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Research Paper

Equivalent conjectures on blowing-ups of $\mathbb{P}^2$ ☆

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We provide a characterization of asymptotical speciality of a nef and big divisor D on an algebraic surface in terms of the arithmetic genus of curves in $D^\perp$. As a consequence we prove that the SHGH conjecture for linear systems on the blowing-up of the projective plane at points in very general position is equivalent to the fact that each nef class is non-special. Finally we prove that if $r < 2^n$ then any nef divisor of the blowing-up of the n-dimensional projective space at r points in very general position is asymptotically non-special.

0. Introduction

Let $\pi\colon X_r^n \to \mathbb{P}^n$ be the blowing-up of the projective space at r points in very general position with exceptional divisors $E_1, \dots, E_r$ and let H be the pullback of a hyperplane. The *virtual dimension* of a divisor $D = dH - \sum_{i=1}^r m_i E_i$ is

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$$v(D) := \binom{d+n}{n} - \sum_{i=1}^{r} \binom{m_i+n-1}{n} - 1$$

and its *expected dimension* is $e(D) := \max\{v(D), -1\}$. The divisor D is *non-special* if $\dim |D| = e(D)$ or equivalently if $h^0(X_r^n, \mathcal{O}(D)) \cdot h^1(X_r^n, \mathcal{O}(D)) = 0$. When $n = 2$ there are three well known conjectures about divisors:

(1) an effective divisor D which has intersection product at least -1 with any (-1)-curve of X_r^2 is *non-special*;
(2) the only negative curves of X_r^2 are (-1)-curves;
(3) the divisor class $\sqrt{r}H - \sum_{i=1}^{r} E_i$ is nef.

The first one is known as the Segre–Harbourne–Gimigliano–Hirschowitz (SHGH) Conjecture, the second statement is known as the "(-1)-Curves Conjecture" or the "Weak SHGH" Conjecture, while the third statement is the famous Nagata Conjecture. It is known that (1) ⇒ (2) ⇒ (3) (see [2]) and that the first conjecture admits other equivalent formulations (see [3]). Our first result is the following.

Theorem 1. *Let r be a positive integer. Then the following are equivalent.*

(1) *The only negative curves of X_r^2 are (-1)-curves.*
(2) *X_r^2 does not contain nef and big divisors which are asymptotically special.*

In order to prove the theorem we show that, on a surface, curves orthogonal to a nef and big divisor that is not asymptotically special have arithmetic genus at most one. This result can be used to bound the Mori cone of the surface in some cases. As a consequence of this result we can prove the following equivalence.

Theorem 2. *Let r be a positive integer. Then the following are equivalent.*

(1) *The SHGH conjecture holds on X_r^2.*
(2) *Each nef class of X_r^2 is non-special.*

The knowledge of the nef cone of X_r^2 is thus of a fundamental importance for all the above conjectures. Recent results about its shape are given in [7,2,4]. We proved similar results to those in [4] in an effort to understand the nef cone in the special case of X_{10}^2, see Remark 2.6. Finally we discuss the higher dimensional case proving the following.

Theorem 3. *If $r < 2^n$ then any nef divisor of X_r^n is asymptotically non-special.*

The paper is organized as follows. Section 1 is devoted to the definition and first properties of asymptotically special nef and big divisors on surfaces. In Section 2 we prove Theorem 1 and Theorem 2, while in Section 3 we prove Theorem 3.

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1. Asymptotically special divisors

Let X be a smooth projective variety and let D be a divisor of X. We say that D is *asymptotically special* if

$$h^1(X, \mathcal{O}_X(nD)) > 0, \quad \text{for } n \gg 0.$$

Similarly we say that D is *asymptotically non-special* if $h^1(X, \mathcal{O}_X(nD)) = 0$ for $n \gg 0$. If X is a surface and $D^2 > 0$ then the quadratic function $D^\perp \to \mathbb{Q}$, defined by $E \mapsto p_a(E) := \frac{1}{2}(E^2+E\cdot K_X)+1$ is bounded from above. Indeed, by the Hodge index theorem, the intersection form on $D^\perp$ is negative definite, so that the function p_a is concave. Thus the following number is well defined

$$p_a(D^\perp) := \begin{cases} 0 & \text{if } D \text{ is ample} \\ \max\{p_a(E) \,:\, E \neq 0 \text{ effective and } E \cdot D = 0\} & \text{otherwise.} \end{cases}$$

The main result of this section is the following.

Proposition 1.1. *Let X be a smooth projective surface and let D be a nef and big Cartier divisor of X.*

- *If $p_a(D^\perp) = 0$, then D is asymptotically non-special.*
- *If $p_a(D^\perp) = 1$, $D \geq 0$ and $\mathcal{O}_X(D)|_C \simeq \mathcal{O}_C$ for an irreducible curve C with $p_a(C) = 1$, then D is asymptotically special.*
- *If $p_a(D^\perp) \geq 2$, then D is asymptotically special.*

We postpone the proof of the proposition until the end of this section and meanwhile we discuss some of its consequences.

