Generalization of the Den Hartog Model and Rule-of-Thumb Formulas for Optimal Tuned Mass Dampers

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ABSTRACT

In recent years, the need of improving safety standards for both existing and new buildings against earthquake and wind loads has created a growing interest in the use of the so-called tuned mass dampers, exploited to control, in active or passive way, the dynamic response of structures. To design and optimize tuned mass damper systems, the effective analytical procedure proposed by Den Hartog in his seminal work [5] has been widely adopted over the years, without including damping of the main structure. However, in many cases of engineering interest, the damping of the primary system plays a key role in the overall mechanical response, with the result of an increase in complexity of the related optimization problem, which are in fact solved in the vast majority by means of *ad hoc* numerical procedures. In the present work, we recover the classical optimization strategy by Den Hartog and generalize it by including the main system's damping, providing new analytical solutions whose results are consistent with the few ones obtained through alternative mathematical methods by Thompson [38] and Krenk [21] and subsequently by Fang et al. [11]. In particular, some closed-form solutions and helpful rule-of-thumb formulas for the optimal setting of the design parameters are derived, by considering two types of incoming excitations, i.e. the force acting on the main mass and the ground motion input, both of interest for engineering applications. Finally, theoretical outcomes are compared with consolidated data from the literature.

1. Introduction

A tuned-mass damper (TMD) is a device –comprising a mass, a spring and a damper– that is attached to a structure in order to control, in an active or passive way, the dynamic response of the so-called primary system. The main idea is based on the dissipation of the vibrational energy of the primary oscillator due to the motion of the secondary mass, through proper calibration of tuning and damping ratios. For design purposes, it is generally assumed that the mass, stiffness and damping of the primary structure are known, while the mechanical and dynamic parameters of the secondary system represent the design variables.

TMDs can be helpfully employed for varied mechanical applications. In civil engineering, they allow to improve the dynamic response of structures undergoing wind and earthquake excitations, resulting very effective in reducing excessive vibration amplitudes in the former cases [16, 20, 23, 25, 28, 35, 36]. For seismic loads, two main limitations in the use of TMDs can be in fact recognized in the optimal setting of the TMD design parameters: *i*) their high sensitivity with respect to the minimization of the maximum displacement and *ii*) their high dependence on both earthquake frequency content and impulsive nature of the seismic excitation [4, 15, 46]. To gain effectiveness and obtain advantages under seismic actions, TMD systems need to involve large secondary masses. This poses new issues and challenges for structural engineers and designers. These TMDs should indeed require large space, which is not always available for their installation [29]. Additionally, since this mass is designed to be in resonance with the supporting structure, systems capable to accommodate large displacements have to be *ad hoc* conceived [43].

Several solutions to the above-mentioned issues have been hypothesized in the literature and some of them have been also actually applied to solve real problems of civil engineering. These solutions are all based on the idea of utilizing part of the building mass as a giant mass damper, e.g. the Mega-Subcontrol system (MSC) for tall buildings proposed by Feng and Mita [12] and the Inter-story Isolation System (IIS) [6, 7, 40, 41, 44].

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From the modelling standpoint, for both classical and so-called non-conventional TMDs (characterized by large mass ratios), the main formulations provided in literature grasp the effect of the design parameters and derive their optimal values utilize reduced order two-degrees-of-freedom (2-DOF) models.
The concept of the conventional TMD has its root in the dynamic vibration absorbers studied as early as 1909 by Frahm [13]. Then, Den Hartog [5] proposed the theory of damped dynamic absorbers attached to undamped main systems, subjected to a sinusoidal excitation. By following the Den Hartog's strategy, simple expressions for the optimal tuning ratio and the damping ratio of the mass damper can be derived, starting from the observation that the curves of the amplification factor of the primary structure, obtained for several values of damping ratios of the absorber, always intersect at two points, named *fixed* or *invariant points*, this allowing to find some minima of the displacement amplitude of the main system. However, when the damping of the primary system is introduced, some difficulties in obtaining analytically optimal parameters are met and numerical methods or approximate formulations are generally called into play. Several scholars indeed employed numerical strategies for damped structures equipped with a TMD, subjected to harmonic or random excitations[18, 30, 39, 45].
Many works and a large number of optimization approaches with various objective functions have been developed over the years, to testify the vivid debate on the use of TMDs in civil engineering applications. Villaverde [42] proposed the application of light TMD for mitigating the seismic response of buildings. Afterwards, Sadek et al. [33] extended the validity of Villaverde's study, by selecting the values of the TMD design parameters in order to result approximately equal to the modal damping ratios for both the primary system and absorber. Furthermore, Moutinho [26] provided an alternative procedure that solves the problem of the optimal parameters obtained by Sadek et al. [33], which gave rise to physically implausible optimal design parameters in the case of large mass ratios. The numerical method provided by Moutinho was based on the direct assignment of the damping ratios of the vibration modes of the two buildings, once the mass ratio is set *a priori*, resulting that optimal damping ratios for TMDs are significantly lower than the one by Sadek et al. [33].
More recently, starting from Lavan [24], Yahyai et al. [47] utilized an optimization process based on two objective functions, that is the minimization of the frequency response function of the TMD primary structure and the minimization of the difference between first and second modal damping ratios of the combined 2-DOF system. An energy-based approach has been instead proposed by Zilletti et al. [49] and then by Reggio and De Angelis [31, 32] who provided a methodology for optimizing TMDs with very large mass ratios, by assuming a white noise input. The authors showed that their optimization procedure leads to approximately equal damping ratios in the two complex modes of vibrations, as observed by Sadek et al. [33].
In the case of damping of the primary system, very few examples of analytical approaches have been however encountered in the literature. A first attempt to provide a closed-form solution was made by Asami et al. [1], who analytically characterized the harmonic response of a dynamic vibration absorber, by proposing a series solution for reducing the magnification factor of the damped main system. Successively, Ghosh and Basu [14] noticed that for low-to-moderate damping ratios in the primary structure, the *fixed points* could be still approximately assumed as independent on the damping ratio of the absorber and so derived, following a Den Hartog approach, an optimal tuning in closed form. A further strategy has been proposed by Krenk and Høgsberg [22], who investigated the tuned mass absorber on damped primary structures subjected to random loads with constant spectral density (i.e. white noise), starting from some previous results by Krenk reported in his seminal work [21]. Therein, by following the procedure firstly proposed by Thompson [38], the author demonstrated that, in case of vanishing damping of the main structure, the optimal frequency tuning is obtained when the two complex loci of the two natural frequencies intersect at a bifurcation point, which corresponds to the maximum modal damping, demonstrating that the optimal value is also the upper limit of the damping to be introduced in the system. Finally, Fang et al. [11] generalized Krenk's approach [21] to the case of damping in the main structure.
With the aim to give a contribution to analytical methods for the optimal tuning of TMDs, this work presents a closed-form solution for minimizing the amplification of the low-moderate damped primary system subjected to harmonic excitation. The proposed procedure extends the Den Hartog's *fixed points* method to general damped TMDs and low-moderate damped structures, by considering arbitrary mass ratios. In particular, in the forced case, by minimizing the displacement amplitude of the main system by means of the proposed procedure, the optimal tuning frequency ratio already derived by Ghosh and Basu [14] is recovered, additionally obtaining in explicit form the optimal damping ratio. Furthermore, by considering the ground motion input, new optimal design parameters are analytically derived, in terms of tuning and damping ratios. In order to show the effectiveness of the proposed strategy, the obtained theoretical outcomes were also compared to literature formulations dealing with TMDs under harmonic excitation. Rule-of-thumb

formulas, based on the analytical solutions and to be used in practical TMD design problems, were also given at the end for both the considered inputs. Finally, insights into the influence of ground motion frequency content on the optimal parameters were provided, highlighting the way for obtaining further results of practical interest in analytical form.

2. Enhanced Den Hartog's Model for damped structure-damped absorber

2.1. Statement of the problem for harmonic input

Tuned Mass Dampers can be described through a simplified two-lumped-degree-of-freedom (2-DOF) model: the degree of freedom representing the main system and the one of the absorber, as sketched in Figure 1.
Mass, stiffness and damping elements of the main system are referred to as m_1, k_1 and c_1, respectively, m_2, k_2 and c_2 being the corresponding parameters of the absorber. By considering harmonic excitations of frequency ω in the form of a force $F_1(t)$ and a ground motion $\ddot{u}_g(t)$, the equations of motion are:

$$\begin{cases} m_1\ddot{x}_1(t) + c_1\dot{x}_1(t) + k_1x_1(t) - c_2(\dot{x}_2(t) - \dot{x}_1(t)) - k_2(x_2(t) - x_1(t)) = F_1(t) - m_1\ddot{u}_g(t) \\ m_2\ddot{x}_2(t) + c_2(\dot{x}_2(t) - \dot{x}_1(t)) + k_2(x_2(t) - x_1(t)) = -m_2\ddot{u}_g(t) \end{cases} \tag{1}$$

where x_1 and x_2 are the displacements of the two masses.
In the frequency domain, by considering the sole $F_1(t)$ input, the dynamic response of the primary mass m_1 can be calculated as:

$$\mathscr{F}\left[x_1(t)\right] = H_{11}(\omega)\,\mathscr{F}\left[F_1(t)\right], \tag{2}$$

where $\mathscr{F}$ represents the Fourier Transform operator, while $H_{11}(\omega)$ is the complex-valued frequency response function (FRF) of the first DOF, given by:

$$H_{11}(\omega) = \frac{N_{11}}{D_{11}}, \tag{3}$$

being N_{11} and D_{11}:

$$\begin{aligned} N_{11} =& -\omega^2 m_2 + i\omega c_2 + k_2; \\ D_{11} =& m_1m_2\omega^4 - i\omega^3\left(c_1m_2 + c_2m_1 + c_2m_2\right) - \omega^2\left(c_1c_2 + k_1m_2 + k_2m_1 + k_2m_2\right) \\ &+ i\omega\left(c_1k_2 + c_2k_1\right) + k_1k_2. \end{aligned} \tag{4}$$

