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# Insights into Topological Spaces: Bounds on the Cardinality of Spaces, Selection Principles involving Networks, and related Games

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# Declaration of Authorship

I, DAVIDE GIACOPELLO, declare that this thesis titled, “Insights into Topological Spaces: Bounds on the Cardinality of Spaces, Selection Principles involving Networks, and related Games” and the work presented in it are my own. I confirm that:

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*"Docendo discimus."*

Seneca

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# *Abstract*

Department of Mathematics and Computer Sciences

Doctor of Philosophy

**Insights into Topological Spaces: Bounds on the Cardinality of Spaces, Selection Principles involving Networks, and related Games**

by DAVIDE GIACOPELLO

In Chapter 1 we present the theory of cardinal invariants and the research in cardinal upper bounds of topological spaces. Then, we deal with the class of Hausdorff spaces having a $\pi$-base whose elements have an H-closed closure. In 2023, Nathan Carlson proved that $|X| \leq 2^{wL(X)\psi_c(X)t(X)}$ for every quasiregular space $X$ with a $\pi$-base whose elements have an H-closed closure. We provide an example of a space $X$ having a $\pi$-base whose elements have an H-closed closure which is not quasiregular (neither Urysohn) such that $|X| > 2^{wL(X)\chi(X)}$ (then $|X| > 2^{wL(X)\psi_c(X)t(X)}$). In the class of spaces with a $\pi$-base whose elements have an H-closed closure, we establish the bound $|X| \leq 2^{wL(X)k(X)}$ for Urysohn spaces and we give an example of an Urysohn space $Z$ such that $k(Z) < \chi(Z)$. Lastly, we present some equivalent conditions to the Martin's Axiom involving spaces with a $\pi$-base whose elements have an H-closed closure and, additionally, we prove that if a quasiregular space has a $\pi$-base whose elements have an H-closed closure then such a space is Choquet (hence Baire).

In Chapter 2 we introduce some new selection principles involving networks, namely, M-*nw*-selective, R-*nw*-selective and H-*nw*-selective. We show that such spaces has countable fan tightness, countable strong fan tightness and the weak Fréchet in strict sense property, respectively, hence they are M-separable, R-separable and H-separable, respectively. Also they are Menger, Rothberger and Hurewicz. We give consistent results and we define *trivial* R-, H-, and M-*nw*-selective spaces the ones with countable netweight having, additionally, the cardinality and the weight strictly less then $cov(\mathcal{M})$, $\mathfrak{b}$, and $\mathfrak{d}$, respectively. Since we establish that spaces having cardinalities more than $cov(\mathcal{M})$, $\mathfrak{b}$, and $\mathfrak{d}$, fail to have the R-, H-, and M-*nw*-selective properties, respectively, non-trivial examples should eventually have weight greater than or equal to these small cardinals. Moreover, using forcing methods, we construct consistent countable non-trivial examples of R-*nw*-selective and H-*nw*-selective spaces. Additionally, we establish some limitations to constructions of non-trivial examples and we consistently prove the existence of two H-*nw*-selective spaces whose product fails to be M-*nw*-selective. Finally, we study some relations between *nw*-selective properties and a strong version of the HFD property.

In Chapter 3 we introduce and investigate two new games called R-*nw*-selective game and the M-*nw*-selective game. These games naturally arise from the corresponding selection principles involving networks introduced by Bonanzinga and Giacopello.

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# Contents

*Dedicated to all my beloved…*

# Chapter 1

# Recent studies on the upper bounds of the cardinality of topological spaces

In this chapter, we will deal with cardinal functions or invariants. These tools represent a very powerful instrument in set-theoretic topology. The study of cardinal functions has opened up the possibility to generalize various topological concepts and explore the cardinality of particular classes of spaces. All uncited results in this chapter are either floklore or can be found in [37]. For more details on the set-theoretical contest and some covering properties see Appendices A and B.

## 1.1 Introduction to cardinal functions and upper bounds on the cardinality of spaces

One of the most primary challenge in topology lies in determining whether two topological spaces are homeomorphic to each other. Explicitly constructing a homeomorphism between two spaces is frequently a tough task. Moreover, establishing that there are no homeomorphisms between spaces is equally difficult. The method of cardinal functions provides some tools to face this problem. Indeed, a cardinal function (or cardinal invariant) is a functor

$$\varphi : TOP \to CARD$$

from the class $TOP$ of topological spaces to the class $CARD$ of cardinal numbers, such that for every pair of homeomorphic spaces $X$ and $Y$, it holds that $\varphi(X) = \varphi(Y)$. Therefore, if one finds out that $\varphi(X) \neq \varphi(Y)$ for some cardinal function $\varphi$, then the spaces $X$ and $Y$ are not homeomorphic to each other.
Cardinal functions also provide an extension to arbitrary cardinalities of numerous fundamental notions in general topology, such as: separability; first and second countability; compactness, and so on. Additionally, they allow for a precise quantitative comparison between sizes of topological objects of a particular space regardless for their scale.
The introduction of such cardinal invariants began in the 1920s by the Russian school of topology (Luzin, Suslin, Alexandroff, Urysohn, etc.) but it fully emerged in the second half of the 1960s with the set-combinatorics theory.
In what follows we will present some cardinal functions. Of course, we cannot provide a list of all the existing ones, we either choose some functions that generalize basic concepts in topology, or some more historically relevant ones. However, numerous other cardinal functions have been introduced over this century.

### 1.1.1 Definitions of cardinal functions and some relations between them

The first cardinal invariant we consider is the *character*, which is the function associated to the concept of local base at one point of a topological space. It exactly measures the minimum cardinality such that each point of the space has a local base of that cardinality. Given a space $X$, the character of $X$ is defined as

$$\chi(X) = \min\{\kappa : \forall x \in X\ \exists \text{ a local base } \mathcal{V}_x \text{ such that } |\mathcal{V}_x| \leq \kappa\}$$

Clearly, a space $X$ is first countable iff $\chi(X) \leq \omega$.
This function is hereditary, which means that if $\chi(Y) \leq \chi(X)$ whenever $Y$ is a subset of $X$. Additionally, it is productive for products that do not exceed the cardinality of the character of each space, i.e., if $\{X_\alpha : \alpha < \lambda\}$ is a collection of spaces such that $\chi(X_\alpha) \leq \kappa$, if $\lambda \leq \kappa$, $\chi(\prod_{\alpha<\lambda} X_\alpha) \leq \kappa$.
Given a space $X$, a local pseudobase at one point $x \in X$ is a collection $\mathcal{V}$ of open neighborhoods of $x$, such that $\{x\} = \bigcap \mathcal{V}$. Then the *pseudocharacter* of a $T_1$ space $X$ is defined as

$$\psi(X) = \min\{\kappa : \forall x \in X\ \exists \text{ a local pseudobase } \mathcal{V}_x \text{ such that } |\mathcal{V}_x| \leq \kappa\}$$

In general for a $T_1$ space $X$, $\psi(X) \leq \chi(X)$ and $\psi(X) \leq |X|$. As the character, the pseudocharacter is a hereditary function.
Usually, since these functions describe local properties, they are locally defined which means $\chi(x, X)$ and $\psi(x, X)$, for every point $x \in X$, namely as the minimum of a local base (or pseudobase, respectively) at $x$ in $X$. Moreover, if $X$ is a Hausdorff locally compact space $\psi(x, X) = \chi(x, X)$ for every $x \in X$.
The *closed pseudocharacter* is defined for Hausdorff spaces as

$$\psi_c(X) = \min\{\kappa : \forall x \in X\ \exists \text{ a family } \mathcal{V}_x \text{ of open neighborhoods of } x$$
$$\text{such that } |\mathcal{V}_x| \leq \kappa \text{ and } \bigcap_{V \in \mathcal{V}_x} \overline{V} = \{x\}\}$$

In general, $\psi(X) \leq \psi_c(X) \leq \chi(X)$ for a Hausdorff space $X$ and $\psi(X) = \psi_c(X)$ for a regular space $X$.

The *tightness* of a space is another local cardinal invariant. It is defined as

$$t(X) = \min\{\kappa : \forall x \in \overline{A}\ \exists B \subseteq A \text{ such that } |B| \leq \kappa \text{ and } x \in \overline{B}\}$$

An important lemma can be proved using this cardinal function.

**Lemma 1.1.1.** Let $X$ be a space such that $t(X) \leq \kappa$ and $\{F_\alpha : \alpha < \lambda\}$ be an increasing sequence of closed subsets of $X$ of lenght $\lambda$, with $cf(\lambda) > \kappa$. Then the set $F = \bigcup_{\alpha<\lambda} F_\alpha$ is closed.

Given a space $X$, a local $\pi$-base at $x$ in $X$ is a collection $\mathcal{V}_x$ of open neighborhoods of $x$, such that for every open neighborhood $U$ of $x$, exists some $V \in \mathcal{V}_x$ such that $V \subseteq U$. The *$\pi$-character* is defined as

$$\pi\chi(X) = \min\{\kappa : \forall x \in X\ \exists \text{ a local } \pi\text{-base } \mathcal{V}_x \text{ such that } |\mathcal{V}_x| \leq \kappa\}$$

Clearly, $\pi\chi(X) \leq \chi(X)$ for every space $X$.
The following proposition gives an example of a space in which $\psi(X) = \omega$ and $\chi(X)$ is arbitrarily large. Reacall given an uncountable cardinal $\kappa$ the character of the space $2^\kappa$ is exactly $\kappa$ and if $D$ is a dense subset of $2^\kappa$ its character is still $\kappa$.

**Proposition 1.1.2.** Let $\kappa$ be a cardinal number. Then there exists a dense subset $D$ of $2^\kappa$ such that $|D| \leq \kappa$ and $\psi(D) \leq \omega$.

*Proof.* For $\kappa \leq 2^\omega$ it is trivial since $2^\kappa$ is separable. Then assume that $\kappa > 2^\omega$. Let $\{s_\alpha : \alpha < \kappa\}$ be an enumeration of finite nonempty partial functions from $\kappa$ to 2. Recursively over $\alpha$ construct a sequence of countable partial functions $\{p_\alpha : \alpha \in \kappa\}$ from $\kappa$ to 2 such that

(1) $p_\alpha$ extends $s_\alpha$, and

(2) $p_\alpha^{-1}(1) \setminus s_\alpha^{-1}(1)$ is an infinite subset of $\kappa \setminus \bigcup_{\beta<\alpha} p_\beta^{-1}(1)$.

Now let $f_\alpha \in 2^\kappa$ be an extension of $p_\alpha$ such that $f_\alpha(\xi) = 0$ for all $\xi \notin dom(p_\alpha)$. It is clear that $D = \{f_\alpha : \alpha < \kappa\}$ is dense, we claim that it also has countable pseudocharacter. Indeed, it follows from (2) that $\{f_\alpha\} = D \cap \bigcap\{U_\xi : p_\alpha(\xi) = 1\}$, where $U_\xi = \{x \in 2^\kappa : x(\xi) = 1\}$. □

The cardinal functions based on the concepts of local base, local $\Psi$-base, and local $\pi$-base find their global versions in the following cardinal functions.
The *weight*

$$w(X) = \min\{\kappa : \exists \text{ a base } \mathcal{B} \text{ such that } |\mathcal{B}| \leq \kappa\}$$

The $\Psi$-*weight* (pseudo-weight) defined for $T_1$ spaces

$$\Psi w(X) = \min\{\kappa : \exists \text{ a } \Psi\text{-base } \mathcal{B} \text{ such that } |\mathcal{B}| \leq \kappa\}$$

where a $\Psi$-base is a family of open subsets such that every point of the space is the intersection of all the members of the family which contains it.
And the $\pi$-*weight*

$$\pi w(X) = \min\{\kappa : \exists \text{ a } \pi\text{-base } \mathcal{B} \text{ such that } |\mathcal{B}| \leq \kappa\}$$

where a $\pi$-base is a family of open subsets such that every open subset of the space there exists a member of the family inside.

A family of subset $\mathcal{N}$ of a space $X$ is called network if for every point $x \in X$ and every open neighborhood $U$ of $x$ there exists $N \in \mathcal{N}$ such that $x \in N \subseteq U$. The *network weight* (or netweight) is defined as

$$nw(X) = \min\{\kappa : \exists \text{ a network } \mathcal{N} \text{ such that } |\mathcal{N}| \leq \kappa\}$$

Clearly, $nw(X) \leq w(X)$ for every space $X$ and for locally compact Hausdorff spaces $nw(X) = w(X)$. Additionally, for $T_1$ spaces $|X| \leq nw(X)^{\psi(X)}$.

The *Lindelöf degree* is the invariant that generalize the compactness or the Lindelöfness of a space. It is defined as

$$L(X) = \min\{\kappa : \forall \text{ open cover } \mathcal{U} \text{ of } X\ \exists \mathcal{V} \subseteq \mathcal{U} \text{ such that } |\mathcal{V}| \leq \kappa \text{ and } \bigcup \mathcal{V} = X\}$$

The Lindelöf degree is not a hereditarily function in general, but for closed subsets. Therefore, it is possible to define the hereditary version of this function. The *hereditary Lindelöf degree* $hL(X) = \sup\{L(Y) : Y \subseteq X\}$. It is straightforrward to prove that $L(X) \leq hL(X) \leq nw(X)$ holds for any space. Also, the hereditary Lindelöf can be equivalently defined as $hL(X) = \sup\{L(Y) : Y \text{ is an open subset of } X\}$.

The cardinal function that generalize the concept of separability of a space is the *density*. It is defined as

$$d(X) = \min\{\kappa : \exists \text{ a dense subset } D \text{ such that } |D| \leq \kappa\}$$

It is straightforward to see that $d(X) \leq nw(X)$ and $d(X) \leq \pi w(X)$ hold for any space. Moreover, if $X$ is a Hausdorff space, $|X| \leq 2^{d(X)}$ and $|X| \leq d(X)^{\chi(X)}$ (called Pospíšil inequality). If a space is regular, then $w(X) \leq 2^{d(X)}$. The hereditary version of the density is defined as $hd(X) = \sup\{d(Y) : Y \subseteq X\}$. The inequalities $t(X) \leq hd(X)$ and $hd(X) \leq nw(X)$ hold for any space.

The *extent* of a space is definded as

$$e(X) = \sup\{|C| : C \text{ is a closed and discrete subset of } X\}$$

It is clear that $e(X) \leq L(X)$.
The hereditary version of the extent is the *spread* defined as

$$s(X) = \sup\{|D| : D \text{ is a discrete subset of } X\}$$

Clearly, $s(X) \leq hd(X)$.

A maximal open family of pairwise disjoint subsets of a space is called cellular. The *cellularity* of a space is defined as

$$c(X) = \sup\{|\mathcal{C}| : \mathcal{C} \text{ is a cellular family in } X\}$$

It is easy to see that $c(X) \leq d(X)$ and $c(X) \leq s(X)$. Moreover, the hereditary version of the cellularity is the *spread*. A space is said to have the countable chain condition (brifly, ccc) if $c(X) = \omega$.

### 1.1.2 Cardinal inequalities

In the previous section, we have examined some inequalities among the introduced cardinal invariants, including some that involve the cardinality of the space. While the majority were straightforward, others required more detailed proofs. Generally, a cardinal inequality gives an upper bound on the cardinality of a particular class of spaces, involving cardinal invariants. Occasionally, we investigate bounds on the cardinality of a specific cardinal function.
The study of cardinal inequalities, as an indipendent line of research, began with the following problem posed in 1923 by the one of the fathers of General Topology, Alexandrov.

**Question 1.1.3.** Is the cardinality of a first countable compact Hausdorff space less than or equal to the continuum?

It seemed like a simple and natural question, but it had been unsolved for almost 50 years until 1969 when the Russian topologist Arhangel'skii provided an affirmative answer.

**Theorem 1.1.4.** [2] Let $X$ be a Hausdorff space. Then $|X| \leq 2^{\chi(X)L(X)}$.

Pol and Shapirovskii improved Arhangel'skii's inequality by replacing the character with the product of the pseudocharacter and the tightness, which in general is less than or equal to the character.

**Theorem 1.1.5.** (Pol-Shapirovskii) Let $X$ be a Hausdorff space. Then $|X| \leq 2^{\psi(X)t(X)L(X)}$.

In 1967, Hajnal and Juhász inspired by the work of Arhangel'skii, proved the following result.

**Theorem 1.1.6.** Let $X$ be a Hausdorff space. Then $|X| \leq 2^{\chi(X)c(X)}$.

The previous two result seemed similar to each other and some mathematicians asked about the possibility to find a common generalization of them. In 1978, Bell, Ginsburg and Woods succeded in this aim in the class of normal Hausdorff spaces by introducing the cardinal function *weak Lindelöf degree* defined as

$$wL(X) = \min\{\kappa : \forall \text{ open cover } \mathcal{U} \text{ of } X\ \exists \mathcal{V} \subseteq \mathcal{U} \text{ such that } |\mathcal{V}| \leq \kappa \text{ and } \overline{\bigcup \mathcal{V}} = X\}$$

It is straightforward to see that $wL(X) \leq L(X)$ and $wL(X) \leq c(X)$.

**Theorem 1.1.7.** [11] Let $X$ be a normal Hausdorff space. Then $|X| \leq 2^{\chi(X)wL(X)}$.

In the same paper the authors provided an example of a Hausdorff space $X$ of arbitrarily large cardinality of countable character and weak Lindelöf degree.
Two issues rised after their work: the first is to find, whether is possible, the Bell, Ginsburg and Woods'inequality under weaker hypotesis. For instance to see if it holds for regular (completely regular or zero-dimensional) Hausdorff spaces. The second is to improve the bound using smaller cardinal functions to get more accurate estimates. A possibility could be replacing $\chi(X)$ with $\psi(X)t(X)$ as it was done by Pol and Shapirovskii.
However, Bell, Ginsburg and Woods posed some limits to the research in both these directions. Indeed, they provided an example of a zero-dimentional Hausdorff (hence completely regular) space $Z$ such that $|Z| > 2^{\psi(Z)t(Z)wL(Z)}$.
In 1982, Dow and Porter achieved a very powerful result in the class of H-closed spaces. Recall that a Hausdorff space $X$ is H-closed if for every open cover $\mathcal{U}$ of $X$ there exists a finite subfamily $\mathcal{V}$ such that $\bigcup_{V \in \mathcal{V}} \overline{V} = X$ (for further information about this class of space see in the Appendix A).

**Theorem 1.1.8.** [33] Let $X$ be an H-closed space. Then $|X| \leq 2^{\psi_c(X)}$.

Bella and Carlson, trying to address the Bell, Ginsburg, and Woods'issue, observed that one of the conditions to add to the "weak" separation axioms to obtain the Bell, Ginsburg, and Woods'inequality is the fact that the space admits a $\pi$-base whose elements have certain properties. Indeed through such $\pi$-base, it is possible to generalize the closing-off argument used to prove this kind of theorems, as we will see later on in Theorem 1.2.7. For simplicity, if a space has a $\pi$-base whose elements have an H-closed (compact) closure, we say that the space has an *H-closed (compact) $\pi$-base*.

**Theorem 1.1.9.** [15] Let $X$ be a regular Hausdorff space with a compact $\pi$-base. Then $|X| \leq 2^{wL(X)t(X)\psi(X)}$

In [16], Bella, Carlson and Gotchev proved that, replacing the pseudocharacter $\psi(X)$ with the closed pseudocharacter $\psi_c(X)$, the same inequality holds also for spaces with a compact $\pi$-base. That is,

**Theorem 1.1.10.** [16] If $X$ is a Hausdorff space with a compact $\pi$-base. Then $|X| \leq 2^{wL(X)t(X)\psi_c(X)}$.

Since H-closedness is a natural generalization of compactness, Bella, Carlson and Gotchev posed the following question.

