B}_x} c(B)$. Then $|X| \leq |D|^\kappa$.*

The following gives a bound for the cardinality of $c(A)$, where A is a subset of a Hausdorff space X.

Proposition 2.2 *Let X be a Hausdorff space and let κ be a cardinal such that $H\psi(X) \leq \kappa$. If $A \subseteq X$, then $|c(A)| \leq |A|^\kappa$.*

Proof Fix $x \in c(A)$. For all $V \in \mathcal{V}_x$, there exists $x_V \in V \cap A$. Let $A_x = \{x_V : V \in \mathcal{V}\}$. Then $|A_x| \leq \kappa$ and $x \in c(A_x)$. We show for all $V \in \mathcal{V}$ that $x \in c(V \cap A_x)$. Let $W \in \mathcal{V}_x$. As $\mathcal{V}_x$ is closed under finite intersections, there exists $U \in \mathcal{V}_x$ such that $x \in U \subseteq W \cap V$. There exists $x_U \in U \cap A_x \subseteq W \cap V \cap A_x$. Thus $x \in c(V \cap A_x)$ for all $V \in \mathcal{V}$. Let $\mathcal{B}_x = \{V \cap A_x : V \in \mathcal{V}_x\}$. Then $\mathcal{B}_x \in \left[[A]^{\leq\kappa}\right]^{\leq\kappa}$ and $x \in \bigcap_{B\in\mathcal{B}_x} c(B)$. Now let $y \neq x$. There exists $V_x \in \mathcal{V}_x$ and $V_y \in \mathcal{V}_y$ such that $V_x \cap V_y = \varnothing$. Thus, $V_y \cap V_x \cap A_x = \varnothing$, and therefore $y \notin c(V_x \cap A_x)$. This says $y \notin \bigcap_{B\in\mathcal{B}_x} c(B)$ and $\{x\} = \bigcap_{B\in\mathcal{B}_x} c(B)$. Now apply Proposition 2.1 to the Hausdorff space $c(A)$ and conclude $|c(A)| \leq |A|^\kappa$. □

As $\overline{A} \subseteq c(A)$ for any subset A of a space X, the following is an immediate consequence of the above Proposition.

Proposition 2.3 *If X is a Hausdorff space then $|X| \leq d(X)^{H\psi(X)}$.*

Proposition 2.4 *If $\{A_\alpha : \alpha < \kappa^+\}$ is a non-decreasing chain of sets, then $c(\bigcup_{\alpha<\kappa^+} A_\alpha) = \bigcup_{\alpha<\kappa^+} c(A_\alpha)$.*

Proof Let $x \in c(\bigcup_{\alpha<\kappa^+} A_\alpha)$. Then for all $V \in \mathcal{V}_x$ there exists $x_V \in V \cap \bigcup_{\alpha<\kappa^+} A_\alpha$. There exists $\alpha < \kappa^+$ such that $\{x_V : V \in \mathcal{V}_x\} \subseteq A_\alpha$. This implies that for all $V \in \mathcal{V}_x$, $V \cap A_\alpha \neq \varnothing$. Thus, $x \in c(A_\alpha)$. This shows, $c(\bigcup_{\alpha<\kappa^+} A_\alpha) \subseteq \bigcup_{\alpha<\kappa^+} c(A_\alpha)$. As $c(\bigcup_{\alpha<\kappa^+} A_\alpha) \supseteq \bigcup_{\alpha<\kappa^+} c(A_\alpha)$, this completes the proof. □

Definition 2.5 Let X be a space and let $A \subseteq X$. A is *c-closed* if $c(A) = A$. Define the *c-closed hull* of A, denoted by $[A]_c$, as the intersection of all c-closed sets containing A.

We show below that not only is the cardinality of $c(A)$ bounded by $|A|^{H\psi(X)}$, but in fact so is the cardinality of the c-closed hull of A.

Proposition 2.6 *For a set $A \subseteq X$, $|[A]_c| \leq |A|^{H\psi(X)}$.*

Proof Let $\kappa = H\psi(X)$. Let $A_0 = A$ and, by transfinite induction, for any $\alpha < \kappa^+$ define $A_\alpha = c\left(\bigcup_{\beta<\alpha} A_\beta\right)$. It is straightforward to see that $\bigcup_{\alpha<\kappa^+} A_\alpha \subseteq [A]_c$.

Now, as $\{A_\alpha : \alpha < \kappa^+\}$ is a non-decreasing chain of sets, by Proposition 2.4 we see that for any $x \in c\left(\bigcup_{\alpha<\kappa^+} A_\alpha\right)$ there exists $\alpha < \kappa^+$ such that $x \in c(A_\alpha) \subseteq A_{\alpha+1} \subseteq \bigcup_{\alpha<\kappa^+} A_\alpha$. Therefore $c\left(\bigcup_{\alpha<\kappa^+} A_\alpha\right) \subseteq \bigcup_{\alpha<\kappa^+} A_\alpha$ and $\bigcup_{\alpha<\kappa^+} A_\alpha$ is c-closed. This implies $[A]_c \subseteq \bigcup_{\alpha<\kappa^+} A_\alpha$ and thus $[A]_c = \bigcup_{\alpha<\kappa^+} A_\alpha$.

To complete the proof, it suffices to show that $\left|\bigcup_{\alpha<\kappa^+} A_\alpha\right| \leq |A|^\kappa$; that is, we need to show that $|A_\alpha| \leq |A|^\kappa$ for every $\alpha < \kappa^+$. Suppose the contrary and let β be the smallest ordinal such that $|A_\beta| > |A|^\kappa$. We have $|A_\gamma| \le |A|^\kappa$ for any $\gamma < \beta$ and therefore $|\bigcup_{\gamma<\beta} A_\gamma| \le |A|^\kappa$. Then, by Proposition 2.2, we have $|A_\beta| = \left|c\left(\bigcup_{\gamma<\beta} A_\gamma\right)\right| \le \left|\bigcup_{\gamma<\beta} A_\gamma\right|^\kappa \le (|A|^\kappa)^\kappa = |A|^\kappa$. This is a contradiction and the proof is complete. □

Observe that the proof of Proposition 2.6 shows that $[A]_c$ can be obtained by iterating the c operator $H\psi(X)^+$-many times.

Proposition 2.7 *A c-closed set is closed.*

Proof Let A be a subset of X such that $c(A) = A$ and pick $x \notin c(A) = A$. Then there exists $V \in \mathcal{V}_x$ such that $V \cap A = \varnothing$, and thus $V \cap c(A) = \varnothing$. □

The following cardinal invariants were introduced by Bella and Spadaro in [12]. Given a set S and a set I of indices we call $\{S_i : i \in I\} \subset \mathcal{P}(S)$ a *decomposition* of S if $S = \bigcup\{S_i : i \in I\}$.

Definition 2.8 Let X be a space. The *piecewise weak Lindelöf degree $pwL(X)$* of X is the least infinite cardinal κ such that for every open cover $\mathcal{U}$ of X and every decomposition $\{\mathcal{U}_i : i \in I\}$ of $\mathcal{U}$, there are families $\mathcal{V}_i \in [\mathcal{U}_i]^{\le\kappa}$ for every $i \in I$ such that $X \subseteq \overline{\bigcup\{\bigcup \mathcal{V}_i : i \in I\}}$. The *piecewise weak Lindelöf degree for closed sets $pwL_c(X)$* of X is the least infinite cardinal κ such that for every closed set $F \subseteq X$, for every open cover $\mathcal{U}$ of F, and every decomposition $\{\mathcal{U}_i : i \in I\}$ of $\mathcal{U}$, there are families $\mathcal{V}_i \in [\mathcal{U}_i]^{\le\kappa}$ for every $i \in I$ such that $F \subseteq \bigcup\{\overline{\bigcup \mathcal{V}_i} : i \in I\}$.

We give our main theorem in this section, which extends the result of Bella and Spadaro in [12].