Similarly, the transfer function of the second DOF can be made explicit as:

$$H_{21}(\omega) = \frac{N_{21}}{D_{21}}, \tag{5}$$

where N_{21} and D_{21} are:

$$\begin{aligned} N_{21} =& i\omega c_2 + k_2; \\ D_{21} =& D_{11}. \end{aligned} \tag{6}$$

By means of some algebraic manipulations, the amplification factors of the main system and of the absorber, respectively $|H_{11}|$ and $|H_{21}|$, can be finally written as:

$$
\begin{aligned}
&|H_{11}\left(\Omega,\, \nu,\, \xi_1,\, \xi_2,\, \mu\right)| = \\
&= \sqrt{\frac{4\nu^2{\xi_2}^2\Omega^2 + \left(\nu^2 - \Omega^2\right)^2}{4\Omega^2\left[\nu\left(\nu\xi_1 + \xi_2\right) - \left[\xi_1 + \nu\xi_2\left(1+\mu\right)\right]\Omega^2\right]^2 + \left[\nu^2 - \left[1 + \nu^2\left(1+\mu\right) + 4\nu\xi_1\xi_2\right]\Omega^2 + \Omega^4\right]^2}}, \\
&|H_{21}\left(\Omega,\, \nu,\, \xi_1,\, \xi_2,\, \mu\right)| = \\
&= \sqrt{\frac{\nu^2\left(\nu^2 + 4\xi_2^2\Omega^2\right)}{4\Omega^2\left[\nu\left(\nu\xi_1 + \xi_2\right) - \left[\xi_1 + \nu\xi_2\left(1+\mu\right)\right]\Omega^2\right]^2 + \left[\nu^2 - \left[1 + \nu^2\left(1+\mu\right) + 4\nu\xi_1\xi_2\right]\Omega^2 + \Omega^4\right]^2}},
\end{aligned}
\tag{7}
$$

where $\mu = {m_1}^{-1}m_2$ is the mass ratio, $\nu = {\omega_1}^{-1}\omega_2$ is the frequency ratio, ω_1 and ω_2 are the natural frequencies of the masses m_1 and m_2, $\Omega = {\omega_1}^{-1}\omega$ is the forced frequency ratio, $\xi_1 = c_1(2m_1\omega_1)^{-1}$ is the damping ratio of the mass m_1 and $\xi_2 = c_2(2m_2\omega_2)^{-1}$ is the damping ratio of the mass m_2.
A different set of results are obtained if the dynamic action is represented only by the ground acceleration $\ddot{u}_g$. In this case, by adopting the same dimensionless parameters utilized in Equations (7), the amplification factors of the first and the second DOF, respectively $|H_{1g}|$ and $|H_{2g}|$, with respect to the ground motion input, are:

$$
\begin{aligned}
&|H_{1g}\left(\Omega,\, \nu,\, \xi_1,\, \xi_2,\, \mu\right)| = \\
&= \sqrt{\frac{4\nu^2{\xi_2}^2\Omega^2(1+\mu)^2 + \left(\nu^2(1+\mu) - \Omega^2\right)^2}{4\Omega^2\left[\nu\left(\nu\xi_1 + \xi_2\right) - \left[\xi_1 + \nu\xi_2\left(1+\mu\right)\right]\Omega^2\right]^2 + \left[\nu^2 - \left[1 + \nu^2\left(1+\mu\right) + 4\nu\xi_1\xi_2\right]\Omega^2 + \Omega^4\right]^2}}, \\
&|H_{2g}\left(\Omega,\, \nu,\, \xi_1,\, \xi_2,\, \mu\right)| = \\
&= \sqrt{\frac{\left((\mu+1)\nu^2 - \Omega^2 + 1\right)^2 + 4\Omega^2\left((\mu+1)\nu\xi_2 + \xi_1\right)^2}{4\Omega^2\left[\nu\left(\upsilon\xi_1 + \xi_2\right) - \left[\xi_1 + \nu\xi_2\left(1+\mu\right)\right]\Omega^2\right]^2 + \left[\nu^2 - \left[1 + \nu^2\left(1+\mu\right) + 4\nu\xi_1\xi_2\right]\Omega^2 + \Omega^4\right]^2}}.
\end{aligned}
\tag{8}
$$

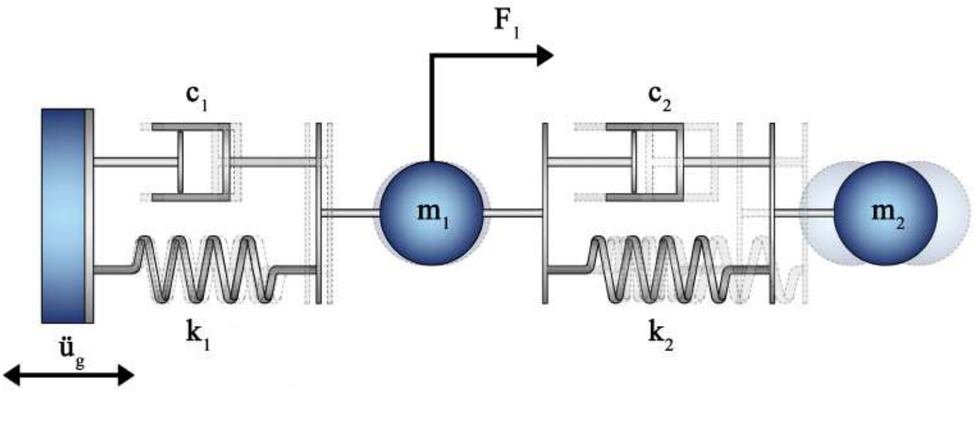

Figure 1: Structural 2-DOF TMD model.

It is worth noticing that the above reported expressions for both the amplification factors are dimensionless and are only functions of dimensionless parameters. For the sake of generality, no constraints were imposed to the range in which these parameters can vary. However, the results from numerical examples shown in the figures were here obtained by considering for the above mentioned parameters values consistent with the ones of interest for the applications, e.g. $\mu \in [0, 2[$; $\nu \in [0, 1]$; $\xi_1 \in [0, 0.1]$; $\xi_2 \in [0, 0.1]$, selecting mass ratios μ in a sufficiently wide interval to include any

possible realistic civil engineering problem. In fact, conventional TMDs are characterized by small mass ratios of the order of 10^{-3}, an example being the steel sphere in Taipei 101, in Taiwan. However, in many IIS buildings, large values of mass ratios can be found, as for instance occurs in the Japanese Iidabashi First Building and Shiodome Sumitomo Building, where values of 0.275 and 2.154 were respectively adopted [10], further studies taking into account mass ratios up to 10 [27, 31, 48]. With respect to the range of values for the damping ratio of the lower structure ξ_1, they were chosen to be compatible with the ones used for steel, reinforced concrete and masonry structures (i.e. 0.02 – 0.05), while values for tuning ratios ν were assumed in line with the optimal ones adopted in literature for both conventional and non-conventional TMDs [4, 5, 12, 14, 15, 26, 29, 33, 34, 39, 42, 45, 47].
As an example, the amplification factors $|H_{11}|$ and $|H_{1g}|$ are reported in Figure 2 as a function of Ω, by assuming two typical values of mass ratios for conventional and non-conventional TMDs, e.g. $\mu = 0.05$ and $\mu = 0.5$, setting $\xi_1 = 0.05$, $\nu = 0.9$, and by varying ξ_2 from 0.01 to 0.20. Figure 2 in particular shows that the curves present two resonant peaks and intersect at two points, namely P and Q, which are independent from the damping factor, confirming the validity of the Den Hartog *fixed points* strategy for low-moderate damping factors, as highlighted by Ghosh and Basu [14]. In fact, as known, thanks to the existence of *fixed points*, Den Hartog proposes to minimize the displacement amplification of the main system subjected to a sinusoidal excitation, by adopting as optimal parameters the tuning frequency ratio, ν_{opt}, and the damping ratio of the absorber, $\xi_{2,opt}$. The amplification factors at the points P and Q are first equated to derive ν_{opt}, then the first derivative of $|H_{11}(\Omega)|$ with respect to Ω at the points P and Q is set equal to zero for defining $\xi_{2,opt}$, as the average of the optimal values obtained at the two *fixed points*.

In addition, Figure 2 highlights that the peaks of the amplification factors are remarkably sensitive to small variations of damping ξ_2, being even more sensitive to perturbations of the damping factor ξ_1, as reported in Figure 3. Herein, the amplification factor of the first degree-of-freedom is plotted by considering the same mass ratio values and tuning ratio of those adopted in Figure 2, by fixing ξ_2 equal to 0.05, while ξ_1 varies from 0.00 to 0.10. For these limit values, the corresponding dimensionless peaks of $|H_{11}(\Omega)|$ and $|H_{1g}(\Omega)|$ drastically decrease, the marked dependence of the static displacement of the main system on slight variations of both the damping factors ξ_1 and ξ_2 suggesting the need of an analytical procedure that ensures the accurate definition of optimal parameters, as provided below.