**Question 1.1.11.** [16] Let $X$ be a Hausdorff space with a $\pi$-base whose elements have H-closed closures. Is it true that $|X| \leq 2^{wL(X)t(X)\psi_c(X)}$?

In [15], some further investigations on spaces having a $\pi$-base with some properties on the closure of the elements have led to Theorem 1.1.12.

Recall that a a space $X$ is called quasiregular if for every open subset $U$ there exists another open subset $V$ such that $\overline{V} \subseteq U$. Also, given a subset $A \subseteq X$ the $\theta$-closure (see [71]) of $A$, $cl_\theta(A)$, is the subset $\{x \in X : \overline{U} \cap A \neq \emptyset$ for every open neighborhood $U$ of $x\}$. Clearly, $\overline{A} \subseteq cl_\theta(A)$ for every $A \subseteq X$. Recall that, given a space $X$, the cardinal invariant $d_\theta(X)$, closely related to the density $d(X)$, is defined as follows. A subspace $D \subseteq X$ is $\theta$-dense in $X$ if $D \cap \overline{U} \neq \emptyset$ for every non-empty open set $U$ of $X$. The $\theta$-density $d_\theta(X)$ is the least cardinality of a $\theta$-dense subspace of $X$. Observe that $d_\theta(X) \leq d(X)$ for any space $X$ (see [20] and [31] for more details about $\theta$-density).

**Theorem 1.1.12.** [15] Let $X$ be a space and $\mathcal{B}$ an open $\pi$-base. Suppose for all $B \in \mathcal{B}$ that $\overline{B}$ is $H$-closed, (or normal, or Lindelöf, or has the ccc). Then $d_\theta(X) \leq 2^{wL(X)\chi(X)}$ and if $X$ is quasiregular or Urysohn then $|X| \leq 2^{wL(X)\chi(X)}$.

Numerous other studies have been conducted in the field of cardinal inequalities, including in spaces with very weak separation axioms (non-Hausdorff). For example, Bonanzinga introduced a class of non-Hausdorff spaces called $n$-Hausdorff in [19] and proved many cardinal inequalities in this class. Recently, these classes of spaces have been investigated in [22] and [23].

## 1.2 On spaces with a $\pi$-base whose elements have an H-closed closure

In this section we focus on spaces with an H-closed $\pi$-base.
First of all, we provide an example of a space having the property we are dealing with and does not have a compact $\pi$-base.

**Example 1.2.1.** An example of a non-quasiregular (hence non-regular) Hausdorff space having an H-closed $\pi$-base with no compact $\pi$-bases.

Consider the space $(\mathbb{R}, \tau_{\mathbb{Q}})$ where $\tau_{\mathbb{Q}}$ is the topology generated by the open set of the form $\{x\} \cup (U \cap \mathbb{Q})$, with $x \in \mathbb{R} \setminus \mathbb{Q}$ and $U$ an open set in the standard topology on $\mathbb{R}$. Let $X = [0, 2]$ with the topology inherited from $(\mathbb{R}, \tau_{\mathbb{Q}})$. $X$ is a Hausdorff non regular nowhere locally compact space. Then it does not have a compact $\pi$-base. In [40] Herrlich proved that $X$ is H-closed then it admits a $\pi$-base whose elements have H-closed closures. Now we prove that there is not a $\pi$-base whose elements have quasiregular closure. Suppose that $\mathcal{B}$ is a $\pi$-base of $X$ having elements with quasiregular closures. Let's take the open set $(0, 2) \cap \mathbb{Q}$, then there exists $U \in \mathcal{B}$ such that $U \subseteq (0, 2) \cap \mathbb{Q}$, by hypothesis $\overline{U}$ is quasiregular. Therefore there is $V \in \mathcal{B}$ such that $V \subseteq \overline{V} \subseteq U$ so $\overline{V} \subseteq \mathbb{Q}$, which is a contradiction. By Corollary 1.2.6 (see below) follows that $X$ is not quasiregular. △

The following example answer in the negative to Question 1.1.11.

**Example 1.2.2.** A Hausdorff space $X$ having an H-closed $\pi$-base such that $|X| > 2^{wL(X)\chi(X)}$ (hence $|X| > 2^{wL(X)t(X)\psi_c(X)}$).

Let $\kappa$ be a cardinal. Consider the space $X = (\mathbb{Q} \times \kappa) \cup (\mathbb{R} \setminus \mathbb{Q})$. If $(q, \alpha) \in \mathbb{Q} \times \kappa$, a basic open neighborhood is $U_n(q, \alpha) = \{(r, \alpha) : r \in \mathbb{Q} \text{ and } |r - q| < \frac{1}{n}\}$ for each $n \in \omega$. If $x \in \mathbb{R} \setminus \mathbb{Q}$, a basic open neighborhood is $U_n(x) = \{x\} \cup \{(r, \alpha) : r \in \mathbb{Q}, \alpha < \kappa \text{ and } |r - x| < \frac{1}{n}\}$. Clearly, $t(X)\psi_c(X) \leq \chi(X) \leq \omega$, moreover $wL(X) \leq \mathfrak{c}$. The collection $\{U_n(q, \alpha) : n \in \omega, q \in \mathbb{Q} \text{ and } \alpha < \kappa\}$ is a $\pi$-base in $X$. We prove that each $\overline{U_n(q, \alpha)}$ is H-closed for each $n \in \omega, q \in \mathbb{Q}$ and $\alpha < \kappa$. Let $\mathcal{U}$ be a basic open cover of $\overline{U_n(q, \alpha)}$. For each $U \in \mathcal{U}$ there exists $W_U$ which is an open subset of $\mathbb{R}$ in the standard topology such that $\overline{U} = \{(q, \alpha) : q \in \overline{W_U} \cap \mathbb{Q}\} \cup ((\mathbb{R} \setminus \mathbb{Q}) \cap \overline{W_U})$. Since $[q - \frac{1}{n}, q + \frac{1}{n}]$ is a compact (hence H-closed) subset of $\mathbb{R}$ and $\{W_U : U \in \mathcal{U}\}$ is an open cover of this interval, there exists a finite subfamily $\{U_1, ..., U_k\}$ such that $[q - \frac{1}{n}, q + \frac{1}{n}] \subseteq \bigcup_{i=1}^{k} \overline{W_{U_i}}$. Therefore $\overline{U_n(q, \alpha)} \subseteq \bigcup_{i=1}^{k} \overline{U_i}$. Considering a cardinal $\kappa > 2^{\mathfrak{c}}$ we have that $|X| > 2^{wL(X)\chi(X)}$ (hence $|X| > 2^{wL(X)t(X)\psi_c(X)}$). △

Notice that the previous example is not Urysohn not even quasiregular, then we pose the following questions motivated by these facts and Theorem 1.1.12.

**Question 1.2.3.** Does the inequality $|X| \leq 2^{wL(X)t(X)\psi_c(X)}$ hold for quasiregular Hausdorff spaces having an H-closed $\pi$-base?

**Question 1.2.4.** Does the inequality $|X| \leq 2^{wL(X)t(X)\psi_c(X)}$ hold for Urysohn spaces having an H-closed $\pi$-base?

We answered Question 1.2.3 in the positive and we proved that the inequality in Question 1.2.4 is true if we use the cardinal function $k(\cdot)$ in the place of $t(\cdot)\psi_c(\cdot)$ (see Theorem 1.2.15).
In order to answer Question 1.2.3, we prove the following results.

**Lemma 1.2.5.** (Carlson) Let $X$ be a Hausdorff space with a $\pi$-base $\mathcal{B}$ whose elements have quasiregular closure. Then $X$ is quasiregular.

*Proof.* Let $U$ be a non-empty open subset of $X$. There exists $V \in \mathcal{B}$ such that $V \subseteq U$ and $\overline{V}$ is quasiregular. The set $V = V \cap \overline{V}$ is open in $\overline{V}$. As $\overline{V}$ is quasiregular, there exists a non-empty open subset $W$ of $X$ such that $\emptyset \neq W \cap \overline{V} \subseteq \overline{(W \cap \overline{V})}^{\overline{V}} = \overline{(W \cap \overline{V})} \cap \overline{V} \subseteq V$.

As $W \cap \overline{V} \neq \emptyset$ then $W \cap V \neq \emptyset$. Furthermore, we have $\overline{(W \cap V)} \subseteq \overline{(W \cap \overline{V})}$ and $\overline{(W \cap V)} \subseteq \overline{V}$. Thus $\overline{(W \cap V)} \subseteq V \subseteq U$. As $W \cap V$ is open in $X$ and non-empty, we conclude $X$ is quasiregular. □

**Corollary 1.2.6.** A Hausdorff space $X$ has a $\pi$-base $\mathcal{B}$ whose elements have quasiregular closure if and only if $X$ is quasiregular.

The previous corollary gives us the motivation to place the hypothesis of quasiregularity on the space instead of putting it on the closure of the elements of a $\pi$-base. Recall that, in [14], Bella and Cammaroto proved that $|X| \leq d(X)^{t(X)\psi_c(X)}$ for every Hausdorff space $X$.

**Theorem 1.2.7.** (Carlson) Let $X$ be a quasiregular Hausdorff space with an H-closed $\pi$-base. Then $|X| \leq 2^{wL(X)t(X)\psi_c(X)}$.

*Proof.* Let $\kappa = wL(X)t(X)\psi_c(X)$ and let $\mathcal{B}$ be a $\pi$-base of non-empty open sets with closures that are H-closed and quasiregular. For each $B \in \mathcal{B}$, as $\overline{B}$ is H-closed, by the Dow-Porter's result (in [33]) we have that $|\overline{B}| \leq 2^{\psi_c(\overline{B})} \leq 2^{\psi_c(X)} \leq 2^{\kappa}$. Since $\psi_c(X) \leq \kappa$, for each $x \in X$ we can fix a collection $\mathcal{V}_x$ of open neighborhoods of $x$

such that $|\mathcal{V}_x| \leq \kappa$ and $\bigcap\{\overline{V} : V \in \mathcal{V}_x\} = \{x\}$. Without loss of generality we may assume that each $\mathcal{V}_x$ is closed under finite intersections.

We will construct by transfinite recursion a non-decreasing chain of open sets $\{U_\alpha : \alpha < \kappa^+\}$ such that

(1) $|\overline{U_\alpha}| \leq 2^\kappa$ for every $\alpha < \kappa^+$, and

(2) if $X \setminus \overline{\bigcup \mathcal{M}} \neq \emptyset$ for some $\mathcal{M} \in [\bigcup\{\mathcal{V}_x : x \in \overline{U_\alpha}\}]^{\leq\kappa}$, then there is $B_{\mathcal{M}} \in \mathcal{B}$ such that $B_{\mathcal{M}} \subset U_{\alpha+1} \setminus \overline{\bigcup \mathcal{M}}$.

Let $B_0 \in \mathcal{B}$ be arbitrary. We set $U_0 = B_0$. Then $|\overline{U_0}| \leq 2^\kappa$. If $\beta = \alpha + 1$, for some $\alpha$, then for every $\mathcal{M} \in [\bigcup\{\mathcal{V}_x : x \in \overline{U_\alpha}\}]^{\leq\kappa}$ such that $X \setminus \overline{\bigcup \mathcal{M}} \neq \emptyset$, we choose $B_{\mathcal{M}} \in \mathcal{B}$ such that $B_{\mathcal{M}} \subseteq X \setminus \overline{\bigcup \mathcal{M}}$. We define $U_\beta = U_\alpha \cup \bigcup\{B_{\mathcal{M}} : \mathcal{M} \in [\bigcup\{\mathcal{V}_x : x \in \overline{U_\alpha}\}]^{\leq\kappa}, X \setminus \overline{\bigcup \mathcal{M}} \neq \emptyset\}$. Therefore, by Bella-Cammaroto's inequality, we have that $|\overline{U_\beta}| \leq 2^\kappa$. If $\beta < \kappa^+$ is a limit ordinal we let $U_\beta = \bigcup_{\alpha<\beta} U_\alpha$. Then clearly $|U_\beta| \leq 2^\kappa$, hence $|\overline{U_\beta}| \leq 2^\kappa$.

Let $F = \bigcup\{\overline{U_\alpha} : \alpha < \kappa^+\}$. Then $|F| \leq 2^\kappa$. Since $t(X) \leq \kappa$, $F$ is closed and therefore $F = \overline{\bigcup\{U_\alpha : \alpha < \kappa^+\}}$. Thus, $F$ is a regular-closed set.

We will show that $X = F$. Suppose that $X \neq F$. As $X \setminus F$ is open, $X$ is quasiregular, and the fact that $\mathcal{B}$ is a $\pi$-base there exists $B \in \mathcal{B}$ such that $\overline{B} \subseteq X \setminus F$. Now, fix $x \in F$. We have $\overline{B} \cap \bigcap\{\overline{V} : V \in \mathcal{V}_x\} = \emptyset$. We will show that there exists $V \in \mathcal{V}_x$ such that $V \cap \overline{B} = \emptyset$. Suppose by way of contradiction that $V \cap \overline{B} \neq \emptyset$ for every $V \in \mathcal{V}_x$. The family $\mathcal{W} = \{V \cap \overline{B} : V \in \mathcal{V}_x\}$ is an open filter base on $\overline{B}$ as it is closed under finite intersections. $\mathcal{W}$ can then be extended to an open ultrafilter $\mathcal{U}$ on $\overline{B}$. As $\overline{B}$ is H-closed, $\mathcal{U}$ must converge to a point $p \in \overline{B}$. Therefore for every $V \in \mathcal{V}_x$ we have $p \in \overline{(V \cap \overline{B})}^{\overline{B}} = \overline{(V \cap \overline{B})} \cap \overline{B} \subseteq \overline{V} \cap \overline{B}$ and thus $p \in \overline{B} \cap \bigcap\{\overline{V} : V \in \mathcal{V}_x\}$. But this is a contradiction as $\overline{B} \cap \bigcap\{\overline{V} : V \in \mathcal{V}_x\} = \emptyset$.

Therefore for every $x \in F$ there exists $V_x \in \mathcal{V}_x$ such that $V_x \cap \overline{B} = \emptyset$. Clearly $\{V_x : x \in F\}$ is an open cover of $F$. Since $wL(X)$ is hereditary with respect to regular-closed sets, there exists $\mathcal{M} \in \{V_x : x \in F\}^{\leq\kappa}$ such that $F \subseteq \overline{\bigcup \mathcal{M}}$. Then there exists $\alpha < \kappa^+$ such that $\mathcal{M} \in [\bigcup\{\mathcal{V}_x : x \in \overline{U_\alpha}\}]^{\leq\kappa}$. As $\overline{B} \cap \bigcup \mathcal{M} = \emptyset$ it follows that $B \subset X \setminus \overline{\bigcup \mathcal{M}}$, hence $X \setminus \overline{\bigcup \mathcal{M}} \neq \emptyset$. Thus, there exists $B_{\mathcal{M}} \in \mathcal{B}$ such that $\emptyset \neq B_{\mathcal{M}} \subseteq U_{\alpha+1} \setminus \overline{\bigcup \mathcal{M}} \subseteq F \setminus \overline{\bigcup \mathcal{M}} = \emptyset$. Since this is a contradiction, we conclude that $X = F$ and the proof is completed. □

Example 1.2.2 witnesses the fact that the hypothesis of quasiregularity in the previous theorem is essential.
Recall the following theorem.

**Theorem 1.2.8.** [16, Theorem 4.18] Let $X$ be a Hausdorff space with an H-closed $\pi$-base. Then $d_\theta(X) \leq 2^{wL(X)t(X)\psi_c(X)}$.

Observe that the same bound cannot hold for the density in all spaces with an H-closed $\pi$-base. Indeed, the Bella-Cammaroto's inequality $|X| \leq d(X)^{t(X)\psi_c(X)}$ for Hausdorff spaces (see [14]), leads to the inequality $|X| \leq 2^{wL(X)t(X)\psi_c(X)}$ for spaces with an H-closed $\pi$-base, which is not true by Example 1.2.2.

We can prove the following result.

**Lemma 1.2.9.** Let $X$ be a Hausdorff space with a $\pi$-base $\mathcal{B}$ whose elements have compact boundaries and $F \subseteq X$ a closed set such that $X \setminus F \neq \emptyset$. Then there is $B \in \mathcal{B}$ such that $\overline{B} \cap F = \emptyset$.

*Proof.* There is $B \in \mathcal{B}$ such that $B \cap F = \varnothing$. We are done if $\overline{B} \cap F = \varnothing$. Otherwise, $(\overline{B} \backslash B) \cap F \neq \varnothing$ and $L = (\overline{B} \backslash B) \cap F$ is compact. Let $y \in B$. As $X$ is Hausdorff, there are open sets $V_y$ and $W_y$ such that $y \in V$, $(\overline{B} \backslash B) \cap F \subseteq W_y$, and $V_y \cap W_y = \varnothing$. There is $C \in \mathcal{B}$ such that $\varnothing \neq C \subseteq B \cap V_y$ implying $\overline{C} \subseteq \overline{B \cap V_y} \subseteq \overline{V_y}$. As $\overline{C} \cap F \subseteq \overline{B} \cap F = (\overline{B} \backslash B) \cap F \subseteq W_y$ and $W \cap \overline{V_y} = \varnothing$, it follows that $\overline{C} \cap F = \varnothing$. □

By Lemma 1.2.9 and by Theorem 1.2.7 we obtain the following results.

**Corollary 1.2.10.** A Hausdorff space with $\pi$-base whose elements have compact boundaries is quasiregular.

**Corollary 1.2.11.** If a Hausdorff space $X$ has an H-closed $\pi$-base whose elements have compact boundaries, then $|X| \leq 2^{wL(X)t(X)\psi_c(X)}$.

In order to answer Question 1.2.4, we start with a well known result that characterizes compactness of the semigularization and we give the proof for sake of completeness.
Recall that a subset $U \subseteq X$ is called regular closed if $U = \overline{int(U)}$ and regular open if $U = int(\overline{U})$. It is known that if a space is H-closed every regular closed subset of it inherits the property. Given a space $(X, \tau)$, the semiregularization of $X$, denoted by $X_s$, is the space $X$ endowed with the topology generated by the basis $\{int(\overline{U}) : U \in \tau\}$ of all the regular open sets of $X$.

**Theorem 1.2.12.** (Katětov) A space $X$ is Urysohn and H-closed if and only if $X_s$ is compact Hausdorff.

*Proof.* Since a space is compact iff it is H-closed and regular it suffices to show that $X_s$ is regular. Let A be a regular-closed subspace of X and $p$ be a point not in $A$. For each $q \in A$, there are disjoint regular-open sets $U_q$ and $V_q$ such that $p \in U_q$, $q \in V_q$, and $\overline{U_q}$ and $\overline{V_q}$ are disjoint. There is a finite subset $F$ of $A$ such that $A$ is contained in $V = \bigcup\{\overline{V_q} : q \in F\}$. Let $U = \bigcap_{q \in F} U_q$. Then $U \cap V = \emptyset$. We have that $V$ is a regular closed set and that $p \in X \setminus \overline{V} \subseteq X \setminus A$. This completes the proof. □

Using the previous theorem, we prove the next result.