Theorem 2.9 *If X is a Hausdorff space then $|X| \le 2^{pwL_c(X)H\psi(X)}$.*

Proof Let $\kappa = pwL_c(X)H\psi(X)$. Declare families $\mathcal{V}_x$ as above. We build a non-decreasing chain $\{A_\alpha : \alpha < \kappa^+\}$ of subsets of X such that for each $\alpha < \kappa^+$,

(1) $|A_\alpha| \le 2^\kappa$, and
(2) Whenever $\{\mathcal{V}_\beta : \beta < \kappa\} \in \left[\left[\bigcup_{x\in A_\alpha} \mathcal{V}_x\right]^{\le\kappa}\right]^{\le\kappa}$ and $X \setminus \bigcup\{\overline{\bigcup \mathcal{V}_\beta} : \beta < \kappa\} \ne \varnothing$, then $A_{\alpha+1} \setminus \bigcup\{\overline{\bigcup \mathcal{V}_\beta} : \beta < \kappa\} \ne \varnothing$.

For limit ordinals $\alpha < \kappa^+$, let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Then $|A_\alpha| \le 2^\kappa$ and $|c(A_\alpha)| \le |A_\alpha|^\kappa \le 2^\kappa$ by Proposition 2.2.

For successor ordinals $\alpha + 1$, for all $\mathcal{V} = \{\mathcal{V}_\beta : \beta < \kappa\} \in \left[\left[\bigcup_{x\in A_\alpha} \mathcal{V}_x\right]^{\le\kappa}\right]^{\le\kappa}$ such that $X \setminus \bigcup\{\overline{\bigcup \mathcal{V}_\beta} : \beta < \kappa\} \ne \varnothing$, let $x_\mathcal{V} \in X \setminus \bigcup\{\overline{\bigcup \mathcal{V}_\beta} : \beta < \kappa\}$. Define $A_{\alpha+1} = c(A_\alpha \bigcup\{x_\mathcal{V} : \mathcal{V} = \{\mathcal{V}_\beta : \beta < \kappa\} \in \left[\left[\bigcup_{x\in A_\alpha} \mathcal{V}_x\right]^{\le\kappa}\right]^{\le\kappa}$ and $X \setminus \bigcup\{\overline{\bigcup \mathcal{V}_\beta} : \beta < \kappa\} \ne \varnothing\})$. Then a counting argument and Proposition 2.2 shows that $|A_{\alpha+1}| \le 2^\kappa$. Also, conditions (1) and (2) above are satisfied.

Let $F = \bigcup_{\alpha<\kappa^+} A_\alpha$, and note $|F| \le 2^\kappa \cdot \kappa^+ = 2^\kappa$. Then, by Proposition 2.4, $c(F) = c(\bigcup_{\alpha<\kappa^+} A_\alpha) = \bigcup_{\alpha<\kappa^+} c(A_\alpha) \subseteq \bigcup_{\alpha<\kappa^+} A_{\alpha+1} = F$. This shows F is c-closed, and by Proposition 2.7, F is closed. We show $X = F$. Suppose by way of contradiction that there exists $x \in X \setminus F$. For all $y \in F$, there exists $V_y \in \mathcal{V}_x$ and $U_y \in \mathcal{V}_y$ such that $V_y \cap U_y = \varnothing$. Let $\mathcal{U} = \{U_y : y \in F\}$ and note $\mathcal{U}$ covers F. For all $V \in \mathcal{V}_x$, let $\mathcal{U}_V = \{U \in \mathcal{U} : V \cap U = \varnothing\}$. Observe that $\mathcal{U} = \bigcup_{V\in\mathcal{V}_x} \mathcal{U}_V$ and that $\{\mathcal{U}_V : V \in \mathcal{V}_x\}$ is a decomposition of $\mathcal{U}$. Since F is closed and $pwL_c(X) \le \kappa$, for all $V \in \mathcal{V}_x$ there exists $\mathcal{W}_V \in [\mathcal{U}_V]^{\le\kappa}$ such that

$F \subseteq \bigcup\{\overline{\bigcup \mathcal{W}_V : V \in \mathcal{V}_x}\}$. As $V \cap \bigcup \mathcal{W}_V = \varnothing$ for every $V \in \mathcal{V}_x$, we see that $x \in X \setminus \overline{\bigcup \mathcal{W}_V}$ for every $V \in \mathcal{V}_x$. This implies $x \in X \setminus \bigcup\{\overline{\bigcup \mathcal{W}_V} : V \in \mathcal{V}_x\} \neq \varnothing$. As $\{\mathcal{W}_V : V \in \mathcal{V}_x\}$ fits condition (2) and $\{\mathcal{W}_V : V \in \mathcal{V}_x\} \in \left[\left[\bigcup_{x \in A_\alpha} \mathcal{V}_x\right]^{\leq\kappa}\right]^{\leq\kappa}$ for some $\alpha < \kappa^+$, we have $\varnothing \neq A_{\alpha+1} \setminus \bigcup\{\overline{\bigcup \mathcal{W}_V} : V \in \mathcal{V}_x\} \subseteq F \subseteq \bigcup\{\overline{\bigcup \mathcal{W}_V : V \in \mathcal{V}_x}\}$. As this is a contradiction, we have $X = F$. This implies $|X| = |F| \leq 2^\kappa$. □

The above proof is a classical "closing-off" argument. Below we give an alternative proof of Theorem 2.9 using the elementary submodels technique.

Proof Let $\kappa = pwL_c(X) \cdot H\psi(X)$ and let θ be a large enough regular cardinal. For every $x \in X$ let $\mathcal{V}_x$ be a family of open neighbourhoods of x witnessing that $H\psi(X) \leq \kappa$ and let $\Phi : X \to \mathcal{P}(\tau)$ be the function defined by $\Phi(x) = \mathcal{V}_x$. Let M be a κ-closed elementary submodel of $H(\theta)$ such that $X, \Phi \in M, \kappa + 1 \subset M$. □

Claim. $X \cap M$ is a closed subset of X.

Proof of Claim 1 It suffices to prove that $X \cap M$ is c-closed. Let $p \in c(X \cap M)$ and let $\{V_\alpha : \alpha < \kappa\}$ be an enumeration of $\mathcal{V}_p$. For every $\alpha < \kappa$ fix a point $x_\alpha \in X \cap M \cap V_\alpha$. Let $A = \{x_\alpha : \alpha < \kappa\}$. Since M is $< \kappa$-closed, $A \in M$. Obviously $p \in c(A)$ and hence $p \in c(A \cap V_\alpha)$ for every $\alpha < \kappa$. Moreover $\bigcap_{\alpha<\kappa} c(A \cap V_\alpha) = \{p\}$. Indeed let $y \in X$ be a point distinct from p. Then there is $V \in \mathcal{V}_x$ and $\alpha < \kappa$ such that $V \cap V_\alpha = \emptyset$ and hence $y \notin c(A \cap V_\alpha)$. Now, applying again κ-closedness of M we see that $A \cap V_\alpha \in M$ and thus, by elementarity $c(A \cap V_\alpha) \in M$. Hence, one final application of κ-closedness of M shows that $\{p\} = \bigcap_{\alpha<\kappa} c(A \cap U_\alpha) \in M$ and therefore $X \cap M$ is c-closed. □

We now claim that $X \subset M$ which implies that $|X| \leq 2^\kappa$. Suppose by contradiction that this is not true and let $z \in X \setminus M$. Let $\{V_\alpha : \alpha < \kappa\}$ be an enumeration of $\mathcal{V}_z$. For every $x \in X \cap M$ choose $V_x \in \mathcal{V}_x$ and $\alpha_x < \kappa$ such that $V_x \cap V_\alpha = \emptyset$. For every $\alpha < \kappa$ let $\mathcal{V}_\alpha = \{V_x : \alpha_x = \alpha\}$. Then $\bigcup\{\mathcal{V}_\alpha : \alpha < \kappa\}$ is an open cover of $X \cap M$ and hence, by $pwL_c(X) \leq \kappa$ we can find a subcollection $\mathcal{U}_\alpha$ of $\mathcal{V}_\alpha$ of cardinality $\leq \kappa$ such that

$$X \cap M \subset \bigcup\left\{\bigcup \mathcal{U}_\alpha : \alpha < \kappa\right\}.$$

Applying repeatedly the κ-closedness of M, we see that the left hand side of the previous equation belongs to M and therefore:

$$M \models X \subset \bigcup\left\{\bigcup \mathcal{U}_\alpha : \alpha < \kappa\right\}$$

Therefore, by elementarity:

$$H(\theta) \models \bigcup\left\{\bigcup \mathcal{U}_\alpha : \alpha < \kappa\right\}$$

But that contradicts the fact that $z \notin \bigcup\{\bigcup \mathcal{U}_\alpha : \alpha < \kappa\}$. □

As $H\psi(X) \leq \chi(X)$, as a corollary we have the following result of Bella and Spadaro.