2.2. Closed-form solutions for the optimal design parameters

Observing the strong influence of the peak amplification factor from the damping ratio of the main system, a new optimized procedure, implemented in Mathematica© [17], is here proposed in order to extend and further generalize the Den Hartog's *fixed points* theory for low-moderate damped system and damped non-conventional TMDs. Under these assumptions, high damping values of the primary structure are a priori excluded, according to the work by Ghosh and Basu [14] in which it is shown that damping ratios over 10% are not physically admissible for main systems in civil engineering applications. In analogy with the Den Hartog's model, by considering the negligible dependence of the amplification factor on the damping ratio of the absorber in the neighbourhood of the fixed points, Ghosh and Basu [14] show that it is possible to calculate the amplification factor $|H_{11}(\Omega)|$ for two extreme values of ξ_2, that is for $\xi_2 \to 0$ and for $\xi_2 \to \infty$. In the Appendix 7, the expressions of the amplification factors of the main system for both the excitations are reported in explicit for these two extreme values. In particular, by equating expressions 19 and 20, four conjugated solutions have been obtained, the two positive ones representing the abscissas of the two *fixed points* P and Q, i.e.:

$$\Omega_{P,Q} = \sqrt{\frac{1 + \nu^2(1+\mu) \mp \sqrt{1 - \nu^2\left[2 - \nu^2(1+\mu)^2\right]}}{2+\mu}}. \tag{9}$$

In the same manner, by equating the two expressions (21) and (22), the abscissas of the two points P and Q can be obtained also in the case of ground motion as:

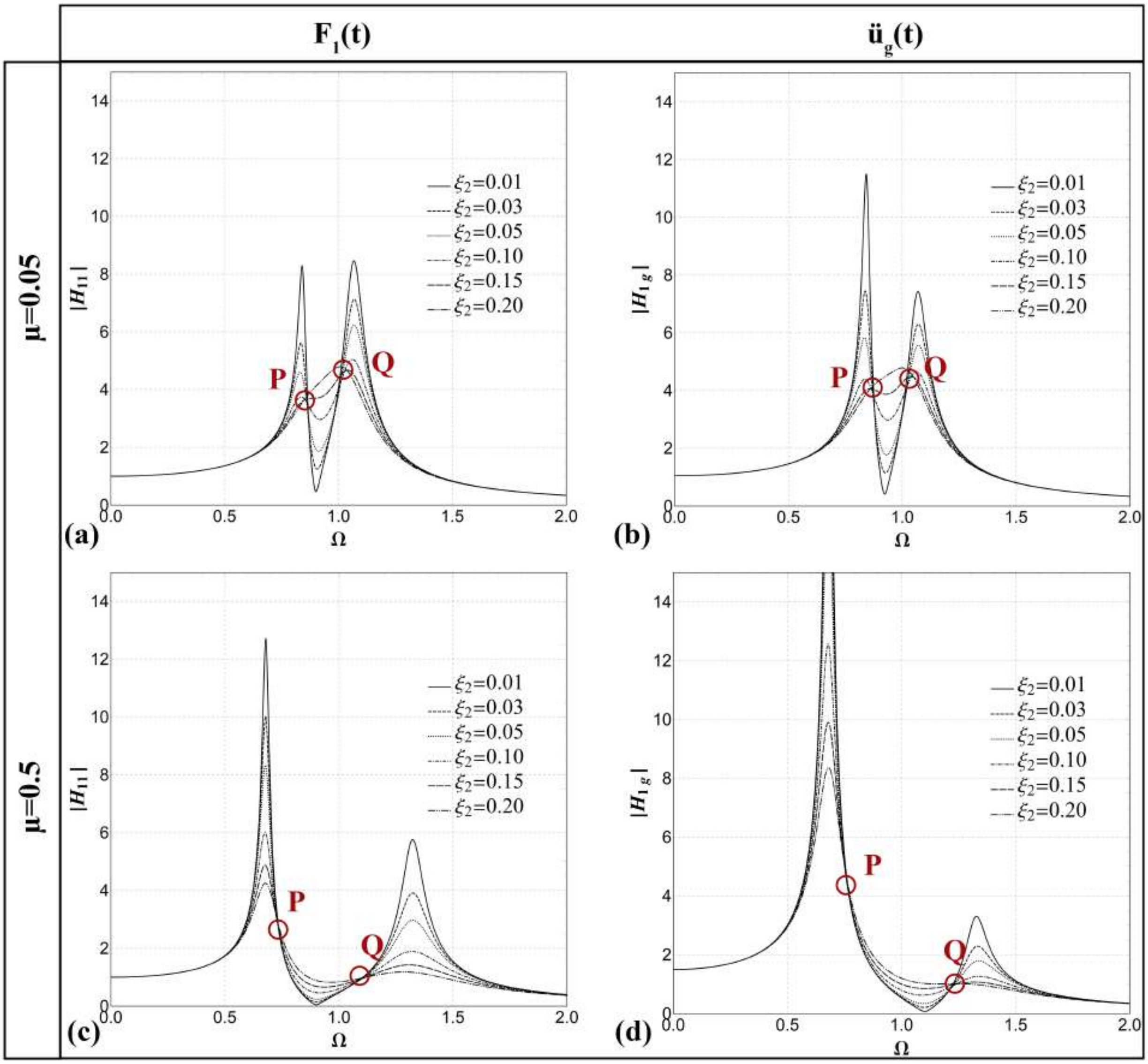

Figure 2: Plot of the amplification factors $|H_{11}(\Omega)|$ and $|H_{1g}(\Omega)|$ by assuming $\nu_=0.9$ and $\xi_1 = 0.05$ and by varying ξ_2 for the two mass ratios.

$$
\begin{aligned}
&\Omega_{P,Q} = \frac{1}{2}\sqrt{\frac{\sqrt{A_{\Omega_{P,Q}}} \mp B_{\Omega_{P,Q}}}{-2(\mu+2)\xi_1^2 + \mu + 1}} \\
&\text{where } A_{\Omega_{P,Q}} \text{ and } B_{\Omega_{P,Q}} \text{ are respectively:} \\
&A_{\Omega_{P,Q}} = 4(\mu+1)^4\nu^4 + 16(\mu+1)\nu^2\xi_1^2\left(4(\mu+1)\nu^2\xi_1^2 - 2(\mu+1)^2\nu^2 + \mu + 2\right) \\
&+ 4(\mu-2)(\mu+1)^2\nu^2 + (\mu+2)^2 \\
&B_{\Omega_{P,Q}} = 8(\mu+1)\nu^2\xi_1^2 - 2(\mu+1)^2\nu^2 - \mu - 2.
\end{aligned}
\tag{10}
$$

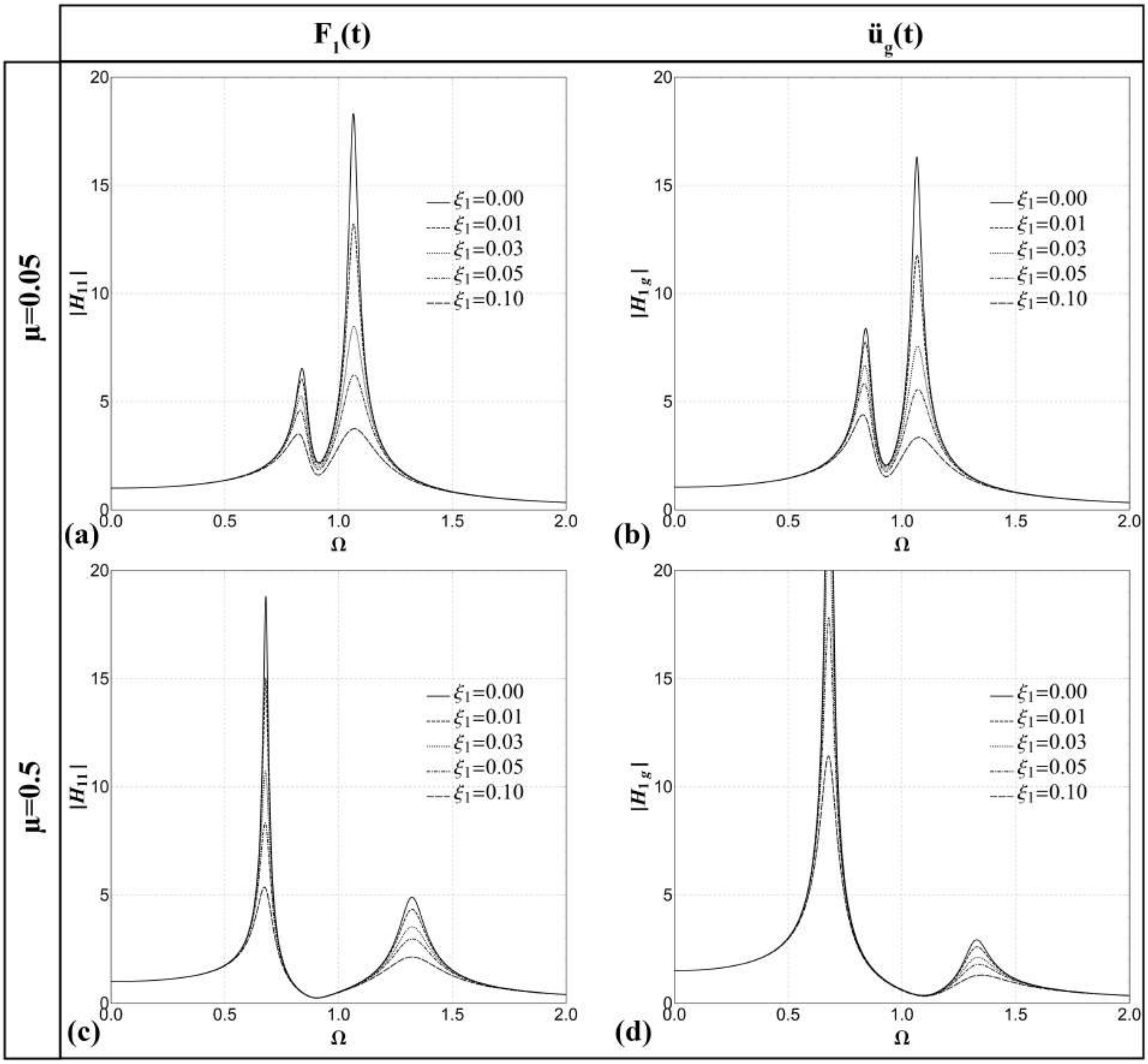

Figure 3: Plot of the amplification factors $|H_{11}(\Omega)|$ and $|H_{1g}(\Omega)|$ by assuming $\nu_{=}0.9$ and $\xi_2 = 0.05$ and by varying ξ_1 for the two mass ratios.

