SINGULAR DOUBLE PHASE PROBLEMS WITH CONVECTION

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Abstract. We consider a nonlinear Dirichlet problem driven by the sum of a p-Laplacian and of a q-Laplacian (double phase equation). In the reaction we have the combined effects of a singular term and of a gradient dependent term (convection) which is locally defined. Using a mixture of variational and topological methods, together with suitable truncation and comparison techniques, we prove the existence of a positive smooth solution.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We study the following Dirichlet (p,q)-equation with a singular term and a gradient dependent perturbation (convection):

$$-\Delta_p u(z) - \Delta_q u(z) = u(z)^{-\eta} + f(z, u(z), \nabla u(z)) \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = 0,\, u > 0. \tag{1}$$

In this problem $1 < q < p < +\infty$, $0 < \eta < 1$. For every $r \in (1, +\infty)$ by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u) \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

The differential operator in problem (1) is the sum of two such operators with different indices (double phase equation) and so it is not homogeneous. This is a source of difficulties in the analysis of problem (1). In the reaction of problem (1) we have the combined effects of two terms of different nature. One is the singular term $u^{-\eta}$ and the other is a perturbation $f(z, u, \nabla u)$ which is a Carathéodory function (that is, for all $(x, y) \in \mathbb{R} \times \mathbb{R}^N$, $z \to f(z, x, y)$ is measurable and for a.a. $z \in \Omega$, $(x, y) \to f(z, x, y)$ is continuous). There are two special features of this perturbation. The first is that it is gradient dependent (convection) and this make the problem nonvariational. The other special feature is that our conditions on $f(z, \cdot, y)$ are only local (near zero). No restrictions are imposed on $x \to f(z, x, y)$ for large $x \geq 0$.

Since the problem is nonvariational (due to the convection), our approach is necessarily topological, based on the fixed point theory. The idea is to freeze the gradient term in the perturbation. This way we have a variational problem which we hope to solve using tools from the critical point theory. However, the presence of the singular term creates problems in this direction since the energy functional is not C^1 and so the minimax methods of critical point theory are not directly applicable on it. We need to find ways to bypass the singularity in order to deal with C^1-functionals. As soon as we do this, we face a new difficulty. We need to find a canonical way to choose a solution

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from each "frozen" problem. If we do that we have a map on which we can apply the fixed point theory. Simpler versions of this approach can be found in the works of Faraci-Motreanu-Puglisi [5] and of Gasiński-Papageorgiou [9]. Both deal with problems with no singular term and in [5] the operator is homogeneous (it is the p-Laplacian). Other methods for different classes of problems with convection, can be found in Papageorgiou-Rădulescu-Repovš [19], Papageorgiou-Vetro-Vetro [26] (Neumann problems driven by the Laplacian) and Hu-Papageorgiou [13] (Dirichlet problems), Papageorgiou-Vetro-Vetro [25] (Robin problems with unilateral constraints) for equations driven by the p-Laplacian. There is no singular term in all the aforementioned works. The only works dealing with nonlinear singular problem with a convection are those of Liu-Motreanu-Zeng [18] (Dirichlet problems) and Papageorgiou-Rădulescu-Repovš [21] (Neumann equations), which consider equations driven by the p-Laplacian. In all these works global growth conditions are imposed on $f(z,\cdot,y)$.

We mention that equations driven by the sum of two differential operators of different nature, arise in many mathematical models of physical processes. We refer to the works of Bahrouni-Rădulescu-Repovš [1] (transonic flow problems), Benci-D'Avenia-Fortunato-Pisani [2] (quantum physics), Cherfils-Il'yasov [3] (reaction diffusion systems) and Zhikov [28] (elasticity theory).

2. Mathematical Background and Hypotheses

The main spaces in the analysis of problem (1) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\}$. By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|\nabla u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is ordered with positive cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\}.$$

Here by $\frac{\partial u}{\partial n}$ we denote the normal derivative of u defined by $\frac{\partial u}{\partial n} = (\nabla u, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

If $x \in \mathbb{R}$, then we set $x^{\pm} = \max\{\pm x, 0\}$. For $u \in W_0^{1,p}(\Omega)$, we define $u^{\pm}(z) = u(z)^{\pm}$ for all $z \in \Omega$. We know that

$$u^{\pm} \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Given $u, v \in W_0^{1,p}(\Omega)$ with $u \leq v$, we define

$$[u,v] = \left\{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\right\}.$$

Let X, Y be Banach spaces and $\varphi : X \to Y$. We say that $\varphi(\cdot)$ is "compact", if it is continuous and maps bounded sets in X to relatively compact sets in Y.

The next theorem is known in the literature as the "Leray-Schauder alternative principle" (see, for example, Gasiński-Papageorgiou [8], Theorem 4.93, p. 642).

Theorem 1. *If X is a Banach space, $\varphi : X \to X$ is a compact map and $\mathcal{D}(\varphi) = \{u \in X : u = \lambda\varphi(u) \text{ for some } 0 < \lambda < 1\}$, then either $\mathcal{D}(\varphi)$ is unbounded or $\varphi(\cdot)$ has a fixed point.*

For every $r \in (1,+\infty)$ by $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^*$ ($\frac{1}{r} + \frac{1}{r'} = 1$) we denote the operator defined by

$$\langle A_r(u), h\rangle = \int_\Omega |\nabla u|^{r-2}(\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,r}(\Omega).$$

This operator is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) (see Gasiński-Papageorgiou [8], Problem 2.192, p. 279).

The hypotheses on the convection perturbation term ($f(z, x, y)$ are the following:

$\underline{H:}$ $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0, y) = 0$ for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$ and

- (i) there exists $\vartheta > 0$ such that $|f(z, x, y)| \le a(z)[1 + |y|^{p-1}]$ for a.a. $z \in \Omega$, all $0 \le x \le \vartheta$, all $y \in \mathbb{R}^N$, with $a \in L^\infty(\Omega)$;
- (ii) $\vartheta^{-\eta} + f(z, \vartheta, y) \le 0$ for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$;
- (iii) there exists $\delta \in (0, \vartheta)$, $1 < \tau < q$ and $c_0 > 0$ such that $c_0 x^{\tau-1} \le f(z, x, y)$ for a.a. $z \in \Omega$, all $0 \le x \le \delta$, all $y \in \mathbb{R}^N$;
- (iv) for every $\mu \in (0, 1)$, we have $f(z, \frac{1}{\mu}x, y) \le \frac{1}{\mu^{p-1}} f(z, x, y)$ for a.a. $z \in \Omega$, all $0 \le x \le \vartheta$, all $y \in \mathbb{R}^N$.