Example 1.2. In case $p_a(D^\perp) = 1$ the divisor D can be neither asymptotically special nor asymptotically non-special. As an example, consider the blow-up X of $\mathbb{P}^2$ at 10 points $p_1 \ldots, p_{10}$ on a smooth plane cubic. Denote by H the pull back of a line, by E_i the exceptional divisor over p_i and let C be the strict transform of the given cubic (i.e. C is the unique element in $|-K_X|$). We consider a divisor $D := 10H - \sum_{i=1}^{10} 3E_i$ which restricts to a 2-torsion class on C and we claim that

$$h^1(X, \mathcal{O}_X(nD)) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

First of all observe that by [8, Thm. III.1], C is contained in the base locus of nD if and only if the restriction of nD to C is non-trivial, i.e. if and only if n is odd. If this is the case, $nD - C$ is effective and satisfies $(nD - C) \cdot (-K_X) = -C^2 = 1$, so that $h^1(X, \mathcal{O}_X(nD)) = 0$ by [8, Thm. III.1]. On the other hand, if n is even, the restriction of nD to C is trivial, and this implies that C is not contained in the base locus of nD. Therefore nD has no fixed part and by [8, Thm. III.1] we conclude that $h^1(X, \mathcal{O}_X(nD)) = 1$.

Corollary 1.3. *Let X be a smooth projective surface and let $C \subseteq X$ be an irreducible and reduced curve with $C^2 < 0$ and $p_a(C) > 1$. Then there exists a nef and big divisor D on X such that $D \cdot C = 0$ and D is asymptotically special.*

Proof. Let A be an ample divisor on X. Set $\lambda = \frac{C^2}{A \cdot C}$ and define

$$D := C - \lambda A.$$

We check that D is nef: by construction $D \cdot C = 0$, and for any other curve $E \neq C$ we have $D \cdot E = C \cdot E - \lambda A \cdot E > 0$, since A is ample and $\lambda < 0$. Moreover

$$D^2 = (C - \lambda A)^2 = C^2 - 2\lambda(C \cdot A) + \lambda^2 A^2 = -C^2 + \lambda^2 A^2 > 0,$$

so D is big. Finally, the orthogonal complement $D^\perp$ in the intersection pairing contains C. Hence $p_a(D^\perp) \geq p_a(C) > 1$. By Proposition 1.1, this implies that D is asymptotically special. □

In order to prove Proposition 1.1 we need some preliminary lemmas. We begin with the following result which is well known in the literature.

Lemma 1.4. *Let X be a smooth projective surface and let $C_1, \ldots, C_n \subseteq X$ be distinct irreducible and reduced curves whose intersection matrix is negative definite. Then the classes $[C_1], \ldots, [C_n]$ are linearly independent in $N^1(X)$. In particular if D is a nef and big divisor of X, then $D^\perp$ contains finitely many classes of irreducible curves.*

Proof. Let $D := \sum_i a_i C_i \sim 0$. Write $D = A - B$, where A is the sum over the positive coefficients of D and B over the negative ones. Then $0 = D^2 = A^2 - 2A \cdot B + B^2$, and the fact that the three summands are all non-positive, implies that both A and B are linearly equivalent to 0. Since both A and B are effective divisors and X is complete, it follows that $A = B = 0$, so that $a_i = 0$ for any i.

To prove the second statement, observe that by the Hodge index theorem and the fact that $D^2 > 0$, the intersection form on $D^\perp$ is negative definite, so that the statement follows. □

A version of the following lemma is proved in [9, Thm. 7.2.1] and we include it here for the sake of completeness of argument.

Lemma 1.5. *Let $C_1, \ldots, C_n$ be irreducible curves on a smooth projective surface whose intersection matrix is negative definite. Then there exists an effective divisor $E := \sum_{i=1}^n a_i C_i$ such that $E \cdot C_i < 0$ for each i.*

Proof. Since the intersection matrix is non-singular there exists a divisor $E = \sum_{i=1}^n a_i C_i$ such that $E \cdot C_i < 0$ for any i. Let $E = E_1 - E_2$, where E_1, E_2 are effective divisors with no common support. Then $0 \geq E \cdot E_2 = (E_1 - E_2) \cdot E_2 = E_1 \cdot E_2 - E_2^2 \geq 0$ implies $E_2 = 0$, so that E is effective. □

Lemma 1.6. *Let X be a smooth projective surface and let D be a nef and big Cartier divisor of X which is not ample. Then there exists an effective divisor E in $D^\perp$ such that*

$$H^1(X, \mathcal{O}_X(nD)) \simeq H^1(E, \mathcal{O}_E(nD)),$$

holds for $n \gg 0$.