In order to find the optimal tuning frequency ratio of the TMD, according to the Den Hartog's procedure, the values assumed by the amplification functions in P and Q, for both the considered excitations, have to be equated. In this way, Ghosh and Basu [14] gave the closed-form for the optimal tuning for the sole case of the force applied to the main structure, by obtaining the following expression:

$$\nu_{opt(F_1)} = \sqrt{\frac{1+\mu-2\,(2+\mu)\,{\xi_1}^2}{(1+\mu)^3}}. \tag{11}$$

By essentially exploiting the same strategy, an analogous closed-form solution for the optimal tuning can be also derived for the case of ground motion as follows:

$$\nu_{opt(\ddot{u}_g)} = \sqrt{\frac{4+\mu}{2(1+\mu)^2} - \frac{2\,(2+\mu)\,\xi_1{}^2}{(1+\mu)^3} - \frac{1}{1+\mu-4\xi_1{}^2}}. \tag{12}$$

Equations (11) and (12) show that the optimal tuning is independent from the damping ratio of the absorber, while it is a function of mass and damping ratios of the primary system. Moreover, when the damping ratio of the main mass tends to zero, the expressions (11) and (12) recover respectively the classical Den Hartog's formula $\nu_{opt} = (1+\mu)^{-1}$ [5] and the Warburton's one $\nu_{opt} = (1-\frac{\mu}{2})^{\frac{1}{2}}(1+\mu)^{-1}$ [45].
Furthermore, the variation of optimal tuning ν_{opt} as a function of the mass ratio μ for selected values of the damping ratio ξ_1 is reported in Figure 4 for both the inputs $F_1(t)$ and $\ddot{u}_g(t)$.

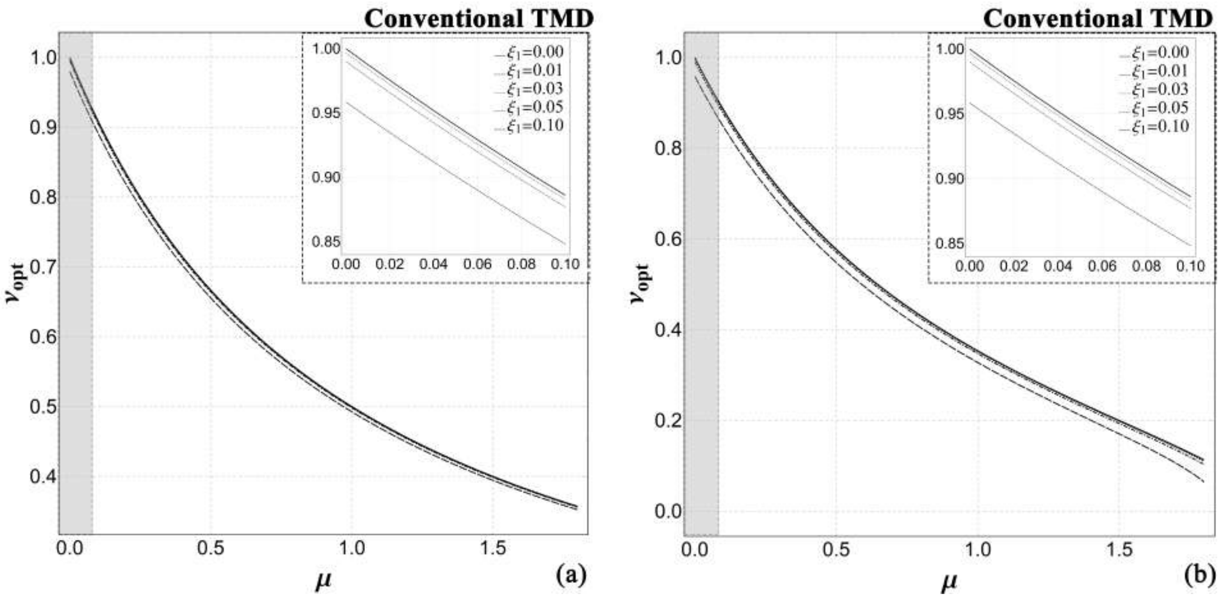

Figure 4: Optimal tuning frequency ratio ν_{opt} vs. the mass ratio μ by varying the damping ξ_1 of the main system for harmonic excitation: (a) $F_1(t)$ input, (b) $\ddot{u}_g(t)$ input. In the inset, the conventional TMD range is highlighted.

Figure 4 highlights that the optimal tuning decreases as the mass ratio grows for both the input cases, exhibiting weak dependence from the damping ratio of the main system when the input is the force (Figure 4a) and resulting slightly more sensitive to very small variation of damping ratio ξ_1 in the case of ground motion input (Figure 4b).
It is worth noticing that Ghosh and Basu [14] do not derive an expression for the optimal damping of the absorber if a force is applied to the primary oscillator. Therefore, to obtain the optimal damping of the absorber, namely $\xi_{2,opt}$, the Den Hartog's procedure is generalized by imposing that $|H_{11}(\Omega)|$ and $|H_{1g}(\Omega)|$ are stationary at both *fixed points* P and Q. After some algebraic manipulations, this leads to the 4$^{\text{th}}$ degree equation in ξ_2:

$$b_0 + b_1\xi_2 + b_2\xi_2{}^2 + b_3\xi_2{}^3 + b_4\xi_2{}^4 = 0, \tag{13}$$

where the coefficients are provided explicitly in the Appendix 8 for both inputs. By substituting Equations (9) and (10) respectively into the expressions of the coefficients $b_0, ..., b_4$ given by (23) and (24) and then using Equation (13) by assuming ν equal to ν_{opt} for each input (Equations (11) and (12)), four roots in closed-form for each *fixed point* P and Q can be obtained by means of Ferrari-Cardano formula. These solutions give $\xi_{2,opt}$ as a function of ξ_1 and μ and can be hence used to design TMDs characterized by a primary system exhibiting a low-moderate damping excited by a harmonic vibration.

The solutions are derived for arbitrary configurations of the 2-DOF TMD. However, if ξ and μ assume the usual values found in civil engineering applications, that is they fall respectively into the intervals $[0, 10^{-1}]$ and $[0, 2[$, one admissible root for Equation (13) is only determined for the points P and Q. As a consequence, there are two optimal damping solutions, $\xi_{2,opt}(\Omega \rightarrow \Omega_P)$ and $\xi_{2,opt}(\Omega \rightarrow \Omega_Q)$, as also occurs for the classical case of undamped structure and damped TMD. Analogously to classical Den Hartog's approach, the optimal damping factor $\xi_{2,opt}$ is here evaluated as the arithmetic mean of $\xi_{2,opt}(\Omega \rightarrow \Omega_P)$ and $\xi_{2,opt}(\Omega \rightarrow \Omega_Q)$. The optimal damping factors $\xi_{2,opt}$ for both the types of excitations as a function of the mass ratio μ is shown in Figure 5, prescribing some values of the damping factor ξ_1. It can be observed that $\xi_{2,opt}$ generally increases with the mass ratio. Also, for mass ratio $\mu \leq 0.1$ (conventional TMD), a discrepancy of about 20% between curves obtained setting ξ_1 equal to 0 and 0.1 is registered, this highlighting the numerical relevance of the results from exact analytical solutions for slightly damped conventional TMDs as well. Furthermore, the outcomes show that, for the case of F_1 input (Figure 5a), when $\mu \geq 1$, the slope of the curves $\xi_{2,opt}$ decreases and their value decreases as ξ_1 increases. A different trend is instead observed for the input $\ddot{u}_g$ input. Indeed, Figure 5b shows that, after a first tract of the curve for $\mu \leq 0.5$, the slope of $\xi_{2,opt}$ grows as the mass ratio increases. Interestingly, due to the fact that no *a priori* restrictions on the range in which the design parameters can vary have been imposed, the optimal damping ratio of the absorber takes values over the unity too, in particular for large mass ratios, thus leading to overdamped systems that however are generally difficult to reach for civil engineering structures under ground motion excitations.

It is worth to highlight that the closed-form solution for the optimal damping factor of TMDs with damped main system is an original result. In fact, scientific works dealing with such damped TMDs, in which the link primary-to-secondary mass is modelled by means of Kelvin-Voigt models, can be grouped in four main categories. In the first one, the TMD is optimized by minimizing the magnitude of the amplification factor of the main system subjected to harmonic excitation or white noise and the related optimal parameters are derived by means of numerical formulations and fitting procedures, this implying the results to be valid only for restricted ranges of mass ratios [2, 12, 14, 15, 26, 29, 33, 34, 39, 42, 45, 47]. The second group is essentially related to the paper by Asami et al. [1] where for the H-∞ optimization a series solution is proposed for the damped primary system both for the force and the motion excitation, by following the Den Hartog's approach and by remarking that *"[...] when damping is present in the primary system [...], there is no exact algebraic solution of the H-∞ optimization"*. For the third group, it can be mentioned the recent paper of Fang et al. [11], in which the authors provide a closed-form solution by enhancing the frequency approach suggested by Krenk [21] to include the sole forced case of damped TMD. Although in that work an analytical solution was proposed, the optimization criterion was based on a strategy different with respect the Den Hartog's one. The main results given in the fourth group can be instead referred to the works by Reggio and De Angelis [4, 31], where a closed-form solution was obtained by means of an energy-based approach, despite the results were only given in the form of charts.

It is finally highlighted that, in the special case of undamped main systems and by making reference to the expressions of the coefficients b_i for the two inputs as reported in Appendix 8, the roots of the equation (13) trace back the already known Den Hartog optimal damping for the forced system [5], giving in closed-form the solution for the optimal damping in case of ground motion as well, that is:

$$\begin{aligned} \xi_{2,opt(F_1)}(\xi_1 \rightarrow 0) &= \sqrt{\frac{3\mu}{8(\mu+1)}}, \\ \xi_{2,opt(\ddot{u}_g)}(\xi_1 \rightarrow 0) &= \frac{\sqrt{\left(\sqrt{2}\sqrt{\mu}+6\right)\mu} + \sqrt{\left(6-\sqrt{2}\sqrt{\mu}\right)\mu}}{4\sqrt{2}\sqrt{-\mu^2+\mu+2}}. \end{aligned} \tag{14}$$

where the second solution in equation (14) is derived consistently with the Den Hartog's approach[1] and differs from the formulas provided in literature by Warburton [45] and Tsai and Lin [39], who all adopt alternative optimization procedures.

