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Self-similarity and probability density function of the transient response of fractional compound motion

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ABSTRACT

In recent years, there has been an increasing use of fractional differential equations due to their ability to effectively represent various engineering phenomena, including viscoelasticity, heat transport, non-local continuum, and others. These equations take into account certain effects that cannot be accurately predicted using classical differential equations.

This paper provides a comprehensive analysis of the fractional compound motion, specifically focusing on the response of a one-term fractional differential equation that is excited by a Poissonian white noise process. The present study introduces a straightforward equation for the probability density function of fractional compound motion. The validity of this equation is subsequently confirmed by the execution of various numerical simulations. Furthermore, a comprehensive analysis is conducted on the self-similarity of fractional compound motion, demonstrating that the phenomenon can be regarded as self-similar in weak sense. This characteristic can be effectively employed to mitigate the loss of Markovianity in fractional differential equations.

1. Introduction

In many cases of engineering interest, the structural excitation is not deterministic and thus it has to be modeled as a stochastic input. For example, ambient vibrations are usually assumed as a Gaussian white noise process [1–5] while vehicular traffic on bridges is usually modeled as a Poissonian white noise process [6–9]. The probabilistic response of linear and non-linear dynamic systems under Poissonian white noise input has been studied by extending Ito calculus to non-normal excitations [10,11], by using path integral methods [12] and by using complex fractional moments [13].

The response of a linear dashpot to a Poissonian white noise process is the classical compound motion (CM) that can be defined as the classical integral of the Poissonian white noise process. Its extension to the fractional case is the so-called fractional compound motion (fCM). In literature there are different definitions of the fCM that extend the classical CM to the fractional case by properly modifying the input process. Specifically, the fractionality is introduced in the counting process that rules the number of event over time. In this way, it is possible to obtain time-fractional compound processes, space-fractional compound processes or space–time-fractional compound processes [14]. However, in many cases of engineering interest, like viscoelastic systems excited by stochastic input [15–17], the fractionality is not in the input process but it is in the differential equation that governs the structural motion. In the latter case, it is possible to model some effects experimentally observed, like long-tail memory of the systems, that cannot be predicted by differential equations of integer order. For this reason, here, the fractionality is introduced in the order of the time derivative present in the equation of motion, and thus the fCM is obtained as the fractional integral of the Poissonian white noise process.

In this paper, for the first time, a simple closed form equation for the probability density function (PDF) of the fCM is proposed showing that, for time instant sufficiently distant from zero, it tends to maintain a proportionality between its statistics calculated in different time axes, i.e. it tends to a self-similar process. Specifically, it is shown that the fCM is not strongly self-similar but it can be considered weakly self-similar.

The proposed equation of the PDF of the fCM is useful in several cases of engineering interest, such as the probabilistic characterization of the response of devices with intermediate behavior between spring and dashpot, i.e. springpot, excited by Poissonian processes. Furthermore, it is the basis for the probabilistic characterization of the response of fractional oscillators, viscoelastic dynamic systems with multiple degrees of freedom and continuous viscoelastic dynamic systems forced by Poissonian processes. Finally, the weak self-similarity

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can be used to overcome the loss of Markovianity of the response of the aforementioned dynamical systems excited by Poissonian processes, allowing to drastically reduce the computational burden required for the calculation of the response statistics both for numerical simulations and for experimental tests and in-situ tests.

The paper is composed as follows: in Section 2 the basic definition of fractional operators, fractional Brownian motion (fBM), self-similarity and fCM is introduced; in Section 3 a simple equation for the PDF of the fCM is proposed and its truthfulness is assessed with the aid of numerical simulations; in Section 4 weak self-similarity of fCM is discussed while, in Section 5, some concluding remarks are reported.

2. Mathematical formulation

In this section some preliminary concepts and definitions on fractional operators, fBM, self-similarity and fCM are reported for introducing appropriate symbology.

2.1. Fractional operators

Fractional operators are convolution integrals with power law kernel [18,19] and they can be considered as the generalization, from the integer order to the order $\beta \in \mathbb{R}^+$ (or even to complex ones with positive real part), of the classical derivatives and integrals. Fractional operators have the same properties of the classical ones (linearity, Leibniz rule, Fourier transform, Laplace transform, semi-group property...) and, for $\beta = 1, 2, \ldots$, they revert into the classical derivatives and integrals of integer order. In literature, there are a lot of representations of fractional operators such as the Riemann–Liouville (RL) fractional derivative and integral, the Caputo fractional derivative and the Grünwald–Letnikov fractional derivative. Other definitions have been proposed by Marchaud, Rietz, Hadamard and others. The RL fractional derivative, labeled as $({}_0D_t^\beta f)(t)$, and the RL fractional integral, labeled as $({}_0I_t^\beta f)(t)$, can be expressed, respectively, in the form

$$({}_0D_t^\beta f)(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\beta-1} f(\tau)\, d\tau; \quad \beta > 0 \tag{1}$$

$$({}_0I_t^\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau)\, d\tau; \quad \beta > 0 \tag{2}$$

where n is the integer part of β and $\Gamma(\beta) = \int_0^\infty e^{-x} x^{\beta-1}\, dx$ is the Euler Gamma function that interpolates all the factorials.