Proof. By Lemma 1.4 there are only finitely many irreducible and reduced curves $C_1, \ldots, C_s$ in $D^\perp$. By Lemma 1.5 there exists an effective divisor E, supported at these curves, such that $E \cdot C_i < 0$ for any i. Up to replacing E with a positive multiple, we can assume E to be Cartier and $(-E - K_X) \cdot C_i > 0$ for any i. For any $n > 0$, we have the following exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(nD - E) \longrightarrow \mathcal{O}_X(nD) \longrightarrow \mathcal{O}_E(nD) \longrightarrow 0.$$

By the above assumption on E the divisor $nD - E - K_X$ is nef and big for $n \gg 0$. Thus we conclude by passing to cohomology and using the Kawamata-Viehweg vanishing Theorem. □

Proof of Proposition 1.1. If D is ample, then the statement follows from the Serre vanishing theorem [10, Thm. 1.2.6]. Assume now that D is not ample. If $p_a(D^\perp) = 0$ then $h^1(E, \mathcal{O}_E) = 0$ for any effective divisor E in $D^\perp$, by [1, Lem. 3.3]. Thus

$$h^1(E, \mathcal{O}_E(nD)) = 0$$

for any $n > 0$ by [1, Cor. 3.6]. We conclude that D is asymptotically non-special by Lemma 1.6.

Assume now $p_a(D^\perp) = 1$, $D \geq 0$ and $\mathcal{O}_X(D)|_C \simeq \mathcal{O}_C$ for an irreducible curve C with $p_a(C) = 1$. For $n \gg 0$ the natural surjection of sheaves $\mathcal{O}_X(nD) \twoheadrightarrow \mathcal{O}_C$, together with the vanishing of $h^2(\mathcal{O}_X(nD - C))$, gives a surjection $H^1(\mathcal{O}_X(nD)) \twoheadrightarrow H^1(\mathcal{O}_C)$. Since $h^1(\mathcal{O}_C) = 1$, the divisor D is asymptotically special.

Assume now $p_a(D^\perp) > 1$. Let E' be an effective divisor in $D^\perp$ such that $p_a(E') = p_a(D^\perp)$. Since $\deg \mathcal{O}_{E'}(nD) = 0$ it follows that

$$\chi(E', \mathcal{O}_{E'}(nD)) = \chi(E', \mathcal{O}_{E'}) = 1 - p_a(E') < 0,$$

where the first equality is by [12, Tag 0BRZ] and the second is by the definition of arithmetic genus. Since $p_a(E') \geq 2$ we deduce $h^1(E', \mathcal{O}_{E'}(nD)) > 0$. The divisor E in Lemma 1.6 can be chosen so that $E' \leq E$. If we denote by I the ideal sheaf of E' in $\mathcal{O}_E$, we have an exact sequence of sheaves

$$0 \longrightarrow I(nD) \longrightarrow \mathcal{O}_E(nD) \longrightarrow \mathcal{O}_{E'}(nD) \longrightarrow 0.$$

Passing to cohomology and using the fact that we are in dimension one we get the surjection $H^1(E, \mathcal{O}_E(nD)) \to H^1(E', \mathcal{O}_{E'}(nD))$, which implies $h^1(E, \mathcal{O}_E(nD)) > 0$, so that D is asymptotically special by Lemma 1.6. $\square$

2. Proof of Theorems 1 and 2

Recall that $\pi\colon X_r^2 \to \mathbb{P}^2$ is the blowing-up of the projective plane at r points in very general position with exceptional divisors $E_1, \dots, E_r$ and H is the pullback of a hyperplane. We denote by $\mathrm{Eff}(X_r^2)$ the cone of classes of effective divisors of X_r^2, by $\overline{\mathrm{Eff}}(X_r^2)$ its closure in the Euclidean topology and by $\mathrm{Nef}(X_r^2)$ the dual of the latter closed cone with respect to the bilinear form defined by the intersection product. Elements of $\overline{\mathrm{Eff}}(X_r^2)$ are called pseudoeffective classes, while elements of $\mathrm{Nef}(X_r^2)$ are nef classes. Let

$$Q(X_r^2) := \{D \in \mathrm{Pic}(X_r^2)_{\mathbb{R}} \,:\, D^2 \geq 0 \text{ and } D \cdot H \geq 0\}$$

be the *positive light cone* of X_r^2. Recall the following inclusions

$$\mathrm{Nef}(X_r^2) \subseteq Q(X_r^2) \subseteq \overline{\mathrm{Eff}}(X_r^2).$$

Let $W(X_r^2)$ be the subgroup of isometries of $\mathrm{Pic}(X_r^2)$ generated by the reflections

$$D \mapsto D + (D \cdot R)R,$$

where R is one of the *fundamental roots* (classes of self-intersection -2): $E_1 - E_2, \dots, E_{r-1} - E_r, H - E_1 - E_2 - E_3$. It is not difficult to show that any such reflection preserves all the cones defined above. The choice of a distinguished representative for an orbit of $W(X_r^2)$ contained in the effective cone leads to the following definition.

Definition 2.1. A class $D := dH - \sum_{i=1}^r m_i E_i \in \mathrm{Pic}(X_r^2)_{\mathbb{R}}$ is *pseudostandard* if

$$d \geq m_1 + m_2 + m_3 \quad \text{and} \quad m_1 \geq \cdots \geq m_r.$$

If in addition $m_r \geq 0$ then the class D is *standard*.