Corollary 2.10 (Bella and Spadaro [12]) *If X is Hausdorff then* $|X| \leq 2^{pwL_c(X)\chi(X)}$.

As $pwL_c(X) \leq L(X)$, as another corollary we have the following 1991 result of Hodel [19].

Corollary 2.11 (Hodel, [19]) *If X is Hausdorff then* $|X| \leq 2^{L(X)H\psi(X)}$.

Also, as $pwL_c(X) \leq c(X)$, we have another corollary of Theorem 2.9.

Corollary 2.12 *If X is Hausdorff then $|X| \leq 2^{c(X)H\psi(X)}$.*

The previous corollary has a direct proof using the Erdős–Rado Theorem. This is essentially the proof given in 2.15(b) in [20]. We give this proof here for completeness.

Proof Let $\kappa = c(X)H\psi(X)$ and assume by way of contradiction that $|X| > 2^\kappa$. For each $x \in X$, let $\{V(x, \alpha) : \alpha < \kappa\}$ be a family of open sets containing x such that whenever $x \neq y \in X$, there exists $\alpha < \kappa$ and $\beta < \kappa$ such that $V(x, \alpha) \cap V(y, \beta) = \emptyset$. Fix a linear ordering $<$ on X. If $x \neq y \in X$ and $x < y$, there exists $\alpha(x, y) \leq \kappa$ and $\beta(x, y) < \kappa$ such that $V(x, \alpha(x, y)) \cap V(y, \beta(x, y)) = \emptyset$.

Define the map $f : [X]^2 \to \kappa \times \kappa$ by $f(x, y) = (\alpha(x, y), \beta(x, y))$. By the Erdős–Rado Theorem, there exists $(\alpha, \beta) \in \kappa$ and a set $Y \in [X]^{\kappa^+}$ such that $\alpha(x, y) = \alpha$ and $\beta(x, y) = \beta)$ for all $\{x, y\} \in [Y]^2$.

For each $x \in Y$, let $U_x = V(x, \alpha) \cap V(x, \beta)$ and note $x \in U_x$, implying U_x is nonempty. Also, for each $x, y \in Y$ and $x < y$, we have $U_x \cap U_y = \emptyset$. Therefore, $\{U_x : x \in Y\}$ is a cellular family with cardinality exactly κ^+. This contradicts that $c(X) \leq \kappa$. We conclude $|X| \leq 2^\kappa$. □

Example 2.13 There is a Hausdorff space X such that $\psi(X) < H\psi(X) < \chi(X)$.

Proof Let $\kappa = \aleph_\omega$ and for every $n < \omega$, let $X_n = \{f \in 2^\kappa : |f^{-1}(1)| \leq n\}$ and define $X = \bigcup\{X_n : n < \omega\}$. For every function σ from a finite subset of κ to 2, let $[\sigma] = \{f \in X : \sigma \subset f\}$. Call a subset A of X *bounded* if there is $n < \omega$ such that $A \subset X_n$ and *unbounded* otherwise. Let A be any bounded subset of X, let F be any countable subset of X and let $\sigma \in Fn(\kappa, 2)$ be any partial function. Define a topology τ on X by declaring $[\sigma] \setminus (A \cup F)$ to be a basic open set.

The topology τ is a refinement of the usual topology on X as a subset of 2^κ and hence it's a Hausdorff topology. □

Claim 1. $\psi(X) = \omega$.

Proof of Claim 1 We will prove much more, namely that every subset of X is a G_δ. Note that this is equivalent to the fact that every subset S of X is an F_σ, but this is clear by decomposing S into $S = \bigcup\{S \cap X_n : n < \omega\}$ and observing that $S \cap X_n$ is closed in X. □

Claim 2. $\chi(X) > \kappa$

Proof of Claim 2 Let $f \in X$ be a point and let $\mathcal{U}$ be a local base at f. Let $\mathcal{C}$ be the set of all countable subsets of X and recall that $|\mathcal{C}| = \kappa^\omega > \kappa$. For every $C \in \mathcal{C}$ there is $U_C \in \mathcal{U}$ such that $f \in U_C \subset X \setminus C$. Without loss of generality we can assume that $U_C = [\sigma_C] \setminus (A_C \cup F_C)$, where $\sigma_C \in Fn(\kappa, 2)$ is a finite partial function, $A_C \subset X$ is a bounded set and $F_C \subset X$ is a countable set. By the pigeonhole principle we can find $\mathcal{D} \subset \mathcal{C}$ such that $|\mathcal{D}| = \kappa^+$ and a positive integer $k < \omega$ such that $A_C \subset X_k$, for every $C \in \mathcal{D}$. Applying the pigeonhole principle again we can find $\mathcal{E} \subset \mathcal{D}$ such that $|\mathcal{E}| = \kappa^+$ and a single partial function $\sigma \in Fn(\kappa, 2)$ such that $\sigma_C = \sigma$, for every $C \in \mathcal{E}$. But since the set of all countable subsets of $[\sigma] \setminus X_k$ has cardinality $\kappa^\omega > \kappa$ the set $\{F_C : C \in \mathcal{E}\}$ must have cardinality κ^+ and hence $\chi(X) \geq \kappa^+$. □

Claim 3. $H\psi(X) = \kappa$.

Proof of Claim 3 We will prove that, for every finite partial function $\sigma \in Fn(\kappa, 2)$, for every bounded set $A \subset X$ and for every countable set $F \subset X$, $cl_\tau([\sigma]\backslash(A \cup F)) = [\sigma]$ and hence $\psi_c(X) = \kappa$. Therefore, since $\kappa = \psi_c(X) \leq H\psi(X) \leq |X| = \kappa$, that will imply that $H\psi(X) = \kappa$.

Start by noting that since τ is an enlargement of the usual topology on X, the set $[\sigma]$ is still closed in τ and therefore $cl_\tau([\sigma] \setminus (A \cup F)) \subset [\sigma]$. Hence it suffices to show that $[\sigma] \subset cl_\tau([\sigma]\backslash(A \cup F))$. To that end, let $x \in [\sigma]$ and let $\sigma' \in Fn(\kappa, 2)$, $A' \subset X$ and $F' \subset X$ be respectively a finite partial function, a bounded set and a countable set such that $x \in [\sigma']\backslash(A' \cup F')$. Note that since $x \in [\sigma] \cap [\sigma'] = [\sigma \cup \sigma']$, this last set is non-empty. A moment's thought reveals that it is actually an unbounded and uncountable set and therefore $([\sigma']\backslash(A' \cup F')) \cap ([\sigma]\backslash(A \cup F)) \neq \emptyset$, whence $x \in cl_\tau([\sigma]\backslash(A \cup F))$. □

It is easy to see that Example 2.13 is Hausdorff non-regular and therefore the following question remains open:

Question 2.14 *Is there a regular space X such that $\psi(X) < H\psi(X) < \chi(X)$?*

Also note that in Example 2.13 we have $H\psi(X) = \psi_c(X)$ and that suggests the following natural question. Obviously an example answering this question cannot be a regular space.