As it happens in classical differential calculus

$$({}_0I_t^\beta ({}_0D_t^\beta f))(t) \neq ({}_0D_t^\beta ({}_0I_t^\beta f))(t) \tag{3}$$

but if $f(t) = 0$ up to $t = 0$, then the inequality in Eq. (3) becomes an identity.

2.2. Fractional Brownian motion and self-similarity

The Brownian motion (BM) [20], labeled as $B(t)$, is the solution of the following differential equation

$$\begin{cases} \dot{B}(t) = W(t) \\ B(0) = 0 \quad w.p.1 \end{cases} \tag{4}$$

in which $w.p.1$ means "with probability one" and $W(t)$ is a zero-mean Gaussian white noise process. The latter is a delta-correlated process having correlation function

$$E[W(t_1)W(t_2)] = q\delta(t_2 - t_1); \quad t_2 > t_1 > 0 \tag{5}$$

in which $E[\cdot]$ represents the expected value, $\delta(\cdot)$ is the Dirac's delta function and $q > 0$ is the constant strength of $W(t)$. Since the BM is a Gaussian process, it can be fully characterized in probabilistic setting by its correlation function, i.e.

$$E[B(t_1)B(t_2)] = \int_0^{t_2} \int_0^{t_1} E[W(\tau_1)W(\tau_2)]\, d\tau_1\, d\tau_2 = qt_1; \quad t_2 > t_1 > 0 \tag{6}$$

and, by putting $t_1 \to t_2 \to t$ in Eq. (6), the variance of $B(t)$ can be obtained in the form

$$\sigma_B^2(t) = E[B^2(t)] = qt. \tag{7}$$

Starting from the variance reported in Eq. (7), the PDF of the BM can be calculated as

$$p_B(b,t) = \frac{1}{\sqrt{2\pi\sigma_B^2(t)}} \exp\left(-\frac{b^2}{2\sigma_B^2(t)}\right) \tag{8}$$

being b the domain of $B(t)$. It is to be stressed that the BM is a first-order Markovian process and thus, to obtain $B(t)$ at a generic time instant $t_j = j\Delta t$, it is sufficient to know only $B(t)$ at $t_{j-1} = (j-1)\Delta t$ being Δt the time sampling step. The same consideration on Markovianity remains valid for the statistics of $B(t)$.

FBM, labeled as $B_\beta(t)$, is an extension of the classical BM and it represents the solution of the following fractional differential equation [21]

$$\begin{cases} ({}_0D_t^\beta B_\beta)(t) = W(t) \\ B_\beta(t) = 0 \quad w.p.1 \quad \forall t \leq 0. \end{cases} \tag{9}$$

Since $B_\beta(t) = 0\ \forall t \leq 0\ w.p.1$, the solution of Eq. (9) can be expressed, according to the RL fractional integral definition, as

$$B_\beta(t) = ({}_0I_t^\beta W)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} W(\tau)\, d\tau. \tag{10}$$

The correlation function of the fBM can be calculated as

$$E[B_\beta(t_1)B_\beta(t_2)] = \frac{1}{\Gamma^2(\beta)} \int_0^{t_2} \int_0^{t_1} (t_1-\tau_1)^{\beta-1}(t_2-\tau_2)^{\beta-1} E[W(\tau_1)W(\tau_2)]\, d\tau_1\, d\tau_2 \tag{11}$$

and its closed form equation can be found in literature [21]. By putting $t_1 \to t_2 \to t$ in Eq. (11), the variance of $B_\beta(t)$ can be obtained as

$$\sigma_{B_\beta}^2(t) = E[B_\beta^2(t)] = \frac{q}{\Gamma^2(\beta)} \frac{t^{2\beta-1}}{2\beta-1}; \quad t \geq 0. \tag{12}$$

The PDF of the fBM can therefore be expressed as

$$p_{B_\beta}(b_\beta,t) = \frac{1}{\sqrt{2\pi\sigma_{B_\beta}^2(t)}} \exp\left(-\frac{b_\beta^2}{2\sigma_{B_\beta}^2(t)}\right) \tag{13}$$

in which b_β is the domain of $B_\beta(t)$. It is to be stressed that the fBM reverts into the classical BM for $\beta = 1$ and that for $\beta \neq 1, 2, 3, \ldots$, $B_\beta(t)$ is not a Markovian process. This means that in order to obtain $B_\beta(t)$ in a generic time instant t_j, it is necessary to know the entire past history of the process. The same consideration on the loss of Markovianity remains valid for the statistics of $B_\beta(t)$, i.e. in order to obtain the statistics of $B_\beta(t)$ in a generic time instant t_j, it is necessary to know the entire past history of the statistics.

Self-similarity was introduced by Mandelbrot [22,23] before the theory of fractal geometry [24]. Particularly, if by changing the temporal scale from t to at, with $a > 0$, the statistics in the time axis at are related to those in the time axis t through a coefficient a^{mH} in which m is the order of the statistic and $H > 0$ is the Hurst index, then the process $X(t)$ is self-similar; i.e.

$$\{X(at_1), \ldots, X(at_m)\} \underline{\underline{d}}\ a^{mH} \{X(t_1), \ldots, X(t_m)\} \tag{14}$$

where the symbol $\underline{\underline{d}}$ means equal in distribution.