[1]Note that Connor [2], starting from the Den Hartog's approach, also gives an analytical expression for the optimal damping in case of ground motion, which contains different terms resulting from some approximations employed by the author. However, at least in the range of parameters usually assumed for applications in civil engineering, the numerical differences between the Connor's formula and the one proposed in the present work are not significant.

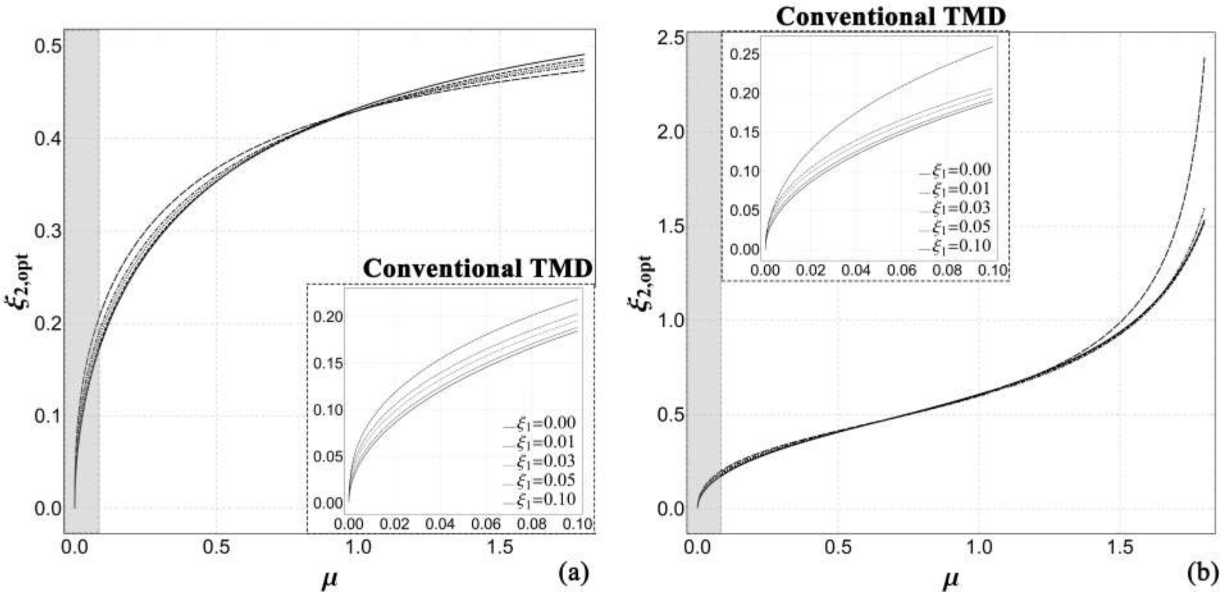

Figure 5: Optimal damping ratio $\xi_{2,opt}$ vs. the mass ratio μ by varying the damping ξ_1 of the main system for harmonic excitation: (a) $F_1(t)$ input, (b) $\ddot{u}_g(t)$ input. In the inset, the conventional TMD range is highlighted.

2.3. Optimization and responses of main system and absorber

The enhanced Den Hartog's theory for the damped main system here proposed leads to a minimization of $|H_{11}|$ and $|H_{1g}|$. In particular, as an example, in Figure 6 the amplification factor of the main system subjected to a force excitation, by considering the optimal parameters previously derived, is plotted as a function of the forced frequency ratio Ω. One can note that the influence of low-moderate damping ξ_1 cannot be assumed negligible, indeed, at increasing of damping factor of the first degree-of-freedom, a reduction of the amplification factor is always registered. Furthermore, as it emerges from the Figure 6, one can observe that increasing mass ratios lead to a greater robustness of the system, for which the optimal parameters and the damping factor ξ_1 do not significantly affect the amplification factor of the main structure.

With respect to the response of the absorber, designed according to the optimal parameters derived from the proposed procedure, it represents a crucial aspect to focus on. In fact, in non-conventional TMDs with large mass ratios, such as the cases of Intermediate Isolation Systems [8–10] and Mega-Sub Controlled configurations [12], the absorber constitutes a part of the building structure, this implying that the behaviour of the second degree of freedom has to be evaluated when the amplification factor of the first degree of freedom is minimized. More in detail, the displacement amplitude of the absorber should be assessed with reference to specific limit values, according to safety and/or serviceability performance requirements.

For conventional TMDs with low mass ratios, the study of the amplification factor of the second degree of freedom is not generally taken into account. Exceptions are the contributions by Feng and Mita [12], referring to Mega–Sub configurations (MSC), Reggio and De Angelis, referring to IISs [31], and Yahyai et al. [47]. In particular, Feng and Mita [12] propose two separate optimization procedures: the first one requires the minimization of the amplification factor of the first degree of freedom –the main mega-structure– subjected to wind excitation or seismic input, whereas the second one derives the optimal parameters by minimizing the acceleration of the absorber, which is the secondary sub-structure. Thus, a joint procedure apt to simultaneously optimize the response of both mega- and the sub-structure would represent a substantial improvement in order to achieve optimal configurations [12]. The algorithm by Yahyai et al. [47] provides instead optimum configurations also taking into account the amplitude of the absorber's acceleration. However, the displacement amplitude of the second degree of freedom is not analysed. Reggio and De Angelis [4, 31], consider a 5-DOF model and, by means of a numerical investigation, they evaluate the responses in terms of displacements, inter-story drifts, accelerations and shear forces of three configurations of IIS systems with different

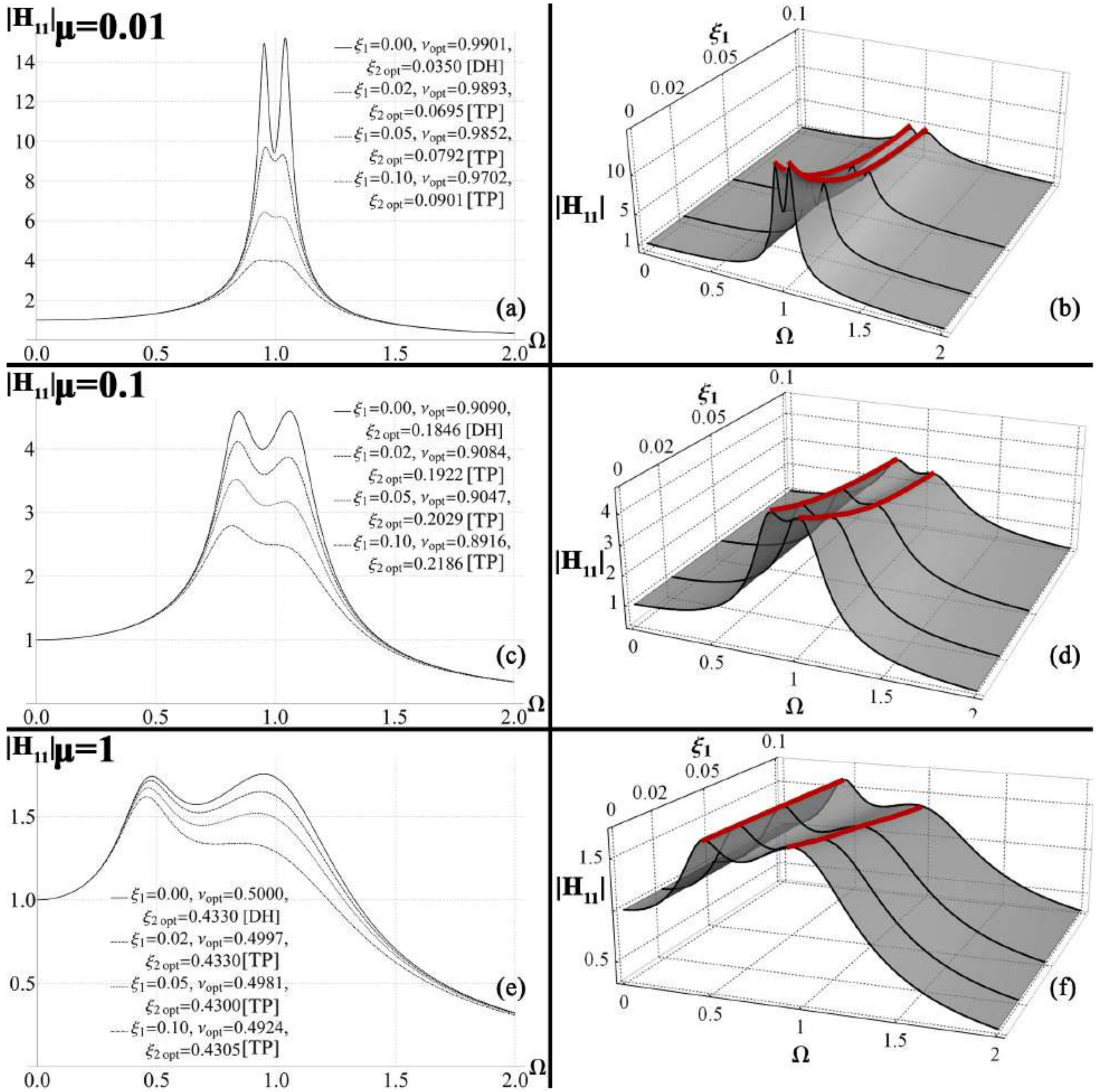

Figure 6: Comparison among the optimized amplification factors of the main system $|H_{11}|$ evaluated for the undamped case with the damped cases proposed by this paper (TP), for different mass ratios μ.

mass ratios and of a fixed-base configuration building taken as term of comparison with the same number of floors. It is worth recalling that Reggio and De Angelis achieve optimum parameters by maximizing an energy performance index (EDI) that contemporary accounts for the reduction of the seismic response of both the substructure and the isolated superstructure. Nonetheless, the incidence of the optimum parameters on the second degree of freedom, by fixing the mass ratio and varying the damping ratio of the absorber starting from the optimum one, is not investigated. By keeping in mind all these aspects about the second degree of freedom in TMD problems, first an optimization procedure based on the minimization of the displacement amplitude of the main system has been here implemented with the aim of maximizing the effectiveness of the mass damping and then the response of the absorber in terms of displacement is controlled. In order to analyse the effect of the optimization procedure on the absorber, as an example, the function $|H_{21}|$, evaluated according to the second expression of (7), is plotted in Figure 7 with respect to the optimal parameters ν_{opt} and $\xi_{2,opt}$, by varying the damping ratio ξ_1 and by considering different mass ratios μ. As shown in Figure 7, a reduction of the amplification factor of the second degree of freedom also occurs by setting the optimal parameters and by increasing the damping ξ_1 if compared to the Den Hartog's undamped case, although the minimization of the amplification factor of main structure is only required and no constraints on the static displacement of the absorber are imposed. Analogous considerations can be made for the function $|H_{2g}|$.