**Lemma 1.2.13.** Let $X$ be a Urysohn space. If there exists an H-closed $\pi$-base in $X$, then there exists a compact $\pi$-base in $X_s$.

*Proof.* Let $\mathcal{B}$ be an H-closed $\pi$-base in $X$. Since $\overline{B} = \overline{int(\overline{B})} = cl_s(int(\overline{B}))$, where $cl_s(A)$ denotes the closure in the semiregularization $X_s$, by Theorem 1.2.12, we have that $cl_s(int(\overline{B}))$ is compact in $X_s$. We have to prove that $\{int(\overline{B}) : B \in \mathcal{B}\}$ is a $\pi$-base in $X_s$. Fix a basic open set $int(\overline{V})$. There exists $B \in \mathcal{B}$ such that $B \subseteq V$. Then $int(\overline{B}) \subseteq int(\overline{V})$. This completes the proof. □

In [1] Alas and Kocinac introduced, for Hausdorff spaces, the cardinal function $k(X)$ that is the least cardinal $\kappa$ such that for every $x \in X$ there exists a family $\mathcal{V}_x$ of open neighborhoods of $x$ such that $|\mathcal{V}_x| \leq \kappa$ and for every regular closed subset $\overline{U}$, containing $x$, there exists $V \in \mathcal{V}_x$ such that $\overline{V} \subseteq \overline{U}$. It is straightforward that $k(X) \leq \chi(X)$. The proof of the following lemma is once again direct.

**Lemma 1.2.14.** [1] $k(X) = \chi(X_s)$ for every Hausdorff space $X$.

**Theorem 1.2.15.** Let $X$ be a Urysohn space with an H-closed $\pi$-base. Then $|X| \leq 2^{wL(X)k(X)}$.

*Proof.* From Lemma 1.2.13 it follows that $X_s$ has a compact $\pi$-base. Since $wL(X_s) = wL(X)$ and by Lemma 1.2.14, we have that $|X| = |X_s| \leq 2^{wL(X_s)\chi(X_s)} = 2^{wL(X)k(X)}$. □

The following result is assumed without proof in [33]. A proof was given in [22].

**Lemma 1.2.16.** [33] Let $X$ be an H-closed space. Then $\chi(X_s) \leq \psi_c(X)$.

*Proof.* Let $\kappa = \psi_c(X)$ and $x \in X$. There is a family $\mathcal{U}$ of open neighbourhood of $x$ of such that $x \in \bigcap_{U \in \mathcal{U}} U \subseteq \bigcap_{U \in \mathcal{U}} \overline{U} = \{x\}$ and $\kappa = |\mathcal{U}|$. Without loss of generality we can assume that $\mathcal{U}$ is closed under finite intersections. We want to show that $\{int(\overline{U}) : U \in \mathcal{U}\}$ is a neighborhood base of $x$ in $X(s)$. Let $T$ be an open neighborhood of $x$ in $X(s)$. As $X(s)$ is semiregular, we can assume that $T = int(\overline{U})$ is regular open. So, $\{x\} = \bigcap_{U \in \mathcal{U}} \overline{U} \subseteq T$ and then $X \backslash T \subseteq X \backslash \bigcap_{U \in \mathcal{U}} \overline{U} = \bigcup_{U \in \mathcal{U}} X \backslash \overline{U}$. Thus, $\{X \backslash \overline{U} : U \in \mathcal{U}\}$ is a family of regular open sets of $X$ that cover $X \backslash T$. Since $X \backslash T$ is a H-set (i.e. a regular closed subset in a H-closed space), there is a finite subset $\mathcal{G} \subseteq \mathcal{U}$ such that $X \backslash T \subseteq \bigcup_{U \in \mathcal{G}} \overline{X \backslash \overline{U}} = \bigcup_{U \in \mathcal{G}} X \setminus int(\overline{U}) = X \setminus \bigcap_{U \in \mathcal{G}} int(\overline{U})$. Then, $\bigcap_{U \in \mathcal{G}} int(\overline{U}) \subseteq T$ implying $x \in int(\overline{\bigcap_{U \in \mathcal{G}} U}) \subseteq int(\bigcap_{U \in \mathcal{G}} \overline{U}) = \bigcap_{U \in \mathcal{G}} int(\overline{U}) \subseteq T$. By the arbitrarity of $x$, we conclude that $\chi(X(s)) \leq \kappa$. □

Since the relations $\psi_c(X) = \psi_c(X_s) \leq \chi(X_s)$ are true for every space $X$, then we can say that $k(X) = \chi(X_s) = \psi_c(X)$ holds for H-closed spaces. Then it is natural to pose the following open question.

**Question 1.2.17.** Is it true that for Urysohn spaces having an H-closed $\pi$-base the equality $k(X) = \psi_c(X)$ holds?

The following example shows that Theorem 1.2.15 is an actual improvement of the bound $|X| \leq 2^{wL(X)\chi(X)}$ for Urysohn spaces having an H-closed $\pi$-base (in [15]).

**Example 1.2.18.** There exist an Urysohn H-closed space $Z$ such that $k(Z) < \chi(Z)$.

Consider the space $Z = 2^\omega$ with the following topology on it: a basic open neighborhood of a point $\alpha$ is of the form $(U \setminus T)$, where $U$ is a basic open subset in $2^\omega$ containing $\alpha$ and $T$ is a subset of $2^\omega$ such that $|T| \leq \omega$ and $\alpha \notin T$. Clearly, $Z$ is Urysohn. We want to show that $Z$ is H-closed. Let $\mathcal{U}$ be an open cover of $Z$ made by basic open sets of the form $B \setminus S$. Since $2^\omega$ is compact there exist $B_1, ..., B_k$ such that $Z = 2^\omega \subseteq \bigcup_{i=1}^m B_i \subseteq \bigcup_{i=1}^m \overline{B_i}^{2^\omega} = \bigcup_{i=1}^m \overline{B_i \setminus S_i}^Z$. This proves that $Z$ is H-closed. Then every $\pi$-base is an H-closed $\pi$-base. As an additional remark, we can say that $wL(Z) < \omega$. It is easy to see that $(Z)_s = 2^\omega$. Then, by Lemma 1.2.14, $k(Z) = \chi(2^\omega) = \omega$. It is straightforward to see that $\chi(Z) = |[2^\omega]^{\leq\omega}| = 2^\omega$. △

Recall that the $\theta$-tightness, $t_\theta(X)$, is the least cardinal $\kappa$ such that if $x \in cl_\theta(A)$, then there exists $B \subseteq A$ such that $|B| \leq \kappa$ and $x \in cl_\theta(B)$. If $X$ is Urysohn is possible to define the $\theta$-pseudocharacter, $\psi_\theta(X)$, that is the least cardinal $\kappa$ such that for each point $x \in X$ there exists a family $\mathcal{V}_x$ of open subsets containing $x$ such that $|\mathcal{V}_x| \leq \kappa$ and $\{x\} = \bigcap_{V \in \mathcal{V}_x} cl_\theta(\overline{V})$. Clearly, $\psi_c(X) \leq \psi_\theta(X)$ for every Urysohn space $X$.
It can be easily proved that $\psi_\theta(X) \leq k(X)$ for every Urysohn space $X$. Then it is worthwhile to pose the following question.

**Question 1.2.19.** Does the inequality $|X| \leq 2^{wL(X)t_\theta(X)\psi_\theta(X)}$ hold for every Urysohn space $X$ with an H-closed $\pi$-base?

In [38] Gotchev has shown that if $X$ is Urysohn then $|X| \leq d_\theta(X)^{t_\theta(X)\psi_\theta(X)}$. Therefore the following proposition gives a partial answer to the previous question.

**Proposition 1.2.20.** Let $X$ be a Urysohn space with an H-closed $\pi$-base. Then $|X| \leq 2^{wL(X)t(X)t_\theta(X)\psi_\theta(X)}$

*Proof.* By combining Gotchev's result, Theorem 1.2.8 and the fact that $\psi_c(X) \leq \psi_\theta(X)$ for every Urysohn space we get that

$$|X| \leq d_\theta(X)^{t_\theta(X)\psi_\theta(X)} \leq (2^{wL(X)t(X)\psi_c(X)})^{t_\theta(X)\psi_\theta(X)} \leq 2^{wL(X)t(X)t_\theta(X)\psi_\theta(X)}.$$

□

## 1.3 Martin's Axiom and quasiregular spaces with a $\pi$-base whose elements have an H-closed closure

Recall the topological definition of Martin's Axiom.

**Definition 1.3.1.** Given a cardinal $\kappa$, $\aleph_0 \leq \kappa < 2^{\aleph_0}$, $MA(\kappa)$ states that for each compact Hausdorff space $X$ with the ccc, if $(U_\alpha : \alpha < \kappa)$ is a family of open dense subsets of $X$, then $\bigcap\{U_\alpha : \alpha < \kappa\} \neq \emptyset$.

The following theorem presents a topological and a set theoretic condition equivalent to Martin's Axiom that are well-known in the literature.

**Theorem 1.3.2.** [55] Let $\aleph_0 \leq \kappa < 2^{\aleph_0}$. The followings are equivalent.

1. $MA(\kappa)$.
2. For each Hausdorff space $X$ with the ccc such that $\{x \in X : x$ has a compact neighborhood$\}$ is dense in $X$, if $(U_\alpha : \alpha < \kappa)$ is a family of open dense subsets of $X$, then $\bigcap\{U_\alpha : \alpha < \kappa\}$ is a dense subset of $X$.
3. For each poset $(\mathbb{P}, \leq, 1)$ with the ccc, if $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ with cardinality $\kappa$ then there exists a filter $G \subseteq \mathbb{P}$ such that for every $D \in \mathcal{D}$ one has $D \cap G \neq \emptyset$.

The following result provides a statement equivalent to Martin's Axiom. We demonstrate that it is possible to replace compactness with a combination of two weaker concepts: quasiregularity and H-closedness.

**Theorem 1.3.3.** Let $\aleph_0 \leq \kappa < 2^{\aleph_0}$. The following is equivalent to the conditions in Theorem 1.3.2.

4. For each quasiregular H-closed space $X$ with the ccc, if $(U_\alpha : \alpha < \kappa)$ is a family of open dense subsets of $X$, then $\bigcap\{U_\alpha : \alpha < \kappa\} \neq \emptyset$.

*Proof.* 3. $\implies$ 4. Let $(X, \tau)$ be a quasiregular H-closed space having the ccc. Let $(U_\alpha : \alpha < \kappa)$ be a family of open dense subsets of $X$. Consider the following poset $(\mathbb{P}, \leq, 1) = (\tau \setminus \{\emptyset\}, \subseteq, X)$. For each $\alpha < \kappa$ construct a family $\mathcal{E}_\alpha = \{U \in \mathbb{P} : \overline{U} \subseteq U_\alpha\}$. For the quasiregularity of $X$, the families $\mathcal{E}_\alpha$, $\alpha < \kappa$, are non-empty and since $U_\alpha$ is an open dense subset of $X$, $\mathcal{E}_\alpha$ is dense in $\mathbb{P}$ in the sense of posets. Then there exist a filter $\mathcal{F} \subseteq \mathbb{P}$, which is an open filter on $X$ such that $\mathcal{E}_\alpha \cap \mathcal{F} \neq \emptyset$ for every $\alpha < \kappa$. Then for every $\alpha < \kappa$ choose $F_\alpha \in \mathcal{E}_\alpha \cap \mathcal{F}$. Since $X$ is H-closed the adherence of $\mathcal{F}$ is non-empty, then there exists $x \in \bigcap\{\overline{F} : F \in \mathcal{F}\}$. Therefore $x \in \bigcap\{\overline{F_\alpha} : \alpha < \kappa\} \subseteq \bigcap\{U_\alpha : \alpha < \kappa\}$. □

Moreover, it is possible to find some interesting other characterizations involving the $\pi$-bases as the following result shows.

**Theorem 1.3.4.** Let $\aleph_0 \leq \kappa < 2^{\aleph_0}$. The followings are equivalent to the statements of Theorem 1.3.2 and Theorem 1.3.3.

5. For each Hausdorff space $X$ with the ccc such that $\{x \in X : x$ has a quasiregular H-closed neighborhood$\}$ is dense in $X$, if $(U_\alpha : \alpha < \kappa)$ is a family of open dense subsets of $X$, then $\bigcap\{U_\alpha : \alpha < \kappa\}$ is a dense subset of $X$.

6. For each Hausdorff space $X$ with the ccc and a compact $\pi$-base, if $(U_\alpha : \alpha < \kappa)$ is a family of open dense subsets of $X$, then $\bigcap\{U_\alpha : \alpha < \kappa\}$ is dense in $X$.

7. For each Hausdorff space $X$ with the ccc and a quasiregular H-closed $\pi$-base, if $(U_\alpha : \alpha < \kappa)$ is a family of open dense subsets of $X$, then $\bigcap\{U_\alpha : \alpha < \kappa\}$ is dense in $X$.

*Proof.* 4. $\Longrightarrow$ 5. Let $X$ be a space with the ccc and let $D = \{x \in X : x$ has a quasiregular H-closed neighborhood$\}$ be a dense subset in $X$. Let $(U_\alpha : \alpha < \kappa)$ be a family of open dense subsets of $X$. Fix a non-empty open subset $W$ of $X$, then there exist $x \in W \cap D$ and a open neighborhood $V$ of $x$ such that $\overline{V}$ is quasiregular and H-closed. Therefore $Y = \overline{V \cap W}$ is a quasiregular H-closed space with the ccc. Moreover $U_\alpha \cap (W \cap V)$, for each $\alpha < \kappa$ is a non-empty open dense subset of $Y$. Then by hypothesis, $\emptyset \neq \bigcap\{U_\alpha \cap (W \cap V) : \alpha < \kappa\} \subseteq \bigcap\{U_\alpha : \alpha < \kappa\} \cap W$.
5. $\Longrightarrow$ 7. Notice that if there exists a quasiregular H-closed $\pi$-base $\mathcal{B}$ then $\bigcup \mathcal{B}$ is a dense subset of $X$ and every point in $\bigcup \mathcal{B}$ has a quasiregular H-closed neighborhood.
7. $\Longrightarrow$ 6. is trivial.
6. $\Longrightarrow$ 2. Notice that if there exists a compact $\pi$-base $\mathcal{B}$ then $\bigcup \mathcal{B}$ is a dense subset of $X$ and every point in $\bigcup \mathcal{B}$ has a compact neighborhood. ☐

Martin's axiom is strictly related to the class of Baire spaces, namely, topological spaces in which the intersection of any countable collection of open dense sets is also dense. Baire spaces are significant in analysis and topology, particularly in the study of spaces of functions. We investigate the connection between quasiregular spaces having an H-closed $\pi$-base and the game associated to the Baire spaces, i.e., the Banach-Mazur game. Recall that the Banach-Mazur game on the space $X$ is played by two players ALICE and BOB in $\omega$-many innings. At the beginning of the game, ALICE chooses a nonempty open set $U_0$ and BOB responds by choosing a nonempty open set $V_0 \subset U_0$. At the $n$-th inning ($n > 0$), ALICE chooses a nonempty $U_n \subset V_{n-1}$ and BOB responds by choosing a nonempty open set $V_n \subset U_n$, and so on. The player BOB wins if and only if $\bigcap_{n \in \omega} V_n \neq \emptyset$. Banach, Mazur and Oxtoby proved that the space $X$ has the Baire property if and only if ALICE does not have a winning strategy in the Banach-Mazur game on $X$. A space $X$ is said Choquet if BOB has a winning strategy in the Banach-Mazur game on $X$. Choquet spaces were introduced in 1975 by White who called them weakly $\alpha$-favorable spaces. Clearly, Choquet spaces are Baire. A Bernstein subset of reals witnesses that Baire spaces need not be Choquet. Bella, Carlson and Gotchev proved the following result.

**Theorem 1.3.5.** [17] A space $X$ with a $\pi$-base whose elements have closures that are compact is Choquet.

It is natural to ask the following question.

**Question 1.3.6.** Is any space $X$ with an H-closed $\pi$-base a Choquet space?

As a corollary of Theorem 1.3.4 we obtain the following.

**Corollary 1.3.7.** If $X$ is a Hausdorff space with the ccc and a quasiregular H-closed $\pi$-base, then $X$ is Baire.

Actually we can remove the hypothesis "ccc"and obtain a stronger result that gives a partial answer to Question 1.3.6. Moreover, by Corollary 1.2.6, we can shift the hypothesis of quasiregularity to the whole space rather than in the closure of the elements of a $\pi$-base. In particular, we can prove the following.

**Theorem 1.3.8.** Let $X$ be a quasiregular Hausdorff space with an H-closed $\pi$-base. Then $X$ is a Choquet space.

*Proof.* Let $\mathcal{B}$ be an H-closed $\pi$-base on $X$. We will construct a winning strategy for BOB in the Banach-Mazur game on $X$. Let $U_0$ be the first choice of ALICE in the game. Since $X$ is quasiregular, it is possible to find an element $B_0 \in \mathcal{B}$ such that $\overline{B_0} \subseteq U_0$. Then $B_0$ will be the first response of BOB. Let $U_1 \subset B_0$ be the second choice of ALICE in the game. Since $X$ is quasiregular, it is possible to find an element $B_1 \in \mathcal{B}$ such that $\overline{B_1} \subseteq U_1$. Then $B_1$ will be the second response of BOB, and so on. The collection $\{B_n : n \in \omega\}$ obtained from the iteration is an open filter base for some open ultrafilter $\mathcal{U}$ on $\overline{B_0}$. Since $\overline{B_0}$ is a regular closed, notice that if $U$ is an open subset of $\overline{B_0}$ then $\overline{U}^{\overline{B_0}} = \overline{U}$. The subset $\overline{B_0}$ is H-closed, therefore there exists $x \in \bigcap\{\overline{U} : U \in \mathcal{U}\}$, hence $x \in \overline{B_n}$ for every $n \in \omega$. Now we prove that $x \in \bigcap_{n\in\omega} B_n$, this will conclude the proof. Suppose not, then there exists $n \in \omega$ such that $x \notin B_n$, but since $x \in \overline{B_{n+1}} \subseteq B_n$, we get a contradiction. □

**Corollary 1.3.9.** Let $X$ be a quasiregular space with an H-closed $\pi$-base. Then $X$ is Baire.

Recall that, given a family of subsets $\mathcal{U}$ and a subset $A$ of a space $X$; the star of $A$ with respect to $\mathcal{U}$ is the set $st(A,\mathcal{U}) = \bigcup\{U : U \in \mathcal{U} \text{ and } U \cap A \neq \emptyset\}$. The star of a one-point set $\{x\}$ with respect to a cover $\mathcal{U}$ is denoted by $st(x,\mathcal{U})$ (for some further properties with this operator see for example [26]). The $n$-star of a subset $A$ with respect to a family $\mathcal{U}$, $st^n(A,\mathcal{U})$, is inductively defined as $st^{n-1}(st(A,\mathcal{U}),\mathcal{U})$. A space $X$ has a $G_\delta$-diagonal provided that there exists a sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of $X$ such that $\bigcap_{n\in\omega} st(x,\mathcal{U}_n) = \{x\}$ for every $x \in X$. Additionally, a space $X$ has a $G_\delta$-diagonal of rank $n$ provided that there exists a sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of $X$ such that $\bigcap_{n\in\omega} st^n(x,\mathcal{U}_n) = \{x\}$ for every $x \in X$.
In [10] the authors showed that if $X$ is a Hausdorff Baire space with a rank 2-diagonal, then $|X| \leq wL(X)^\omega$ and if $X$ is a Hausdorff Baire space with a $G_\delta$-diagonal, then $d(X) \leq wL(X)^\omega$. Therefore, as corollaries of these results and Theorem 1.3.8 we have the following:

**Corollary 1.3.10.** Let $X$ be a space with a quasiregular H-closed $\pi$-base.

(a) If $X$ has a $G_\delta$-diagonal, then $d(X) \leq wL(X)^\omega$.