Question 2.15 *Is there an example of a Hausdorff space X satisfying the following chain of strict inequalities?*

$$\psi(X) < \psi_c(X) < H\psi(X) < \chi(X)$$

Let X be a Urysohn space and let the Urysohn pseudocharacter $U\psi(X)$ be bounded above by a cardinal κ. For all $x \in X$ fix families $\mathcal{V}_x$ of open sets containing x such that $|\mathcal{V}_x| \leq \kappa$ and for all $x \neq y$ there exist $U \in \mathcal{V}_x$ and $V \in \mathcal{V}_y$ such that $\overline{U} \cap \overline{V} \neq \varnothing$. Furthermore, assume the families $\mathcal{V}_x$ are closed under finite intersections. For $A \subseteq X$, define $c'(A) = \{x \in X : \overline{V} \cap A \neq \varnothing$ for all $V \in \mathcal{V}_x\}$.

Definition 2.16 A subset A of a Urysohn space X is c'-*closed* if $c'(A) = A$.

The following is a straightforward modification of Proposition 2.4 in [15].

Proposition 2.17 *Let X be a Urysohn space, $D \subseteq X$, and suppose there exists a cardinal κ such that for all $x \in X$ there exists $\mathcal{B}_x \in \left[[D]^{\leq\kappa}\right]^{\leq\kappa}$ such that $\{x\} = \bigcap_{B\in\mathcal{B}_x} c'(B)$. Then $|X| \leq |D|^\kappa$.*

The following two propositions have proofs similar to that of Propositions 2.2 and 2.4.

Proposition 2.18 *Let X be a Urysohn space and let $U\psi(X) \leq \kappa$. If $A \subseteq X$, then $|c'(A)| \leq |A|^\kappa$.*

Proposition 2.19 *If $\{A_\alpha : \alpha < \kappa^+\}$ is a non-decreasing chain of sets, then $c'(\bigcup_{\alpha<\kappa^+} A_\alpha) = \bigcup_{\alpha<\kappa^+} c'(A_\alpha)$.*

By using the above propositions and suitably modifying the proof of Theorem 2.9, we can obtain the following result for Urysohn spaces. The proof is a closing-off argument left to the reader.

Theorem 2.20 *If X is a Urysohn space then $|X| \leq 2^{pwL(X)U\psi(X)}$.*

As $pwL(X) \leq aL(X)$ as mentioned in [12], we have as a corollary the following result of Stavrova.

Corollary 2.21 (Stavrova [23]) *If X is Urysohn then $|X| \leq 2^{aL(X)U\psi(X)}$.*

As $U\psi(X) \leq \chi(X)$ for a Urysohn space X, we have as another corollary of Theorem 2.20 the following result of Bella and Spadaro.

Corollary 2.22 (Bella and Spadaro [12]) *If X is Urysohn then $|X| \leq 2^{pwL(X)\chi(X)}$.*

3 Linearly Lindelöf spaces

The notion of a linearly Lindelöf space is an important weakening of Lindelöfness. It is equivalent to the assertion that every set of regular uncountable cardinality has a complete accumulation point. There are examples of first-countable or locally compact linearly Lindelöf spaces which are not Lindelöf. An outstanding open problem is to find a normal linearly Lindelöf space which is not Lindelöf. Such a space would be a strong kind of Dowker space.

In this section we show that if X is a Hausdorff linearly Lindelöf space with $H\psi(X) = \omega$ then $|X| \leq 2^\omega$, under the assumption that either $2^{<\mathfrak{c}} = \mathfrak{c}$ or $\mathfrak{c} < \aleph_\omega$.

Definition 3.1 A space is *linearly Lindelöf* provided that every increasing open cover has a countable subcover.

The following is Lemma 3.2 in [6].

Lemma 3.2 ([6, Lemma 3.2]) *Assume either $\mathfrak{c} < \aleph_\omega$ or $2^{<\mathfrak{c}} = \mathfrak{c}$. Let X be a linearly Lindelöf T_1 space. If $\psi(X) \leq \mathfrak{c}$, then any closed set of cardinality at most $\mathfrak{c}$ is the intersection of $\mathfrak{c}$-many open sets.*

Definition 3.3 A space is *ω_1-Lindelöf* if every open cover of size at most ω_1 has a countable subcover.

The following is straightforward to prove.

Lemma 3.4 *Every linearly Lindelöf space is ω_1-Lindelöf.*

Theorem 3.5 *Let X be a Hausdorff space satisfying $H\psi(X) = \omega$. If X is ω_1-Lindelöf and $\psi(C, X) \leq 2^\omega$ for every c-closed set C with $|C| \leq 2^\omega$, then $|X| \leq 2^\omega$.*

Proof Let $\{\mathcal{U}_x : x \in X\}$ be a collection witnessing $H\psi(X) = \omega$ and c the associated operator. For any c-closed set H with $|H| \leq 2^\omega$ fix a family $\mathcal{V}(H)$ of open sets satisfying $|\mathcal{V}(H)| \leq 2^\omega$ and $H = \bigcap \mathcal{V}(H)$.

We will construct by transfinite induction a nondecreasing family $\{H_\alpha : \alpha < \omega_1\}$ of c-closed subsets of X satisfying:

(1) $|H_\alpha| \leq 2^\omega$;
(2) if $X \setminus \bigcup \mathcal{V} \neq \varnothing$ for some countable family $\mathcal{V} \subseteq \bigcup\{\mathcal{V}(H_\beta) : \beta \leq \alpha\}$, then $H_{\alpha+1} \setminus \bigcup \mathcal{V} \neq \varnothing$.

Fix a choice function ϕ and assume we have already defined H_β for each $\beta < \alpha$. If α is a limit ordinal, put $H_\alpha = [\bigcup\{H_\beta : \beta < \alpha\}]_c$. If $\alpha = \gamma + 1$, put $H_\alpha = [H_\gamma \cup \{\phi(X \setminus \bigcup \mathcal{V}) : \mathcal{V}$

countable subfamily of $\bigcup\{\mathcal{V}(H_\beta) : \beta \leq \gamma\}]_c$. After the induction, put $H = \bigcup\{H_\alpha : \alpha < \omega_1\}$. Because $H\psi(X) = \omega$, we have that H is c- closed.

Since $|H| \leq 2^\omega$, it suffices to check that $H = X$. If not, fix $p \in X \setminus H$. For any α choose $V_\alpha \in \mathcal{V}(H_\alpha)$ such that $p \notin V_\alpha$. Since H is c-closed, it is also closed in X. Therefore, H is ω_1-Lindelöf and we may then fix a countable subcover $\mathcal{V} \subseteq \{V_\alpha : \alpha < \omega_1\}$. Since there exists an ordinal $\gamma < \omega_1$ such that $\mathcal{V} \subseteq \bigcup\{\mathcal{V}(H_\alpha) : \alpha \leq \gamma\}$, we reach a contradiction with the definition of $H_{\gamma+1}$. This completes the proof. □

Now, we immediately get the following as a consequence of the above theorem and Lemmas 3.2 and 3.4.

Corollary 3.6 *Assume either $2^{<\mathfrak{c}} = \mathfrak{c}$ or $\mathfrak{c} < \aleph_\omega$. If X is a Hausdorff linearly Lindelöf space with $H\psi(X) = \omega$, then $|X| \leq 2^\omega$.*

It is still unknown, even for first countable spaces, whether the above corollary is true in ZFC. However, Arhangel′skiĭ and Buzyakova proved this works for Tychonoff spaces. The crucial point in [1, Lemma 2.10] is a proof that lemma holds in ZFC for Tychonoff spaces. Their argument uses the fact that in a first countable linearly Lindelöf space every free sequence is countable. As Example 3.7 shows, this may fail at least for Hausdorff spaces. We do not know what happens for Tychonoff spaces, but formally Lemma 2.10 in [1] cannot be immediately adapted to spaces of countable Hausdorff pseudocharacter.

Example 3.7 Refine the topology of the Double Alexandroff of the unit interval so as to make every countable set closed discrete. The resulting space X is a Lindelöf space of countable Hausdorff pseudocharacter and any ω_1-sized discrete subset of X is a free sequence.