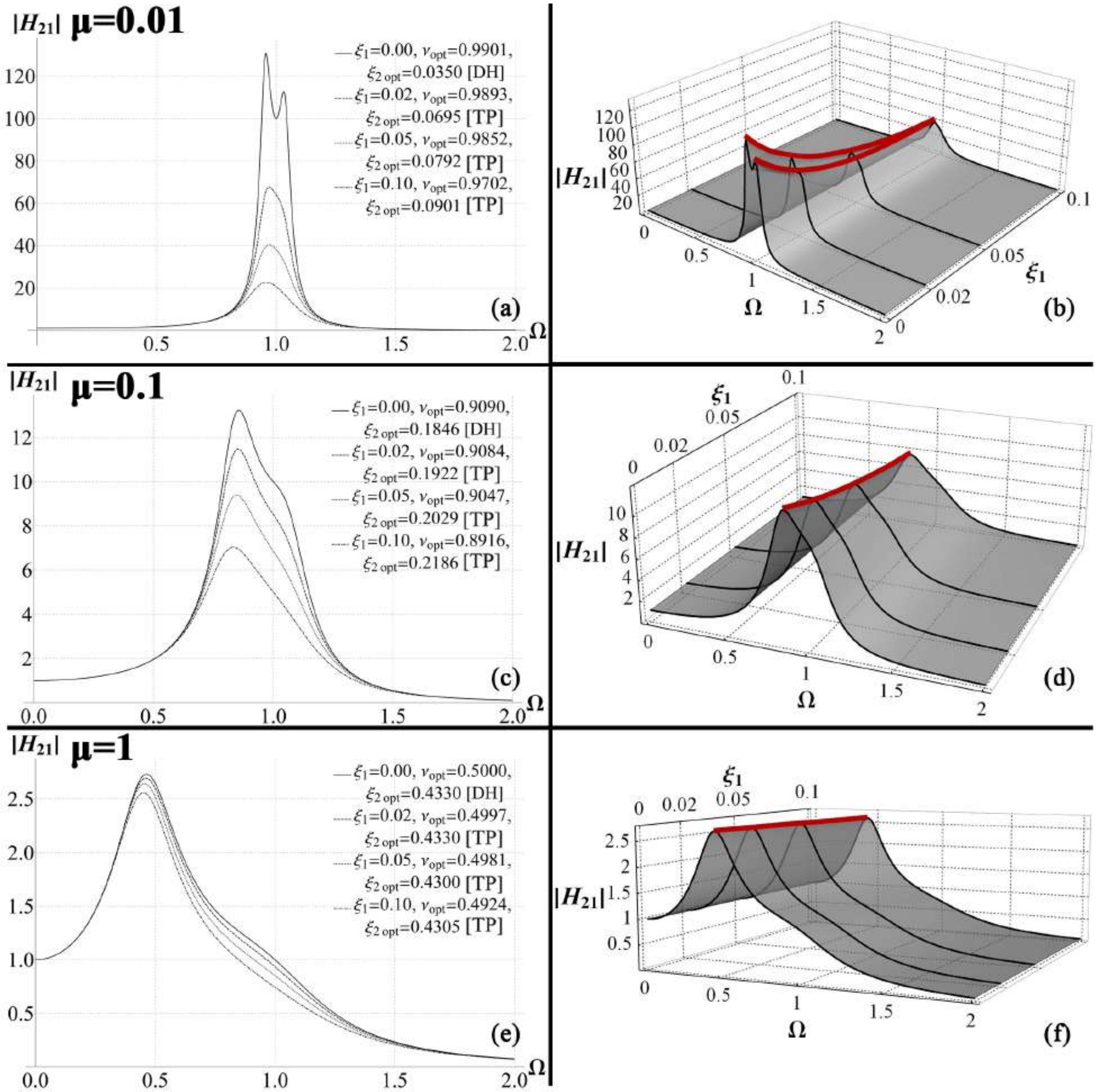

Figure 7: Comparison among the optimized amplification factors of the secondary system $|H_{21}|$ evaluated for Den Hartog's case (DH) and the damped cases proposed by this paper (TP), for different mass ratios μ.

3. Optimal design and rule-of-thumb formulas

The analytical solution of the 4th degree equation (13) here proposed has been calculated by means of the symbolic computational software Mathematica© [17], readily deriving the expression of optimal damping, which is not explicitly reported for sake of brevity. More manageable functions that well fit the analytical solutions of equation (13) for both the excitation $F_1(t)$ and $\ddot{u}_g(t)$ are however given in the following for deriving the optimal damping as a function of μ and ξ_1:

$$\xi_{2,opt(F_1)} = \sqrt{\frac{3\mu\left(1.1043\xi_1^{0.726891}+1\right)}{8\left(\mu\left(2.21626\xi_1^{0.703942}+1\right)+1\right)}},$$
$$\xi_{2,opt(\ddot{u}_g)} = \frac{\sqrt{\sqrt{36-2\mu}+6}\sqrt{1-1.11457\sqrt{\xi_1}}\sqrt{\mu\left(\frac{4.27658\xi_1}{\mu^{0.470186}}+1\right)}}{4\sqrt{\mu+1}\sqrt{-0.629905\mu\xi_1-\mu+2}}, \tag{15}$$

where the coefficients have been chosen on the basis of a standard error minimization procedure in the physically reasonable ranges of variability indicated in **section 2.1**, with a maximum error of 10% in the interval of interest. The expressions (15), together with the well-established equation (13) with coefficients reported explicitly in (23) and (24), can be used as design formulas for TMDs characterised by arbitrary mass ratios and any physically reasonable value (for instance with reference to civil engineering applications) of damping ratio of the main system. In fact, the optimum parameters provided in the major part of the literature, are usually reported either by means of numerical tables or through graphical representations, and are frequently referred to a restricted range of mass ratios (generally taking μ lower than 1). In some cases, curve-fitting formulations are derived from numerical analyses, thus restricting their validity to the considered range of values. Moreover, while many literature works assume discrete values of damping ratio ξ_1, only few papers account for a continuous variation of the damping of the main structure from 0 to 0.10, such as the ones by Asami et al. [1], Reggio and De Angelis [4, 31] and Fang et al. [11]. As above-mentioned, the first ones derive a series solution based on the Den Hartog's approach, Reggio and De Angelis present their outcomes by means of graphical representation without reporting analytical expressions and Fang et al. provide a closed-form solution, although their optimization procedure is based on the frequency analysis of the TMD and not directly on the minimization of the amplification factor. In the present paper, by taking advantage from the consistency of a fully analytical strategy, a unique formulation is given for each considered excitation, which allows to calculate optimal damping ratio of the absorber for any μ and for ξ_1 covering the vast majority of the actual values to be employed for real buildings and civil engineering applications.

4. Synoptic comparison between literature procedures and proposed analytical method for harmonic inputs

In order to compare the proposed procedure with the main literature formulations, several numerical tests have been carried out by varying the damping factor ξ_1 and the mass ratio μ of the simplified 2-DOF model (Figure 1). As already discussed in **section 3**, the literature works considered in this study can be somehow categorized on the basis of the adopted criteria, i.e. the minimization of the displacement amplitude of the main system carried out by means of analytical or numerical approaches [1, 5, 12, 14, 15, 18, 26, 29, 30, 33, 34, 39, 42, 45, 47], the frequency analyses by Fang et al. [11] and Krenk [21] and those ones deriving optimum parameters by maximizing the Energy Dissipation Index (EDI) [4, 31]. However, among the works following the first strategy, the comparison is here performed only with the ones that consider the minimization of the static displacement of the first degree-of-freedom subjected to a harmonic input (see the synoptic comparison reported in Table (1). Some of these papers refer to the not-damped main system case, except for Ioi and Ikeda [18], Randall et al. [30], Tsai and Lin [39], Rana and Soong [29], Asami et al. [1], Ghosh and Basu [14], Salvi and Rizzi [34] and Fang et al. [11]. In particular, Den Hartog [5] derives a closed-form solution only for systems with null ξ_1 subjected to harmonic input. Even if its theory is provided for TMD with very low mass ratio and low-damped absorber, no limit on the variation of mass ratio and damping ratio ξ_2 is imposed. Warburton [45] analyses the 2-DOF system subjected to force and motion excitation modelled as harmonic

Table 1
Comparison between the optimization procedures provided in literature and the proposed strategy by adopting harmonic input([5] p1; [45] p2; [18] p3; [30] p4; [39] p5; [29] p6; [21] p7; [1] p8; [14] p9; [34] p10; [11] p11; This Paper p12).