Remark 1. Since we look for positive solutions and all the above hypotheses concern the positive semiaxis, without any loss of generality, we may assume that

$$f(z, x, y) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \le 0, \text{ all } y \in \mathbb{R}^N. \tag{2}$$

We point out that all restrictions on $f(z, \cdot, y)$ are on the interval $[0, \vartheta]$. We do not impose any restriction on $f(z, \cdot, y)$ on the half line $[\vartheta, +\infty)$ (see (2)). Hypothesis H (iv) is satisfied if for a.a. $z \in \Omega$ and all $y \in \mathbb{R}^N$, the quotient function $x \to \frac{f(z,x,y)}{x^{p-1}}$ is nonincreasing on $(0, \vartheta)$. An example of a perturbation which satisfies hypotheses H is the following function (for the sake of simplicity we drop the z-dependence):

$$f(x, y) = [x^{\tau-1} - x^{r-1}](1 + |y|^{p-1}) \quad \text{for all } 0 \le x \le 1, \text{ all } y \in \mathbb{R}^N, \text{ with } 1 < \tau < q, r.$$

As we already explained in the Introduction, in the implementation of the frozen variable technique, the presence of the singular term prevents us from the use of variational tools on the "frozen problem", since the corresponding energy functional is not C^1. We need to find a way to isolate the singularity and consider an auxiliary Dirichlet problem with a C^1-energy functional.

For this purpose, we consider the following parametric (p, q)-Dirichlet problem:

$$-\Delta_p u(z) - \Delta_q u(z) = \lambda c_0 u(z)^{\tau-1} \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = 0,\ u > 0,\ \lambda > 0. \tag{3_λ}$$

Proposition 1. *For every $\lambda > 0$ problem (3_λ) has a unique positive solution $\overline{u}_\lambda \in \operatorname{int} C_+$, the map $\lambda \to \overline{u}_\lambda$ from $(0, +\infty)$ into $C_0^1(\overline{\Omega})$ is nondecreasing, that is, $0 < \lambda_1 < \lambda_2$ implies $\overline{u}_{\lambda_1} \le \overline{u}_{\lambda_2}$ and $\overline{u}_\lambda \to 0$ in $C_0^1(\overline{\Omega})$ as $\lambda \to 0^+$.*

Proof. We consider the C^1-functional $\psi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda c_0}{\tau}\|u^+\|_\tau^\tau \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Since $1 < \tau < q < p$, we see that
$$\psi_\lambda(\cdot) \text{ is coercive.}$$
Also using the Sobolev embedding theorem, we see that
$$\psi_\lambda(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$
By the Weierstrass-Tonelli theorem, we can find $\overline{u}_\lambda \in W_0^{1,p}(\Omega)$ such that
$$\psi_\lambda(\overline{u}_\lambda) = \min\left[\psi_\lambda(u) : u \in W_0^{1,p}(\Omega)\right]. \tag{4}$$
Let $u \in C_+ \setminus \{0\}$ and $t \in (0,1)$. We have
$$\begin{aligned}\psi_\lambda(tu) &= \frac{t^p}{p}\|\nabla u\|_p^p + \frac{t^q}{q}\|\nabla u\|_q^q - \frac{\lambda c_0 t^\tau}{\tau}\|u\|_\tau^\tau \\ &\leq c_1 t^q - c_2 t^\tau \quad \text{for some } c_1 = c_1(u) > 0,\ c_2 = c_2(u) > 0 \text{ (recall that } 1 < q < p).\end{aligned}$$
Since $\tau < q$, choosing $t \in (0,1)$ even smaller if necessary, we can have that
$$\begin{aligned}&\psi_\lambda(tu) < 0, \\ \Rightarrow\quad &\psi_\lambda(\overline{u}_\lambda) < 0 = \psi_\lambda(0) \quad \text{(see (4))}, \\ \Rightarrow\quad &\overline{u}_\lambda \neq 0.\end{aligned}$$
From (4) we have
$$\begin{aligned}&\psi'_\lambda(\overline{u}_\lambda) = 0, \\ \Rightarrow\quad &\langle A_p(\overline{u}_\lambda), h\rangle + \langle A_q(\overline{u}_\lambda), h\rangle = \int_\Omega \lambda c_0 (\overline{u}_\lambda^+)^{\tau-1} h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega).\end{aligned} \tag{5}$$
In (5) we choose $h = -u_\lambda^- \in W_0^{1,p}(\Omega)$. We obtain
$$\begin{aligned}&\|\overline{u}_\lambda^-\|^p \leq 0, \\ \Rightarrow\quad &\overline{u}_\lambda \geq 0, \quad \overline{u}_\lambda \neq 0.\end{aligned}$$
Then from (5) it follows that
$$-\Delta_p \overline{u}_\lambda(z) - \Delta_q \overline{u}_\lambda(z) = \lambda c_0 \overline{u}_\lambda(z)^{\tau-1} \quad \text{for a.a. } z \in \Omega, \quad \overline{u}_\lambda\Big|_{\partial\Omega} = 0. \tag{6}$$
Theorem 7.1, p. 286, of Ladyzhenskaya-Uralt'seva [15], implies that $\overline{u}_\lambda \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [17] says that $\overline{u}_\lambda \in C_+ \setminus \{0\}$. From (6) we have
$$\begin{aligned}&\Delta_p \overline{u}_\lambda(z) + \Delta_q \overline{u}_\lambda(z) \leq 0 \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow\quad &\overline{u}_\lambda \in \operatorname{int} C_+ \quad \text{(see Pucci-Serrin [27], pp. 111, 120).}\end{aligned}$$
Next we show that this positive solution is unique. To this end, we consider the function $G_0 : \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ = [0, +\infty)$) defined by
$$G_0(t) = \frac{1}{p}t^p + \frac{1}{q}t^q \quad \text{for all } t \geq 0.$$
Evidently $G_0(\cdot)$ is increasing and $t \to G_0(t^{1/q})$ is convex (recall that $1 < q < p$). We set
$$G(y) = G_0(|y|) \quad \text{for all } y \in \mathbb{R}^N.$$