Question 3.8 *Let X be a Tychonoff linearly Lindelöf space with $H\psi(X) = \omega$. Is it true that $\psi(C, X) \leq \mathfrak{c}$ for every c-closed set C satisfying $|C| \leq \mathfrak{c}$?*

Question 3.9 *Let X be a Tychonoff linearly Lindelöf space with $H\psi(X) = \omega$. Is $|X| \leq 2^\omega$?*

4 Game-theoretic bounds

For an infinite cardinal κ, let $G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$ be the two-player game in κ-many innings where at inning $\alpha < \kappa$, player I plays an open cover $\mathcal{O}_\alpha$ of X and player II responds by picking an open set $O_\alpha \in \mathcal{O}_\alpha$. Player II wins if $\overline{\bigcup\{O_\alpha : \alpha < \kappa\}} = X$. Obviously, if player II has a winning strategy in $G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$ then $wL(X) \leq \kappa$, so the countable version of this game provides a natural game-theoretic strengthening of the weak Lindelöf property.

Let $G_o^p(\kappa)$ be the two-player game in κ many innings where at inning $\alpha < \kappa$ player I chooses a point $p_\alpha \in X$ and player II responds by picking an open neighbourhood O_α of p_α and player I wins if $\overline{\bigcup\{O_\alpha : \alpha < \kappa\}} = X$. This game is a variant of the classical point-open game defined by Galvin in [18].

It was shown in [3] that the games $G_o^p(\kappa)$ and $G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$ are dual, that is, player I (respectively, player II) has a winning strategy in the first game if and only if player II (respectively, player I) has a winning strategy in the second game.

Theorem 4.1 *Let (X, τ) be a regular space such that player two has a winning strategy in the game $G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$, $H\psi(X) < \kappa$ and $\chi(X) \leq 2^{<\kappa}$. Then $|X| \leq 2^{<\kappa}$.*

Proof Fix a winning strategy σ for the first player in the game $G_o^p(\kappa)$. By the assumption on the Hausdorff pseudocharacter, for every $x \in X$ we can fix a family $\mathcal{V}_x$ of open sets such that $|\mathcal{V}_x| < \kappa$ and for every couple of distinct points $x, y \in X$ there are $V_x \in \mathcal{V}_x$ and $V_y \in \mathcal{V}_y$ such that $V_x \cap V_y = \emptyset$. Without loss we can assume that each family $\mathcal{V}_x$ is closed under finite intersections. Let $\Phi : X \to \mathcal{P}(\tau)$ be the map $\Phi(x) = \mathcal{V}_x$.

Let θ be a large enough regular cardinal and M be a $< \kappa$-closed elementary submodel of $H(\theta)$ such that $X, \sigma, \Phi \in M$ and $|M| \leq 2^{<\kappa}$. □

Claim 1. $X \cap M$ is a closed subset of X.

Proof of Claim 1 It suffices to prove that $X \cap M$ is c-closed. Let $p \in c(X \cap M)$ and let $\{V_\alpha : \alpha < \mu\}$ be an enumeration of $\mathcal{V}_p$, where $\mu < \kappa$. For every $\alpha < \mu$ fix a point $x_\alpha \in X \cap M \cap V_\alpha$. Let $A = \{x_\alpha : \alpha < \mu\}$. Since M is $< \kappa$-closed, $A \in M$. Obviously $p \in c(A)$ and hence $p \in c(A \cap V_\alpha)$ for every $\alpha < \mu$. Moreover $\bigcap_{\alpha<\mu} c(A \cap V_\alpha) = \{p\}$. Indeed let $y \in X$ be a point distinct from p. Then there is $V \in \mathcal{V}_x$ and $\alpha < \mu$ such that $V \cap V_\alpha = \emptyset$ and hence $y \notin c(A \cap V_\alpha)$. Now, applying again $< \kappa$-closedness of M we see that $A \cap V_\alpha \in M$ and hence, by elementarity $c(A \cap V_\alpha) \in M$. Hence, one final application of $< \kappa$-closedness of M shows that $\{p\} = \bigcap_{\alpha<\mu} c(A \cap U_\alpha) \in M$ and therefore $X \cap M$ is c-closed. □

Claim 2. $X \cap M$ is dense in X.

Proof of Claim 2 Suppose this is not true, then there is a non-empty open subset V of X such that $\overline{V} \cap (X \cap M) = \emptyset$. We play a game of $G_o^p(\kappa)$ where player II uses the strategy σ.

In the first inning player I picks the point $x_0 = \sigma(\emptyset) \in X \cap M$. By elementarity we can fix a local base $\mathcal{U} \in M$ for x_0 having cardinality $2^{<\kappa}$. Since $2^{<\kappa} + 1 \subset M$ we actually have $\mathcal{U} \subset M$ and hence, since $x_0 \notin \overline{V}$, we can choose an open neighbourhood $U_0 \in M$ of x_0 such that $U_0 \cap V = \emptyset$.

Let now $\beta < \kappa$ and, for every $\alpha < \beta$, suppose player II picked an open set $U_\alpha \in M$ such that $U_\alpha \cap V = \emptyset$ at inning α. Since M is $< \kappa$-closed $x_\beta = \sigma(\{U_\alpha : \alpha < \beta\}) \in M$ and therefore, reasoning as before, player II can choose an open neighbourhood U_β of x_β such that $U_\beta \cap V = \emptyset$.

Eventually, since σ is a winning strategy for player I in $G_o^p(\kappa)$, the set $\bigcup\{U_\alpha : \alpha < \kappa\}$ is dense in X, but this contradicts the fact that $V \cap U_\alpha = \emptyset$, for every $\alpha < \kappa$. □

Since $X \cap M$ is both closed and dense in X we have $X = X \cap M$ and hence $|X| \leq 2^{<\kappa}$.

As pointed out by the referee, Claim 2 of the above theorem is related to Galvin's theorem that if player I has a winning strategy in the point-open game of countable length on a space X with points G_δ then X is countable. Indeed, if we assume that player I has a winning strategy in the point-open game, then by a similar argument as the one proving Claim 2 (replacing *local base* with *local pseudobase*) we see that $X = X \cap M$. The countable case of Claim 2 is also a consequence of Theorem 9 of [4] (which is attributed there to Tkachuk. See also [10]).

Lemma 4.2 *Every c'-closed set is θ-closed.*

Proof Let A be a c'-closed set and let $x \notin A$. Then there is $V \in \mathcal{V}_x$ such that $\overline{V} \cap A = \emptyset$ and hence $x \notin cl_\theta(A)$. This proves that $cl_\theta(A) \subset A$ and hence A is θ-closed. □

The next theorem improves Theorem 3 in [2] (see [13] for a related result).

Theorem 4.3 *Let (X, τ) be a Urysohn space such that player two has a winning strategy in the game $G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$, $U\psi(X) < \kappa$ and $\chi(X) \leq 2^{<\kappa}$. Then $|X| \leq 2^{<\kappa}$.*

Proof Fix a winning strategy σ for the first player in the game $G^p_o(\kappa)$. By the assumption on the Urysohn pseudocharacter, for every $x \in X$ we can fix a family $\mathcal{V}_x$ of open sets such that $|\mathcal{V}_x| < \kappa$ and for every couple of distinct points $x, y \in X$ there are $V_x \in \mathcal{V}_x$ and $V_y \in \mathcal{V}_y$ such that $\overline{V_x} \cap \overline{V_y} = \emptyset$. Without loss we can assume that each family $\mathcal{V}_x$ is closed under finite intersections. Let $\Phi : X \to \mathcal{P}(\tau)$ be the function $\Phi(x) = \mathcal{V}_x$.

Let θ be a large enough regular cardinal and M be a $< \kappa$-closed elementary submodel of $H(\theta)$ such that $X, \sigma, \Phi \in M$ and $|M| \leq 2^{<\kappa}$. □

Claim 1. $X \cap M$ is a θ-closed subset of X.