The first inequality in the definition is equivalent to $D \cdot (H - E_1 - E_2 - E_3) \geq 0$, while the ordering of the multiplicities comes from $D \cdot (E_i - E_{i+1}) \geq 0$. In what follows, with abuse of notation, we will denote by $-K_s$ the class $3H - \sum_{i=1}^{s} E_i$, for any $s \leq r$.

Lemma 2.2. *Let $D \in \operatorname{Pic}(X_r^2)_{\mathbb{R}}$ be a standard class. If $D^2 \leq 0$ and $D \cdot K \leq 0$ then D is a positive multiple of either $H - E_1$ or $-K_9$.*

Proof. Let $D = dH - \sum_{i=1}^{r} m_i E_i$. Since D is standard we have that $d \geq m_1 + m_2 + m_3$ and $m_1 \geq \cdots \geq m_r \geq 0$. The inequalities $D^2 \leq 0$ and $D \cdot K_r \leq 0$ imply $D^2 + m_3 D \cdot K_r \leq 0$, that is

$$
\begin{aligned}
& d(d - 3m_3) - \sum_{i=1}^{r} (m_i^2 - m_3 m_i) \leq 0 \\
\Rightarrow\ & d(d - 3m_3) - (m_1^2 - m_1 m_3) - (m_2^2 - m_2 m_3) \leq \sum_{i=3}^{r} (m_i^2 - m_3 m_i) \leq 0.
\end{aligned}
$$

Since $d \geq m_1 + m_2 + m_3$ the left hand side is greater than or equal to $(m_1 + m_2 + m_3)(m_1 + m_2 + m_3 - 3m_3) - (m_1^2 - m_1 m_3) - (m_2^2 - m_2 m_3) = 2(m_1 m_2 - m_3^2)$, from which we deduce

$$
0 \leq 2(m_1 m_2 - m_3^2) \leq \sum_{i=3}^{r} (m_i^2 - m_3 m_i) \leq 0.
$$

Therefore either $m_2 = \cdots = m_r = 0$ or all the multiplicities are equal. In the first case $D^2 \leq 0$ imply $d^2 \leq m_1^2$, so that $d = m_1$ and D is a multiple of $H - E_1$. In the second case $D = dH - \sum_{i=1}^{r} mE_i$. The inequalities $0 \leq d^2 - 3md = D^2 + mD \cdot K_r \leq 0$ imply $d = 3m$ and $D^2 = D \cdot K_r = 0$, which proves the statement. □

Corollary 2.3. *The surface X_r^2 does not contain (-2)-curves and the classes of (-1)-curves form an orbit for $W(X_r^2)$.*

Proof. Let C be a (-1) or (-2)-curve. Since the action of $W(X_r^2)$ preserves the effective cone, we can assume the class $[C] := dH - \sum_{i=1}^{r} m_i E_i$ to be effective and pseudostandard. By Lemma 2.2 this class cannot be standard. This immediately implies $d = 0$, so that by the irreducibility of C one deduces $m_1 = \cdots = m_{r-1} = 0$ and $m_r = -1$, which proves the statement. □

Remark 2.4. The following result has been proved also by T. de Fernex in [7, Prop. 2.4]. That paper contains other results towards the (-1)-Curves Conjecture: specifically, it proves the conjecture for the blow-up of $\mathbb{P}^2$ at 9 very general points [7, Prop. 2.3] and for curves whose image in $\mathbb{P}^2$ has a singularity of multiplicity 2 at one of the centers of the blow-up [7, Thm. 0.2].

Lemma 2.5. *If C is an irreducible and reduced curve of X_r^2 with $C^2 < 0$ and $p_a(C) = 0$, then C is a (-1)-curve.*

Proof. Since $p_a(C) = 0$, the curve C is smooth and rational. By Corollary 2.3, it suffices to show that C^2 cannot be smaller than -2. Assume, by contradiction, that $C^2 < -2$.

Consider a flat family $\mathcal{X} \to \Delta$ whose general fiber is of type X_r^2 and whose special fiber Y over $0 \in \Delta$ is an anticanonical rational surface containing a smooth irreducible member $E \in |-K_Y|$. The existence of such a degeneration is guaranteed by specializing the r points to lie on a smooth plane cubic.

The smooth rational curve C degenerates to an effective divisor Γ on Y. Let $\pi : \mathcal{C} \to \Delta$ be a flat family with general fiber C and special fiber Γ. Since $p_a(C) = 0$, we have

$$\Gamma \cdot E = \Gamma \cdot (-K_Y) = C \cdot (-K_r) = C^2 + 2 < 0,$$

so E must be a component of Γ. Let $\eta : \tilde{\mathcal{C}} \to \mathcal{C}$ be a resolution of singularities. We obtain the commutative diagram

$$\begin{tikzcd} \tilde{\mathcal{C}} \arrow[rr, "\eta"] \arrow[dr, "\tilde{\pi}"'] & & \mathcal{C} \arrow[dl, "\pi"] \\ & \Delta & \end{tikzcd}$$