We consider the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\|\nabla u^{1/q}\|_p^p + \frac{1}{q}\|\nabla u^{1/q}\|_q^q & \text{if } u \geq 0,\ u^{1/q} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of $j(\cdot)$) and let $u_1, u_2 \in \operatorname{dom} j$. We set $v = [tu_1 + (1-t)u_2]^{1/q}$ with $t \in [0,1]$. From Díaz-Saá [4] (see the proof of Lemma 1), we have

$$\begin{aligned} & |\nabla v(z)| \leq \left[t|\nabla u_1(z)^{1/q}|^q + (1-t)|\nabla u_2(z)^{1/q}|^q\right]^{1/q} \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow\quad & G_0(|\nabla v|) \leq G_0\left(\left[t|\nabla u_1^{1/q}|^q + (1-t)|\nabla u_2^{1/q}|^q\right]^{1/q}\right) \quad \text{(since } G_0(\cdot) \text{ is increasing)} \\ & \leq tG_0\left(|\nabla u_1^{1/q}|\right) + (1-t)G_0\left(|\nabla u_2^{1/q}|\right) \quad \text{(since } t \to G_0(t^{1/q}) \text{ is convex)}, \\ \Rightarrow\quad & G(\nabla v) \leq tG(\nabla u_1^{1/q}) + (1-t)G(\nabla u_2^{1/q}), \\ \Rightarrow\quad & j(\cdot) \text{ is convex.} \end{aligned}$$

Suppose that $\widetilde{u}_\lambda$ is another positive solution of (3_λ). Again we have that $\widetilde{u}_\lambda \in \operatorname{int} C_+$. We set $h = \overline{u}_\lambda^q - \widetilde{u}_\lambda^q \in C^1(\overline{\Omega})$. Then for $|t| \leq 1$ small, we have

$$\overline{u}_\lambda^q + th \in \operatorname{dom} j \quad \text{and} \quad \widetilde{u}_\lambda^q + th \in \operatorname{dom} j.$$

Hence the functional $j(\cdot)$ is Gâteaux differentiable at $\overline{u}_\lambda^q$ and at $\widetilde{u}_\lambda^q$ in the direction h. Using the nonlinear Green's identity (see Papageorgiou-Rădulescu-Repovš [23], Corollary 1.5.17, p. 35), we obtain

$$\begin{aligned} j'(\overline{u}_\lambda^q)(h) =& \frac{1}{q}\int_\Omega \frac{-\Delta_p \overline{u}_\lambda - \Delta_q \overline{u}_\lambda}{\overline{u}_\lambda^{q-1}} h dz = \frac{\lambda c_0}{q}\int_\Omega \frac{1}{\overline{u}_\lambda^{q-\tau}} h dz, \\ j'(\widetilde{u}_\lambda^q)(h) =& \frac{1}{q}\int_\Omega \frac{-\Delta_p \widetilde{u}_\lambda - \Delta_q \widetilde{u}_\lambda}{\widetilde{u}_\lambda^{q-1}} h dz = \frac{\lambda c_0}{q}\int_\Omega \frac{1}{\widetilde{u}_\lambda^{q-\tau}} h dz. \end{aligned}$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. So, we have

$$\begin{aligned} & 0 \leq \int_\Omega \left[\frac{1}{\overline{u}_\lambda^{q-\tau}} - \frac{1}{\widetilde{u}_\lambda^{q-\tau}}\right](\overline{u}_\lambda^q - \widetilde{u}_\lambda^q) dz \leq 0, \\ \Rightarrow\quad & \overline{u}_\lambda = \widetilde{u}_\lambda. \end{aligned}$$

This proves the uniqueness of the positive solution $\overline{u}_\lambda \in \operatorname{int} C_+$.

Now we consider the map $\lambda \to \overline{u}_\lambda$ from $\mathring{\mathbb{R}}_+ = (0, +\infty)$ into $C_0^1(\overline{\Omega})$ and we show that it is nondecreasing.

Let $0 < \beta < \lambda$ and let $\overline{u}_\lambda \in \operatorname{int} C_+$ be the unique positive solution of problem (3_λ). We consider the Carathéodory function $k_\beta : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_\beta(z,x) = \begin{cases} \beta c_0 (x^+)^{\tau-1} & \text{if } x \leq \overline{u}_\lambda(z), \\ \beta c_0 \overline{u}_\lambda(z)^{\tau-1} & \text{if } \overline{u}_\lambda(z) < x. \end{cases} \tag{7}$$

We set $K_\beta(z,x)=\int_0^x k_\beta(z,s)ds$ and consider the C^1-functional $\sigma_\beta : W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\sigma_\beta(u)=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{q}\|\nabla u\|_q^q-\int_\Omega K_\beta(z,u)dz \quad \text{for all } u\in W_0^{1,p}(\Omega).$$

From (7) it is clear that $\sigma_\beta(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_\beta\in W_0^{1,p}(\Omega)$ such that

$$\sigma_\beta(\widetilde{u}_\beta)=\min\left[\sigma_\beta(u) : u\in W_0^{1,p}(\Omega)\right]. \tag{8}$$

Since $\overline{u}_\lambda\in \operatorname{int} C_+$, given $u\in C_+\setminus\{0\}$, we can find $t\in(0,1)$ small such that

$$tu\le \overline{u}_\lambda \tag{9}$$

(see Papageorgiou-Rădulescu-Repovš [23], Proposition 4.1.22, p. 274). From (9), (7) and since $1<\tau<q<p$, choosing $t\in(0,1)$ even smaller if necessary, we have

$$\begin{aligned}
&\sigma_\beta(tu)<0,\\
\Rightarrow\quad &\sigma_\beta(\widetilde{u}_\beta)<0=\sigma_\beta(0) \quad \text{(see (8))},\\
\Rightarrow\quad &\widetilde{u}_\beta\ne 0.
\end{aligned}$$