Proof of Claim 1 In view of Lemma 4.2 it suffices to prove that $X \cap M$ is c'-closed. Let $p \in c'(X \cap M)$ and let $\{V_\alpha : \alpha < \mu\}$ be an enumeration of $\mathcal{V}_p$, where $\mu < \kappa$. For every $\alpha < \mu$ fix a point $x_\alpha \in X \cap M \cap \overline{V_\alpha}$. Let $A = \{x_\alpha : \alpha < \mu\}$. Since M is $< \kappa$-closed, $A \in M$. Obviously $p \in c'(A)$ and hence $p \in c'(A \cap \overline{V_\alpha})$ for every $\alpha < \mu$. Moreover $\bigcap_{\alpha<\mu} c'(A \cap \overline{V_\alpha}) = \{p\}$. Indeed let $y \in X$ be a point distinct from p. Then there is $V \in \mathcal{V}_x$ and $\alpha < \mu$ such that $\overline{V} \cap \overline{V_\alpha} = \emptyset$ and hence $y \notin c'(A \cap \overline{V_\alpha})$. Now, applying again $< \kappa$-closedness of M we see that $A \cap \overline{V_\alpha} \in M$ and hence, by elementarity $c'(A \cap \overline{V_\alpha}) \in M$. Hence, one final application of $< \kappa$-closedness of M shows that $\{p\} = \bigcap_{\alpha<\mu} c'(A \cap \overline{U_\alpha}) \in M$ and therefore $X \cap M$ is c'-closed. □

Claim 2. $X \cap M$ is a θ-dense subset of X.

Proof of Claim 2 The proof is the same as that of Claim 2 of the previous theorem. □

Since $X \cap M$ is both θ-closed and θ-dense in X we have $X = X \cap M$ and hence $|X| \leq 2^{<\kappa}$.

5 Almost discretely Lindelöf spaces

In this section we show that a theorem proved in [11] for first countable spaces continues to hold for spaces with countable Hausdorff pseudocharacter. A space is *almost discretely Lindelöf* provided that every discrete set is contained in a Lindelöf subspace.

Theorem 5.1 *[$2^{<\mathfrak{c}} = \mathfrak{c}$] If X is a Hausdorff almost discretely Lindelöf space satisfying $H\psi(X) = \omega$, then $|X| \leq 2^\omega$.*

Proof Let c be the operator associated to a witness $\{\mathcal{V}_x : x \in X\}$ for $H\psi(X) = \omega$. Let L be a left-separated subset of X.

We will construct by trasfinite induction a non decreasing family $\{H_\alpha : \alpha < \mathfrak{c}\}$ of c-closed subsets of X satisfying:

(1) $|H_\alpha| \leq \mathfrak{c}$;
(2) if $L \setminus \bigcup \mathcal{V} \neq \emptyset$ for some family $\mathcal{V} \in [\bigcup\{\mathcal{V}_x : x \in H_\alpha\}]^{<\mathfrak{c}}$, then $L \cap H_{\alpha+1} \setminus \bigcup \mathcal{V} \neq \emptyset$.

Fix a choice function ϕ and assume we have already defined H_β for each $\beta < \alpha$. If α is a limit ordinal, let $H_\alpha = [\bigcup\{H_\beta : \beta < \alpha\}]_c$. If $\alpha = \gamma + 1$, then let $H_\alpha = [H_\gamma \cup \{\phi(L \setminus \bigcup \mathcal{V}) : \mathcal{V} \in [\bigcup\{\mathcal{V}_x : x \in H_\gamma\}]^{<\mathfrak{c}}\}]_c$. Proposition 2.6 and our set-theoretic assumption guarantee that $|H_{\alpha+1}| \leq \mathfrak{c}$. After the induction let $H = \bigcup\{H_\alpha : \alpha < \mathfrak{c}\}$. It is clear that $|H| \leq \mathfrak{c}$. Moreover, as $H\psi(X) = \omega$, we have that H is c-closed.

We will check that $L \subset H$. If not, fix $p \in L \setminus H$. For any $x \in H$ choose $V_x \in \mathcal{V}_x$ such that $p \notin V_x$ and let $\mathcal{U} = \{V_x : x \in H\}$. □

Claim. There is a $<\mathfrak{c}$-sized subcollection $\mathcal{V}$ of $\mathcal{U}$ covering $L \cap H$.

Proof of Claim Of course, if $|L \cap H| < \mathfrak{c}$ there is nothing to prove. So, assume $|L \cap H| = \mathfrak{c}$. Let $\{U_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{U}$ in type $\mathfrak{c}$ and set $V_\alpha = U_\alpha \setminus \bigcup\{U_\beta : \beta < \alpha\}$. Suppose by way of contradiction that the statement of the Claim is not true. Then the set $S = \{\alpha < \mathfrak{c} : V_\alpha \cap (L \cap H) \neq \emptyset\}$ has cardinality continuum.

Pick a point $x_\alpha \in V_\alpha \cap L$, for every $\alpha \in S$. Then $R = \{x_\alpha : \alpha \in S\}$ is a set of size continuum which is both right-separated and left-separated. So by 2.12 of [20], the set R contains a discrete set D having cardinality continuum.

Since X is almost discretely Lindelöf, we can find a Lindelöf subspace $Y \subset X$. such that $D \subset Y$. Now, H being c-closed and hence closed in X, the set $Y \cap H$ is also Lindelöf, and since $\mathcal{U}$ covers $Y \cap H$, we can find an ordinal $\delta < \mathfrak{c}$ such that $D \subset Y \cap H \subset \bigcup\{U_\alpha : \alpha < \delta\}$. But since D has cardinality continuum, there must be $\gamma > \delta$ such that $D \cap V_\gamma \neq \emptyset$ and this contradicts the fact that V_γ is disjoint from $\bigcup\{U_\alpha : \alpha < \delta\}$. □

Fix a subcollection $\mathcal{V} \subset \mathcal{U}$ of cardinality smaller than the continuum such that $L \cap H \subset \bigcup \mathcal{V}$. $2^{<\mathfrak{c}} = \mathfrak{c}$ implies that $\mathfrak{c}$ is a regular cardinal and so there exists $\gamma < \mathfrak{c}$ such that $\mathcal{V} \in [\bigcup\{\mathcal{V}_x : x \in H_\gamma\}^{<\mathfrak{c}}$. Since $L \setminus \bigcup \mathcal{V} \neq \emptyset$, according to the way $H_{\gamma+1}$ was constructed, we should have $L \cap H_{\gamma+1} \setminus \bigcup \mathcal{V} \neq \emptyset$. But, this is a contradiction because we have $L \cap H_{\gamma+1} \subset L \cap H \subset \bigcup \mathcal{V}$.

What has been proved so far shows that every left-separated subset of X has size not exceeding $\mathfrak{c}$. In particular, this means that $d(X) \leq \mathfrak{c}$. Thus, by Proposition 2.3, we finally have $|X| \leq d(X)^{H\psi(X)} \leq \mathfrak{c}^\omega = \mathfrak{c}$.

We now present a shorter proof of the above theorem using elementary submodels.

Proof Let μ be a large enough regular cardinal. Let M be a $<\mathfrak{c}$-closed elementary submodel of $H(\mu)$ such that $|M| = \mathfrak{c}$, $\mathfrak{c} + 1 \subset M$ and $X \in M$. □

Claim 1. $X \cap M$ is a closed subset of X.

Proof The proof is the same as that of Claim 1 of Theorem 4.1 □

In view of Proposition 2.3 it suffices to prove that $d(X) \leq \mathfrak{c}$.

Suppose by contradiction that $d(X) \geq \mathfrak{c}^+$. Using that, it is easy to find a left-separated subset L of X having cardinality $\mathfrak{c}^+$. Without loss we can assume that $L \in M$.

Since L has cardinality larger than the continuum, we can pick a point $p \in L \setminus M$. Fix a point $x \in X \cap M$. Since $\psi(X) \leq H\psi(X)$, we can find a countable family of open sets $\mathcal{U}_x \in M$ such that $\bigcap \mathcal{U}_x = \{x\}$. Since $\omega + 1 \subset M$ we actually have that $\mathcal{U}_x \subset M$. Hence, for every $x \in X \cap M$, we can find an open set $U_x \in M$ such that $x \in U_x$ and $p \notin U_x$.