Since the base Δ has dimension 1, the morphism $\tilde{\pi}$ is still flat. Let $\tilde{\Gamma}$ be the special fiber of $\tilde{\pi}$. As $E \subset \Gamma$, the fiber $\tilde{\Gamma}$ contains a smooth genus–one curve $\tilde{E}$. By [11, Prop. (6.4.2)], the morphism $\tilde{\pi}$ is cohomologically flat, hence $h^0(\mathcal{O}_{\tilde{\Gamma}}) = 1$. By invariance of the arithmetic genus, it follows that $h^1(\mathcal{O}_{\tilde{\Gamma}}) = 0$. The natural surjection $\mathcal{O}_{\tilde{\Gamma}} \twoheadrightarrow \mathcal{O}_{\tilde{\Gamma}_{\mathrm{red}}}$ then induces a surjection

$$H^1(\mathcal{O}_{\tilde{\Gamma}}) \twoheadrightarrow H^1(\mathcal{O}_{\tilde{\Gamma}_{\mathrm{red}}}),$$

showing that $p_a(\tilde{\Gamma}_{\mathrm{red}}) = 0$. Hence $\tilde{\Gamma}_{\mathrm{red}}$ consists only of smooth rational curves, contradicting the presence of the elliptic component $\tilde{E}$. This contradiction proves that $C^2 \geq -2$. $\square$

Proof of Theorem 1. We first observe that for $r \leq 9$, the SHGH Conjecture holds (see, e.g., [8]), implying the validity of the two statements of the theorem. Thus, we henceforth assume $r > 9$.

We prove (1) $\Rightarrow$ (2). Let D be a nef and big divisor of X_r^2. By hypothesis the only negative curves in $D^\perp$ are (-1)-curves. By the Hodge index theorem and the fact that $D^2 > 0$, the intersection matrix of these curves is negative definite. In particular these curves are disjoint, since $(E + E')^2 \geq 0$ if E, E' are (-1)-curves with $E \cdot E' > 0$. Contracting these curves we get a birational morphism $X_r^2 \to X_s^2$, with $s \leq r$, and D is the pullback of an ample divisor of X_s, so that it is not asymptotically special.

We prove (2) $\Rightarrow$ (1). Assume there is a negative curve C on X_r^2 which is not a (-1)-curve. By Lemma 2.5 we have $p_a(C) > 0$. If $p_a(C) > 1$, then by Corollary 1.3 the surface X_r^2 would contain an asymptotically special divisor, a contradiction.

We may then assume that $p_a(C) = 1$. By adjunction, $\mathcal{O}_{X_r}(K_r + C)|_C \simeq \omega_C$. Since C is a divisor on a smooth surface, it is Gorenstein, so that ω_C is an invertible sheaf. Moreover, $\deg(\omega_C) = 2p_a(C) - 2 = 0$, and $h^0(C, \omega_C) = p_a(C) = 1$. Since ω_C is a line bundle of degree 0 admitting a non-zero global section, this section is nowhere vanishing, yielding $\omega_C \cong \mathcal{O}_C$. We then have the following exact sequence

$$0 \to \mathcal{O}_{X_r}(K_r) \to \mathcal{O}_{X_r}(K_r + C) \to \mathcal{O}_C \to 0.$$

From the fact that $h^i(\mathcal{O}_{X_r}(K_r)) = 1$ for $i = 2$ and 0 otherwise, and $h^2(\mathcal{O}_{X_r}(K_r + C)) = 0$ we deduce

$$K_r + C \sim D \geq 0, \qquad h^0(\mathcal{O}_{X_r}(D)) = 1, \qquad h^1(\mathcal{O}_{X_r}(D)) = 0.$$

Let $C_1, \ldots, C_k$ be the irreducible components of the support of D. Since $D - C \sim K_r$ is not effective, the divisor D cannot contain C in its support. Hence the condition $\mathcal{O}_{X_r}(D)|_C \simeq \mathcal{O}_C$ forces $C_i \cap C = \emptyset$ for all i.

Let us assume that $p_a(C_i) = 0$ for some i. If $C_i^2 \geq 0$, then $1 < h^0(\mathcal{O}_{X_r}(C_i)) \leq h^0(\mathcal{O}_{X_r}(D)) = 1$, a contradiction. Thus $C_i^2 < 0$, and by Lemma 2.5 such a curve must be a (-1)-curve. Since the curves C_i are disjoint from C, we can contract them to obtain a new surface X_s. The image of C in X_s retains the same arithmetic genus and self-intersection. On X_s, we consider the new effective divisor equivalent to $K_{X_s} + C$. If this new divisor contains rational components, they must again be (-1)-curves. We repeat the argument: since the Picard rank decreases at each step, this process terminates, leading to a situation where all irreducible components have positive arithmetic genus.