From (8) we have

$$\begin{aligned}
&\sigma_\beta'(\widetilde{u}_\beta)=0,\\
\Rightarrow\quad &\langle A_p(\widetilde{u}_\beta),h\rangle+\langle A_q(\widetilde{u}_\beta),h\rangle=\int_\Omega k_\beta(z,\widetilde{u}_\beta)hdz \quad \text{for all } h\in W_0^{1,p}(\Omega).
\end{aligned} \tag{10}$$

Choosing $h=-\widetilde{u}_\beta^-\in W_0^{1,p}(\Omega)$ in (10), we obtain

$$\begin{aligned}
&\|\widetilde{u}_\beta^-\|^p\le 0 \quad \text{(see (7))},\\
\Rightarrow\quad &\widetilde{u}_\beta\ge 0,\quad \widetilde{u}_\beta\ne 0.
\end{aligned}$$

Next in (10) we choose $h=(\widetilde{u}_\beta-\overline{u}_\lambda)^+\in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\langle A_p(\widetilde{u}_\beta),(\widetilde{u}_\beta-\overline{u}_\lambda)^+\rangle+\langle A_q(\widetilde{u}_\beta),(\widetilde{u}_\beta-\overline{u}_\lambda)^+\rangle\\
&=\int_\Omega \beta\overline{u}_\lambda^{\tau-1}(\widetilde{u}_\beta-\overline{u}_\lambda)^+dz\\
&\le\int_\Omega \lambda\overline{u}_\lambda^{\tau-1}(\widetilde{u}_\beta-\overline{u}_\lambda)^+dz \quad (\text{since } \beta<\lambda)\\
&=\langle A_p(\overline{u}_\lambda),(\widetilde{u}_\beta-\overline{u}_\lambda)^+\rangle+\langle A_q(\overline{u}_\lambda),(\widetilde{u}_\beta-\overline{u}_\lambda)^+\rangle\\
\Rightarrow\quad &\widetilde{u}_\beta\le\overline{u}_\lambda.
\end{aligned}$$

So, we have proved that

$$\widetilde{u}_\beta\in[0,\overline{u}_\lambda],\ \widetilde{u}_\beta\ne 0. \tag{11}$$

From (11), (7), (10), it follows that $\widetilde{u}_\beta$ is a positive solution of (3_β). Hence $\widetilde{u}_\beta=\overline{u}_\beta\in\operatorname{int} C_+$ (from the uniqueness of the positive solution, see the first part of the proof). Therefore we have

$$\overline{u}_\beta\le\overline{u}_\lambda \quad \text{(see (11))},$$

$$\Rightarrow \quad \lambda \to \overline{u}_\lambda \text{ is nondecreasing from } \mathring{\mathbb{R}}_+ \text{ into } C_0^1(\overline{\Omega}).$$

Finally we show that $\overline{u}_\lambda \to 0$ in $C_0^1(\overline{\Omega})$ as $\lambda \to 0^+$. So, let $0 < \lambda \leq 1$ and consider $\overline{u}_\lambda \in \operatorname{int} C_+$ the unique positive solution of problem (3_λ). We have

$$-\Delta_p \overline{u}_\lambda(z) - \Delta_q \overline{u}_\lambda(z) = \lambda c_0 \overline{u}_\lambda(z)^{\tau-1} \quad \text{for a.a. } z \in \Omega, \quad \overline{u}_\lambda\Big|_{\partial\Omega} = 0, \tag{12}$$

$$0 \leq \overline{u}_\lambda \leq \overline{u}_1 \quad \text{for all } 0 < \lambda \leq 1. \tag{13}$$

From (12), (13) and the nonlinear regularity theory of Lieberman [17], we know that there exist $\alpha \in (0,1)$ and $c_3 > 0$ such that

$$\overline{u}_\lambda \in C_0^{1,\alpha}(\overline{\Omega}) = C^{1,\alpha}(\overline{\Omega}) \cap C_0^1(\overline{\Omega}), \quad \|\overline{u}_\lambda\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c_3 \quad \text{for all } 0 < \lambda \leq 1.$$

The compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and (12), (13) imply that $\overline{u}_\lambda \to 0$ in $C_0^1(\overline{\Omega})$ as $\lambda \to 0^+$. $\square$

On account of Proposition 1, we can find $\lambda_0 \in (0,1]$ such that

$$0 \leq \overline{u}_\lambda(z) \leq \delta \quad \text{for all } z \in \overline{\Omega}, \text{ all } 0 < \lambda \leq \lambda_0, \tag{14}$$

where $\delta > 0$ is as postulated by hypothesis $H\,(iii)$.

Consider the Banach space $C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\}$. The order cone for this space is

$$K_+ = \left\{u \in C_0(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} K_+ = \left\{u \in K_+ : c_u \widehat{d} \leq u \text{ for some } c_u > 0\right\},$$

with $\widehat{d}(z) = d(z, \partial\Omega)$ for all $z \in \overline{\Omega}$. From Lemma 14.16, p. 335, of Gilbarg-Trudinger [11], we know that there exists $\delta_0 > 0$ such that $\widehat{d} \in C^2(\Omega_{\delta_0})$ where $\Omega_{\delta_0} = \{z \in \overline{\Omega} : \widehat{d}(z) < \delta_0\}$. It follows that $\widehat{d} \in \operatorname{int} C_+$ and then by Proposition 4.1.22, p. 274, of Papageorgiou-Rădulescu-Repovš [23], we can find $0 < c_4 < c_5$ such that

$$\begin{aligned} & c_4 \widehat{d} \leq \overline{u}_\lambda \leq c_5 \widehat{d}, \\ \Rightarrow \quad & \overline{u}_\lambda \in \operatorname{int} K_+. \end{aligned} \tag{15}$$