Now $\mathcal{U} = \{U_x : x \in X \cap M\}$ is an open cover of $X \cap M$.

Claim 2. There is a $<\mathfrak{c}$-sized subcollection of $\mathcal{U}$ covering $L \cap M$.

Proof of Claim If the set $L \cap M$ had cardinality smaller than the continuum, this would be trivially true. Hence we may assume that $|L \cap M| = \mathfrak{c}$.

Let $\{U_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{U}$ in type $\mathfrak{c}$ and set $V_\alpha = U_\alpha \setminus \bigcup\{U_\beta : \beta < \alpha\}$. Suppose by contradiction that the statement of Claim 2 is not true. Then the set $S = \{\alpha < \mathfrak{c} : V_\alpha \cap (L \cap M) \neq \emptyset\}$ has cardinality continuum.

Pick a point $x_\alpha \in V_\alpha \cap L$, for every $\alpha \in S$. Then $R = \{x_\alpha : \alpha \in S\}$ is a set of size continuum which is both right-separated and left-separated. So by 2.12 of [20] the set R contains a discrete set D having cardinality continuum.

Since X is almost discretely Lindelöf, we can find a Lindelöf subspace $Y \subset X$. such that $D \subset Y$. Now, $X \cap M$ being closed, the set $Y \cap M$ is also Lindelöf, and since $\mathcal{U}$ covers $Y \cap M$, we can find an ordinal $\delta < \mathfrak{c}$ such that $D \subset Y \cap M \subset \bigcup\{U_\alpha : \alpha < \delta\}$. But since D has cardinality continuum, there must be $\gamma > \delta$ such that $D \cap V_\gamma \neq \emptyset$ and this contradicts the fact that V_γ is disjoint from $\bigcup\{U_\alpha : \alpha < \delta\}$. □

Fix a subcollection $\mathcal{V} \subset \mathcal{U}$ of cardinality smaller than the continuum such that $L \cap M \subset \bigcup \mathcal{V}$.

Since M is $< \mathfrak{c}$-closed we have that $\mathcal{V} \in M$ and since L is also an element of M it follows that: $M \models L \subset \bigcup \mathcal{V}$.

By elementarity $H(\mu) \models L \subset \bigcup \mathcal{V}$, but that is a contradiction because $p \in L \setminus \bigcup \mathcal{V}$.

6 Spaces with a compact π-base

We say that a space X has a *compact π-base* if X has a π-base $\mathcal{B}$ such that $\overline{B}$ is compact for each $B \in \mathcal{B}$. Observe that every locally compact space and every space with a dense set of isolated points has a compact π-base. It was shown by Bella, Carlson, and Gotchev in [9] that the cardinality of a Hausdorff space with a compact π-base is at most $2^{wL(X)t(X)\psi_c(X)}$. Spaces with a compact π-base were further studied in [8] where improvements and variations were made to this cardinality bound. Using the Hausdorff pseudocharacter $H\psi(X)$ we obtain a further bound for these spaces in Theorem 6.1.

Recall that a space is *quasiregular* if every nonempty open set contains the closure of a nonempty open set. It was shown in [9] that any space with a compact π-base is quasiregular. This fact is used in the next theorem.

Theorem 6.1 *If X is a Hausdorff space with a compact π-base, then $|X| \leq 2^{wL(X)H\psi(X)}$.*

Proof Let $\kappa = wL(X)H\psi(X)$ and let $\mathcal{B}$ be a π-base of non-empty open sets with compact closures.

For each $x \in X$ fix a collection $\mathcal{V}_x$ of open neighborhoods of x such that $|\mathcal{V}_x| \leq \kappa$ witnessing $H\psi(X) \leq \kappa$ and let $c(\cdot)$ be the associated operator. Without loss of generality we may assume that each $\mathcal{V}_x$ is closed under finite intersections.

Since $\psi(\overline{B}) \leq H\psi(X)$ and $\overline{B}$ is compact, we have $|\overline{B}| \leq 2^{\psi(\overline{B})} \leq 2^{H\psi(X)} \leq 2^\kappa$. In particular, every element of $\mathcal{B}$ has cardinality not exceeding 2^κ.

We will construct by transfinite recursion a non-decreasing chain of open sets $\{U_\alpha : \alpha < \kappa^+\}$ such that

(1) $|U_\alpha| \leq 2^\kappa$ for every $\alpha < \kappa^+$, and
(2) if $X \setminus [\bigcup \mathcal{M}]_c \neq \varnothing$ for some $\mathcal{M} \in [\bigcup\{\mathcal{V}_x : x \in [U_\alpha]_c\}]^{\leq\kappa}$, then there is $B_\mathcal{M} \in \mathcal{B}$ such that $B_\mathcal{M} \subset U_{\alpha+1} \setminus [\bigcup \mathcal{M}]_c$.

Let $B_0 \in \mathcal{B}$ be arbitrary and set $U_0 = B_0$. If $\beta = \alpha + 1$, for some α, then for every $\mathcal{M} \in [\bigcup\{\mathcal{V}_x : x \in [U_\alpha]_c\}]^{\leq\kappa}$ such that $X \setminus [\bigcup \mathcal{M}]_c \neq \varnothing$, we choose $B_\mathcal{M} \in \mathcal{B}$ such that $B_\mathcal{M} \subseteq X \setminus [\bigcup \mathcal{M}]_c$. We define $U_\beta = U_\alpha \cup \bigcup\{B_\mathcal{M} : \mathcal{M} \in [\bigcup\{\mathcal{V}_x : x \in [U_\alpha]_c\}]^{\leq\kappa}, X \setminus [\bigcup \mathcal{M}]_c \neq \varnothing\}$. Since $|[U_\alpha]_c| \leq |U_\alpha|^{H\psi(X)} \leq (2^\kappa)^\kappa = 2^\kappa$, we see that $|U_\beta| \leq 2^\kappa$ as required. If $\beta < \kappa^+$ is a limit ordinal we let $U_\beta = \bigcup_{\alpha<\beta} U_\alpha$.

Let $F = \bigcup\{[U_\alpha]_c : \alpha < \kappa^+\}$. Then $|F| \leq 2^\kappa$. Since $H\psi(X) \leq \kappa$, we have $F = [\bigcup\{U_\alpha : \alpha < \kappa^+\}]_c = [W]_c$, where $W = \bigcup\{U_\alpha : \alpha < \kappa^+\}$.

We will show that $X = F$. Suppose that $X \neq F$. Since F is closed and a Hausdorff space with a compact π-base is quasiregular, there is $B \in \mathcal{B}$ such that $\overline{B} \subseteq X \setminus F$. Then for every $x \in F$ and $y \in \overline{B}$ there is $V_x(y) \in \mathcal{V}_x$ such that $y \notin \overline{V_x(y)}$. Therefore, using the compactness of $\overline{B}$ and the fact that each $\mathcal{V}_x$ is closed under finite intersections, for every $x \in F$ we can find $V_x \in \mathcal{V}_x$ such that $\overline{V_x} \cap \overline{B} = \emptyset$; hence $\overline{B} \cap V_x = \emptyset$. Clearly $\{V_x : x \in F\}$ is an open cover of F and hence $\{V_x : x \in F\} \cup \{X \setminus F\}$ is an open cover of X. Since $wL(X) \leq \kappa$, there exists $\mathcal{M} \in \{V_x : x \in F\}^{\leq\kappa}$ such that $X = \overline{\bigcup \mathcal{M} \cup X\setminus F}$. As $W \subseteq F$ and $X \setminus W$ is closed, we have $\overline{X\setminus F} \subseteq X\setminus W$ and so $W \subseteq \overline{\bigcup \mathcal{M}}$. Observe that for any set S we always have $\overline{S} \subseteq c(S)$ and consequently $[S]_c \subseteq [\overline{S}]_c \subseteq [c(S)]_c = [S]_c$. From this we obtain $F = [W]_c \subseteq [cl\bigcup \mathcal{M}]_c = [\bigcup \mathcal{M}]_c$. Then there exists $\alpha < \kappa^+$ such that $\mathcal{M} \in [\bigcup\{\mathcal{V}_x : x \in [U_\alpha]_c\}]^{\leq\kappa}$. We claim that $B \cap [\bigcup \mathcal{M}]_c = \emptyset$. To this end, fix $x \in B$. Since $\overline{B}$ is compact Hausdorff (and hence regular), there is an open set O satisfying $x \in O \subseteq \overline{O} \subseteq B$. From $\{x\} = \bigcap\{\overline{V} : V \in \mathcal{V}_x\}$ and the compactness of $\overline{O}$ we deduce the existence of $V \in \mathcal{V}_x$ such that $V \subseteq \overline{V} \subseteq \overline{O} \subseteq B$. This means $x \notin c(X\setminus B)$. We then have $c(X\setminus B) \subseteq X\setminus B$ implying $X\setminus B$ is c-closed. Thus $[X \setminus B]_c = X \setminus B$. Now, from $\bigcup \mathcal{M} \subseteq X\setminus B$, it follows that $[\bigcup \mathcal{M}]_c \subseteq [X\setminus B]_c = X\setminus B$, as required.