So from now on we assume $p_a(C_i) > 0$ for any i. For any such C_i, Serre duality gives $H^2(\mathcal{O}_{X_r}(D - C_i)) \simeq H^0(\mathcal{O}_X(K_{X_r} - D + C_i)) = H^0(\mathcal{O}_{X_r}(C_i - C)) = 0$, where the last equality is due to $h^0(\mathcal{O}_{X_r}(C_i)) \leq h^0(\mathcal{O}_{X_r}(D)) = 1$. From the long exact sequence of

$$0 \to \mathcal{O}_{X_r}(D - C_i) \to \mathcal{O}_{X_r}(D) \to \mathcal{O}_{C_i}(D) \to 0$$

and $h^1(\mathcal{O}_{X_r}(D)) = 0$ it follows that $h^1(\mathcal{O}_{C_i}(D)) = 0$. Hence $D \cdot C_i \geq 0$ by the generalized Riemann-Roch Theorem for singular curves [12, Tag 0BRZ]. In particular D is nef. If D were big, it would be asymptotically special by Proposition 1.1, a contradiction.

It remains to consider the case $D^2 = 0$. We have

$$D \cdot K_r = (K_r + C) \cdot K_r = (K_r + C)^2 = D^2 = 0.$$

Thus, by Lemma 2.2, after applying an isometry ϕ of the Picard group which takes D to its standard form, we get $\phi(D) = -mK_9$, with $m > 0$. Moreover $\phi(K_r) = K_r$. Thus $K_r + \phi(C) \sim -mK_9$, i.e.

$$\phi(C) \sim -(m+1)K_9 - E'_{10} - \cdots - E'_r,$$

where E'_j are exceptional classes. Since $h^0(-(m+1)K_9) = 1$, imposing additional base points we obtain a divisor that can not be effective. Hence $\phi(C)$ (and therefore C) cannot exist. This contradiction completes the proof. □

Before entering the proof of Theorem 2 we recall that the virtual dimension of a divisor $D = dH - \sum_{i=1}^n m_i E_i$ is

$$v(D) = \binom{d+2}{2} - \sum_{i=1}^{r} \binom{m_i+1}{2} - 1 = \frac{1}{2}(D^2 - D \cdot K_r).$$

Proof of Theorem 2. The implication (1) ⇒ (2) is obvious.

We prove (2) ⇒ (1). Since every nef class is non-special, it is also asymptotically non-special, so that, by Theorem 1, the only negative curves of X_r^2 are (-1)-curves. Let D be a divisor such that $D \cdot E \geq -1$ for any (-1)-curve E. Then $D \sim M + F$, where F is the sum of all (-1)-curves which have intersection product -1 with D. Observe that if E, E' are two distinct (-1)-curves in F then both curves are in the base locus of $|D|$ so that $E \cdot E' = 0$. It follows that F is a reduced divisor (each curve in its support appears with coefficient 1) and $M \cdot F = 0$. Thus

$$\begin{aligned} v(D) &= \frac{1}{2}(D^2 - D \cdot K_r) \\ &= \frac{1}{2}((M+F)^2 - (M+F) \cdot K_r) \\ &= v(M) + v(F) + M \cdot F \\ &= v(M). \end{aligned}$$

Observe that M is nef because the only negative curves of X_r^2 are (-1)-curves. Thus M is non-special and we deduce $\dim |D| = \dim |M| = v(M) = v(D)$, which shows that D is non-special as well. □

Remark 2.6. As a consequence of Lemma 2.2 one can show that any divisor class of $\mathrm{Pic}(X_{10}^2)_{\mathbb{R}}$ satisfying the equalities $D^2 = D \cdot K_{10} = 0$ is nef (see [7] and [4]). The argument goes like this. First of all observe that for any $r \geq 10$ a class $D \in \mathrm{Pic}(X_r^2)$ satisfying $D^2 = D \cdot K_r = 0$ is effective. Indeed, by Riemann-Roch either D or $K_r - D$ is effective but $(K_r - D) \cdot H < 0$. If D is in pseudostandard form, we can write $D = D_0 + m_1 E_1 + \cdots + m_s E_s$, with D_0 effective, $D_0 \cdot E_i = 0$ and $m_i > 0$ for any i. If we denote by $\pi \colon X_r^2 \to X_{r-s}^2$ the contraction of $E_1, \ldots, E_s$, then $D_0 = \pi^*(B)$ for an effective divisor class $B \in \mathrm{Pic}(X_{r-s}^2)$. Thus

$$0 = D \cdot K_r = \left(D_0 + \sum_{i=1}^{s} m_i E_i\right) \cdot K_r = B \cdot K_{r-s} - \sum_{i=1}^{s} m_i$$

implies that $B \cdot K_{r-s} = \sum_{i=1}^{s} m_i > 0$. Thus $-K_{r-s}$ is not nef, so that $r - s \geq 10$. We deduce that if $r = 10$ then s must be zero or in other words D is a standard divisor class. By Lemma 2.2 we conclude that D is nef. To conclude one observes that such classes define a quadric in $\mathrm{Pic}(X_{10}^2)_{\mathbb{R}}$ whose rational points are dense.

Example 2.7. Let $B := dH - \sum_{i=1}^{8} m_i E_i$ be such that $b := \frac{1}{4}(2B^2 - (B \cdot K_8)^2) > 0$ and let $a := -\frac{1}{2} B \cdot K_8$. Then the following class

$$D := B - (a + \sqrt{b})E_9 - (a - \sqrt{b})E_{10}$$

satisfies $D^2 = D \cdot K_{10} = 0$ so that it is nef.