FBM is a relevant example of self-similar process [21,23], in fact, by putting $t_1 \to at_1$ and $t_2 \to at_2$ in Eq. (11), it can be easily noted that

$$E[B_\beta(at_1)B_\beta(at_2)] = a^{2\beta-1} E[B_\beta(t_1)B_\beta(t_2)]. \tag{15}$$

The same result can be obtained in terms of variance. In fact, by changing the temporal scale from t to at, Eq. (12) becomes

$$\sigma_{B_\beta}^2(at) = E[B_\beta^2(at)] = \frac{q}{\Gamma^2(\beta)} \frac{(at)^{2\beta-1}}{2\beta-1}; \quad t \geq 0. \tag{16}$$

By comparing Eqs. (12) and (16), it is possible to state that

$$\sigma^2_{B_\beta}(at) = a^{2\beta-1}\sigma^2_{B_\beta}(t). \tag{17}$$

Since fBM is a Gaussian process, then it is sufficient that the self-similarity is verified only in terms of second order correlation (or in terms of variance) in order to state that the process is strongly self-similar and thus, from Eqs. (15) and (17), it is clear that fBM is a strongly self-similar process having Hurst index $H = \beta - 1/2$. It is to be stressed that, if the variance is known in a generic time instant t_j, then Eq. (17) can be used to calculate the variance in another time instant at_j and, by varying the value of a, the variance can be obtained for each time instant desired. For this reason, self-similarity is a very important property that allows to overcome the loss of Markovianity in fractional differential equations.

2.3. Fractional compound motion

The CM [20], labeled as $C(t)$, can be defined as the solution of the following differential equation

$$\begin{cases} \dot{C}(t) = W_P(t) \\ C(0) = 0 \quad w.p.1 \end{cases} \tag{18}$$

in which $W_P(t)$ is a zero-mean Poissonian white noise process. The latter is a stochastic process constituted by a train of impulses with random amplitudes Y_k occurring at random time instants T_k distributed according to the Poissonian distribution. This means that the time instants T_k at which the spikes occur are such that the increments $N(t_1, t_2)$, labeled as Poisson counting process of the stochastic process $T(t)$, take an integer value giving the number of events in $[t_1, t_2)$. The probability $p\{N(t_1, t_2) = g\} = p_N(t_1, t_2)$ is assumed to satisfy:

1. the random variables $N(t_1, t_2)$, $N(t_2, t_3)$, …, $N(t_{j-1}, t_j)$ with $t_1 < t_2 < \cdots < t_j$ are mutually independent;
2. for $t_j - t_{j-1} = \Delta t_j$ sufficiently small, $p_N(t_{j-1}, t_j) = \lambda \Delta t_j + \mathcal{O}(\Delta t_j)$ being λ the number of impulses per unit time.

If $\Delta t_j \to 0$, then $\mathcal{O}(\Delta t_j) \to 0$ and this implies that the probability of the occurrence of two or more random instants in which the impulse occur in an infinitesimal interval is zero. The probability of occurrence of g events in $[t_r, t_s)$ for the process $T(t)$ is given as

$$p_N(t_r, t_s) = \frac{(\lambda(t_r - t_s))^g}{g!}\exp(-\lambda(t_r - t_s)); \quad t_r > t_s > 0 \tag{19}$$

that is just the Poisson distribution. The correspondent mean number of events in the same time interval is $E[N(t_r, t_s)] = \lambda(t_r - t_s)$, the variance of $N(t_r) - N(t_s)$ is also $E[N(t_r, t_s)] = \lambda(t_r - t_s)$ and the correlation function $R_N(t_r, t_s)$ is given as

$$R_N(t_r, t_s) = \lambda \min(t_r, t_s). \tag{20}$$

Therefore, the Poissonian white noise process can be expressed as

$$W_P(t) = \sum_{k=1}^{N(t)} Y_k \delta(t - T_k) \tag{21}$$

in which $N(t)$ represents the Poisson counting process that gives the number of spikes in the interval $[0, t)$. The solution of Eq. (18) can be expressed as

$$C(t) = \int_0^t W_P(\tau)\, d\tau; \tag{22}$$

and thus

$$C(t) = \sum_{k=1}^{N(t)} Y_k U(t - T_k) \tag{23}$$

being $U(t)$ the unit step function. It is to emphasize that the counting process $N(t)$ may be considered as the CM in the case in which $Y_k = 1\ \forall k$.

The fCM, labeled as $C_\beta(t)$, is an extension of the classical CM and it represents the solution of the following differential equation

$$\begin{cases} ({}_0D_t^\beta C_\beta)(t) = W_P(t) \\ C_\beta(t) = 0 \quad w.p.1 \quad \forall t \le 0. \end{cases} \tag{24}$$

Since $C_\beta(t) = 0\ \forall t \le 0\ w.p.1$, the solution of Eq. (24) can be expressed, according to the RL fractional integral definition, as

$$C_\beta(t) = ({}_0I_t^\beta W_P)(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} W_P(\tau)\, d\tau \tag{25}$$

and thus

$$C_\beta(t) = \sum_{k=1}^{N(t)} \frac{Y_k (t - T_k)^{\beta-1} U(t - T_k)}{\Gamma(\beta)}. \tag{26}$$

It is to be stressed that for $\beta = 1$, Eq. (26) coalesces with the classical CM $C(t)$.