Let $s > N$ and consider the positive, L^p-normalized principal eigenfunction $\widehat{u}_1(p) \in W_0^{1,p}(\Omega)$ of $(-\Delta_p, W_0^{1,p}(\Omega))$. The nonlinear regularity theory and the nonlinear maximum principle imply that $\widehat{u}_1(p) \in \operatorname{int} C_+$. Thus $\widehat{u}_1(p)^{1/s} \in K_+$ and so (15) implies that there exists $c_6 > 0$ such that

$$\begin{aligned} & 0 \leq \widehat{u}_1(p)^{1/s} \leq c_6 \overline{u}_\lambda \quad \text{(see [23], p. 274)}, \\ \Rightarrow \quad & 0 \leq \overline{u}_\lambda^{-\eta} \leq c_7 \widehat{u}_1(p)^{-\eta/s} \quad \text{for some } c_7 > 0. \end{aligned}$$

The Lemma in Lazer-McKenna [16] implies that

$$\begin{aligned} & \widehat{u}_1(p)^{-\eta/s} \in L^s(\Omega), \\ \Rightarrow \quad & \overline{u}_\lambda^{-\eta} \in L^s(\Omega),\ s > N. \end{aligned} \tag{16}$$

3. Positive Solution

In this section we use the frozen variable method and the Leray-Schauder alternative principle, to produce a positive solution for problem (1).

We fix $v \in C_0^1(\overline{\Omega})$ and consider the Carathéodory function $g_v(z,x)$ defined by

$$g_v(z,x) = f(z,x,\nabla v(z)).$$

Then we consider the following singular Dirichlet (p,q)-problem:

$$-\Delta_p u(z) - \Delta_q u(z) = u(z)^{-\eta} + g_v(z,u(z)) \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = 0,\ u > 0,\ 1 < q < p,\ 0 < \eta < 1. \tag{17}$$

Let S_v be the set of positive solutions of (17) in the order interval $[0,\vartheta]$.

Proposition 2. *If hypotheses $H\,(i),(ii),(iii)$ hold, then $\emptyset \neq S_v \subseteq [0,\vartheta] \cap \operatorname{int} C_+$.*

Proof. Problem (17) although variational, it does not have a C^1-energy functional due to the singular term. As we indicated in the Introduction, we will overcome this difficulty by using the solutions from Proposition 1 to bypass the singularity.

We fix $\lambda \in (0,\lambda_0]$ (see (14)) and consider the unique solution $\overline{u}_\lambda \in \operatorname{int} C_+$ of problem (3_λ) (see Proposition 1). On account of (14) (recall that $0 < \delta < \vartheta$, see hypothesis $H\,(iii)$) and of (16), we can introduce the following Carathéodory function:

$$\widehat{g}_v(z,x) = \begin{cases} \overline{u}_\lambda(z)^{-\eta} + g_v(z,\overline{u}_\lambda(z)) & \text{if } x < \overline{u}_\lambda(z), \\ x^{-\eta} + g_v(z,x) & \text{if } \overline{u}_\lambda(z) \leq x \leq \vartheta, \\ \vartheta^{-\eta} + g_v(z,\vartheta) & \text{if } \vartheta < x. \end{cases} \tag{18}$$

We set $\widehat{G}_v(z,x) = \int_0^x \widehat{g}_v(z,s)ds$ and introduce the functional $\widehat{\varphi}_v : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_v(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega \widehat{G}_v(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We have that $\widehat{\varphi}_v \in C^1(W_0^{1,p}(\Omega))$ (see also Papageorgiou-Smyrlis [24], Proposition 3). From (18) it is clear that this functional is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} & \widehat{\varphi}_v(u_0) = \min[\widehat{\varphi}_v(u) : u \in W_0^{1,p}(\Omega)], \\ \Rightarrow\ & \widehat{\varphi}_v'(u_0) = 0, \\ \Rightarrow\ & \langle A_p(u_0),h\rangle + \langle A_q(u_0),h\rangle = \int_\Omega \widehat{g}_v(z,u_0)hdz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{19}$$

In (19), we choose $h = (\overline{u}_\lambda - u_0)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} & \langle A_p(u_0),(\overline{u}_\lambda - u_0)^+\rangle + \langle A_q(u_0),(\overline{u}_\lambda - u_0)^+\rangle \\ & = \int_\Omega \left[\overline{u}_\lambda^{-\eta} + f(z,\overline{u}_\lambda,\nabla v)\right](\overline{u}_\lambda - u_0)^+dz \quad \text{(see (18))} \\ & \geq \int_\Omega c_0\overline{u}_\lambda^{\tau-1}(\overline{u}_\lambda - u_0)^+dz \quad \text{(see (14) and hypothesis } H\,(iii)) \end{aligned}$$

$$
\begin{aligned}
&\geq \int_\Omega \lambda c_0 \overline{u}_\lambda^{\tau-1}(\overline{u}_\lambda - u_0)^+ dz \quad (\text{since } 0<\lambda\leq\lambda_0\leq 1)\\
&= \langle A_p(\overline{u}_\lambda), (\overline{u}_\lambda - u_0)^+\rangle + \langle A_q(\overline{u}_\lambda), (\overline{u}_\lambda - u_0)^+\rangle \quad (\text{see Proposition 1}),\\
\Rightarrow\ &\overline{u}_\lambda \leq u_0.
\end{aligned}
$$

Next in (19) we choose $h = (\overline{u}_0 - \vartheta)^+ \in W_0^{1,p}(\Omega)$. We have

$$
\begin{aligned}
&\langle A_p(u_0), (\overline{u}_0 - \vartheta)^+\rangle + \langle A_q(u_0), (\overline{u}_0 - \vartheta)^+\rangle\\
&= \int_\Omega \left[\vartheta^{-\eta} + f(z,\vartheta,\nabla v)\right](u_0-\vartheta)^+ dz \quad (\text{see (18)})\\
&\leq 0 \quad (\text{see hypothesis } H\,(iii)),\\
\Rightarrow\ &\overline{u}_0 \leq \vartheta.
\end{aligned}
$$

So, we have proved that

$$u_0 \in [\overline{u}_\lambda, \vartheta]. \tag{20}$$

From (20), (18) and (19), we infer that $u_0 \in S_v \neq \emptyset$. From Theorem B.1 of Giacomoni-Saoudi [10] (see also Lieberman [17]), we have that $S_v \subseteq [0,\vartheta]\cap C_+$. Let $\widehat{\xi}_\vartheta > 0$ such that $f(z,x,\nabla v(z)) + \widehat{\xi}_\vartheta x^{p-1} \geq 0$ for a.a. $z\in\Omega$, all $0\leq x\leq\vartheta$ (see hypothesis $H\,(i)$). Then for every $u \in S_v$, we have

$$
\begin{aligned}
&\Delta_p u(z) + \Delta_q u(z) \leq \widehat{\xi}_\vartheta u(z)^{p-1} \quad \text{for a.a. } z\in\Omega,\\
\Rightarrow\quad & u \in \operatorname{int} C_+ \quad (\text{see Pucci-Serrin [27], pp. 111, 120}).
\end{aligned}
$$

We conclude that $S_v \subseteq [0,\vartheta]\cap \operatorname{int} C_+$. ☐

The next proposition gives a canonical way to select an element from each set S_v, $v\in C_0^1(\overline{\Omega})$.