(b) If $X$ has a rank 2-diagonal, then $|X| \leq wL(X)^\omega$.

# Chapter 2

# Recent studies on Selection Principles

The systematic study of selection principles began with the work on covering properties by Scheepers [61]. His methodical cataloguing gives rise to the use of this approach to describe selective properties involving some other topological objects, not just collections of coverings of some type (see [62, 44, 58, 59, 12, 13, 25]). All uncited results in this chapter are either trivial remarks or can be found in [24] and [27]. For more details on covering properties, their relations with particular cardinal numbers (small cardinals), and a gentle introduction to Forcing methods see Appendices A and C.

## 2.1 Some old and new Selection Principles

In [61] a systematic approach was considered to describe covering properties. This type of new approach has led to catalog these properties within so-called "Selection Principles". In particular, given two collections $\mathcal{A}$ and $\mathcal{B}$ of some topological objects on a space $X$, Scheepers introduced this notation:

$S_1(\mathcal{A}, \mathcal{B})$ : For every sequence $(\mathcal{U}_n : n \in \omega)$ of elements of $\mathcal{A}$ there exists $U_n \in \mathcal{U}_n$, $n \in \omega$, such that $\{U_n : n \in \omega\}$ belongs to $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ : For every sequence $(\mathcal{U}_n : n \in \omega)$ of elements of $\mathcal{A}$ there exists a finite subset $\mathcal{F}_n \in \mathcal{U}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} \mathcal{F}_n$ belongs to $\mathcal{B}$.

$U_{fin}(\mathcal{A}, \mathcal{B})$ : For every sequence $(\mathcal{U}_n : n \in \omega)$ of elements of $\mathcal{A}$ there exists a finite subset $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \omega$, such that $\{\bigcup \mathcal{F}_n : n \in \omega\}$ belongs to $\mathcal{B}$.

Recall that it is called $\gamma$-cover of a space $X$ a particular cover such that each point of $X$ belongs to all but finitely many members of the cover. If one denotes by $\mathcal{O}$ and $\Gamma$ the family of all open covers and the family of all $\gamma$-covers of a space $X$, respectively, it follows that the Rothberger property can be expressed by $S_1(\mathcal{O}, \mathcal{O})$, the Menger property by $S_{fin}(\mathcal{O}, \mathcal{O})$ and the Hurewicz property by $U_{fin}(\mathcal{O}, \Gamma)$.
Inspired by the previous selective variation of Lindelöfness, many mathematicians (see for instance [62], [13], and [12]) introduced and studied some selection principles that are strengthening of separability.

We denote by $\mathcal{D}$ the family of all dense subsets of a space $X$ and by $\mathcal{D}_\Gamma$ the collection of all families $\mathcal{F}$ of subsets such that every nonempty open set intersects all but finitely many members of $\mathcal{F}$. We call R-separable a space satisfying $S_1(\mathcal{D}, \mathcal{D})$, M-separable a space sastisfying $S_{fin}(\mathcal{D}, \mathcal{D})$, and H-separable a space satisfying $U_{fin}(\mathcal{D}, \mathcal{D}_\Gamma)$.

Note that “M-”, “R-”, and “H-”are motived by analogy with Menger, Rothberger, and Hurewicz properties, respectively.

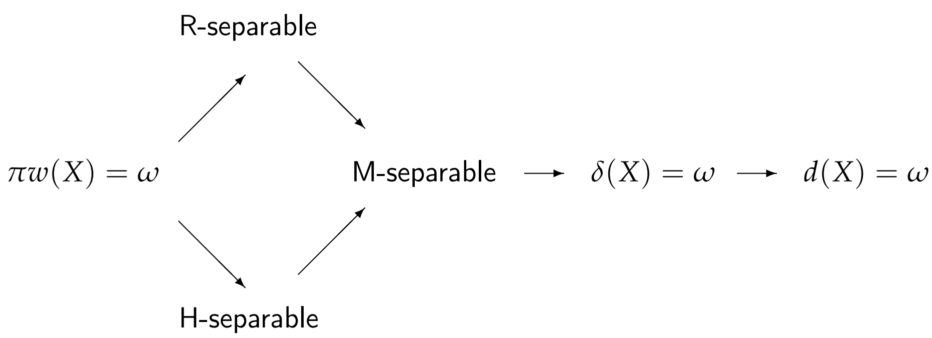
In [62] it is shown that every space with countable $\pi$-weight is R-separable (hence M-separable). Actually, in [62] it was proved that having countable $\pi$-weight is equivalent to a stronger property that comes from a topological game (we will deal with in Chapter 3).

Also in [13] it is observed that every space with countable $\pi$-weight is H-separable.

In [72] it is introduced the cardinal function strong density, i.e., $\delta(X) = \sup\{d(Y) : Y \text{ is dense in } X\}$.

**Proposition 2.1.1.** [72] $d(X) \leq \delta(X) \leq \pi w(X)$ for any space $X$.

Also, it is straightforward to prove that M-separability implies $\delta(X) = \omega$ for any space. Indeed, in an M-separable space every dense subset is separable.
Therefore we have the following implications.




Moreover, Juhasz and Shelah in [42] proved that in the class of compact Hausdorff spaces $\pi w(X) = \delta(X)$.
Therefore the following result can be formulated.

**Proposition 2.1.2.** For a compact Hausdorff space $X$. The followings are equivalent.

1. $\pi w(X) = \omega$;
2. $X$ is R-separable;
3. $X$ is H-separable;
4. $X$ is M-separable;
5. $\delta(X) = \omega$.

In [13] and [12] it is also shown that

- if $\delta(X) = \omega$ and $\pi w(X) < \mathfrak{d}$, then $X$ is M-separable (a little stronger version of this fact in terms of games is estabilished in [62]);
- if $\delta(X) = \omega$ and $\pi w(X) < \mathrm{cov}(\mathcal{M})$, then $X$ is R-separable (a little stronger version of this fact in terms of games is estabilished in [62]);
- if $\delta(X) = \omega$ and $\pi w(X) < \mathfrak{b}$, then $X$ is H-separable.

We saw that every space having a countable base is M-, R- and H-separable. However, it is clear that not every space with countable netweight is M-separable: consider any countable not M-separable space (see, for example, Example 2.14 in [13]).

Then, it is natural to pose the following question.

**Question 2.1.3.** Under what conditions must a space with countable netweight be M-separable?

We introduced and studied the following selection principles involving networks.

**Definition 2.1.4.** A space $X$ is

M-nw-selective (we read M-network-selective) if $nw(X) = \omega$ and for every sequence $(\mathcal{N}_n : n \in \omega)$ of countable networks for $X$ one can select finite $\mathcal{F}_n \subset \mathcal{N}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} \mathcal{F}_n$ is a network for $X$.

R-nw-selective (we read R-network-selective) if $nw(X) = \omega$ and for every sequence $(\mathcal{N}_n : n \in \omega)$ of countable networks for $X$ one can pick $F_n \in \mathcal{N}_n$, $n \in \omega$, such that $\{F_n : n \in \omega\}$ is a network for $X$.

H-nw-selective (we read H-network-selective) if $nw(X) = \omega$ and for every sequence $(\mathcal{N}_n : n \in \omega)$ of countable networks for $X$ one can select finite $\mathcal{F}_n \subset \mathcal{N}_n$, $n \in \omega$, such that for any $x \in X$ and any open neighbourhood $U$ of $x$, there exists some $\kappa \in \omega$ such that for any $n \geq \kappa$ there exists $A \in \mathcal{F}_n$ with $x \in A \subseteq U$.

Note that if the networks $\mathcal{N}_n$, $n \in \omega$, in the previous definitions were uncountable then the space must be countable, and the definitions become trivial. Indeed, in an uncountable space the sequence of networks consisting of all singletons witnesses that the space is not M-nw-selective.

In 1925 Hurewicz [40] proved that a basis property formulated by Menger [51] in 1924 is equivalent to the Menger's property. In particular Hurewicz proved the following proposition (we give the proof for sake of completness). Then, replacing "countable network"with "base"in the definition of M-nw-selectivity, ones obtain a property equivalent to the Menger property in the class of metrizable spaces.
Recall that if $A$ is a subset of a space $X$ and $\mathcal{B}$ is a family of subsets of $X$, we say that $A$ refines $\mathcal{B}$ if $A$ is a subset of some element of $\mathcal{B}$; in this case we write $A \prec \mathcal{B}$.

**Proposition 2.1.5.** [40] Let $X$ be a metrizable space. $X$ is Menger iff for every sequence $(\mathcal{B}_n : n \in \omega)$ of bases for $X$ one can select finite $\mathcal{F}_n \subset \mathcal{B}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} \mathcal{F}_n$ is a base for $X$.

*Proof.* Let $(X, d)$ be Menger and $\xi = (\mathcal{B}_n : n \in \omega)$ a sequence of bases for $X$. Reenumerate $\xi$ as $(\mathcal{B}_{n,m} : n, m \in \omega)$ . We may assume that $\mathcal{B}_{n,m}$ consist of sets of diameter $< \frac{1}{2^n}$. For each $n$, pick finite $\mathcal{F}_{n,m} \subset \mathcal{B}_{n,m}$, $m \in \omega$, such that $\bigcup_{m\in\omega} \mathcal{F}_{n,m}$ is a cover of $X$. Then $\bigcup_{n,m\in\omega} \mathcal{F}_{n,m}$ is a base for $X$. Indeed, every point is contained in a set of diameter $< \frac{1}{2^n}$. Now let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers of $X$. For every $n \in \omega$, put $\mathcal{B}_n = \{U : U \text{ is an open in } X \text{ and } U \prec \mathcal{U}_n\}$. Then $(\mathcal{B}_n : n \in \omega)$ is a sequence of bases for $X$ and by hypothesis we conclude the proof. □

Since every space with countable netweight is hereditarily Lindelöf, M-nw-selectivity is a strengthening of the hereditary Lindelöf property. We also prove that M-nw-(resp., R-nw-, H-nw-)selectivity is a common strengthening of Menger (resp., Rothberger and Hurewicz) property and M-(resp., R- and H-)separability.

**Proposition 2.1.6.** If $X$ is M-nw-selective, then $X$ is Menger.

*Proof.* Let $X$ be M-nw-selective and $(\mathcal{U}_n : n \in \omega)$ a sequence of open covers of $X$. Fix a countable network $\mathcal{N}$ for $X$. For every $n \in \omega$, put $\mathcal{N}_n = \{N \in \mathcal{N} : N$ refines $\mathcal{U}_n\}$. For every $n \in \omega$, $\mathcal{N}_n$ is a countable network for $X$ (in fact, let $W$ be an open subset of $X$ and $x \in W$. Since $\mathcal{U}_n$ covers $X$, there exists $V \in \mathcal{U}_n$ such that $x \in V$. Then $V \cap W$ in an open set containing $x$. Then there exits $N \in \mathcal{N}$ such that $x \in N \subset V \cap W \subset V$. Hence $N \in \mathcal{N}_n$). Then $(\mathcal{N}_n : n \in \omega)$ is a sequence of countable networks for $X$. By hypothesis, there exist finite $\mathcal{F}_n \subset \mathcal{N}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} \mathcal{F}_n$ is a network for $X$. For every $N \in \mathcal{F}_n$, pick $U_{N,n} \in \mathcal{U}_n$ such that $N \subset U_{N,n}$ and put $\mathcal{A}_n = \{U_{N.n} : N \in \mathcal{F}_n\}$. Then $\mathcal{A}_n$, $n \in \omega$, is a finite subfamily of $\mathcal{U}_n$ such that $\bigcup_{n\in\omega} \bigcup \mathcal{A}_n = X$. □

**Proposition 2.1.7.** If $X$ is R-nw-selective (H-nw-selective), then $X$ is Rothberger (Hurewicz).

*Proof.* The proof is similar to the proof of Proposition 2.1.6. □

The converse of propositions 2.1.6 and 2.1.7 is not true as the following example shows.

**Example 2.1.8.** A Rothberger and Hurewicz space which is not M-nw-selective (hence not R-nw-, H-nw-selective).

In [12, Example 2.14] it is proved the existence of a countable subspace $X$ of $C_p(\omega^\omega)$ which is not M-separable, where $C_p(X)$ denotes the space of all the continuous real functions from $X$ with the pointwise convergence topology on it. By next Proposition 2.1.13, the space $X$ is not M-nw-selective. Of course, $nw(X) = \omega$ and $X$ is Rothberger and Hurewicz. △

In order to prove that the nw-selective properties implies the M-(R- and H-) separability we give the following definition of selective properties that are strengthenings of countable tightness and countable dense tightness. Recall that the dense tightness, denoted by $t_d(X)$, is the minimum cardinal $\kappa$ such that for every point $x \in X$ and every dense subset $D \subseteq X$ there exists a subset $A \subseteq D$ such that $|A| \leq \kappa$ and $x \in \overline{A}$.

**Definition 2.1.9.** (see [4], [58], [59], and [12])

A space $X$ has countable fan tightness if for every sequence $(A_n : n \in \omega)$ of subspaces of $X$ with $x \in \overline{A_n}$ for every $n \in \omega$, one can choose finite $F_n \subset A_n$ so that $x \in \overline{\bigcup\{F_n : n \in \omega\}}$.

A space $X$ has countable strong fan tightness if for every sequence $(A_n : n \in \omega)$ of subspaces of $X$ with $x \in \overline{A_n}$ for every $n \in \omega$, one can choose finitea point $x_n \in A_n$ so that $x \in \overline{\{x_n : n \in \omega\}}$.

A space $X$ is weakly Fréchet in the strict sence if for every sequence $(A_n : n \in \omega)$ of subspaces of $X$ with $x \in \overline{A_n}$ for every $n \in \omega$, there are finite $F_n \subset A_n$ such that every neighborhood of $x$ intersects all but finitely many $F_n$'s.

A space $X$ has countable fan tightness with respect to dence subsets if for every sequence $(D_n : n \in \omega)$ of dense subspaces of $X$ one can choose finite $F_n \subset D_n$ so that $x \in \overline{\bigcup\{F_n : n \in \omega\}}$.

A space $X$ has countable strong fan tightness with respect to dence subsets if for every sequence $(A_n : n \in \omega)$ of subspaces of $X$ one can choose a point $x_n \in D_n$ so that $x \in \overline{\{x_n : n \in \omega\}}$.

A space $X$ is weakly Fréchet in the strict sence with respect to dense subspaces if for every sequence $(D_n : n \in \omega)$ of dense subspaces of $X$ and every $x \in X$ there are finite $F_n \subset D_n$ such that every neighborhood of $x$ intersects all but finitely many $F_n$.

We prove the following proposition.

**Proposition 2.1.10.** If $X$ is M-nw-selective, then $X$ has countable fan tightness.

*Proof.* Let $X$ be M-nw-selective, $\mathcal{M}$ be a countable network for $X$, $x \in X$ and $(A_n : n \in \omega)$ be a sequence of subsets of $X$ such that $x \in \overline{A_n}$, for every $n \in \omega$. Every space with countable network is hereditarily separable and thus has countable tightness. Then we may assume that the sets $A_n$ are countable. Let $Y = \{x\} \cup \bigcup_{n\in\omega} A_n$. $Y$ is a countable subset of $X$ and by Proposition 2.1.24, $Y$ is M-nw-selective. For every $n \in \omega$, put $\mathcal{M}_n = \{\{y\} : y \in Y \setminus \{x\}\} \cup \{\{x,a\} : a \in A_n\}$. Since $x \in \overline{A_n}$ for every $n \in \omega$, $(\mathcal{M}_n : n \in \omega)$ is a sequence of countable networks for $Y$. Then one can select finite $\mathcal{F}_n \subset \mathcal{M}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} \mathcal{F}_n$ is a network for $Y$. Put $B_n = \{a \in A_n : \{x,a\} \in \mathcal{F}_n\}$, $n \in \omega$. Then, for every $n \in \omega$, $B_n$ is a finite subset of $A_n$ and $x \in \overline{\bigcup\{B_n : n \in \omega\}}$. □

The converse of the previous result does not hold, as the following example shows.

**Example 2.1.11.** A space having countable fan tightness which is not M-nw-selective.

Consider the space $C_p(I)$, where $I = [0,1]$. Since a space $C_p(X)$ is Menger iff $X$ is finite [4], by Proposition 2.1.6, we have that $C_p(I)$ is not M-nw-selective. Arhangelskii, ([3], Theorem 2.2.2 in [4]) proved that $C_p(X)$ has countable fan tightness iff all finite powers of X are Menger. Then $C_p(I)$ has countable fan tightness.

Recall the following

**Proposition 2.1.12.** [13, Proposition 2.3] Every separable space having countable fan tightness is M-separable.

Then, by Proposition 2.1.10, we obtain

**Proposition 2.1.13.** If $X$ is M-nw-selective, then $X$ is M-separable.

The converse of the previous proposition is not true. Indeed, the space $C_p(I)$, where $I = [0,1]$ is M-separable [13, Example 2.14] and we have proved that it is not M-nw-selective.

Also, recall the following

**Proposition 2.1.14.** [63, Lemma 30] Every separable space having countable strong fan tightness is R-separable.

We can prove that

**Proposition 2.1.15.** If $X$ is R-nw-selective, then $X$ has countable strong fan tightness.

*Proof.* The proof is similar to the proof of Proposition 2.1.10. □

The converse of the previous result is not true as the following example shows. Recall that $iw(X) = min\{w(Y) : Y$ is the continuous bijective image of $X\}$ is the injective weight of $X$. It is known that $iw(X) \leq nw(X) \leq w(X)$, and in the class of compact Hausdorff spaces $iw(X) = nw(X) = w(X)$ (see [7]).

**Example 2.1.16.** A space having countable strong fan tightness which is not M-nw-selective, hence not R-nw-selective.

Consider the space Tychonoff plank $T = (\omega_1 + 1) \times (\omega + 1)$. It is known that a compact space is Rothberger iff it is scattered (see, for a proof, [21, Proposition 34]). Since a $C_p(X)$ space has countable strong fan tightness iff all finite powers of X are Rothberger [58], we have that $C_p(T)$ has countable strong fan tightness. However $nw(C_p(T)) > \omega$, hence $C_p(T)$ is not M-nw-selective. Note that, since $C_p(X)$ is R-separable iff $iw(X) = \omega$ and all finite powers of $X$ are Rothberger [12, Theorem 57], in fact $C_p(T)$ is not R-separable.

By the previous proposition, we have the following result.

**Proposition 2.1.17.** If $X$ is R-nw-selective, then $X$ is R-separable.

**Proposition 2.1.18.** If $X$ is H-nw-selective, then $X$ is weakly Fréchet in the strict sence.

*Proof.* The proof is similar to the proof of Proposition 2.1.10. □

The converse of the previous result does not hold, as the following example shows.