Paper	Par. Range μ	ξ_1	Input	ν_{opt}	$\xi_{2,opt}$
p1	$\forall\mu$	0.00	$\mathbf{F_1(t)}$	closed-form	closed-form
	-	-	$\mathbf{\ddot{u}_g(t)}$	-	-
p2	$\forall\mu$	0.00	$\mathbf{F_1(t)}$	closed-form	closed-form
	]0,2[	0.00	$\mathbf{\ddot{u}_g(t)}$	closed-form	closed-form
p3	$\forall\mu$	$\forall\xi 1$	$\mathbf{F_1(t)}$	numerical procedure	numerical procedure
	-	-	$\mathbf{\ddot{u}_g(t)}$	-	-
p4	]0,0.40[	[0,0.5]	$\mathbf{F_1(t)}$	numerical procedure	numerical procedure
	-	-	$\mathbf{\ddot{u}_g(t)}$	-	-
p5	$\forall\mu$	$\forall\xi 1$	$\mathbf{F_1(t)}$	numerical procedure	numerical procedure
	]0,2[	$]0,\frac{\sqrt{2}}{2}]$	$\mathbf{\ddot{u}_g(t)}$	numerical procedure	numerical procedure
p6	0.02, 0.06, 0.10	]0,0.1]	$\mathbf{F_1(t)}$	numerical procedure	numerical procedure
	0.02, 0.06, 0.10	]0,0.1]	$\mathbf{\ddot{u}_g(t)}$	numerical procedure	numerical procedure
p7	$\forall\mu$	0.00	$\mathbf{F_1(t)}$	closed-form	closed-form
	-	-	$\mathbf{\ddot{u}_g(t)}$	-	-
p8	]0,1]	[0,0.30]	$\mathbf{F_1(t)}$	series solution	series solution
	]0,1]	[0,0.30]	$\mathbf{\ddot{u}_g(t)}$	series solution	series solution
p9	$\forall\mu$	]0,0.1]	$\mathbf{F_1(t)}$	closed-form	-
	-	-	$\mathbf{\ddot{u}_g(t)}$	-	-
p10	-	-	$\mathbf{F_1(t)}$	-	-
	]0,0.1]	0.05	$\mathbf{\ddot{u}_g(t)}$	numerical procedure	numerical procedure
p11	$\forall\mu$	$\forall\xi_1$	$\mathbf{F_1(t)}$	closed-form	closed-form
	-	-	$\mathbf{\ddot{u}_g(t)}$	-	-
p12	$\forall\mu$	[0,0.1]	$\mathbf{F_1(t)}$	closed-form	closed-form
	]0,2]	[0,0.1]	$\mathbf{\ddot{u}_g(t)}$	closed-form	closed-form

inputs. By following the Den Hartog's method and neglecting the damping in the main structure, optimal parameters are provided in closed-form also for the motion excitation case. In [18, 29, 30, 34, 39] ν_{opt} and $\xi_{2,opt}$ are obtained by utilizing a curve-fitting formulation and numerical procedures, optimal parameters being finally provided by means of design tables and graphs. Moreover, while Asami et al. [1] proposed series solutions for both the two excitation types, Krenk [21] and Fang et al. [11] derived closed-form optimal formulations starting from a frequency analysis of the TMD. However, although the analytical formulation proposed by Fang et al. [11] considers large ranges of mass ratio and damping ratio ξ_1, it is valid only in the forced case, no optimal solution being presented for the motion excitation. Also referring to the F_1 input, it is worth recalling that Ghosh and Basu [14] derive only ν_{opt} in closed-form for any mass ratio, without taking into account the role of optimal damping. Therefore, as it emerges from the comparative analysis of the literature, the major part of the optimization procedures is focussed only on a specific type of input or otherwise limited to study cases in which ξ_1 is null. As a matter of fact, if damping of the main system is included in the models and both the two types of excitations are considered, only numerical or approximate solutions can be found in the literature [1, 29, 39], the proposed analytical formulation thus becoming an effective tool for design purposes

5. Some insights on optimal TMDs by adopting a stochastic approach for the ground motion input

In the previous sections, the optimization of TMD has been provided by modelling both the two excitations, i.e. $F_1(t)$ and $\ddot{u}_g(t)$, as deterministic inputs. However, it is well-known that a ground motion is often considered as a stochastic process and generally modelled as a Gaussian random process, with a zero mean, described by a constant power spectral density (PSD), i.e. a white noise [4, 22, 31, 45]. Despite these processes neglect the dependency on the excitation frequency content, they are considered as acceptable for deriving optimal parameters in an initial design phase [4]. For more faithfully grasping the seismic excitation, Kanai-Tajimi model [19, 37] can be for instance employed for filtering the white noise ground acceleration with the actual stiffness and damping characteristics of the soil deposit, thus leading to a non-uniform PSD of the ground motion signal. With the aim to explore how the optimization procedure presented above can be adapted to consider the frequency content in $\ddot{u}_g$, the Den Hartog's problem can be reformulated by requiring to minimize the following integral:

$$\int_{\Omega_1}^{\Omega_2} \phi(\Omega) \left| H_{1g}(\Omega) \right|^2 d\Omega, \tag{16}$$

where $\phi(\Omega)$ is a weight function that describes the frequency distribution of the seismic excitation and $|H_{1g}(\Omega)|$ is the amplification factor of the main structure (8). The design variables of the minimization procedure are again the tuning ratio ν and the damping ratio of the absorber ξ_2, while μ and ξ_1 are the varying parameters. When $\phi(\Omega)$ is a unitary constant function and Ω_1 and Ω_2 tend respectively to $-\infty$ e $+\infty$, the provided optimization method degenerates into the minimization of the variance of the displacement of the primary system subjected to a white noise input [3]. By then considering a unitary weight function $\phi(\Omega)$, the indefinite integral in (16) can be found in explicit form and written in the following compact way:

$$\int |H_{1g}(\Omega)| d\Omega = -\sum_{i=1}^{4} \left\{ \frac{\left[G_i \left(a_1 + G_i\right) + a_0\right] \tanh^{-1}\left(\sqrt{\frac{\Omega}{G_i}}\right)}{\sqrt{G_i} \prod_{j=1}^{4} \left[G_i - \left(1 - \delta_{ij}\right) G_j\right]} \right\} \tag{17}$$

where $G_{i,j}$ are the four poles of the amplification factor $|H_{1g}|$ (8), δ_{ij} is the Kronecker delta and a_i have the following expressions:

$$\begin{aligned} a_0 &= \nu^4 (1+\mu)^2; \\ a_1 &= 2\nu^2 (1+\mu) \left(2(1+\mu)\xi_2{}^2 - 1\right). \end{aligned} \tag{18}$$

By imposing that the design variables assume only non-negative values, thanks to the analytical expression (17), standard numerical minimization algorithms –here implemented in the Mathematica© [17] environment– can be applied to provide the optimal parameters ν_{opt} and $\xi_{2,opt}$ as functions of μ and ξ_1. Figure 3 shows that the range of the dimensionless frequency ratios Ω, where the two peaks of the amplification factor $|H_{1g}|$ occur, varies between $\Omega = 0.25$ and $\Omega = 1.5$, the results of the numerical optimization strategy in this frequency interval being illustrated in Figure 8.

Note that, when the whole frequency range is considered, i.e. $\Omega_1 \to -\infty$ and $\Omega_2 \to +\infty$, the optimization procedure consistently falls into the minimization of the root mean square of the displacement of the main structure subjected to a white noise signal with unitary PSD, as shown in Figure 9.

The comparison among results in Figures 8 and 9 with those shown in Figures 4b and 5b also highlights that optimal parameters are significantly influenced by the approach adopted for optimizing TMD systems when subjected to ground accelerations.

6. Conclusions

The increasing need to improve safety standards of existing buildings and enhance the performance of modern civil structures is stimulating a vivid debate on how to conceive and design systems to integrate with these constructions and capable to control their dynamic response under different types of actions. To this aim, growing attention has been paid in the last years for optimizing TMD configurations, the vast majority of the proposed solutions being based

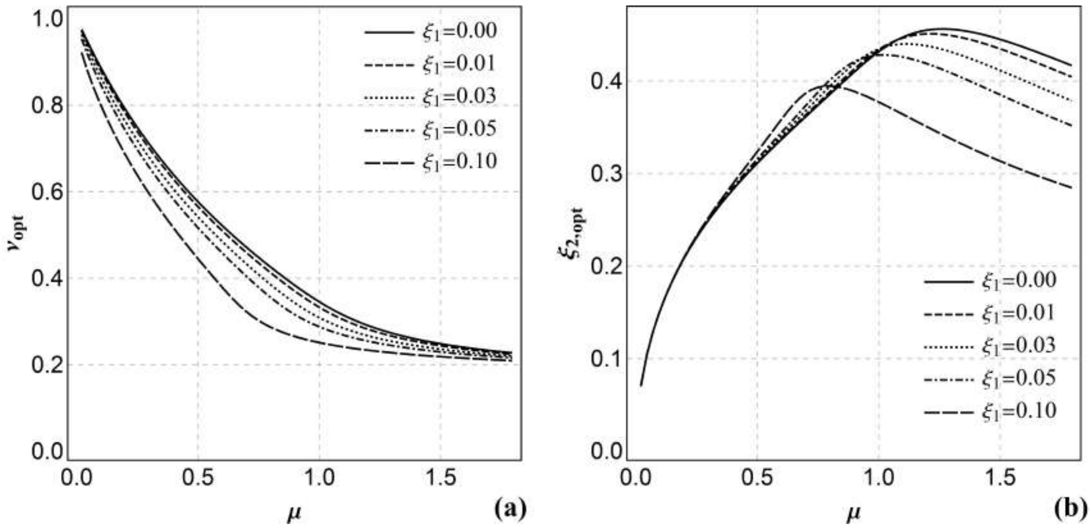

Figure 8: Results of the minimization of the definite integral (16) with $\Omega_1 = 0.25$ and $\Omega_2 = 1.5$: optimal tuning ratio ν_{opt} (a) and optimal damping ratio $\xi_{2,opt}$ (b) vs. the mass ratio μ by varying the damping ξ_1 of the main system.