Proposition 3. *If hypotheses $H\,(i),(ii),(iii)$ hold, then the solution set S_v has a smallest element $u_v^* \in \operatorname{int} C_+$ (that is, $u_v^*\leq u$ for all $u\in S_v$).*

Proof. Using Lemma 3.10, p. 178, of Hu-Papageorgiou [12], we can find $\{u_n\}_{n\geq1}\subseteq S_v$ such that

$$0\leq u_n\leq\vartheta \quad \text{for all } n\in\mathbb{N},\ \inf_{n\geq1} u_n = \inf S_v.$$

As before the nonlinear regularity theory (see [10] and [17]) implies that there exists $\alpha\in(0,1)$ and $c_8>0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}), \quad \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c_8 \quad \text{for all } n\in\mathbb{N}.$$

The compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ implies that at least for a subsequence, we have

$$u_n \to u_v^* \text{ in } C_0^1(\overline{\Omega}) \text{ as } n\to+\infty. \tag{21}$$

Suppose that $u_v^*=0$. From (21) we see that we can find $n_0\in\mathbb{N}$ such that

$$u_n(z)\in[0,\delta] \quad \text{for all } z\in\overline{\Omega},\text{ all } n\geq n_0. \tag{22}$$

Then for $n\geq n_0$, we have

$$\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle$$

$$= \int_\Omega \left[u_n^{-\eta} + f(z, u_n, \nabla v)\right] hdz \quad (\text{since } u_n \in S_v)$$

$$\geq \int_\Omega c_0 u_n^{\tau-1} hdz \quad \text{for all } h \in W_0^{1,p}(\Omega),\ h \geq 0 \text{ (see (22) and hypothesis } H\,(iii)). \tag{23}$$

We fix $n \in \mathbb{N}$, $n \geq n_0$ and $0 < \lambda \leq \lambda_0 \leq 1$ (see (14)) and consider the Carathéodory function defined by

$$e_\lambda(z, x) = \begin{cases} \lambda c_0 (x^+)^{\tau-1} & \text{if } x \leq u_n(z), \\ \lambda c_0 u_n(z)^{\tau-1} & \text{if } u_n(z) < x. \end{cases} \tag{24}$$

We set $E_\lambda(z, x) = \int_0^x e_\lambda(z, s)ds$ and consider the C^1-functional $\sigma_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega E_\lambda(z, u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

As in the proof of Proposition 1, using the direct method of the calculus of variations, we can find $\widetilde{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\sigma_\lambda(\widetilde{u}_\lambda) = \min\left[\sigma_\lambda(u) :\ u \in W_0^{1,p}(\Omega)\right] < 0 = \sigma_\lambda(0) \quad (\text{recall that } 1 < \tau < q < p), \tag{25}$$

$$\widetilde{u}_\lambda \in [0, u_n] \quad (\text{see (24)}). \tag{26}$$

From (25) and (26), it follows that $\widetilde{u}_\lambda$ is a positive solution for problem (3_λ), hence $\widetilde{u}_\lambda = \overline{u}_\lambda \in \operatorname{int} C_+$ (see Proposition 1). Therefore we have

$$\overline{u}_\lambda \leq u_n \quad \text{for all } n \geq n_0,$$

a contradiction to our hypothesis that $u_n \to 0$ in $C_0^1(\overline{\Omega})$.

We infer that $u_v^* \neq 0$. For every $n \in \mathbb{N}$ we have

$$\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle = \int_\Omega [u_n^{-\eta} + f(z, u_n, \nabla v)]hdz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}.$$

We pass to the limit as $n \to +\infty$ and use (21). We obtain

$$\langle A_p(u_v^*), h\rangle + \langle A_q(u_v^*), h\rangle = \int_\Omega [(u_v^*)^{-\eta} + f(z, u_v^*, \nabla v)]hdz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

$$\Rightarrow \quad u_v^* \in S_v, \quad u_v^* = \inf S_v.$$

□

We can introduce the map $\xi : C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ defined by

$$\xi(v) = u_v^* \quad (\text{see Proposition 3}).$$

We will show that $\xi(\cdot)$ is compact. To this end, we will use the following proposition.

Proposition 4. *If hypotheses $H\,(i), (ii), (iii)$ hold, $v_n \to v$ in $C_0^1(\overline{\Omega})$ and $u \in S_v$, then we can find $u_n \in S_{v_n}$, $n \in \mathbb{N}$, such that $u_n \to u$ in $C_0^1(\overline{\Omega})$ as $n \to +\infty$.*

Proof. We know that $u \in S_v \subseteq [0,\vartheta] \cap \operatorname{int} C_+$. Then Proposition 1 implies that we can find $\lambda \in (0,1]$ small such that $\overline{u}_\lambda(z) \leq \min\{\delta, u(z)\}$ for all $z \in \overline{\Omega}$. Let $n \in \mathbb{N}$ and consider the Carathéodory function $\widehat{g}_{v_n}(z,x)$ from the proof of Proposition 2 (see (18)). We consider the following Dirichlet (p,q)-problem:

$$-\Delta_p y(z) - \Delta_q y(z) = \widehat{g}_{v_n}(z,u(z)) \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = 0. \tag{27}$$