3. Higher dimension

We recall that a divisor on a projective variety is said to be *semiample* if it admits a base point free multiple. In this section we consider the asymptotical behavior of semiample divisors in X_r^n with the purpose of studying possible generalizations of the conjectures described in the Introduction. A possible generalization of these conjectures could be:

Conjecture 3.1. *The blow-up X_r^n of $\mathbb{P}^n$ at r points in very general position does not admit nef divisors which are asymptotically special.*

We are going to prove that Conjecture 3.1 holds if $r < 2^n$ (Theorem 3), but first we need some preliminary results.

Definition 3.2. Given a semiample Cartier divisor D on a normal variety X let

$$m_D := \gcd(n \in \mathbb{N} \, : \, nD \text{ is base point free}).$$

For any positive integer m denote by $\phi_m \colon X \dashrightarrow Y_m$ the rational map defined by the linear system $|mD|$. By [10, Thm. 2.1.27] there is a morphism with connected fibers

$$f_D \colon X \to Y$$

onto a normal variety Y, such that $\phi_m = f_D$ and $Y_m = Y$ for any sufficiently big multiple m of m_D. Moreover $m_D D = f_D^* A$ for some ample Cartier divisor A on Y.

Remark 3.3. The semiample and big divisor D of Example 1.2 has $m_D = 2$. Indeed any odd multiple of D restricts to a non-trivial degree zero class on the elliptic curve $C \in |-K_X|$, while the even multiples restrict trivially to C. The latter fact implies that C is not in the base locus of $2D$ and by [8, Thm. III.1] we conclude that $2D$ is base point free.

Proposition 3.4. *Let X be a normal projective variety and let D be a semiample big Cartier divisor of X with $m_D = 1$. If $R^1 f_{D*}\mathcal{O}_X$ is trivial then D is asymptotically non-special and otherwise it is asymptotically special.*

Proof. Let $f := f_D$ and recall that $D = f^*A$, with A ample on Y. By Serre's vanishing [10, Thm. 1.2.6] the higher cohomology groups of $\mathcal{O}_Y(mA)$ vanish for all $m \gg 0$. Since the Grothendieck-Leray spectral sequence

$$E_2^{p,q}(D) := H^p(Y, R^q f_*\mathcal{O}_X(D)) \Rightarrow H^{p+q}(X, \mathcal{O}_X(D))$$

is a first quadrant one, by [9, Prop. 3.6.2] we have the following five terms exact sequence:

$$0 \to E_2^{1,0}(mD) \to H^1(\mathcal{O}_X(mD)) \to E_2^{0,1}(mD) \to E_2^{2,0}(mD) \to H^2(\mathcal{O}_X(mD)).$$

By the projection formula we have $f_*\mathcal{O}_X(mD) = f_*\mathcal{O}_X \otimes \mathcal{O}_Y(mA) \simeq \mathcal{O}_Y(mA)$, so that the first and fourth cohomology groups $E_2^{i,0}(mD) = H^i(Y, f_*\mathcal{O}_X(mD))$ vanish for all $m \gg 0$. Thus the second and third cohomology groups are isomorphic. Again by the projection formula we have $R^1 f_*\mathcal{O}_X(mD) \simeq R^1 f_*\mathcal{O}_X \otimes \mathcal{O}_Y(mA)$ so that

$$H^1(X, \mathcal{O}_X(mD)) \simeq H^0(Y, R^1 f_*\mathcal{O}_X \otimes \mathcal{O}_Y(mA)) \qquad \text{for all } m \gg 0.$$

If $R^1 f_*\mathcal{O}_X$ is trivial we conclude that D is asymptotically non-special. On the other hand if $R^1 f_*\mathcal{O}_X$ is non-trivial, since it is a coherent sheaf and A is ample, by Serre's Theorem [10, Thm. 1.2.6] the sheaf $R^1 f_*\mathcal{O}_X \otimes \mathcal{O}_Y(mA)$ is generated by global sections for all $m \gg 0$. We conclude that D is asymptotically special. $\square$

Before proving Theorem 3 we are now going to discuss an example of a nef and big divisor which is special but asymptotically non-special, on the blow up of $\mathbb{P}^4$ at 14 points.

Example 3.5. Let D be the divisor $2H - \sum_{i=1}^{14} E_i$ on X^4_{14}. The virtual dimension of mD, for $m = 1, 2, 3$ is $0, -1, -1$ respectively, so that $2D$ and $3D$ are special. Moreover a computer calculation shows that $4D$ is non-special of dimension 4 and its base locus is zero-dimensional. Thus D is semiample by [10, Rem. 2.1.32] and it is big because $D^4 = 2$. We now show that mD is non-special for $m \geq 5$ using a degeneration argument. Let $\mathcal{X} \to \mathbb{A}^1$ be a flat family whose general fiber is X^4_{14} and whose special fiber Y is the blowing-up of $\mathbb{P}^4$ at 14 general points on the complete intersection of three general quadrics. Denote by $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq Y_4 = Y$ the strict transforms of the complete intersections of three, two and one quadrics respectively, plus the central fiber of the degeneration. By abuse of notation we denote by D the specialized divisor on Y. Observe that for any $2 \leq i \leq 4$, $Y_{i-1} \in |D|_{Y_i}|$, so that we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{Y_i}((m-1)D) \longrightarrow \mathcal{O}_{Y_i}(mD) \longrightarrow \mathcal{O}_{Y_{i-1}}(mD) \longrightarrow 0.$$