**Example 2.1.19.** A weakly Fréchet in the strict sense space which is not H-nw-selective.

Consider the space $C_p(I)$. Recall that $C_p(X)$ is weakly Fréchet in the strict sense iff all finite powers of X are Hurewicz ([47], stated as in [59]) . Then $C_p(I)$ is weakly Fréchet in the strict sense but is is not M-nw-selective (cfr. Example 2.1.11), hence not H-nw-selective. △

**Proposition 2.1.20.** [12, Proposition 35] A separable space is H-separable iff it is weakly Fréchet in the strict sense with respect to dense subspaces.

**Corollary 2.1.21.** Every separable weakly Fréchet in the strict sense space is H-separable.

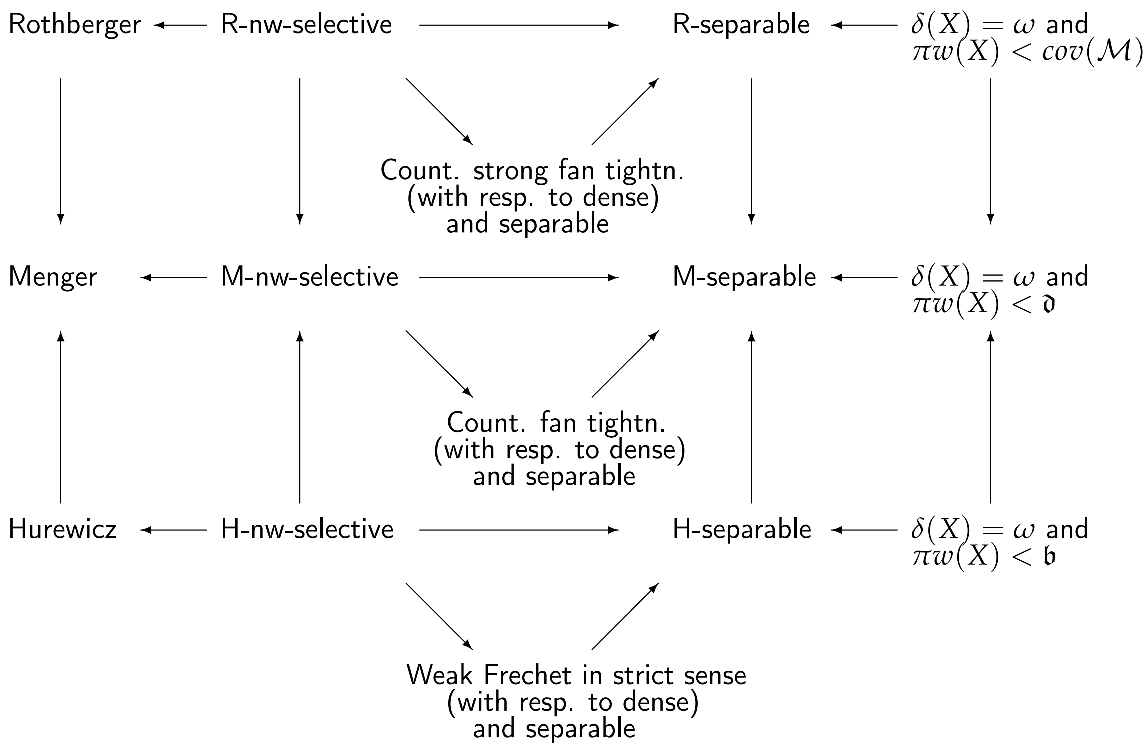
Then, by Proposition 2.1.18, we have the following.

**Proposition 2.1.22.** If $X$ is H-nw-selective, then $X$ is H-separable.

**Example 2.1.23.** A countable space with countable $\pi$-weight (hence R-separable, H-separable, and M-separable), which is not M-nw-selective space (hence not R-nw-selective, not H-nw-selective).

Consider the product of the usual convergent sequence $\omega + 1$ with the discrete space $\omega$. The quotient space of it obtained by identifying all non isolated points is called Fréchet-Urysohn fan space or sequencial fan space. This space, denoted by $S_\omega$, is a typical example of a countable space of countable $\pi$-weight having uncountable fan-tightness, hence $S_\omega$ does not have the nw-selective properties. △

The following diagram sums up all the implications between the previous properties.




Recall that M-, R- and H- separability are not preserved by arbitrary subspaces, but they are preserved by open subspaces, and by dense subspaces (see [13] for M-separability).

We prove that M-mw-, R-nw- and H-nw- separability are preserved by arbitrary subspaces.

**Proposition 2.1.24.** M-nw-separability is a hereditary property.

*Proof.* Let $X$ be M-nw-selective and $Y \subset X$. In particular, $nw(X) = \omega$ and then $nw(Y) = \omega$. Let $\mathcal{M}$ be a countable network for $X$. Then $(M \cap Y : M \in \mathcal{M})$ is a countable network for $Y$. Let $(\mathcal{N}_n : n \in \omega)$ be a sequence of countable networks for $Y$. For every $n \in \omega$, put $\mathcal{M}_n = \mathcal{N}_n \cup \{M \setminus Y : M \in \mathcal{M}\}$. Then $(\mathcal{M}_n : n \in \omega)$ is a sequence of countable networks for $X$. By hypothesis, one can select finite $\mathcal{H}_n \subset \mathcal{M}_n$, $n \in \omega$ such that $\bigcup_{n\in\omega} \mathcal{H}_n$ is a network for $X$. For every $n \in \omega$, put $\mathcal{F}_n = \mathcal{H}_n \cap \mathcal{N}_n$. Then $\mathcal{F}_n$ is a finite subset of $\mathcal{N}_n$ and $\bigcup_{n\in\omega} \mathcal{F}_n$ is a network for $Y$. □

**Proposition 2.1.25.** R-nw-selective and H-nw-selective are hereditary property.

*Proof.* The proof is similar to the proof of Proposition 2.1.24. □

Recall that a space is "analytic"if it is a continuous image of the spaces of irrationals [45]. In [4] the author wrote that Arhangel'skii [3] and J. Calbrix have shown the following result.

**Proposition 2.1.26.** Every analytic Menger space is $\sigma$-compact.

Now we prove that following.

**Proposition 2.1.27.** Every analytic subset of a M-nw-selective Tychonoff space is countable.

*Proof.* By contradiction, assume there exists a M-nw-selective space having an uncountable analytic subspace $Y$. By Proposition 2.1.24, $Y$ is M-nw-selective and then, by Proposition 2.1.13, it is Menger. So, by Proposition 2.1.26, $Y$ is $\sigma$-compact and then $Y$ contains an uncountable compact space $H$. By hypothesis and compactness of $H$, we have that $w(H) = nw(H) = \omega$. Then, by Aleksandroff-Urysohn metrization's theorem, $H$ is metrizable. Hence, since any uncountable compact metrizable space contains a copy of the space of irrationals, we have that $Y$ contains a copy of the space of irrationals. Since irrationals are not Menger, hence by Proposition 2.1.13, not M-nw-selective, we conclude that $Y$ is not M-nw-selective; a contradiction. □

**Corollary 2.1.28.** Every analytic subset of an R-nw- or H-nw- selective Tychonoff space is countable.

*Proof.* The proof is similar to the proof of Proposition 2.1.27 using respectively Proposition 2.1.17 and Proposition 2.1.22 instead of Proposition 2.1.13, and Proposition 2.1.25 instead of Proposition 2.1.24. □

Recall the following result.

**Theorem 2.1.29.** [4, Proposition II.2.11] If $X$ is a compact space of countable weight, then $C_p(X)$ is an analytic space.

**Corollary 2.1.30.** If $X$ is a compact space of countable weight, then $C_p(X)$ is not M-nw-selective.

Using the previous result we can say that, for example $C_p(2^\omega)$ and $C_p(I)$ are not M-nw-selective. Recall that $C_p(2^\omega)$ is H-separable.

Now we provide some results on operations with this classes of spaces and some questions are posed.
It is well-known that Menger, Rothberger and Hurewicz properties are preserved by countable unions. In [39] it is proved that M- and R- separability is preserved by finite unions; it is an open question if H-separability is preserved by finite unions.
We will prove the following results.

**Theorem 2.1.31.** Let $X = \bigoplus_{n\in\omega} X_n$, where $\bigoplus$ denotes the direct sum. If $X_n$ is a M-nw-selective space for every $n \in \omega$, then $X$ is a M-nw-selective space.

*Proof.* Of course, countable network is preserved by countable direct sums. Let $(\mathcal{N}_k : k \in \omega)$ be a sequence of countable networks for $X$. For every $n \in \omega$ consider $\{N \cap X_n : N \in \mathcal{N}_k \text{ and } k \geq n\}$ that is a sequence of countable networks for $X_n$. Since $X_n$ is M-nw-selective for every $n \in \omega$, there exists $(\mathcal{F}_{n,k} : k \geq n)$ with $\mathcal{F}_{n,k}$ a finite subfamily of $\mathcal{N}_k$ for every $k \geq n$ such that $\bigcup_{k\geq n}\{F \cap X_n : F \in \mathcal{F}_{n,k}\}$ is a network for $X_n$. We put $\mathcal{F}_k = \bigcup\{\mathcal{F}_{n,k} : k \geq n\}$ that is a finite subfamily of $\mathcal{N}_k$ for every $k \in \omega$. We can easily see that $\bigcup_{k\in\omega} \mathcal{F}_k$ is a network for $X$. This means $X$ is M-nw-selective. □

**Theorem 2.1.32.** Let $X = \bigoplus_{n\in\omega} X_n$. If $X_n$ is a R-nw-selective space for every $n \in \omega$, then $X$ is a R-nw-selective space.

*Proof.* Of course, countable network is preserved by countable direct sums. Let $(\mathcal{N}_k : k \in \omega)$ be a countable sequence of networks for $X$. Divide the sequence of networks into countably many pairwise disjoint sequences of countable networks $(\mathcal{M}_{k,n} : k, n \in \omega)$. For every $n \in \omega$, $(\mathcal{M}_{k,n} : k \in \omega)$ is a sequence of countable networks of $X_n$. Since $X_n$, $n \in \omega$ is R-nw-selective, there exist $F_{k,n} \in \mathcal{M}_{k,n}$, $n \in \omega$, such that $\{F_{k,n} : n \in \omega\}$ is a network for $X_n$. Then $\{F_{k,n} : k, n \in \omega\}$ is a network for $X$. □

It is natural to pose the following question.

**Question 2.1.33.** Is the countable (or finite) direct sum of H-nw-selective spaces H-nw-selective?

In [64] the author proves that Menger property is not finitely productive. In [39] it is proved that under CH, there is a countable regular maximal space $X$ which is R-separable but $X^2$ is not M-separable.

**Question 2.1.34.** Is the product of two M-nw-selective spaces M-nw-selective? (or, at least, Menger or M-separable?)

In the next section, Theorem 2.2.41 provides a consistent answer to the previous question by showing that the product of two countable H-nw-selective spaces is not M-nw-selective.

## 2.2 Cardinality and weight of M-nw-, R-nw- and H-nw-selective spaces

The aim of this section is to improve the following easy proposition and establish which is the wider class of spaces that can satisfy the nw-selective properties.

**Proposition 2.2.1.** If $X$ is countable second countable space, then $X$ is R-nw-selective.

*Proof.* Of course, $nw(X) = \omega$. Let $X = (x_n : n \in \omega)$, $\mathcal{B} = (B_n : n \in \omega)$ be a base for $X$ and $(\mathcal{N}_n : n \in \omega) = (\mathcal{N}_{n,m} : n, m \in \omega)$ be a sequence of countable networks for $X$. For each $n, m \in \omega$, if $x_n \in B_m$, then take $A_{n,m} \in \mathcal{N}_{n,m}$ such that $x_n \in A_{n,m} \subset B_m$; if $x_n \notin B_m$, then take any $A_{n,m} \in \mathcal{N}_{n,m}$. Then $(A_{n,m} : n, m \in \omega)$ is a network for $X$. □

This fact motivated us to pose the following question we will answer thoughout this section.

**Question 2.2.2.** [24] Are there uncountable M-nw-selective (R-nw-selective or H-nw-selective) spaces?

Note that answering Question 3.2.9 allows us to achive another important goal, i.e., to give a consistent positive answer to the following question.

**Question 2.2.3.** Are there M-nw-selective spaces which are not R-nw-selective or not H-nw-selective?

**Proposition 2.2.4.** Let $X$ be a space such that $nw(X) = \omega$, $|X| < \mathfrak{d}$ and $w(X) < \mathfrak{d}$. Then $X$ is M-nw-selective.

*Proof.* Let $\kappa, \lambda < \mathfrak{d}$ be two cardinals such that $X = \{x_\alpha : \alpha < \kappa\}$ and $\mathcal{B} = \{B_\beta : \beta < \lambda\}$ is a base for $X$. Let $\langle \mathcal{N}_n : n \in \omega \rangle$ be a sequence of countable networks on $X$, where $\mathcal{N}_n = \{N^n_m : m \in \omega\}$. For every $\alpha < \kappa$ and $\beta < \lambda$ such that $x_\alpha \in B_\beta$ consider the function $f_{\alpha,\beta} \in \omega^\omega$ defined by $f_{\alpha,\beta}(n) = \min\{m : x_\alpha \in N^n_m \subseteq B_\beta\}$. The family $\{f_{\alpha,\beta} : \alpha < \kappa, \beta < \lambda, x_\alpha \in B_\beta\}$ is not dominating, and hence there exists a function $f \in \omega^\omega$ such that $f \not\leq^* f_{\alpha,\beta}$ for every $\alpha < \kappa$ and $\beta < \lambda$ such that $x_\alpha \in B_\beta$. A direct verification shows that $\{N^n_m : n \in \omega, m \leq f(n)\}$ is a network for X. □

In an analogous way it is possible to prove the following two propositions, for the first one using Theorem A.0.1.

**Proposition 2.2.5.** Let $X$ be a space such that $nw(X) = \omega$, $|X| < cov(\mathcal{M})$ and $w(X) < cov(\mathcal{M})$. Then $X$ is R-nw-selective.

**Proposition 2.2.6.** Let $X$ be a space such that $nw(X) = \omega$, $|X| < \mathfrak{b}$ and $w(X) < \mathfrak{b}$. Then $X$ is H-nw-selective.

In what follows we shall call spaces satisfying the assumptions of Proposition 2.2.4, 2.2.6 and 2.2.5 *trivial examples* of M-nw-selective, H-nw-selective and R-nw-selective spaces, respectively. The reason for this terminology is that such spaces have these properties solely due to cardinality considerations, and not because of some specific structure etc.

**Proposition 2.2.7.** Let $X$ be a space such that $|X| \geq \mathfrak{d}$. Then $X$ is not M-nw-selective.

*Proof.* Suppose that $nw(X) = \omega$, pick $Y \subseteq X$ such that $|Y| = \mathfrak{d}$ and let $h : Y \to D$ be a bijection, where $D \subseteq \omega^\omega$ is dominating. Without loss of generality we can assume that for every $f \in \omega^\omega$ there exists $g \in D$ such that $f(n) \leq g(n)$ for all $n \in \omega$ (in what follows we shall write $f \leq g$ in such cases). For any $n, k \in \omega$ put $Y^n_k = \{y \in Y : h(y)(n) = k\}$ and consider the countable cover $\mathcal{A}_n = \{Y^n_k : k \in \omega\} \cup \{X \setminus Y\}$ of $X$. For any $n \in \omega$ and every $\mathcal{B}_n \in [\mathcal{A}_n]^{<\omega}$, $\bigcup_{n\in\omega} \bigcup \mathcal{B}_n \not\supseteq Y$. Indeed, pick $g \in \omega^\omega$ such that $\mathcal{B}_n \subseteq \{Y^n_k : k \leq g(n)\} \cup \{X \setminus Y\}$. Pick $y \in Y$ such that $h(y)(n) > g(n)$ for every $n \in \omega$. Then $y \notin Y^n_k$ for every $k \leq g(n)$, so $y \notin \bigcup_{n\in\omega} \bigcup \mathcal{B}_n$. Let $\mathcal{N}$ be a countable network of $X$. For each $n \in \omega$ consider the networks

$$\mathcal{N}_n = \mathcal{N} \wedge \mathcal{A}_n = \{N \cap A : N \in \mathcal{N}, A \in \mathcal{A}_n\}.$$

It follows that if $\mathcal{F}_n \in [\mathcal{N}_n]^{<\omega}$ for each $n \in \omega$, then $X$ is not covered by the family $\bigcup_{n\in\omega} \mathcal{F}_n$, so it cannot be a network for $X$. □

**Corollary 2.2.8.** Let $X$ be a space with $nw(X) = \omega$ and $w(X) < \mathfrak{d}$. Then $X$ is M-nw-selective iff $|X| < \mathfrak{d}$. In particular, this equivalence holds for metrizable separable spaces.

By Propositions 2.2.4 and 2.2.7, it is possible to formulate the following result.

**Corollary 2.2.9.** The following are equivalent facts.

1. $\omega_1 < \mathfrak{d}$;
2. Every space $X$ with $|X| = \omega_1$, $w(X) = \omega_1$, and $nw(X) = \omega$, is M-nw-selective.

The following fact is analogous to Proposition 2.2.7, we present its proof for the sake of completeness.

**Proposition 2.2.10.** Let $X$ be a topological space such that $|X| \geq cov(\mathcal{M})$. Then $X$ is not R-nw-selective.

*Proof.* Suppose that $nw(X) = \omega$, pick $Y \subseteq X$ such that $|Y| = cov(\mathcal{M})$ and let $h : Y \to F'$ be a bijection, where $F' \subseteq \omega^\omega$ is such that for every $g \in \omega^\omega$ there exists $f \in F'$ such that $g(n) \neq f(n)$ for all $n \in \omega$. Such an $F'$ exists, e.g., we could take $F$ satisfying Theorem A.0.1 and set $F' = \{z \in \omega^\omega : \exists f \in F (z =^* f)\}$. For any $n, k \in \omega$ put $Y^n_k = \{y \in Y : h(y)(n) = k\}$ and consider the countable cover

$\mathcal{A}_n = \{Y^n_k : k \in \omega\} \cup \{X \setminus Y\}$ of $X$. For any $g \in \omega^\omega$ we have $\bigcup_{n\in\omega} Y^n_{g(n)} \not\supseteq Y$. Indeed, pick $f \in F'$ with $f(n) \neq g(n)$ for all $n \in \omega$, and let $y \in Y$ be such that $h(y) = f$. Then $y \notin Y^n_{g(n)}$ for every $n \in \omega$ because $y \in Y^n_{f(n)}$ and $Y^n_{f(n)} \cap Y^n_{g(n)} = \emptyset$, since the family $\{Y^n_k : k \in \omega\}$ is disjoint by the definition. Let $\mathcal{N}$ be a countable network of $X$. For each $n \in \omega$ consider the networks

$$\mathcal{N}_n = \mathcal{N} \wedge \mathcal{A}_n = \{N \cap A : N \in \mathcal{N}, A \in \mathcal{A}_n\}.$$

It follows that if $N_n \in \mathcal{N}_n$ for each $n \in \omega$, then $Y$ is not covered by the family $\{N_n : n \in \omega\}$, so it cannot be a network for $X$. □

**Corollary 2.2.11.** Let $X$ be a space with $nw(X) = \omega$ and $w(X) < cov(\mathcal{M})$. Then $X$ is R-nw-selective iff $|X| < cov(\mathcal{M})$. In particular, this equivalence holds for metrizable separable spaces.

By Propositions 2.2.5 and 2.2.10, it is possible to formulate the following result.