$B \subseteq X \setminus [\bigcup \mathcal{M}]_c$ implies $X \setminus [\bigcup \mathcal{M}]_c \neq \emptyset$. Thus, there exists $B_{\mathcal{M}} \in \mathcal{B}$ such that $\emptyset \neq B_{\mathcal{M}} \subseteq U_{\alpha+1}\setminus[\bigcup \mathcal{M}]_c \subseteq F\setminus[\bigcup \mathcal{M}]_c = \emptyset$. Since this is a contradiction, we conclude that $X = F$ and the proof is completed. □

As every space with a dense set of isolated points has a compact π-base, we have the following corollary.

Corollary 6.2 *If X is a Hausdorff space with a dense set of isolated points then $|X| \leq 2^{wL(X)H\psi(X)}$.*

Recall that Bell, Ginsburg and Woods [5] showed that if X is normal then $|X| \leq 2^{wL(X)\chi(X)}$. In light of the above theorem, we ask the following.

Question 6.3 *If X is normal, is $|X| \leq 2^{wL(X)H\psi(X)}$?*

This is related to Question 4.1 in [5], which asks if $|X| \leq 2^{wL(X)t(X)\psi_c(X)}$ for normal spaces X. However, note that if X is normal then it is easily seen that $pwL_c(X) = pwL(X)$ and hence $|X| \leq 2^{pwL(X)H\psi(X)}$ by Theorem 2.9. This represents an improvement over Theorem 2.20 in the case where X is normal.

While we do not know the answer to Question 6.3, the answer is negative in the case when X is Tychonoff as the following example demonstrates.

Example 6.4 A weakly Lindelöf zero-dimensional T_1 (hence Tychonoff) space X with $H\psi(X) = \omega$ whose cardinality may be arbitrarily large.

Let κ be any cardinal, let $\mathbb{Q}$ denote the rationals, and let A be any countable dense subset of the space of irrational numbers. Let Y be the set $(\mathbb{Q} \times \kappa) \cup A$ with the following topology. If $q \in \mathbb{Q}$ and $\alpha < \kappa$ then a neighborhood base at (q, α) is $\{U_n(q, \alpha) : n = 1, 2, \ldots\}$ where $U_n(q, \alpha) = \{(r, \alpha) : r \in \mathbb{Q} \text{ and } |r - q| < 1/n\}$. If $a \in A$, $n \in \mathbb{N}$, and $F \in [\kappa]^{<\omega}$, let $V_{n,F}(a) = \{b \in A : |b - a| \leq 1/n\} \cup \{(q, \alpha) \in \mathbb{Q} \times \kappa : |q - a| < 1/n \text{ and } \alpha \notin F\}$. Then $\{V_{n,F}(a) : n \in \mathbb{N} \text{ and } F \in [\kappa]^{<\omega}\}$ is a neighborhood base at a.

A shown in [5], the space Y is a weakly Lindelöf zero-dimensional T_1 space. Let us check that $H\psi(Y) = \omega$. As a witness of it, we take the collection $\{\{U_n(q, \alpha) : n = 1, 2, \ldots\} : (q, \alpha) \in \mathbb{Q} \times \kappa\} \cup \{\{V_{n,\emptyset}(a) : n = 1, 2, \ldots\} : a \in A\}$.

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References

1. Arhangel'skiĭ, A.V., Buzyakova, R.Z.: On linearly Lindelöf and strongly discretely Lindelöf spaces. Proc. Am. Math. Soc. **127**(8), 2449–2458 (1999)
2. Aurichi, L., Bella, A., Spadaro, S.: Cardinal estimates involving the weak Lindelöf game. Revista de la Real Academia de Ciencias Exactas Físicas y Naturales. Serie A. Matemáticas **116**, 5 (2022)
3. Aurichi, L., Spadaro, S., Zdomskyy, L.: Selective versions of chain-condition type properties. Acta Math. Hung. **148**, 1–16 (2016)
4. Babinkostova, L., Pansera, B.A., Scheepers, M.: Weak covering properties and infinite games. Topol. Appl. **159**, 3644–3657 (2012)
5. Bell, M., Ginsburg, J., Woods, G.: Cardinal inequalities for topological spaces involving the weak Lindelöf number. Pac. J. Math. **79**(1), 37–45 (1978)
6. Bella, A.: Observations on some cardinality bounds. Topol. Appl. **228**, 355–362 (2017)
7. Bella, A., Bonanzinga, M., Matveev, M.: Sequential + separable vs sequentially separable and another variation on selective separability. Cent. Eur. J. Math. **11**, 530–538 (2013)
8. Bella, A., Carlson, N., Gotchev, I.: On spaces with a π-base with elements with compact closure **(preprint)**
9. Bella, A., Carlson, N., Gotchev, I.: More on cardinality bounds involving the weak Lindelöf degree. Quaest. Math. (2022). https://doi.org/10.2989/16073606.2022.2040634
10. Bella, A., Spadaro, S.: Infinite games and cardinal properties of topological spaces. Houst. J. Math. **41**, 1063–1077 (2015)
11. Bella, A., Spadaro, S.: On the cardinality of almost discretely Lindelöf spaces. Monatsh. Math. **186**(2), 345–353 (2018)
12. Bella, A., Spadaro, S.: A common extension of Arhangel'skiĭ's theorem and the Hajnal–Juhász inequality. Can. Math. Bull. **63**(1), 197–203 (2020)
13. Bella, A., Chiozini de Souza, L., Spadaro, S.: Cardinal inequalities involving the weak Rothberger and cellularity games **(preprint)**
14. Bonanzinga, M., Cuzzupé, M.V., Pansera, B.: On the cardinality of n-Urysohn and n-Hausdorff spaces. Cent. Eur. J. Math. **12**, 330–336 (2014)
15. Carlson, N.: On the weak tightness and power homogeneous compacta. Topol. Appl. **249**, 103–111 (2018)
16. Engelking, R.: General Topology. PWN, Warsaw (1977)
17. Fedeli, A.: On the cardinality of Hausdorff spaces. Comment. Math. Univ. Carol. **39**, 581–585 (1998)
18. Galvin, F.: Indeterminacy of point-open games. Bull Acad. Polon. Sci. Sér. Sci. Math. Astron. Phys. **26**, 445–449 (1978)
19. Hodel, R.: Combinatorial set theory and cardinal function inequalities. Proc. Am. Math. Soc. **111**, 567–575 (1991)
20. Juhász, I.: Cardinal Functions in Topology—Ten Years Later, 2nd edn. Mathematical Centre Tracts, vol. 123. Mathematisch Centrum, Amsterdam (1980)
21. Kortezov, I.: Cardinal inequalities using weak spread and cellularity. Quest. Answ. Gen. Topol. **14**, 209–215 (1996)
22. Kunen, K.: Set Theory, Studies in Logic, vol. 34. College Publications, London (2011)
23. Stavrova, D.: Separation pseudo-character and the cardinality of topological spaces. Topol. Proc. **25**, 333–343 (2000)

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