Considering that $W_P(t)$ is a delta-correlated process, the variance of $C_\beta(t)$, labeled as $\sigma^2_{C_\beta}(t)$, can be easily calculated in the form

$$\sigma^2_{C_\beta}(t) = \frac{\lambda E[Y_k^2]}{\Gamma^2(\beta)}\int_0^t (t^2 - 2t\tau_1 + \tau_1^2)^{\beta-1}\, d\tau_1 = \frac{\lambda E[Y_k^2]}{\Gamma^2(\beta)}\frac{t^{2\beta-1}}{2\beta-1}; \quad t \ge 0. \tag{27}$$

From Eq. (27) it can be noted that the variance of the fCM coalesces with the variance of a fBM having strength $q(t) = \lambda E[Y_k^2]\ \forall t \ge 0$. This means that, by changing the temporal scale from t to at, Eq. (27) becomes

$$\sigma^2_{C_\beta}(at) = \frac{\lambda E[Y_k^2]}{\Gamma^2(\beta)}\frac{(at)^{2\beta-1}}{2\beta-1}; \quad t \ge 0 \tag{28}$$

and thus

$$\sigma^2_{C_\beta}(at) = a^{2\beta-1}\sigma^2_{C_\beta}(t). \tag{29}$$

From Eq. (29) it can be noted that self-similarity is verified for the variance and that the Hurst index is $H = \beta - 1/2$. However, it is not sufficient to state that $C_\beta(t)$ is strongly self-similar since, in the limit of validity of the central limit theorem, the fCM is Gaussian only asymptotically. For the statistics of order higher than two, self-similarity is not verified since the Hurst index changes with the order of the statistics with law

$$H_m = \beta - \frac{m-1}{m}. \tag{30}$$

For this reason, fCM cannot be considered strongly self-similar. However, it is to be stressed that if $\lambda \to \infty$ and $\lambda E[Y_k^2] \to c$, with $c \in \mathbb{R}^+$, then the fCM reverts into a fBM having strength $q(t) = \lambda E[Y_k^2]\ \forall t \ge 0$ and only in this case it is a strongly self-similar process.

3. Probability density function of the fractional compound motion

In this section, a simple equation for the PDF of the fCM is proposed. Specifically, in the first subsection, the mathematical form of the proposed PDF is described in detail while, in the second subsection, the proposed equation of the PDF of the fCM is compared with the PDF obtained from the numerical simulations performed.

3.1. Proposed equation of the PDF of the fCM

The proposed equation of the PDF of the fCM is composed by two different parts. The first part is a Dirac's delta $\delta(c_\beta)$ multiplied by a modulating function $\alpha_1(j\Delta t)$ that represents the probability that a sample of the process $C_\beta(t)$ is still quiescent at a generic time instant t_j. The function $\alpha_1(j\Delta t)$ can be easily calculated, by putting $g = 0$, $t_s = 0$ and $t_r = t_j = j\Delta t$ in Eq. (19), as

$$\alpha_1(j\Delta t) = \frac{(\lambda j \Delta t)^0}{0!}\exp(-\lambda j\Delta t) = \exp(-\lambda j \Delta t). \tag{31}$$

Therefore, the probability that a sample of the process $C_\beta(t)$ is not quiescent at a generic time instant $t_j = j\Delta t$ is

$$\alpha_2(j\Delta t) = 1 - \alpha_1(j\Delta t) = 1 - \exp(-\lambda j\Delta t). \tag{32}$$

The second part of the proposed equation of the PDF of the fCM depends not only on $\alpha_2(j\Delta t)$ but also on a PDF $\tilde{p}_{C_\beta}(c_\beta, j\Delta t)$ in turn dependent on the distribution of spikes Y_k. However, in this paper it is assumed as a Gaussian distribution, and thus it assumes the form

$$\tilde{p}_{C_\beta}(c_\beta, j\Delta t) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2_{C_\beta}(j\Delta t)}} \exp\left(-\frac{c_\beta^2}{2\tilde{\sigma}^2_{C_\beta}(j\Delta t)}\right). \tag{33}$$

Since in $t = 0\Delta t = 0$ all the samples of the fCM are quiescent, then the proposed PDF of the fCM in the same time instant is

$$p_{C_\beta}(c_\beta, 0) = \alpha_1(0)\delta(c_\beta) = \delta(c_\beta). \tag{34}$$

In $t = \Delta t$, the PDF of the fCM can be expressed in the form

$$p_{C_\beta}(c_\beta, \Delta t) = \alpha_1(\Delta t)\delta(c_\beta) + \alpha_2(\Delta t)\tilde{p}_{C_\beta}(c_\beta, \Delta t) \tag{35}$$

in which $\alpha_1(\Delta t)$ represents the probability that a sample of the fCM process is quiescent in $t = \Delta t$, $\alpha_2(\Delta t)$ represents the probability that a sample of the fCm process starts its motion in $t = \Delta t$, while $\tilde{p}_{C_\beta}(c_\beta, \Delta t)$ represents the PDF of the samples that start their motion in the considered time instant, i.e. $t = \Delta t$.