**Corollary 2.2.12.** The following are equivalent facts.

1. $\omega_1 < cov(\mathcal{M})$;
2. Every space $X$ with $|X| = \omega_1$, $w(X) = \omega_1$, and $nw(X) = \omega$, is R-nw-selective.

By Propositions 2.2.9 and 2.2.12, if $\omega_1 = cov(\mathcal{M}) < \mathfrak{d}$ holds, each space of cardinality and weight equal to $\omega_1$ and countable netweight is M-nw-selective not R-nw-selective.

As in the case of Proposition 2.2.10, the next fact is also analogous to Proposition 2.2.7, but we nonetheless present its proof for the sake of completeness.

**Proposition 2.2.13.** Let $X$ be a space such that $|X| \geq \mathfrak{b}$. Then $X$ is not H-nw-selective.

*Proof.* Suppose that $nw(X) = \omega$, pick $Y \subseteq X$ such that $|Y| = \mathfrak{b}$ and let $h : Y \to B$ be a bijection, where $B \subseteq \omega^\omega$ is unbounded. Let $Y^n_k$ and $\mathcal{A}_n$ be defined in the same way as in the proof of Proposition 2.2.7. For any $n \in \omega$ and every $\mathcal{B}_n \in [\mathcal{A}_n]^{<\omega}$ there exists $I \in [\omega]^\omega$ with $\bigcup_{n\in I} \cup\mathcal{B}_n \not\supseteq Y$. Indeed, pick $g \in \omega^\omega$ such that $\mathcal{B}_n \subseteq \{Y^n_k : k \leq g(n)\} \cup \{X \setminus Y\}$. Pick $y \in Y$ such that $h(y)(n) > g(n)$ for infinitely many $n \in \omega$, and let $I$ be the set of all such $n$. Then $y \notin Y^n_k$ for every $k \leq g(n)$ and $n \in I$, so $y \notin \bigcup_{n\in I} \bigcup \mathcal{B}_n$. Let $\mathcal{N}$ be a countable network of $X$. For each $n \in \omega$ consider the networks $\mathcal{N}_n$ defined in the same way as in the proof of Proposition 2.2.7 and note that if $\mathcal{F}_n \in [\mathcal{N}_n]^{<\omega}$ for each $n \in \omega$, then $y$ is not covered by the family $\bigcup_{n\in I} \mathcal{F}_n$. Thus, for every $m \in \omega$ there exists $n \geq m$ (namely $\min(I \setminus m)$) such that no $F \in \mathcal{F}_n$ contains $y$, which implies that $X$ is not H-nw-selective. □

**Corollary 2.2.14.** Let $X$ be a space with $nw(X) = \omega$ and $w(X) < \mathfrak{b}$. Then $X$ is H-nw-selective iff $|X| < \mathfrak{b}$. In particular, this equivalence holds for metrizable separable spaces.

By Propositions 2.2.6 and 2.2.13 it is possible to formulate the following result.

**Corollary 2.2.15.** The following are equivalent facts.

1. $\omega_1 < \mathfrak{b}$;
2. Every space $X$ with $|X| = \omega_1$, $w(X) = \omega_1$, and $nw(X) = \omega$, is H-nw-selective.

By Propositions 2.2.9 and 2.2.15, if $\omega_1 = \mathfrak{b} < \mathfrak{d}$ holds, each space of cardinality and weight equal to $\omega_1$ and countable netweight is M-nw-selective not H-nw-selective. By Propositions 2.2.12 and 2.2.15, if $\omega_1 = cov(\mathcal{M}) < \mathfrak{b}$ holds, each space of cardinality and weight equal to $\omega_1$ and countable netweight is H-nw-selective not R-nw-selective. By Propositions 2.2.12 and 2.2.15, if $\omega_1 = \mathfrak{b} < cov(\mathcal{M})$ holds, each space of cardinality and weight equal to $\omega_1$ and countable netweight is R-nw-selective not H-nw-selective. Additionally, Corollaries 2.2.8, 2.2.11 and 2.2.14 allow us to find consistent examples of sets of reals distinguishing between the corresponding properties: If $cov(\mathcal{M}) < \mathfrak{d}$ (resp. $\mathfrak{b} < \mathfrak{d}$, $cov(\mathcal{M}) < \mathfrak{b}$, $\mathfrak{b} < cov(\mathcal{M})$), then any $X \in [\mathbb{R}]^{cov(\mathcal{M})}$ (resp. $X \in [\mathbb{R}]^{\mathfrak{b}}$, $X \in [\mathbb{R}]^{cov(\mathcal{M})}$, $X \in [\mathbb{R}]^{\mathfrak{b}}$) is M-nw-selective but not R-nw-selective (resp. M-nw-selective but not H-nw-selective, H-nw-selective but not R-nw-selective, R-nw-selective but not H-nw-selective).

However, we do not know of any examples in ZFC distinguishing these properties, because at the moment it is not even known whether there are in ZFC non-trivial countable spaces which are M-nw-selective, H-nw-selective, or R-nw-selective. More precisely, the following question (in fact, each of the 6 subquestions it naturally comprises) is still open.

**Question 2.2.16.**
1. Is there a ZFC example of a non-trivial M-nw-selective (resp. R-nw-selective, H-nw-selective) space $X$? What about countable spaces?
2. Is the existence of a non-trivial uncountable M-nw-selective (resp. R-nw-selective, H-nw-selective) space $X$ consistent with ZFC?

### 2.2.1 nw-Selectivity of countable subspaces of $C_p(X,2)$: sufficient conditions and consistent non-trivial examples

For a topological space $X$ we denote by

- $\mathcal{B}(X)$ the family of all countable Borel covers of $X$;
- $\mathcal{B}_\Omega(X)$ the family of all countable Borel $\omega$-covers of $X$;
- $\mathcal{B}_\Gamma(X)$ the family of all countable Borel $\gamma$-covers of $X$.

We omit $X$ from these notations if it is clear from the context.

**Theorem 2.2.17.** Suppose that $X \subseteq 2^\omega$ is such that $X^n$ is $S_1(\mathcal{B},\mathcal{B})$ for every $n \in \omega$. Let $Y \subseteq C_p(X,2)$ be a countable subset. Then $Y$ is R-nw-selective. Moreover, if $Y$ is dense, then $w(Y) = |X|$.

*Proof.* Clearly, $|X| \geq w(Y) \geq \chi(Y)$, and if $Y$ is dense, then additionally we have[1] $\chi(Y) = \chi(C_p(X,2)) = |X|$, so in this case $w(Y) = |X|$.

Let $\{y_j : j \in \omega\}$ be an enumeration of $Y$ and $\mathcal{N}_k = \{N^k_m : m \in \omega\} \subset \mathcal{P}(Y)$ a countable network for each $k \in \omega$. Given a basic open subset of $C_p(X,2)$ of the form

$$[\vec{x},\vec{\epsilon}] = \{f \in C_p(X,2) : f(x_0) = \epsilon_0, ..., f(x_{n-1}) = \epsilon_{n-1}\},$$

where $\vec{x} \in X^n, \vec{\epsilon} \in 2^n$, and $j \in \omega$, set $A_{j,\vec{\epsilon}} = \{\vec{x} \in X^n : y_j \in [\vec{x},\vec{\epsilon}]\}$. Let $h_{j,\vec{\epsilon}} : A_{j,\vec{\epsilon}} \to \omega^\omega$ be a function defined by

$$h_{j,\vec{\epsilon}}(\vec{x})(k) = \min\{m : y_j \in N^k_m \subseteq [\vec{x},\vec{\epsilon}]\}.$$

[1] This part does not use any additional properties of $X$ like $S_1(\mathcal{B},\mathcal{B})$.

$A_{j,\vec{\epsilon}}$ satisfies $S_1(\mathcal{B},\mathcal{B})$ because this property is hereditary by [65, Theorem 13], and therefore $h_{j,\vec{\epsilon}}[A_{j,\vec{\epsilon}}]$ is Rothberger by [65, Theorem 14] because the function $h_{j,\vec{\epsilon}}$ is clearly Borel. Then $R := \bigcup_{n\in\omega,j\in\omega,\vec{\epsilon}\in 2^n} h_{j,\vec{\epsilon}}[A_{j,\vec{\epsilon}}]$ is Rothberger, being a countable union of Rothberger spaces. So there exists $h \in \omega^\omega$ such that $\{k \in \omega : r(k) = h(k)\}$ is infinite for every $r \in R$. As a result. $\{N^k_{h(k)} : k \in \omega\}$ is a network for $Y$. Indeed, pick a basic open set $[\vec{x},\vec{\epsilon}]$ and a point $y_j \in [\vec{x},\vec{\epsilon}]$. Then $\vec{x} \in A_{j,\vec{\epsilon}}$, and therefore there exists $k \in \omega$ such that $h_{j,\vec{\epsilon}}(\vec{x})(k) = h(k)$, hence

$$y_j \in N^k_{h_{j,\vec{\epsilon}}(\vec{x})(k)} = N^k_{h(k)} \subseteq [\vec{x},\vec{\epsilon}],$$

which completes our proof. □

One of the ways to get non-trivial (namely those having size at least $cov(\mathcal{M})$) examples of spaces $X$ like in Theorem 2.2.17 is using forcing. This approach is not new and in a slightly different form was used in [29], so the next fact may be thought of as folklore. We present its proof since we were unable to find it published elsewhere in the form we need.

**Proposition 2.2.18.** Let $X = \{c_\alpha : \alpha < \lambda\}$ be the set of Cohen generic reals over the ground model $V$ added by the standard poset $Fin(\lambda \times \omega, 2)$ consisting of finite partial functions from $\lambda \times \omega$ to 2, where $\lambda$ is an uncountable cardinal. Let $G$ be $Fin(\lambda \times \omega, 2)$-generic filter giving rise to $X$. Then in $V[G]$, for any $k \in \omega$ and a sequence $\langle \mathcal{B}_n : n \in \omega\rangle$ of countable Borel covers of $X^k$, for each $n \in \omega$ there is $B_n \in \mathcal{B}_n$ such that $X^k \subseteq \bigcup_{n\in\omega} B_n$. I.e., all finite power of $X$ satisfy $S_1(\mathcal{B},\mathcal{B})$.

We shall need the following standard fact whose proof we add for the sake of completeness.

**Lemma 2.2.19.** Let $\lambda, G, X$ be such as in Proposition 2.2.18 and suppose that $D \subset 2^\omega$ is a Borel non-meager set coded in the ground model $V$. Then there exists $\beta < \lambda$ such that $c_\beta \in D$.

*Proof.* Since $D$ is non-meager, there exist $s \in 2^{<\omega}$ such that $D \cap [s]$ is comeager in $[s]$, i.e., $[s] \setminus D$ is meager. (Recall that $[s] = \{z \in 2^\omega : z \restriction |s| = s\}$.) Fix $p \in Fin(\lambda \times \omega, 2)$ and $\beta \in \lambda$ such that $dom(p) \cap (\{\beta\} \times \omega) = \emptyset$. Let $q = p \cup r$, where $dom(r) = \{\beta\} \times |s|$ and $r(\beta, j) = s(j)$ for every $j \in |s|$. Then

$$q \Vdash c_\beta \in [s] \wedge c_\beta \text{ lies in every comeager set coded in } V,$$

and hence $q$ also forces $c_\beta \in D \cap [s]$. Now the statement of the lemma follows by the density argument. □

*Proof of Proposition 2.2.18.* We will prove it by induction on $k$. For $k = 0$ there is nothing to prove. Assuming that it is true for any natural number $\leq k$, we will prove our statement for $k+1$. Consider $\mathcal{B}_n = \{B^n_i : i \in \omega\} \in \mathcal{B}(X^{k+1})$, for every $n \in \omega$. Let $A \in [\lambda]^\omega$ be such that $\langle\langle B^n_i : i \in \omega\rangle : n \in \omega\rangle$ is coded in $V[\{c_\alpha : \alpha \in A\}]$. Let $\omega = I_0 \sqcup I_1$ be a partition into two infinite disjoint sets. The set $(2^\omega)^{k+1} \setminus \bigcup \mathcal{B}_n$ is meager in $(2^\omega)^{k+1}$ for every $n \in \omega$. Indeed, suppose that contrary to our claim there exists $n \in \omega$ such that $K := (2^\omega)^{k+1} \setminus \bigcup \mathcal{B}_n$ is non-meager. Then Lemma 2.2.19 implies that there exists an injective sequence $\langle \beta_i : i < k+1\rangle$ of ordinals in $\lambda \setminus A$ such that $\langle c_{\beta_i} : i < k+1\rangle \in K$, which is impossible because $\mathcal{B}_n$ covers $X^{k+1}$.

For every $n \in I_0$ pick $i_n \in \omega$ such that $\bigcup_{n\in I_0} B^n_{i_n}$ is comeager in $(2^\omega)^{k+1}$. This could be done in $V[\{c_\alpha : \alpha \in A\}]$ as follows: Given an enumeration $\{\vec{s}_n : n \in$

$I_0\}$ of $(2^{<\omega})^{k+1}$, let $i_n$ be such that $B^n_{i_n} \cap [\vec{s}_n]$ is non-meager in $[\vec{s}_n]$, $n \in I_0$, where $[\vec{s}_n] = \prod_{j\le k}[s^n_j]$. Then the union $\bigcup_{n\in I_0} B^n_{i_n}$ is comeager in $(2^\omega)^{k+1}$, because its intersection with each clopen subset of $(2^\omega)^{k+1}$ is non-meager. Fix any mutually different $\alpha_0, ..., \alpha_k \in \lambda \setminus A$. Then $\langle c_{\alpha_0}, ..., c_{\alpha_k}\rangle \in \bigcup_{n\in I_0} B^n_{i_n}$ because any such $\langle c_{\alpha_0}, ..., c_{\alpha_k}\rangle$ lies in every comeager subset of $(2^\omega)^{k+1}$ coded in $V[c_\alpha : \alpha \in A]$. From the above we conclude that

$$Y := X^{k+1} \setminus \bigcup_{n\in I_0} B^n_{i_n} \subset \big\{\langle c_{\alpha_0}, ..., c_{\alpha_k}\rangle : \exists j \le k\, (\alpha_j \in A) \vee \exists j_1, j_2 \le k\, (\alpha_{j_1} = \alpha_{j_2})\big\}.$$

Thus $Y$ is covered by a countable union of homeomorphic copies of $X^j$ with $j \le k$, hence by our assumption we can conclude the proof by covering $Y$ with suitably chosen $B^n_{i_n}$'s for $n \in I_1$. □

Combining the results above and the fact that $cov(\mathcal{M}) = \mathfrak{d} = \mathfrak{c} = \lambda$ after adding $\lambda$-many Cohen reals to a ground model of GCH, where $\lambda$ is a cardinal of uncountable cofinality, we get a consistent non-trivial example of a R-nw-selective space which is also a non-trivial example of a M-nw-selective space.

**Corollary 2.2.20.** Suppose that GCH holds in the ground model $V$. Let $\lambda$ be a cardinal of uncountable cofinality and $G, X$ such as in Proposition 2.2.18. Finally, in $V[G]$ let $Y \subset C_p(X, 2)$ be a countable dense subspace. Then $Y$ is a R-nw-selective (and hence M-nw-selective) and $w(Y) = cov(\mathcal{M}) = \mathfrak{d}$.

The above corollary motivates the following question, which is related to Question 2.2.16.

**Question 2.2.21.** Is there a consistent example of a countable R-nw-selective space of weight $> cov(\mathcal{M})$?

Theorem 2.2.17, Proposition 2.2.18 and Lemma 2.2.19 have their counterparts for random reals, with "Cohen" and "meager" replaced with "random" and "measure zero", respectively. We omit proofs which are completely analogous, i.e., those of Theorem 2.2.22 and Lemma 2.2.24.

Again, this approach of using random reals could be traced back in some sense to [29] and hence may be thought of as folklore. We refer the reader to [9, Section 3.1] for more information about the random forcing.

**Theorem 2.2.22.** Suppose that $X \subseteq 2^\omega$ is such that $X^n$ is $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ for every $n \in \omega$. Let $Y \subseteq C_p(X, 2)$ be a countable subset. Then $Y$ is H-nw-selective. Moreover, if $Y$ is dense, $w(Y) = |X|$.

**Proposition 2.2.23.** Let $X = \{r_\alpha : \alpha < \lambda\}$ be the set of generic random reals over the ground model $V$ added by the standard poset $B(\lambda) = Bor(2^{\lambda\times\omega})/\mathcal{Z}_\lambda$, where $\mathcal{Z}_\lambda$ is the ideal of subsets of $2^{\lambda\times\omega}$ which have measure 0 with respect to the usual product probability measure on $2^{\lambda\times\omega}$. Let also $G$ be $B(\lambda)$-generic over $V$ giving rise to $X$. Then in $V[G]$, for any $k \in \omega$ and a sequence $\langle \mathcal{B}_n : n \in \omega\rangle$ of Borel $\gamma$-covers of $X^k$, for each $n \in \omega$ there is $B_n \in \mathcal{B}_n$ such that $\{B_n : n \in \omega\}$ is a $\gamma$-cover of $X^k$. I.e., all finite power of $X$ satisfy $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.

The key part of the proof of Proposition 2.2.23 relies on the following

**Lemma 2.2.24.** Suppose that $D \subset 2^\omega$ is a Borel non-measure zero set coded in the ground model $V$ and $G, X$ are such as in Proposition 2.2.23. Then there exists $\beta < \lambda$ such that $r_\beta \in D$.

*Proof of Proposition 2.2.23.* We will prove it by induction on $k$ that $X^k$ satisfies $U_{fin}(\mathcal{B}, \mathcal{B}_\Gamma)$, which is equivalent to $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ by [65, Theorem 1].

For $k = 0$ there is nothing to prove. Assuming that it is true for any natural number $\leq k$, we will prove our statement for $k+1$. Consider $\mathcal{B}_n = \{B^n_i : i \in \omega\} \in \mathcal{B}(X^{k+1})$, for every $n \in \omega$. Let $A \in [\lambda]^\omega$ be such that $\langle\langle B^n_i : i \in \omega\rangle : n \in \omega\rangle$ is coded in $V[\{r_\alpha : \alpha \in A\}]$. The set $(2^\omega)^{k+1} \setminus \bigcup \mathcal{B}_n$ has measure 0 in $(2^\omega)^{k+1}$ for every $n \in \omega$. Indeed, suppose that contrary to our claim there exists $n \in \omega$ such that $L := (2^\omega)^{k+1} \setminus \bigcup \mathcal{B}_n$ has positive measure. Then Lemma 2.2.24 implies that there exists an injective sequence $\langle \beta_i : i < k+1\rangle$ of ordinals in $\lambda \setminus A$ such that $\langle r_{\beta_i} : i < k+1\rangle \in L$, which is impossible because $\mathcal{B}_n$ covers $X^{k+1}$.