Since Y_1 is a smooth curve of genus 5 and $\deg(D|_{Y_1}) = D \cdot Y_1 = 2$ it follows that $mD|_{Y_1}$ is non-special for $m \geq 5$. From the above exact sequence we deduce that $h^1(\mathcal{O}_{Y_2}(mD)) \leq h^1(\mathcal{O}_{Y_2}((m-1)D)) \cdots \leq h^1(\mathcal{O}_{Y_2}(5D))$. A computer calculation shows that $5D|_{Y_i}$ is non-special for $2 \leq i \leq 4$. A repeated use of the above exact sequence shows that for any $m \geq 5$, mD is non-special on Y_4 and, by semicontinuity of cohomology, it is non-special on X^4_{14} too.

Proof of Theorem 3. Let $H, E_1, \dots, E_r$ be a basis of $\mathrm{Pic}(X^n_r)$ consisting of the pullback of a hyperplane together with the exceptional divisors. If we denote by $h, e_1, \dots, e_r$ the dual basis in $\mathrm{N}_1(X^n_r)$, we have that the Mori cone of X^n_r is polyhedral, generated by the classes $h - e_i - e_j$ and e_k, for $i, j, k \in \{1, \dots, r\}$ and $i < j$. (see [5]). The proof of this fact goes as follows: the cone generated by these classes is dual to the cone generated by the classes

$$H, \quad H - E_i, \quad 2H - \sum_{i \in I} E_i,$$

for any subset $I \subseteq \{1, \dots, r\}$. These classes are all semiample (and thus nef) on the variety Y^n_r obtained by degenerating the r points to a complete intersection of n general smooth quadrics. Then one concludes by observing that the nef cone can only become smaller by degeneration.

By semicontinuity of cohomology it is now enough to show that any nef class of Y^n_r is asymptotically non-special. Observe that any $H - E_i$ is asymptotically non-special because it is pullback of a class of X^n_1 which is asymptotically non-special by Kawamata-Viehweg vanishing. All the nef classes of Y^n_r, which are not multiples of $H - E_i$, are big. Thus by Proposition 3.4 it suffices to show that for any nef and big divisor class D of Y^n_r the sheaf $R^1 f_{D*}\mathcal{O}_{Y^n_r}$ is trivial. This is a consequence of the fact that f_D can contract only exceptional divisors and strict transforms of lines through any two of the blown-up points. In the first case one gets a smooth point while in the second case the image of f_D has a rational singularity because the morphism is locally toric and one can apply [6, Thm. 11.4.2]. □

Data availability

No data was used for the research described in the article.

References

[1] Lucian Bădescu, Algebraic Surfaces, Universitext, Springer-Verlag, New York, 2001, Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author.
[2] Ciro Ciliberto, Brian Harbourne, Rick Miranda, Joaquim Roé, Variations of Nagata's conjecture, in: A Celebration of Algebraic Geometry, in: Clay Math. Proc., vol. 18, Amer. Math. Soc., Providence, RI, 2013, pp. 185–203.

[3] Ciro Ciliberto, Rick Miranda, The Segre and Harbourne-Hirschowitz conjectures, in: Applications of Algebraic Geometry to Coding Theory, Physics and Computation, Eilat, 2001, in: NATO Sci. Ser. II Math. Phys. Chem., vol. 36, Kluwer Acad. Publ., Dordrecht, 2001, pp. 37–51.

[4] Ciro Ciliberto, Rick Miranda, Joaquim Roé, Quadric cones on the boundary of the Mori cone for very general blowups of the plane, Collect. Math. (2024).

[5] Izzet Coskun, John Lesieutre, John Christian Ottem, Effective cones of cycles on blowups of projective space, Algebra Number Theory 10 (9) (2016) 1983–2014.

[6] David A. Cox, John B. Little, Henry K. Schenck, Toric Varieties, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

[7] T. Fernex, Negative Curves on Very General Blow-Ups of P2, De Gruyter, Berlin, New York, 2005, pp. 199–208.

[8] Brian Harbourne, Anticanonical rational surfaces, Trans. Am. Math. Soc. 349 (3) (1997) 1191–1208.

[9] Shihoko Ishii, Introduction to Singularities, second edition, Springer, Tokyo, 2018.

[10] Robert Lazarsfeld, Positivity in Algebraic Geometry. I, Ergebnisse der Mathematik und Ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.

[11] Michel Raynaud, Spécialisation du foncteur de Picard, Inst. Hautes Études Sci. Publ. Math. 38 (1970) 27–76.

[12] The Stacks Project Authors, Stacks Project, https://stacks.math.columbia.edu, 2018.