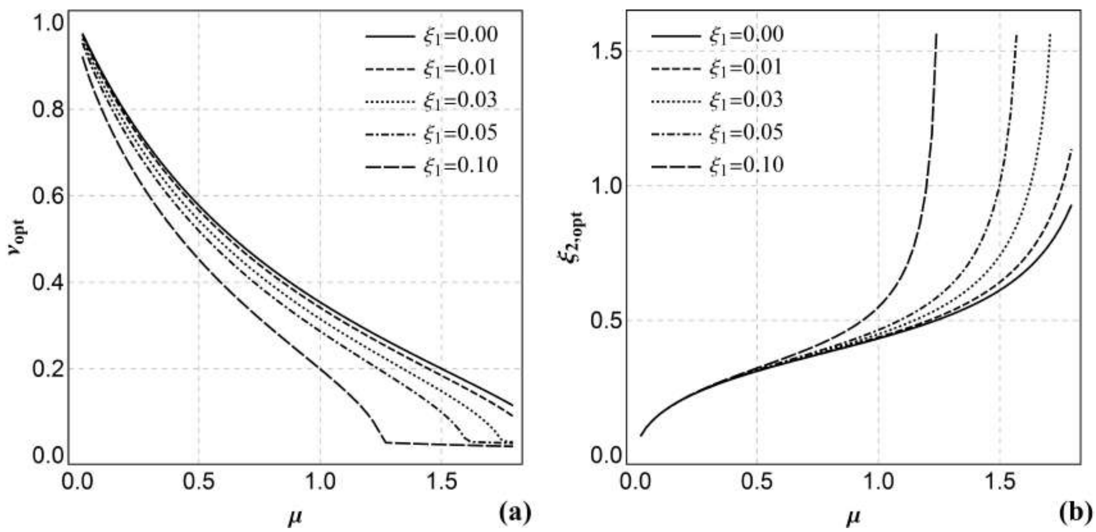

Figure 9: Results of the minimization of the definite integral (16) with $\Omega_1 \to -\infty$ and $\Omega_2 \to +\infty$: optimal tuning ratio ν_{opt} (a) and optimal damping ratio $\xi_{2,opt}$ (b) vs. the mass ratio μ by varying the damping ξ_1 of the main system.

on the pioneering strategy suggested by Den Hartog in 1947. Despite a large number of models implementing this strategy, their use is partially still limited by some difficulties, for example the possibility to have at one disposal formulas to predict optimal setting of TMD parameters in cases in which the damping ratio of both primary and absorber systems have to be taken into account or both force and motion excitations can expected. To contribute in overcoming these limitations and give insights into optimized design of tuned mass dampers in presence of damping of the primary system, in the present work we generalized the Den Hartog's method extending it to the optimization of low-moderate damped structures and non-conventional damped TMDs, also including ground motion inputs. This purpose is achieved by considering an effective 2-DOF model and thus providing a fully analytical strategy for evaluating optimal parameters, finally obtaining closed-form solutions that give optima in terms of tuning frequency and damping, for a wide range of mass ratios values of possible interest in civil engineering applications. This analytical strategy, which avoids the use of fully numerical procedures that in some cases could risk to obscure the role of the design variables in the TMD optimization process, can be in fact seen as design tool, as benchmark for other numerical procedures or as an alternative method to ones based on the root locus analysis of the TMD first proposed by Thompson [38], then by Krenk [21] and finally enhanced by Fang et al. in 2019 for the damped main system case [11]. Furthermore, the proposed generalized Den Hartog approach allows to analytically derive optimal parameters also for the motion excitation modelled as a deterministic input, which is not considered in the above-mentioned papers [11, 21, 38]. As ancillary result, it is in fact shown that the method also allows to trace back optimal findings obtained through both analytical and numerical strategies by other scholars. The robustness of the proposed approach has been in particular demonstrated by providing several numerical examples that are designed according to the optimum parameters, starting from the object function of minimizing the amplification factor of the main system. Relevance is also given to the displacement amplitude of the second degree of freedom since involving high values of mass ratios implies to face important problems related to absorbers that usually constitute a large part of a structure, as in the cases of IIS buildings and MSC configurations for tall buildings.

At the end, the validation of the proposed approach is also provided through the direct comparison with other optimization procedures, showing that the optimal parameters theoretically found by means of the present approach are capable to trace back all the configurations obtained in the literature and large ranges of variation of parameters for both types of inputs.

Additionally, by utilizing a stochastic approach for modelling seismic excitations, a further analytical optimization strategy was implemented in order to show how optimal parameters can be strongly affected by the frequency content of the ground acceleration input.

It must be finally underlined that, by recovering and extending the Den Hartog methodology, a linear model was considered for both the two degrees of freedom of the main system and the absorber. Although non-linear tuned mass dampers are currently investigated and exploited, the authors believe that the analysis of the linear behaviour and the use of simple (analytical) expressions of the optimal parameters are still indispensable tools for sizing the structures at the initial design phase and for predicting key features of their dynamic response.

Acknowledgments

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7. Appendix A.

The expressions of the amplification factors $|H_{11}|$ and $|H_{1g}|$ at the two extreme values, $\xi_2 \to 0$ and $\xi_2 \to \infty$ are here provided:

$$|H_{11,\xi_{2-1}}| = |H_{11}(\xi_2 \to 0)| = \sqrt{\frac{\left(\upsilon^2 - \Omega^2\right)^2}{4\Omega^2\left(\upsilon^2\xi_1 - \xi_1\Omega^2\right)^2 + \left[\upsilon^2 - \left[1 + \upsilon^2\left(1+\mu\right)\right]\Omega^2 + \Omega^4\right]^2}}, \tag{19}$$

$$|H_{11,\xi_{2-2}}| = |H_{11}(\xi_2 \to \infty)| = \sqrt{\frac{1}{1 - 2\left(1 + \mu - 2{\xi_1}^2\right)\Omega^2 + (1+\mu)^2\Omega^4}}, \tag{20}$$

$$|H_{1g,\xi_{2-1}}| = |H_{1g}(\xi_2 \to 0)| = \sqrt{\frac{\left(\Omega^2 - (\mu+1)\nu^2\right)^2}{\left(-\Omega^2\left((\mu+1)\nu^2 + 1\right) + \nu^2 + \Omega^4\right)^2 + 4\xi_1^2\Omega^2\left(\Omega^2 - \nu^2\right)^2}}, \tag{21}$$

$$|H_{1g,\xi_{2-2}}| = |H_{11}(\xi_2 \to \infty)| = \sqrt{\frac{(\mu+1)^2}{\left((\mu+1)\Omega^2-1\right)^2+4\xi_1^2\Omega^2}}. \tag{22}$$

8. Appendix B.

The coefficients of Equation (13) for F_1 input particularize as:

$$\begin{aligned}
b_0 &= -2\Omega\left(\nu^2-\Omega^2\right)[-\Omega^6\left(-1+2\xi_1{}^2+\Omega^2\right)+\\
&\nu^4\Omega^2\left(3+\mu-6\xi_1{}^2-3\Omega^2-3\mu\Omega^2\right)+\nu^2\Omega^4\left(-3+6\xi_1{}^2+(3+\mu)\,\Omega^2\right)+\\
&+\nu^6\left[-1+2\xi_1{}^2+\Omega^2+\mu^2\Omega^2+\mu\left(-1+2\Omega^2\right)\right]];\\
b_1 &= -8\nu\mu\xi_1\Omega^5\left(3\nu^4-4\nu^2\Omega^2+\Omega^4\right);\\
b_2 &= 4\nu^2\Omega^3[2\nu^2\Omega^2\left(-4-3\mu+8\xi_1{}^2+4\Omega^2+6\mu\Omega^2+2\mu^2\Omega^2\right)+\\
&-\Omega^4\left[-4+8\xi_1{}^2+\left(4+2\mu+\mu^2\right)\Omega^2\right]+\\
&-4\nu^4\left[-1+2\xi_1{}^2+\Omega^2+\mu^2\Omega^2+\mu\left(-1+2\Omega^2\right)\right]];\\
b_3 &= -64\nu^3\mu\xi_1\Omega^7;\\
b_4 &= 32\nu^4\Omega^5\left[1+\mu-2\xi_1{}^2-(1+\mu)^2\Omega^2\right],
\end{aligned} \tag{23}$$

while the coefficients for $\ddot{u}_g$ input are:

$$\begin{aligned}
b_0 &= -4\Omega\left(\nu^2(1+\mu)-\Omega^2\right)(-\Omega^6\left(-1+2\xi_1{}^2+\Omega^2\right)+\\
&+\nu^2\Omega^2\left(\mu+\mu\left(-3+6\xi_1{}^2\right)\Omega^2+3\mu\Omega^4+3\Omega^2\left(-1+2\xi_1{}^2+\Omega^2\right)\right)+\\
&+\nu^6(1+\mu)\left(-1+2\xi_1{}^2+\Omega^2+\mu^2\Omega^2+\mu\left(-1+2\Omega^2\right)\right)+\\
&-\nu^4\left(\mu+\mu(-5+8\xi_1{}^2)\,\Omega^2+6\mu\Omega^4+3\Omega^2\left(-1+2\xi_1{}^2+\Omega^2\right)+\right.\\
&\mu^2\Omega^2\left(-2+3\Omega^2\right)));\\
b_1 &= -16\nu\mu\xi_1\Omega^5\left(3\nu^4(1+\mu)^2-4\nu^2(1+\mu)\,\Omega^2+\Omega^4\right);\\
b_2 &= -32\nu^6(1+\mu)^2\Omega^3\left(-1+2\xi_1{}^2+\Omega^2+\mu^2\Omega^2+\mu\left(-1+2\Omega^2\right)\right)+\\
&+16\nu^4(1+\mu)\,\Omega^5(4\left(-1+2\xi_1{}^2+\Omega^2\right)+\mu^2\left(-1+4\Omega^2\right)+\\
&\mu\left(-5+4\xi_1{}^2+8\Omega^2\right))+\\
&-8\nu^2\Omega^5(4\Omega^2(-1+2\xi_1{}^2+\Omega^2)+2\mu(1+(-4+8\xi_1{}^2)\Omega^2+4\Omega^4)+\\
&\mu^2(1+(-4+8\xi_1{}^2)\Omega^2+4\Omega^4));\\
b_3 &= -128\nu^3\mu(1+\mu)^2\xi_1\Omega^7;\\
b_4 &= 64\nu^4(1+\mu)^2\Omega^5\left(1+\mu-2\xi_1{}^2-(1+\mu)^2\Omega^2\right).
\end{aligned} \tag{24}$$

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