In $V[\{r_\alpha : \alpha \in A\}]$, for every $n \in \omega$ pick $i_n \in \omega$ such that $\nu(\bigcup_{i \leq i_n} B^n_i) \geq 1 - \frac{1}{2^n}$ and note that $\nu(Z) = 1$, where $\nu$ is the Lebesgue measure on $(2^\omega)^{k+1}$ and

$$Z = \bigcup_{m \in \omega} \bigcap_{n \geq m} \bigcup_{i \leq i_n} B^n_i.$$

Fix any mutually different $\alpha_0, ..., \alpha_k \in \lambda \setminus A$. Then $\langle r_{\alpha_0}, ..., r_{\alpha_k}\rangle \in Z$ because any such $\langle r_{\alpha_0}, ..., r_{\alpha_k}\rangle$ lies in every measure 1 subset of $(2^\omega)^{k+1}$ coded in $V[r_\alpha : \alpha \in A]$. From the above we conclude that

$$Y := X^{k+1} \setminus Z \subset \big\{\langle r_{\alpha_0}, ..., r_{\alpha_k}\rangle : \exists j \leq k\,(\alpha_j \in A) \vee \exists j_1, j_2 \leq k\,(\alpha_{j_1} = \alpha_{j_2})\big\}.$$

Thus $Y$ is covered by a countable union of homeomorphic copies of $X^j$ with $j \leq k$, hence by our assumption we can find $\langle j_n : n \in \omega\rangle \in \omega^\omega$ such that $\{\bigcup_{i \leq j_n} B^n_i : n \in \omega\} \in \mathcal{B}_\Gamma(Y)$. Since $\{\bigcup_{i \leq i_n} B^n_i : n \in \omega\} \in \mathcal{B}_\Gamma(Z)$ by the choice of $\langle i_n : n \in \omega\rangle$, we conclude that

$$\big\{ \bigcup_{i \leq \max\{i_n, j_n\}} B^n_i : n \in \omega\big\} \in \mathcal{B}_\Gamma(Z \cup Y) = \mathcal{B}_\Gamma(X^{k+1}),$$

which completes our proof. □

Combining the results above and the fact that $\mathfrak{d} = \omega_1$ after adding $\lambda$-many random reals to a ground model of GCH, we get a consistent non-trivial example of a H-nw-selective space which is also a non-trivial example of a M-nw-selective space.

**Corollary 2.2.25.** Suppose that GCH holds in the ground model $V$ and $\lambda$ is an uncountable cardinal. Let $G, X$ be such as in Proposition 2.2.23. Finally, in $V[G]$ let $Y \subset C_p(X, 2)$ be a countable dense subspace. Then $Y$ is H-nw-selective (and hence M-nw-selective) and $w(Y) = \lambda = \mathfrak{c} \geq \mathfrak{d} = \omega_1$.

The corollary above among others shows that the counterparts of Question 2.2.21 for H-nw-selective and M-nw-selective spaces have affirmative answers, i.e., there are consistent examples of countable H-nw-selective (resp. M-nw-selective) spaces with weight $> \mathfrak{b}$ (resp. $> \mathfrak{d}$).

### 2.2.2 Various kinds of nw-selectivity of "standard"countable dense subspaces of $C_p(X, 2)$: necessary conditions

In this section we establish some limitations to constructions of non-trivial examples by the methods developed in Section 2.2.1. More precisely, we consider certain specific countable dense subspaces of $C_p(X)$ defined before Theorem 2.2.28, where $X$ is a metrizable separable spaces, and show that these having *nw*-selectivity properties implies $X$ having quite strong combinatorial covering properties with respect to the family of all countable closed covers.

We start by showing that the properties we consider are equivalent to their local counterparts in the realm of countable spaces. We call $\mathcal{N}$ a *network for a space $X$ at $x \in X$*, if for every neighbourhood $U \ni x$ there exists $N \in \mathcal{N}$ with $x \in N \in \mathcal{N}$. Thus, $\mathcal{N}$ is a network for $X$ if and only if it is a network for $X$ at each $x \in X$.

**Definition 2.2.26.** A space $X$ is

locally M-nw-selective if for every $x \in X$ and sequence $\langle \mathcal{N}_m : m \in \omega \rangle$ of networks for $X$ at $x$, there exists a sequence $\langle \mathcal{L}_m : m \in \omega \rangle$ such that $\mathcal{L}_m \in [\mathcal{N}_m]^{<\omega}$ for all $m$ and $\bigcup_{m\in\omega} \mathcal{L}_m$ is a networks for $X$ at $x$;

locally R-nw-selective if for every $x \in X$ and sequence $\langle \mathcal{N}_m : m \in \omega \rangle$ of networks for $X$ at $x$, there exists a sequence $\langle N_m : m \in \omega \rangle$ such that $N_m \in \mathcal{N}_m$ for all $m$ and $\{N_m : m \in \omega\}$ is a networks for $X$ at $x$.

locally H-nw-selective if for every $x \in X$ and sequence $\langle \mathcal{N}_m : m \in \omega \rangle$ of networks for $X$ at $x$, there exists a sequence $\langle \mathcal{L}_m : m \in \omega \rangle$ such that $\mathcal{L}_m \in [\mathcal{N}_m]^{<\omega}$ for all $m$ and $\bigcup_{m\in I} \mathcal{L}_m$ is a networks for $X$ at $x$ for any $I \in [\omega]^{\omega}$;

**Lemma 2.2.27.**

1. If $X$ is locally M-nw-selective (resp. H-nw-selective, R-nw-selective) and $|X| = \omega$, then it is M-nw-selective (resp. H-nw-selective, R-nw-selective).
2. If $X$ is M-nw-selective (resp. H-nw-selective, R-nw-selective), then it is locally M-nw-selective (resp. H-nw-selective, R-nw-selective).

*Proof.* The first item is rather obvious. For the second one it suffices to note that if $\mathcal{M}$ is a (countable) network for $X$ and $\mathcal{N}$ is a (countable) network for $X$ at $x \in X$, then

$$\mathcal{N} \cup \{M \setminus \{x\} : M \in \mathcal{M}\}$$

is a (countable) network for $X$. □

In what follows we shall call a sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of finite families of subsets of $X$ a *$\gamma_{fs}$-sequence*[2] *on* $X$, if for every $F \in [X]^{<\omega}$ there exists $n \in \omega$ such that for all $k \geq n$ there exists $U \in \mathcal{U}_n$ containing $F$.

For a subset $A$ of $X$ the *characteristic function of $A$* is $\chi_A : X \to 2$ such that $\chi_A(x) = 0$ iff $x \in A$. Let $X$ be a metrizable separable zero-dimensional space and $\mathcal{S}$ a base for $X$ closed under finite unions and complements of its elements. Then we denote by $Y_{\mathcal{S}}$ the set $\{\chi_S : S \in \mathcal{S}\} \subset C_p(X, 2)$.

**Theorem 2.2.28.** Let $X$ be a metrizable separable zero-dimensional space and $\mathcal{S}$ a countable clopen base of $X$ closed under finite unions and complements. If $Y = Y_{\mathcal{S}}$ is H-nw-selective as a subspace of $C_p(X, 2)$, then for every sequence $\langle \mathcal{C}_n : n \in \omega \rangle$ of countable closed $\omega$-covers of $X$ there exists a $\gamma_{fs}$-sequence $\langle \mathcal{D}_n : n \in \omega \rangle$ on $X$ such that $\mathcal{D}_n \in [\mathcal{C}_n]^{<\omega}$.

*Proof.* Note that the constant 0 function (which we denote by 0) belongs to $Y$: Given any $S \in \mathcal{S}$, we have that $X \setminus S \in \mathcal{S}$, and hence $X = S \cup (X \setminus S) \in \mathcal{S}$, which yields $0 = \chi_X \in Y$.

[2]"fs" is the abbreviation of *finite subsets*

For every $C \subset X$ we denote by $[C, 0]$ the set $\{f \in C_p(X) : f \restriction C \equiv 0\}$. Let $\langle \mathcal{C}_n : n \in \omega \rangle$ be a sequence of countable closed $\omega$-covers of $X$. It is easy to check that

$$\mathcal{N}_n := \{[C, 0] \cap Y : C \in \mathcal{C}_n\}$$

is a network for $Y$ at 0 for every $n \in \omega$. By Lemma 2.2.27 we know that $Y$ is locally H-nw-selective, and hence there exists a sequence $\langle \mathcal{D}_n : n \in \omega \rangle$ such that $\mathcal{D}_n \in [\mathcal{C}_n]^{<\omega}$ for all $n \in \omega$ and

$$\mathcal{N} := \{[C, 0] \cap Y : C \in \mathcal{D}_n, n \in I\}$$

is a network for $Y$ at 0 for any infinite $I \subset \omega$.

We claim that $\langle \mathcal{D}_n : n \in \omega \rangle$ is a $\gamma_{fs}$-sequence of subsets of $X$. Indeed, suppose towards a contradiction, that there exists $I \in [\omega]^\omega$ and $A \in [X]^{<\omega}$ such that $A \not\subset C$ for any $C \in \bigcup_{n \in I} \mathcal{D}_n$. Set $O = [A, 0] \cap Y$ and note that $O$ is an open neighbourhood of 0 in $Y$. Thus there exists $n \in I$ and $C \in \mathcal{D}_n$ such that $O \supset [C, 0] \cap Y$. However, there exists $x \in A \setminus C$, and hence there exists $S \in \mathcal{S}$ with[3] $C \subset S$ and $x \notin S$. It follows that $\chi_S \in [C, 0] \cap Y$ and $\chi_S \notin [A, 0] \cap Y = O$, which gives the desired contradiction. □

For a topological space $X$ we make the following notation:

- $\mathcal{C}(X)$ is the family of all countable closed covers of $X$;
- $\mathcal{C}_\Omega(X)$ is the family of all countable closed $\omega$-covers of $X$;
- $\mathcal{C}_\Gamma(X)$ is the family of all countable closed $\gamma$-covers of $X$;
- $\mathcal{C}^o(X)$ is the family of all countable clopen covers of $X$;
- $\mathcal{C}^o_\Omega(X)$ is the family of all countable clopen $\omega$-covers of $X$;
- $\mathcal{C}^o_\Gamma(X)$ is the family of all countable clopen $\gamma$-covers of $X$.

Recall from [46] that a countable family $\mathcal{U}$ of subsets of $X$ is *$\omega$-groupable*, if there exists a sequence $\langle \mathcal{D}_n : n \in \omega \rangle$ of mutually disjoint finite subsets of $\mathcal{U}$ such that the set $\{n \in \omega : x \notin \cup \mathcal{D}_n\}$ is finite for all $x \in X$. We extend our list of notation for specific covers of a space $X$ as follows:

- $\mathcal{C}_{\omega-gp}(X)$ is the family of all closed $\omega$-groupable covers of $X$;
- $\mathcal{C}^o_{\omega-gp}(X)$ is the family of all clopen $\omega$-groupable covers of $X$;
- $\mathcal{B}_{\omega-gp}(X)$ is the family of all Borel $\omega$-groupable covers of $X$.

**Corollary 2.2.29.** Let $X$ be a metrizable separable zero-dimensional space and $\mathcal{S}$ a countable clopen base of $X$ closed under finite unions and complements. If $Y = Y_\mathcal{S}$ is H-nw-selective as a subspace of $C_p(X, 2)$, then all finite powers of $X$ satisfy $U_{fin}(\mathcal{O}, \Gamma)$ (i.e., are Hurewicz) and $X$ satisfies $U_{fin}(\mathcal{B}, \mathcal{B}_\Gamma)$ (i.e., is Hurewicz with respect to all countable Borel covers), which is equivalent to $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.

*Proof.* Let $\langle \mathcal{C}_n : n \in \omega \rangle$ be a sequence of countable closed $\omega$-covers of $X$. Theorem 2.2.28 yields a a $\gamma_{fs}$-sequence $\langle \mathcal{D}_n : n \in \omega \rangle$ on $X$ such that $\mathcal{D}_n \in [\mathcal{C}_n]^{<\omega}$. It follows that $\{\cup \mathcal{D}_n : n \in \omega\} \in \mathcal{C}_\Gamma(X)$. Thus, $X$ satisfies $U_{fin}(\mathcal{C}_\Omega, \mathcal{C}_\Gamma)$, which is obviously equivalent to $U_{fin}(\mathcal{C}, \mathcal{C}_\Gamma)$, i.e., the Hurewicz covering property with respect to countable closed covers. [30, Theorem 5.2] implies that $X$ satisfies the Hurewicz

[3] Here we use that $Y = Y_\mathcal{S}$ and not just arbitrary countable dense subset of $C_p(X, 2)$.

property with respect to countable Borel covers, which is equivalent to $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ by [65, Theorem 1].

Finally, to see that all finite powers of $X$ are Hurewicz, note that Theorem 2.2.28 implies that for every sequence $\langle \mathcal{C}_n : n \in \omega \rangle$ of countable closed $\omega$-covers of $X$ there exists a $\gamma_{fs}$-sequence $\langle \mathcal{D}_n : n \in \omega \rangle$ on $X$ such that $\mathcal{D}_n \in [\mathcal{C}_n]^{<\omega}$ and $\mathcal{D}_{n_0} \cap \mathcal{D}_{n_1} = \emptyset$ for any natural numbers $n_0 \neq n_1$. Indeed, for this it is enough replace $\langle \mathcal{C}_n : n \in \omega \rangle$ with a sequence $\langle \mathcal{C}'_m : m \in \omega \rangle$ of countable closed $\omega$-covers of $X$ such that for every cofinite subset $C$ of some $\mathcal{C}_n$, there exists $m \in \omega$ with $\mathcal{C} = \mathcal{C}'_m$. Theorem 2.2.28 implies that there exists a $\gamma_{fs}$-sequence $\langle \mathcal{D}'_m : m \in \omega \rangle$ on $X$ such that $\mathcal{D}'_m \in [\mathcal{C}'_m]^{<\omega}$. Now it is easy to see that one can choose a subsequece $\langle \mathcal{D}_n : n \in \omega \rangle$ of $\langle \mathcal{D}'_m : m \in \omega \rangle$ consisting of mutually disjoint elements and such that $\mathcal{D}_n \in [\mathcal{C}_n]^{<\omega}$.

Using the consequence of Theorem 2.2.28 established in the paragraph above for sequences of countable clopen covers of $X$, and the obvious fact that if a $\gamma_{fs}$-sequence $\langle \mathcal{D}_n : n \in \omega \rangle$ consists of mutually disjoint finite sets, then $\bigcup_{n\in\omega} \mathcal{D}_n$ is an $\omega$-groupable cover of $X$, we conclude that $X$ satisfies $S_{fin}(\mathcal{C}_\Omega, \mathcal{C}_{\omega-gp})$, and hence also $S_{fin}(\mathcal{C}^o_\Omega, \mathcal{C}^o_{\omega-gp})$. By [46, Theorem 16] all finite powers of $X$ are Hurewicz. □

By arguments similarly to (but easier than) those used in the proofs of Theorem 2.2.28 and Corollay 2.2.29, we can also get necessary conditions for countable dense subsets of $C_p(X)$ of the form $Y_\mathcal{B}$ to be M-nw-selective and R-nw-selective.

**Theorem 2.2.30.** *Let $X$ be a metrizable separable zero-dimensional space and $\mathcal{S}$ a countable clopen base of $X$ closed under finite unions and complements. If $Y = Y_\mathcal{S}$ is M-nw-selective as a subspace of $C_p(X, 2)$, then $X$ satisfies $S_{fin}(\mathcal{C}_\Omega, \mathcal{C}_\Omega)$.*

**Corollary 2.2.31.** *Let $X, \mathcal{S}$ be such as in Theorem 2.2.30. If $Y = Y_\mathcal{S}$ is M-nw-selective as a subspace of $C_p(X, 2)$, then all finite powers of $X$ are Menger, and also $X$ has the Menger property with respect to countable closed covers.*

*Proof.* Clearly, $S_{fin}(\mathcal{C}_\Omega, \mathcal{C}_\Omega)$ implies $U_{fin}(\mathcal{C}, \mathcal{C})$, i.e., the Menger property with respect to all countable closed covers

Also, $S_{fin}(\mathcal{C}_\Omega, \mathcal{C}_\Omega)$ implies $S_{fin}(\mathcal{C}^o_\Omega, \mathcal{C}^o_\Omega)$, which for zero-dimensional spaces is equivalent to all finite powers having the Menger property $U_{fin}(\mathcal{O}, \mathcal{O})$ by [44, Theorem 3.9]. □

**Theorem 2.2.32.** Let $X, \mathcal{S}$ be such as in Theorem 2.2.30. If $Y = Y_\mathcal{S}$ is R-nw-selective as a subspace of $C_p(X, 2)$, then $X$ satisfies $S_1(\mathcal{C}_\Omega, \mathcal{C}_\Omega)$.

**Corollary 2.2.33.** Let $X, \mathcal{S}$ be such as in Theorem 2.2.30. If $Y = Y_\mathcal{S}$ is R-nw-selective as a subspace of $C_p(X, 2)$, then all finite powers of $X$ satisfy $S_1(\mathcal{O}, \mathcal{O})$, i.e., are Rothberger, and also $X$ has the Rothberger property $S_1(\mathcal{C}, \mathcal{C})$ with respect to countable closed covers.

*Proof.* By Theorem 2.2.32 we know that $X$ satisfies $S_1(\mathcal{C}_\Omega, \mathcal{C}_\Omega)$, and hence it also satisfies satisfies $S_1(\mathcal{C}_\Omega, \mathcal{C})$, which is equivalent to $S_1(\mathcal{C}, \mathcal{C})$ by the same argument as in the proof of [61, Theorem 17], which asserts that $S_1(\Omega, \mathcal{O})$ and $S_1(\mathcal{O}, \mathcal{O})$ are equivalent.

Also, by Theorem 2.2.32 the space $X$ satisfies $S_1(\mathcal{C}_\Omega, \mathcal{C}_\Omega)$, and hence also $S_1(\mathcal{C}^o_\Omega, \mathcal{C}^o_\Omega)$, which is obviously equivalent to $S_1(\Omega, \Omega)$ because $X$ is zero-dimensional. Finally, $S_1(\Omega, \Omega)$ is equivalent to all finite powers being Rothberger, see [58, Lemma, p. 918] or [44, Theorem 3.8]. □

The necessary conditions proved above motivate the following question.

**Question 2.2.34.** Let $X$ be a metrizable separable zero-dimensional space.

1. Suppose that $X$ is Menger with respect to countable closed covers. Is $X$ Menger with respect to countable Borel covers?
2. Suppose that $X$ is Rothberger with respect to countable closed covers. Is $X$ Rothberger with respect to countable Borel covers?

As we have already mentioned in the proof of Corollary 2.2.29, by [30, Theorem 5.2] the answer to the analogous question for the Hurewicz property is affirmative. Regarding the Rothberger part of Question 2.2.34, in the Laver model all Rothberger metrizable spaces are countable, hence the affirmative answer is consistent, which means that this question is interesting in models where the Borel conjecture fails, e.g., models of CH. Below we show that also for the Menger part of Question 2.2.34 the affirmative answer is consistent.

**Proposition 2.2.35.** In the Miller model, if $X \subset 2^\omega$ satisfies $U_{fin}(\mathcal{C},\mathcal{C})$, then $|X| < \mathfrak{d}$, and hence $X$ satisfies the Menger property with respect to arbitrary countable covers.