In $t = 2\Delta t$, the PDF of the fCM assumes the form

$$\begin{aligned} p_{C_\beta}(c_\beta, 2\Delta t) = \alpha_1(2\Delta t)\delta(c_\beta) + \alpha_2(\Delta t)\tilde{p}_{C_\beta}(c_\beta, 2\Delta t) + \\ (\alpha_2(2\Delta t) - \alpha_2(\Delta t))\tilde{p}_{C_\beta}(c_\beta, \Delta t) \end{aligned} \tag{36}$$

in which $\alpha_1(2\Delta t)$ represents the probability that a sample of the fCM process is quiescent in $t = 2\Delta t$, $(\alpha_2(2\Delta t) - \alpha_2(\Delta t))$ represents the probability that a sample of the fCM process starts its motion in $t = 2\Delta t$, while $\tilde{p}_{C_\beta}(c_\beta, \Delta t)$ and $\tilde{p}_{C_\beta}(c_\beta, 2\Delta t)$ represent the PDF of the samples that starts their motion, respectively, in the considered time instant, i.e. $t = 2\Delta t$, and in the time instant preceding the one considered, i.e. $t = \Delta t$. Similarly, in $t = 3\Delta t$, the PDF of the fCM can be expressed as

$$\begin{aligned} p_{C_\beta}(c_\beta, 3\Delta t) = \alpha_1(3\Delta t)\delta(c_\beta) + \alpha_2(\Delta t)\tilde{p}_{C_\beta}(c_\beta, 3\Delta t) + \\ (\alpha_2(2\Delta t) - \alpha_2(\Delta t))\tilde{p}_{C_\beta}(c_\beta, 2\Delta t) + (\alpha_2(3\Delta t) - \alpha_2(2\Delta t))\tilde{p}_{C_\beta}(c_\beta, \Delta t). \end{aligned} \tag{37}$$

Therefore, the proposed PDF of the fCM in a generic time instant $t = j\Delta t$ is obtained as

$$p_{C_\beta}(c_\beta, j\Delta t) = \alpha_1(j\Delta t)\delta(c_\beta) + \sum_{u=1}^{j}(\alpha_2(u\Delta t) - \alpha_2((u-1)\Delta t))\tilde{p}_{C_\beta}(c_\beta, (j-u+1)\Delta t). \tag{38}$$

In order to calculate the variance $\tilde{\sigma}^2_{C_\beta}(j\Delta t)$ present in Eq. (33), it is possible to proceed with a similar reasoning to that already done for $p_{C_\beta}(c_\beta, j\Delta t)$. Specifically, by multiplying Eq. (35) for c_β^2 and by integrating between $-\infty$ and ∞ with respect to c_β, Eq. (35) becomes

$$\sigma^2_{C_\beta}(\Delta t) = \alpha_2(\Delta t)\tilde{\sigma}^2_{C_\beta}(\Delta t) \tag{39}$$

and thus

$$\tilde{\sigma}^2_{C_\beta}(\Delta t) = \frac{\sigma^2_{C_\beta}(\Delta t)}{\alpha_2(\Delta t)}. \tag{40}$$

By multiplying Eq. (36) for c_β^2 and by integrating between $-\infty$ and ∞ with respect to c_β, Eq. (36) becomes

$$\sigma^2_{C_\beta}(2\Delta t) = \alpha_2(\Delta t)\tilde{\sigma}^2_{C_\beta}(2\Delta t) + (\alpha_2(2\Delta t) - \alpha_2(\Delta t))\tilde{\sigma}^2_{C_\beta}(\Delta t) \tag{41}$$

and thus

$$\tilde{\sigma}^2_{C_\beta}(2\Delta t) = \frac{1}{\alpha_2(\Delta t)}\left[\sigma^2_{C_\beta}(2\Delta t) - (\alpha_2(2\Delta t) - \alpha_2(\Delta t))\tilde{\sigma}^2_{C_\beta}(\Delta t)\right]. \tag{42}$$

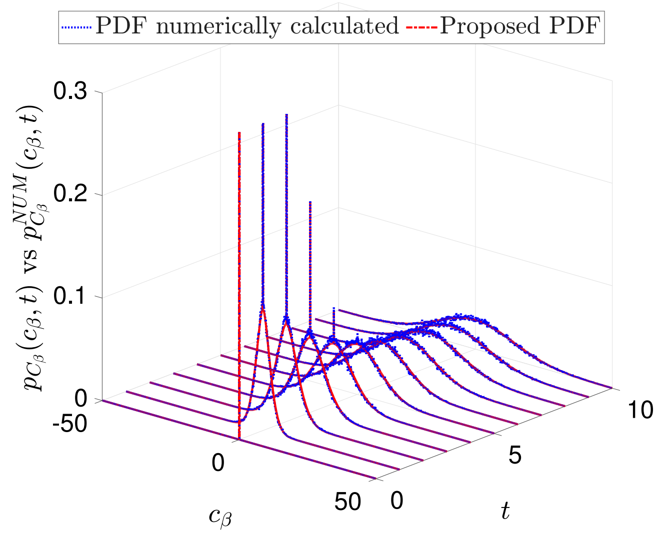

Fig. 1. Comparison between the proposed PDF of fCM ($p_{C_\beta}(c_\beta,t)$) and the PDF numerically calculated ($p^{NUM}_{C_\beta}(c_\beta,t)$) for Gaussian distribution of Y_k.