We have $g_v(\cdot,u(\cdot)) \neq 0$ and $\widehat{g}_{v_n}(\cdot,u(\cdot)) \to \widehat{g}_v(\cdot,u(\cdot))$ in $L^{p'}(\Omega)$ as $n \to +\infty$ (see hypothesis $H\,(i)$, we infer that $\widehat{g}_{v_n}(\cdot,u(\cdot)) \neq 0$ for all $n \geq n_0$. So, without any loss of generality we may say that $\widehat{g}_{v_n}(\cdot,u(\cdot)) \neq 0$ for all $n \in \mathbb{N}$. Note that $\widehat{g}_{v_n}(\cdot,u(\cdot)) \in L^s(\Omega)$, $s > N$. From $s > N$, we have $s' < N' = \frac{N}{N-1} < p^*$ (recall that for $r \in (1,+\infty)$, $\frac{1}{r} + \frac{1}{r'} = 1$ and p^* is the critical Sobolev exponent for p, that is, $p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p \end{cases}$).

So, we have $L^s(\Omega) \hookrightarrow W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ (see Gasiński-Papageorgiou [6], Lemma 2.2.27, p. 141). Consider the nonlinear map $V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$V(u) = A_p(u) + A_q(u) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We know that $V(\cdot)$ is continuous, strictly monotone (hence maximal monotone too) and coercive. Therefore by Corollary 2.8.8, p. 135, of Papageorgiou-Rădulescu-Repovš [22], we can find $y_n \in W_0^{1,p}(\Omega)$, $y_n \neq 0$ such that

$$V(y_n) = \widehat{g}_{v_n}(\cdot,u(\cdot)).$$

In fact this solution is unique due to the strict monotonicity of the operator $V(\cdot)$. Moreover, using (18) as in the proof of Proposition 2, we show that

$$y_n \in [\overline{u}_\lambda, \vartheta] \quad \text{for all } n \in \mathbb{N}.$$

Let $\gamma_n(\cdot) = \widehat{g}_{v_n}(\cdot,u(\cdot))$. Then $\{\gamma_n(\cdot)\}_{n\geq 1} \subseteq L^s(\Omega)$ is bounded. We consider the following auxiliary linear Dirichlet problem:

$$-\Delta w(z) = \gamma_n(z) \quad \text{in } \Omega, \quad w\Big|_{\partial\Omega} = 0,\ n \in \mathbb{N}. \tag{28}$$

Using Theorem 9.16, p. 241 and Lemma 9.17, p. 242, of Gilbarg-Trudinger [11], we see that problem (28) has a unique solution w_n such that

$$w_n \in W^{2,s}(\Omega), \quad \|w_n\|_{W^{2,s}(\Omega)} \leq c_9 \quad \text{for some } c_9 > 0, \text{ all } n \in \mathbb{N}.$$

From the Sobolev embedding theorem, we have

$$W^{2,s}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega}), \quad \alpha = 1 - \frac{N}{s} \in (0,1) \text{ compactly}.$$

So, we have that the sequence $\{w_n\}_{n\geq 1} \subseteq C^{1,\alpha}(\overline{\Omega})$ is relatively compact and then so is $\{l_n(\cdot) = \nabla w_n(\cdot)\}_{n\geq 1} \subseteq C^{0,\alpha}(\overline{\Omega},\mathbb{R}^N)$. We rewrite (27) as follows

$$-\operatorname{div}\left[|\nabla y_n|^{p-2}\nabla y_n + |\nabla y_n|^{q-2}\nabla y_n - l_n\right] = 0, \quad n \in \mathbb{N}.$$

Then the nonlinear regularity theory of Lieberman [17] implies that there exist $\alpha_0 \in (0,1)$ and $c_{10} > 0$ such that

$$y_n \in C^{1,\alpha_0}(\overline{\Omega}), \quad \|y_n\|_{C^{1,\alpha_0}(\overline{\Omega})} \leq c_{10} \quad \text{for all } n \in \mathbb{N}.$$

From the compact embedding of $C^{1,\alpha_0}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we infer that

$$\{y_n\}_{n\geq 1} \subseteq C_0^1(\overline{\Omega}) \text{ is relatively compact.}$$

By passing to a subsequence if necessary, we may assume that

$$\begin{aligned} & y_n \to y \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to +\infty, \\ \Rightarrow \quad & -\Delta_p y - \Delta_q y = \widehat{g}_v(z,u) \quad \text{in } \Omega, \quad y\Big|_{\partial\Omega} = 0 \quad (\text{see (27)}), \\ \Rightarrow \quad & y = u \quad (\text{recall } u \in S_v). \end{aligned}$$

Therefore for the original sequence we have

$$y_n \to u \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to +\infty.$$

Next we consider the following Dirichlet (p,q)-problem:

$$-\Delta_p u(z) - \Delta_q u(z) = \widehat{g}_{v_n}(z, y_n(z)) \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = 0,\, n \in \mathbb{N}.$$

Similarly as before, this problem has a unique solution $y_n^1 \in [\overline{u}_\lambda, \vartheta] \cap \operatorname{int} C_+$ and we have

$$y_n^1 \to u \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to +\infty.$$

We continue this way and produce functions $y_n^k \in [\overline{u}_\lambda, \vartheta] \cap \operatorname{int} C_+$ for all $k, n \in \mathbb{N}$ such that

$$y_n^k \to u \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to +\infty, \text{ for all } k \in \mathbb{N}. \tag{29}$$

For fixed $n \in \mathbb{N}$ and since $y_n^k \in [\overline{u}_\lambda, \vartheta] \cap \operatorname{int} C_+$ for all $k \in \mathbb{N}$, as above, we have that $\{y_n^k\}_{k\geq 1} \subseteq C_0^1(\overline{\Omega})$ is relatively compact. So, we may assume that

$$y_n^k \to u_n \text{ in } C_0^1(\overline{\Omega}) \text{ as } k \to +\infty. \tag{30}$$

Using (30), in the limit as $k \to +\infty$, we obtain

$$\begin{aligned} & -\Delta_p u_n - \Delta_q u_n = \widehat{g}_{v_n}(z, u_n) \quad \text{in } \Omega, \quad u_n\Big|_{\partial\Omega} = 0,\, n \in \mathbb{N}, \\ & \overline{u}_\lambda \leq u_n \leq \vartheta \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Then from (18) it follows that $u_n \in S_{v_n}$ for all $n \in \mathbb{N}$. The nonlinear regularity theory (see [17]) implies that $\{u_n\}_{n\geq 1} \subseteq C_0^1(\overline{\Omega})$ is relatively compact. So, from this fact, (29) and the double limit lemma (see Gasiński-Papageorgiou [7], Problem 1.175, p. 61), we infer that $u_n \to u$ in $C_0^1(\overline{\Omega})$ as $n \to +\infty$. $\square$

Using this proposition, we can now prove the compactness of the map $\xi(\cdot)$.