Finally, by multiplying Eq. (37) for c_β^2 and by integrating between $-\infty$ and ∞ with respect to c_β, Eq. (37) becomes

$$\begin{aligned} \sigma^2_{C_\beta}(3\Delta t) = \alpha_2(\Delta t)\tilde{\sigma}^2_{C_\beta}(3\Delta t) + (\alpha_2(2\Delta t) - \alpha_2(\Delta t))\tilde{\sigma}^2_{C_\beta}(2\Delta t) + \\ (\alpha_2(3\Delta t) - \alpha_2(2\Delta t))\tilde{\sigma}^2_{C_\beta}(\Delta t) \end{aligned} \tag{43}$$

and thus

$$\begin{aligned} \tilde{\sigma}^2_{C_\beta}(3\Delta t) = \frac{1}{\alpha_2(\Delta t)}\Big[\sigma^2_{C_\beta}(3\Delta t) - (\alpha_2(2\Delta t) - \alpha_2(\Delta t))\tilde{\sigma}^2_{C_\beta}(2\Delta t) - \\ (\alpha_2(3\Delta t) - \alpha_2(2\Delta t))\tilde{\sigma}^2_{C_\beta}(\Delta t)\Big]. \end{aligned} \tag{44}$$

Therefore, the variance $\tilde{\sigma}^2_{C_\beta}(j\Delta t)$ can be expressed as

$$\begin{aligned} \tilde{\sigma}^2_{C_\beta}(j\Delta t) = \frac{1}{\alpha_2(\Delta t)}\Bigg[\sigma^2_{C_\beta}(j\Delta t) - \\ \sum_{u=2}^{j}(\alpha_2(u\Delta t) - \alpha_2((u-1)\Delta t))\tilde{\sigma}^2_{C_\beta}((j-u+1)\Delta t)\Bigg]. \end{aligned} \tag{45}$$

The same result reported in Eq. (45) can be obtained by multiplying Eq. (38) for c_β^2 and by integrating between $-\infty$ and ∞ with respect to c_β. Furthermore, it is to be stressed that Eq. (38) remains valid also from distributions of Y_k that are not Gaussian. In the latter case, a Gaussian distribution of $\tilde{p}_{C_\beta}(c_\beta, t)$ provides perfect results for time instant sufficiently distant from zero but, in order to have a perfect probabilistic representation also for time instants near to zero, $\tilde{p}_{C_\beta}(c_\beta, t)$ has to be properly defined in order to take into account the distribution of Y_k.

3.2. Numerical results

In order to assess the truthfulness of the proposed PDF of the fCM, a Monte Carlo simulation has been performed considering a Gaussian distribution of Y_k. Specifically, a total of 10^5 samples of the process $C_\beta(t)$ having a duration equal to 10 s discretized with a time sampling step $\Delta t = 0.01$ s have been generated considering $\beta = 1.25$, $\lambda = 1.44$ and $E[Y_k^2] = 5.4289$. For each time instant, the PDF of the fCM has been numerically calculated and it has been compared with the proposed PDF of fCM taking into account that, for numerical applications, $\delta(0)$ has to be substituted with $1/\Delta c_\beta$ in Eq. (38) being $\Delta c_\beta = 0.1$ the sampling step of the domain c_β. The results obtained are reported in Fig. 1. From the results reported in Fig. 1, it is clear that the proposed equation of the PDF perfectly matches the PDF obtained from

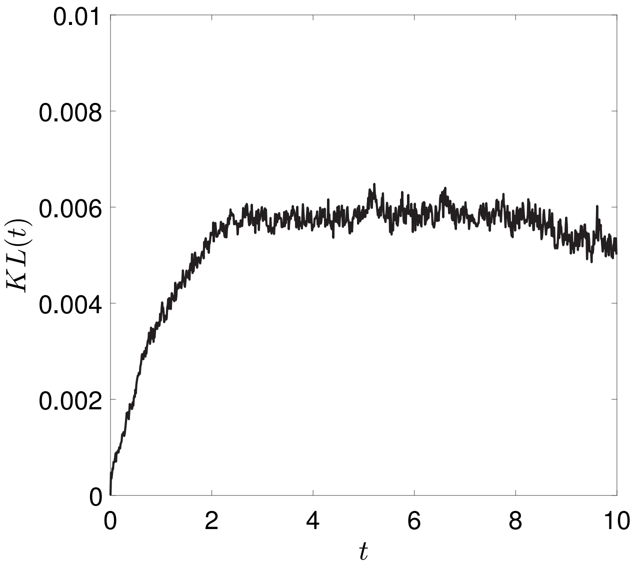

Fig. 2. Kullback–Leibler divergence between the proposed PDF of fCM and the PDF numerically calculated (Gaussian distribution of Y_k).

the numerical results. An additional comparison has been performed by calculating, for each considered time instant, the Kullback–Leibler (KL) divergence (also called relative entropy) between the proposed equation of the PDF and the PDF numerically calculated. It is a type of statistical distance that is calculated in the form

$$KL(t) = \sum_{l=-\infty}^{\infty} p_{c_\beta}^{NUM}(l\Delta c_\beta, t) \ln\left(\frac{p_{c_\beta}^{NUM}(l\Delta c_\beta, t)}{p_{c_\beta}(l\Delta c_\beta, t)}\right) \tag{46}$$

being $p_{c_\beta}^{NUM}(c_\beta, t)$ the PDF numerically calculated. The results obtained in terms of KL divergence are reported in Fig. 2. From Fig. 2 it is clear that the KL divergence assumes very low values and thus the proposed PDF of fCM gives a perfect statistical representation of the results numerically obtained.

In order to verify if the PDF of the fCM tends to a Gaussian distribution, a Gaussianity test has been performed by using the Kurtosis coefficient. The latter is expressed as

$$K_C(t) = \frac{E\left[\left(C_\beta(t) - E[C_\beta(t)]\right)^4\right]}{\sigma_{C_\beta}^4(t)} \tag{47}$$

in which the numerator represents the fourth central moment that, in this case (zero-mean process), coalesces with the fourth order moment. If the Kurtosis coefficient is equal to three, then the distribution is perfectly Gaussian. The results obtained in terms of Kurtosis coefficient are reported in Fig. 3. From Fig. 3 it is clear that, as expected, the PDF of fCM tends to be Gaussian for time instants sufficiently distant from zero.

In order to assess the truthfulness of the proposed PDF of the fCM also for distributions of Y_k different than Gaussian, a Monte Carlo simulation has been performed considering Y_k uniformly distributed between −3 and 3. Also in this case, a total of 10^5 samples of the process $C_\beta(t)$ having a duration equal to 10 s discretized with a time sampling step $\Delta t = 0.01$ s have been generated considering $\beta = 1.25$ and $\lambda = 1.44$. The comparison between the proposed PDF of fCM and the PDF numerically calculated is reported in Fig. 4 while the Kullback–Leibler divergence between the proposed closed form of the PDF and the PDF numerically calculated and the Kurtosis coefficient are reported, respectively, in Figs. 5 and 6.

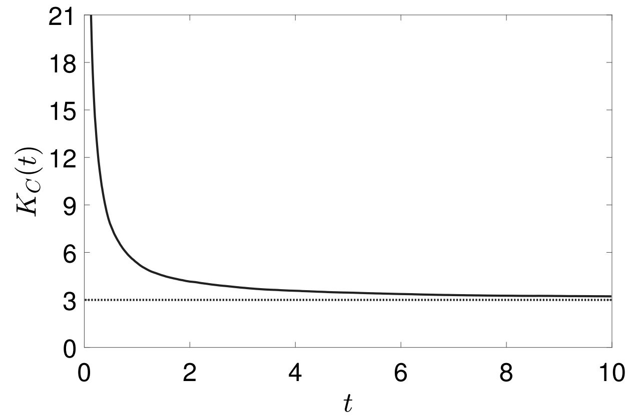

Fig. 3. Kurtosis coefficient (Gaussian distribution of Y_k).

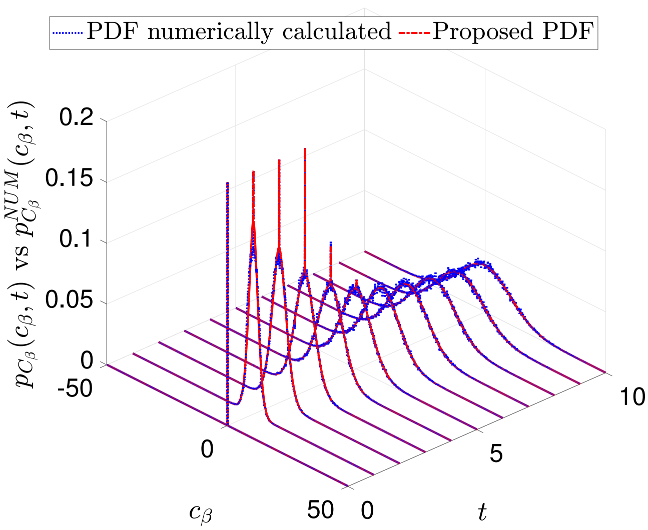

Fig. 4. Comparison between the proposed PDF of fCM ($p_{C_\beta}(c_\beta, t)$) and the PDF numerically calculated ($p_{C_\beta}^{NUM}(c_\beta, t)$) for uniform distribution of Y_k.

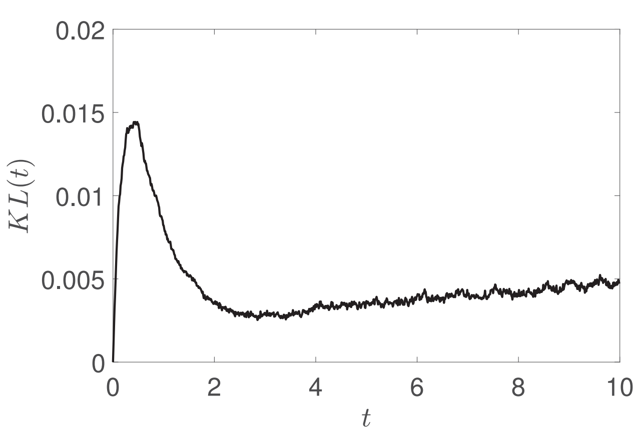

Fig. 5. Kullback–Leibler divergence between the proposed PDF of fCM and the PDF numerically calculated (uniform distribution of Y_k).