Proposition 5. *If hypotheses $H\,(i), (ii), (iii)$ hold, then the map $\xi : C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ is compact.*

Proof. Let $D \subseteq C_0^1(\overline{\Omega})$ be bounded. We have $\xi(D) \subseteq [0,\vartheta] \cap \operatorname{int} C_+$ and so using hypothesis $H(i)$ and the nonlinear regularity theory (see [10] and [17]), we have that $\xi(D) \subseteq C_0^1(\overline{\Omega})$ is relatively compact.

Next we show that $\xi(\cdot)$ is continuous. Let $v_n \to v$ in $C_0^1(\overline{\Omega})$. On account of Proposition 4, we can find $u_n \in S_{v_n}$, $n \in \mathbb{N}$, such that $u_n \to u_v^*$ in $C_0^1(\overline{\Omega})$ as $n \to +\infty$. From the first part of the proof, we have that $\{u_n^*\}_{n\geq 1} \subseteq C_0^1(\overline{\Omega})$ is relatively compact. So, we may assume that

$$u_n^* \to \widetilde{u}_v \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to +\infty.$$

Clearly $\widetilde{u}_v \in S_v$. Since $u_n^* \leq u_n$ for all $n \in \mathbb{N}$, we have $\widetilde{u}_v \leq u_v^*$, hence $\widetilde{u}_v = u_v^*$. So for the original sequence we have

$$\begin{aligned} & \xi(v_n) = u_n^* \to u_v^* = \xi(v) \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to +\infty, \\ \Rightarrow\quad & \xi(\cdot) \text{ is compact.} \end{aligned}$$

∎

Let $\mathcal{D}(\xi) = \{u \in C_0^1(\overline{\Omega}) : u = \lambda\xi(u),\, 0 < \lambda < 1\}$.

Proposition 6. *If hypotheses H hold, then $\mathcal{D}(\xi) \subseteq C_0^1(\overline{\Omega})$ is bounded.*

Proof. Let $u \in \mathcal{D}(\xi)$. Then $\frac{1}{\lambda}u = \xi(u)$ with $0 < \lambda < 1$. We have

$$\begin{aligned} & \langle A_p(\frac{1}{\lambda}u), h\rangle + \langle A_q(\frac{1}{\lambda}u), h\rangle \\ & = \int_\Omega \left[\frac{\lambda^\eta}{u^\eta} + f(z, \frac{1}{\lambda}u, \nabla u)\right] h dz \\ & \leq \int_\Omega \left[\frac{\lambda^\eta}{u^\eta} + \frac{1}{\lambda^{p-1}} f(z,u,\nabla u)\right] h dz \quad \text{for all } h \in W_0^{1,p}(\Omega),\, h \geq 0 \\ & \qquad\qquad (\text{see hypothesis } H\,(iv)). \end{aligned} \tag{31}$$

We have $u \geq 0$. So, in (31) we can take $h = u$ and obtain

$$\begin{aligned} & \frac{1}{\lambda^{p-1}}\|u\|^p \leq \int_\Omega \left[\lambda u^{1-\eta} + \frac{1}{\lambda^{p-1}} f(z,u,\nabla u)u\right] dz \\ \Rightarrow\quad & \|u\|^p \leq \int_\Omega \left[u^{1-\eta} + f(z,u,\nabla u)u\right] dz \quad (\text{since } 0 < \lambda < 1) \\ & \leq \int_\Omega \left[\vartheta^{1-\eta} + c_{11}(1 + |\nabla u|^{p-1})\right] dz \quad \text{for some } c_{11} > 0 \\ & \qquad\qquad (\text{see } H(i) \text{ and recall } u \in [0,\vartheta]), \\ \Rightarrow\quad & \|u\|^p \leq c_{12}[1 + \|u\|^{p-1}] \quad \text{for some } c_{12} > 0, \end{aligned}$$

$$\Rightarrow\quad \mathcal{D}(\xi) \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{32}$$

From (32) and Kumar-Sreenadh [14] (Lemma 2.2), we have that $\mathcal{D}(\xi) \subseteq L^\infty(\Omega)$ is bounded. Then using Theorem B.1 of Giacomoni-Saoudi [10] (see also Lieberman [17]), we have that $\mathcal{D}(\xi) \subseteq C_0^1(\overline{\Omega})$ is bounded (in fact relatively compact). ∎

Now we are ready for the existence theorem.

Theorem 2. *If hypotheses H hold, then problem* (1) *admits a positive solution* $\widehat{u} \in [0, \vartheta] \cap \operatorname{int} C_+$.

Proof. Propositions 5 and 6 permit the use of Theorem 1 (the Leray-Schauder alternative principle). So, we can find $\widehat{u} \in C_0^1(\overline{\Omega})$ such that

$$\begin{aligned} & \widehat{u} = \xi(\widehat{u}), \\ \Rightarrow \quad & \widehat{u} \in [0, \vartheta] \cap \operatorname{int} C_+ \text{ solves problem (1).} \end{aligned}$$

□

Remark 2. If we strengthen hypothesis $H\,(ii)$ in the following way

$$\vartheta^{-\eta} + f(z, \vartheta, y) \le -\widehat{c} < 0 \quad \text{for a.a. } z \in \Omega, \text{ all } y \in \mathbb{R}^N,$$

then using Proposition 6 of Papageorgiou-Rădulescu-Repovš [22], we have

$$\widehat{u}(z) < \vartheta \quad \text{for all } z \in \overline{\Omega}.$$

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