From Figs. 4 and 5 it is clear that, for time instants sufficiently distant from zero, also in case of uniform distribution of Y_k, the proposed equation of the PDF perfectly matches the PDF obtained from the numerical results while, for time instants near to zero, there is a little discrepancy between the proposed equation of the PDF and

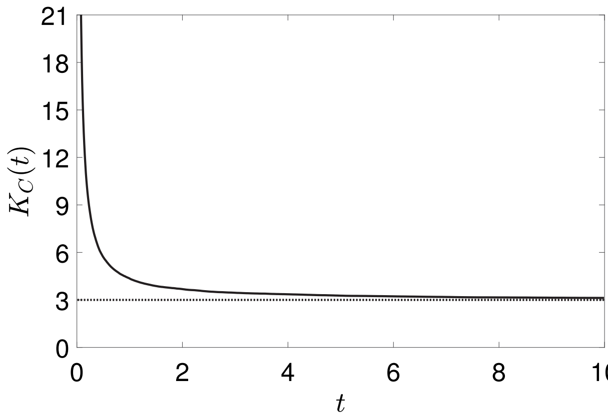

Fig. 6. Kurtosis coefficient (uniform distribution of Y_k).

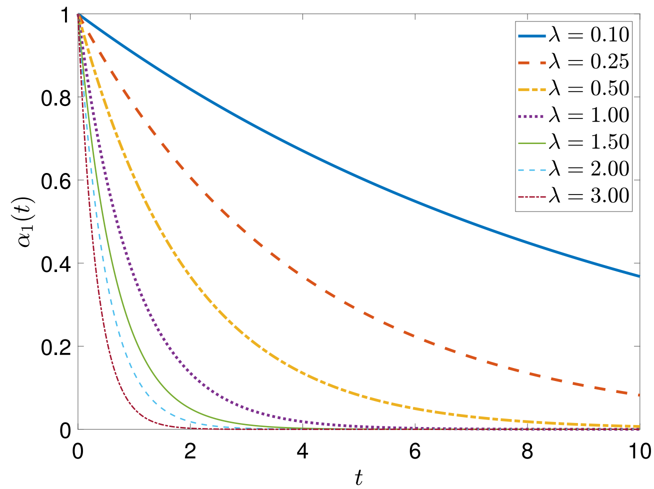

Fig. 7. Modulating function $\alpha_1(t)$ for different values of λ.

the PDF numerically calculated due to the fact that $\tilde{p}_{C_\beta}(c_\beta, t)$ has been approximated with a Gaussian distribution. From Fig. 6 it can be noted that the PDF of the fCM, as expected, tends to be Gaussian for time instants sufficiently distant from zero also for uniform distribution of Y_k.

4. Remarks on weak self-similarity of fractional compound motion

As previously reported, fCM cannot be considered strongly self-similar since the Hurst index changes with the order of statistics. However, from the proposed PDF of fCM and from the results of the numerical simulations performed, it is clear that the more the time increases and the more the fCM tends to be a Gaussian process fully described in probabilistic setting by its variance. Furthermore, by comparing Eqs. (12) and (27), it can be observed that the variance of the fCM is equal to the variance of a fBM having strength $q(t) = \lambda E[Y_k^2]$ $\forall t \geq 0$. Therefore, it can be stated that the more the time increases and the more the fCM tends to revert into a fBM having the aforementioned strength. Since the fBM is self-similar with Hurst index $H = \beta - 1/2$, it is clear that the more the time increases and the more the fCM tends to become self-similar with Hurst index $H = \beta - 1/2$. The time instant starting from which fCM can be treated as a self-similar process strongly depends on how quickly the Dirac's delta function becomes negligible in Eq. (38), i.e. on how quickly the modulating function $\alpha_1(t)$ tends to zero. In Fig. 7 the modulating function $\alpha_1(t)$ is reported for different values of λ. From Fig. 7 it is clear that the larger λ is, the faster the modulating function $\alpha_1(t)$ tends to zero and thus it can be stated that the larger λ is, the faster fCM tends to revert into a self-similar process with Hurst index $H = \beta - 1/2$.

Weak self-similarity of fCM can therefore be used to overcome the loss of Markovianity. In fact, starting from the statistics of the fCM calculated in the time instant from which $\alpha_1(t)$ can be neglected, it is possible to use the self-similarity in order to calculate the future statistics of the fCM.

5. Concluding remarks

In this paper, a simple equation of the probability density function of the fractional compound motion process has been proposed. The proposed equation is composed by two different parts: the first one is a Dirac's delta function multiplied by a modulating function, while the second one depends on the aforementioned modulating function and on a probability density function in turn dependent on the distribution of the spikes of the Poisson process. Two different numerical simulations have been performed in order to assess the truthfulness of the proposed equation and the results show that the proposed equation perfectly matches the probability density function numerically calculated. Furthermore, it has been show that, after a time instant that depends on the mean number of impulses per unit time, the fractional compound motion reverts into a fractional Brownian motion, i.e. it tends to become a self-similar process with Hurst index $H = \beta - 1/2$. After the aforementioned time instant, the loss of Markovianity can be overcomed by exploiting the advantages of the self-similar processes.

CRediT authorship contribution statement

Salvatore Russotto: Writing – review & editing, Writing – original draft, Validation, Investigation, Formal analysis, Data curation, Conceptualization. **Mario Di Paola:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Investigation, Conceptualization. **Antonina Pirrotta:** Writing – review & editing, Validation, Supervision, Methodology, Investigation, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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