*Proof.* First we shall show that any $G \subset X$ is Menger. Indeed, let $\langle \mathcal{U}_n : n \in \omega\rangle$ be a sequence of covers of $G$ by open subsets of $X$. For every $n$ let $\mathcal{A}_n$ be a countable cover of $G$ by open subsets of $X$ such that for every $A \in \mathcal{A}_n$ there exists $U \in \mathcal{U}_n$ with $\bar{A} \subset U$, the closure being taken in $X$. Set

$$\omega_n = \{\bar{A} : A \in \mathcal{A}_n\} \cup \{F_n\}, \text{ where } F_n = X \setminus \cup\mathcal{A}_n,$$

and note that $\omega_n$ is a countable closed cover of $X$. The Menger property for countable closed covers applied to $X$ yields a sequence $\langle \omega_n' : n \in \omega\rangle$ such that $\omega_n' \in [\omega_n]^{<\omega}$ and $G \subset \bigcup_{n\in\omega} \cup\omega_n'$. Since each $F_n$ is disjoint from $G$, we get $G \subset \bigcup_{n\in\omega}\cup\omega_n''$, where $\omega_n'' = \omega_n' \setminus \{F_n\}$. It follows that there exists a finite $\mathcal{A}_n'' \subset \mathcal{A}_n$ such that $\omega_n'' = \{\bar{A} : A \in \mathcal{A}_n''\}$. Consequently, there exists a finite $\mathcal{U}_n'' \subset \mathcal{U}_n$ such that

$$\forall A \in \mathcal{A}_n''\ \exists \mathcal{U} \in \mathcal{U}_n''\ (\bar{A} \subset U).$$

Putting all together, we get $G \subset \bigcup_{n\in\omega}\cup\mathcal{U}_n''$, and therefore $G$ is Menger[4].

In the Miller model for every Menger space $Z \subset 2^\omega$ and a $G_\delta$-subset $H \subset 2^\omega$, if $Z \subset H$, then there is a family $\mathcal{K}$ of compact subspaces of $H$ of size $|\mathcal{K}| \le \omega_1$ and such that $Z \subset \cup\mathcal{K}$, see [53, Theorem 4.4]. As a result, if $Q \in [2^\omega]^\omega$ is disjoint from $Z$, then there exists a $G_{\omega_1}$-subset (i.e., an intersection of $\omega_1$-many open sets) $R$ of $2^\omega$ such that $Q \subset R$ and $R \cap Z = \emptyset$.

Since $X$ is hereditarily Menger, we conclude that for every $Q \in [X]^\omega$ there exists a $G_{\omega_1}$-subset $R(Q)$ with $R(Q) \cap X = Q$. Now a direct application of [73, Lemma 2.5] gives that there exists a family $\mathsf{Q} \subset [2^\omega]^\omega$ of size $|\mathsf{Q}| = \omega_1$ and such that

$$X = \bigcup_{Q\in\mathsf{Q}} (R(Q) \cap X) = \bigcup_{Q\in\mathsf{Q}} Q = \cup\mathsf{Q},$$

which yields $|X| \le \omega_1 < \mathfrak{d}$. Finally, the fact that any space of size $< \mathfrak{d}$ has the Menger property with respect to the family of all countable covers is straightforward. □

The next statement is a consequence of [52, Corollary 4.4].

**Proposition 2.2.36.** In the Laver model, if $X \subset 2^\omega$ satisfies $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$, i.e., is Hurewicz with respect to the family of countable Borel covers, then $|X| < \mathfrak{b}$.

[4] Let us note that this part did not require any assumptions beyond ZFC.

As a conclusion we have the following fact showing that countable spaces considered in Theorems 2.2.28, 2.2.30 and 2.2.32 *cannot* give non-trivial examples satisfying corresponding *nw*-selectivity properties in ZFC.

**Corollary 2.2.37.** In the Miller (resp. Laver) model, let $X$ be a metrizable separable zero-dimensional space and $\mathcal{S}$ a countable clopen base of $X$ closed under finite unions and complements. If $Y_{\mathcal{S}}$ is M-nw-selective (resp. H-nw-selective or R-nw-selective), then it is trivial, i.e., $w(Y) = |X| \leq \omega_1 < \mathfrak{d}$ (resp. $w(Y) = |X| \leq \omega_1 < \mathfrak{b}$ or $w(Y) = |X| \leq \omega < cov(\mathcal{M}) = \omega_1$).

Even though spaces of the form $Y_{\mathcal{S}}$ seem to be one's first inclination to construct countable dense subspaces of $C_p(X,2)$, there are also other countable dense subspaces of $C_p(X,2)$, and we do not have any efficient way of analyzing their *nw*-selectivity properties in terms of (covering) properties of $X$.

**Question 2.2.38.** Are there ZFC examples of metrizable separable zero-dimensional spaces $X$ of size $\geq \mathfrak{d}$ (resp. $\geq \mathfrak{b}$, $\geq cov(\mathcal{M})$) and countable dense subspaces $Y$ of $C_p(X,2)$ which are M-nw-selective (resp. H-nw-selective, R-nw-selective)?

The following fact has been established at the beginning of the proof of Proposition 2.2.35 for Menger spaces without using any additional assumptions beyond ZFC, and the same argument also works in two other cases.

**Corollary 2.2.39.** Let $X$ be a metrizable separable space. If $X$ satisfies the Menger (resp. Hurewicz, Rothberger) property for countable closed covers, then it is hereditarily Menger (resp. Hurewicz, Rothberger).

In [24] it is provided an example distinguishing countable fan tightness and *M-nw*-selectivity which is uncountable, now we can provide a countable one.

**Proposition 2.2.40.** The space $X = C_p(2^\omega, 2)$ is countable $H$-separable and weakly Fréchet in the strict sense, but it is not M-nw-selective.

*Proof.* $2^\omega$ is not hereditarily Menger since $[\omega]^\omega \subset 2^\omega$ is not Menger being a copy of the Baire space. Thus, $C_p(2^\omega, 2)$ is not M-nw-selective by Corollary 2.2.39.

The other properties of $C_p(2^\omega, 2)$ directly follow from [12, Theorem 40]. □

### 2.2.3 Non-preservation by products

This section is devoted to the proof of the following theorem that consistently answer to Question 2.1.34.

**Theorem 2.2.41.** It is consistent that there exist two countable H-nw-selective spaces with non-M-nw-selective product.

We need the following variation of Proposition 2.2.23. Let us note that $2^\omega = \{0,1\}^\omega$ with the operation $+$ of the coordinate-wise addition modulo 2 is a compact Boolean topological group.

**Proposition 2.2.42.** Suppose that $V$ is a ground model of CH and $\{z_\alpha : \alpha < \omega_1\}$ is an enumeration in $V$ of $[\omega]^\omega \subset 2^\omega$. Let $X = \{r_\alpha : \alpha < \omega_1\} \subset 2^\omega$ be the set of generic random reals over $V$ added by $B(\omega_1)$. Let also $G$ be $B(\omega_1)$-generic over $V$ giving rise to $X$. Then in $V[G]$, all finite power of

$$X_1 = \{r_\alpha + z_\alpha : \alpha < \omega_1\} \subset 2^\omega$$

satisfy $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.

Proposition 2.2.42 is a consequence of the following fact which could be established in the same way as Lemma 2.2.19, basically replacing "Cohen" and "meager" with "random" and "measure 0".

**Lemma 2.2.43.** We use notation from Proposition 2.2.42. Suppose that $D \subset 2^\omega$ is a Borel non-measure zero set coded in $V$. Then there exists $\beta < \lambda$ such that $r_\beta + z_\alpha \in D$.

*Proof of Theorem 2.2.41.* We use notation from Proposition 2.2.42 and work in $V[G]$. By the definitions of $X$ and $X_1$ we have that

$$z_\alpha = (r_\alpha + z_\alpha) + r_\alpha \in X_1 + X$$

for all $\alpha \in \omega_1$, and hence $[\omega]^\omega \cap V \subset X + X_1$. On the other hand, since for $\alpha_0 \neq \alpha_1$ the sum $r_{\alpha_0} + r_{\alpha_1}$ cannot lie in $V$, we conclude that $X + X_1 \subset [\omega]^\omega$. Thus

$$[\omega]^\omega \cap V \subset X + X_1 \subset [\omega]^\omega.$$

Since $B(\omega_1)$ does not add unbounded reals, $[\omega]^\omega \cap V$ is dominating, where each infinite subset $a$ of $\omega$ is identified with the increasing function in $\omega^\omega$ whose range is $a$. It follows that $X \times X_1$ is not Menger since it has a dominating continuous image in $[\omega]^\omega$, namely $X + X_1$. Consequently, $[X \sqcup X_1]^2$ is not Menger because it has a closed topological copy of $X \times X_1$.

Now, by Corollary 2.2.31, if $\mathcal{S}$ is a countable clopen base for $X \sqcup X_1$ closed under finite unions and complements, than $Y_\mathcal{S}$ is not M-nw-selective as a subspace of $C_p(X \sqcup X_1, 2) = C_p(X, 2) \times C_p(X_1, 2)$.

Let $\mathcal{S}(X)$ and $\mathcal{S}(X_1)$ be countable clopen bases closed under finite unions and complements for $X$ and $X_1$, respectively. Then $Y_{\mathcal{S}(X)}$ and $Y_{\mathcal{S}(X_1)}$ are countable dense H-nw-selective subspaces of $C_p(X, 2)$ and $C_p(X_1, 2)$ by Theorem 2.2.22, respectively.

On the other hand, set

$$\mathcal{S} = \{U \cup W : U \in \mathcal{S}(X), W \in \mathcal{S}(X_1)\}$$

and observe that $\mathcal{S}$ is a countable clopen base for $X \sqcup X_1$ closed under finite unions and complements, and hence $Y_\mathcal{S}$ is not M-nw-selective as a subspace of $C_p(X \sqcup X_1, 2)$. It remains to note that $Y_\mathcal{S}$ is a homeomorphic copy of $Y_{\mathcal{S}(X)} \times Y_{\mathcal{S}(X_1)}$. $\mathcal{B}ox$

The analogous strategy with random reals replaced by Cohen reals does not seem to give anything interesting: Unlike in the random model, $[\omega]^\omega \cap V$ is known to satisfy $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ in the Cohen model, so the approach above based on $[\omega]^\omega \cap V$ being "big" in a suitable sense (e.g., dominating in the random model) does not work. This motivates the following

**Question 2.2.44.** Is the existence of two countable R-nw-selective spaces with non-R-nw-selective (or even non-M-nw-selective) product consistent?

On the other hand, we do not know whether countable spaces like in Theorem 2.2.41 could be constructed in ZFC.

**Question 2.2.45.** Is it consistent that the product of two countable M-nw-selective (resp. H-nw-selective, R-nw-selective) spaces is again M-nw-selective (resp. H-nw-selective, R-nw-selective)?

### 2.2.4 HFD's and R-$nw$-selectivity

HFD spaces where introduced in order to construct $S$-spaces, i.e., hereditarily separable spaces which are not Lindelöf, see [41, 43] and references therein. In this section we show that stronger version of the HFD spaces are R-$nw$-selectivity. The following definition is taken from [41].

**Definition 2.2.46.** Let $\lambda$ be an uncountable cardinal. A subset $X \subset 2^\lambda$ is called HFD (abbreviated from Hereditarily Finally Dence) if $X$ is infinite and for every $A \in [X]^\omega$ there is a $B \in [\lambda]^\omega$ such that $A$ (i.e., $A \upharpoonright (\lambda \backslash B)$) is dense in $2^{\lambda \backslash B}$.

We use the following notation of [43]. Given some $\varepsilon \in Fin(I, 2)$, where $Fin(I, 2)$ denotes the collection of all finite partial functions on $I$ to 2, we set $[\varepsilon] = \{f \in 2^I : \varepsilon \subset f\}$. Thus, $[\varepsilon]$ is a standard basic clopen subset of $2^I$. Now suppose that $I$ is a set of ordinals, $b \in [I]^{<\omega}$, $b = \{\beta_i : i \in n = |b|\}$ is an increasing enumeration, and $\varepsilon \in 2^n$. In this case we denote by $\varepsilon * b$ the element of $Fin(I, 2)$ which has $b$ as its domain and satisfies $\varepsilon * b\,(\beta_i) = \varepsilon(i)$ for all $i \in n$.

For any infinite cardinal $\mu$ and $r \in \omega$ we denote by $\mathcal{D}^r_\mu(I)$ the collection of all sets $B \in [[I]^r]^\mu$ such that the members of $B$ are pairwise disjoint. We shall write

$$\mathcal{D}_\mu(I) = \bigcup \left\{ \mathcal{D}^r_\mu(I) : r \in \omega \right\}$$

If $B \in \mathcal{D}_\mu(I)$ then $n(B) = |b|$ for any $b \in B$. Now, if $B \in \mathcal{D}_\mu(I)$ and $\varepsilon \in 2^{n(B)}$ then

$$[\varepsilon, B] = \bigcup \{[\varepsilon * b] : b \in B\}$$

is called a $\mathcal{D}_\mu$-set in $2^I$. The most important instance of the above is $\mu = \omega$, in this case we shall often omit the lower index $\omega$ of $\mathcal{D}$, i.e., a $\mathcal{D}$-set in $2^I$ is a $\mathcal{D}_\omega$-set. Clearly, any $\mathcal{D}$-set is open dense and has product (Lebesgue) measure 1 in $2^I$. Recall that a map $F : \kappa \times \lambda \to 2$ with $\kappa \geqslant \omega, \lambda \geqslant \omega_1$ is called an HFD matrix (see [43]) if for every $A \in [\kappa]^\omega, B \in \mathcal{D}_{\omega_1}(\lambda)$ and $\varepsilon \in 2^{n(B)}$ there are $\alpha \in A$ and $b \in B$ such that

$$f_\alpha = F(\alpha, -) \supset \varepsilon * b.$$

The latter inclusion means that $F(\alpha, \beta_i) = \varepsilon(i)$ for each $i < n(B) = |b|$, where $\beta_i$ is the $i$-th member of $b$ in its increasing enumeration. The following fact was established in [43].

**Proposition 2.2.47.** $X \subset 2^\lambda$ is an HFD space if and only if there exists an HFD matrix $F : \kappa \times \lambda \to 2$ such that $X = \{f_\alpha : \alpha \in \kappa\}$.

In this case we say that $F$ *represents* $X$. Following [43], for an HFD space $X \subset 2^\lambda$ and $A \in [X]^\omega$ we set

$$\mathcal{J}(A) = \left\{ I \in [\lambda]^\omega : \forall \varepsilon \in Fin(I, 2) \left( |A \cap [\varepsilon]| = \omega \;\Rightarrow\; A \cap [\varepsilon] \text{ is dense in } 2^{\lambda \backslash I} \right) \right\}.$$

**Proposition 2.2.48.** [43] If $X \subset 2^\lambda$ is HFD and $A \in [X]^\omega$ then $\mathcal{J}(A)$ is closed and unbounded in $[\lambda]^\omega$.

**Proposition 2.2.49.** If $\mathcal{N}$ is a countable network in a HFD space $X \subset 2^\lambda$, then $\mathcal{N} \cap [X]^{<\omega}$ is a network in $X$ as well, and hence $X$ is countable.

*Proof.* Set $\mathcal{A} = \mathcal{N} \cap [X]^{\geq\omega}$, i.e., $\mathcal{A}$ is a family of all infinite elements of $\mathcal{N}$. For every $A \in \mathcal{A}$ fix a countable infinite $C(A) \subset A$ and pick $J \in \bigcap_{A \in \mathcal{A}} \mathcal{J}(C(A))$. Fix any $\alpha \in J$,

$\beta \in \lambda \setminus J$, $x \in X$ and set $\epsilon_0 = \{\langle \alpha, x(\alpha) \rangle\} \in Fin(J, 2)$, $\epsilon_1 = \{\langle \alpha, x(\alpha) \rangle, \langle \beta, 1 - x(\beta) \rangle\} \in Fin(\lambda, 2)$, and $\epsilon_2 = \{\langle \alpha, x(\alpha) \rangle, \langle \beta, x(\beta) \rangle\} \in Fin(\lambda, 2)$.

We claim that $C(A) \not\subset [\epsilon_2]$ for any $A \in \mathcal{A}$, which would clearly imply that $\mathcal{N} \cap [X]^{<\omega}$ must be a network for $X$. Fix $A \in \mathcal{A}$. If $C(A) \not\subset [\epsilon_0]$ there is nothing to prove, so assume that $C(A) \subset [\epsilon_0]$. But then $C(A) \cap [\epsilon_1] \neq \emptyset$ because $J \in \mathcal{J}(C(A))$, and hence $C(A) \not\subset [\epsilon_2]$. □

Scheepers [62] proved that any HFD is R-separable. Now we show that the following stronger version of HFD spaces introduced in [66] implies R-*nw*-selectivity.

**Definition 2.2.50.** A set $X \subset 2^{\omega_1}$ is a *very strong HFD* if for each sequence $\{A_n : n \in \omega\}$ of pairwise disjoint, non-empty finite subsets of $X$ there is $\beta < \omega_1$ such that for all $s \in Fin(\omega_1 \setminus \beta, 2)$ there are infinitely many $n$ with $A_n \subset [s]$.

Obviously, we get an equivalent notion if we require in the definition above only that $\{A_n : n \in \omega\} \cap [[s]]^{<\omega} \neq \emptyset$.

Recall that a dense set $D \subseteq X$ is called *groupable* if it admits a partition $\mathcal{A} = \{A_n : n < \omega\}$ into finite sets such that every non-empty open subset of $X$ meets all but finitely elements of $\mathcal{A}$. Every very strong HFD space cannot contain a groupable dense subset [66]. On the other hand, every H-separable space has a groupable dense subset. Therefore a very strong HFD space cannot be H-separable (hence not H-*nw*-selective).

**Theorem 2.2.51.** Every countable very strong HFD space $X$ is R-*nw*-selective.

*Proof.* Let $(\mathcal{N}_{n,m,k} : n, m, k \in \omega)$ be an enumeration of countably many networks of $X$. By Proposition 2.2.49 there is no loss of generality in assuming that each $\mathcal{N}_{n,m,k}$ consists of finite subsets of $X$.

Let $\vartheta$ be a large enough regular cardinal and $M$ a countable elementary submodel of $\mathcal{H}_\vartheta$ which contains $X$ and $(\mathcal{N}_{n,m,k} : n, m, k \in \omega)$. Set $\beta = M \cap \omega_1$ and enumerate $Fin(\beta, 2)$ as $\{s_n : n \in \omega\}$. Let $L = \{n \in \omega : [s_n] \cap X \neq \emptyset\}$ and for all $n \in L$ fix a (non-necessary injective) enumeration $\{x^n_k : k \in \omega\}$ of $[s_n] \cap X$. Given $n \in L$ and $k \in \omega$, by induction on $m \in \omega$ choose $N_{n,m,k} \in [[s_n]]^{<\omega} \cap \mathcal{N}_{n,m,k}$ such that

- $x^n_k \in N_{n,m,k}$,
- $(N_{n,m_0,k} \setminus \{x^n_k\}) \cap (N_{n,m_1,k} \setminus \{x^n_k\}) = \emptyset$ for any $m_0 < m_1$, and
- the function $m \mapsto N_{n,m,k}$ is in $M$.

We claim that $\{N_{n,m,k} : n \in L, m, k \in \omega\}$ is a network in $X$. Indeed, let $s \in Fin(\omega_1, 2)$, $x \in [s]$, and note that $s \restriction_\beta = s_n$ for some $n$. Thus $n \in L$. Fix $k \in \omega$ such that $x = x^n_k$. The sequence

$$(N_{n,m,k} \setminus \{x^n_k\} : m \in \omega)$$

lies in $M$ and consists of mutually disjoint finite subsets of $X$, so there is $\beta_n \in M$ such that

$$[[t]]^{<\omega} \cap \{N_{n,m,k} \setminus \{x^n_k\} : m \in \omega\}$$

is infinite for all $t \in Fin(\omega_1 \setminus \beta_n, 2)$ with $x = x^n_k \in [t]$. This is a direct consequence of $X$ being a very strong HFD. It follows that

$$[[t]]^{<\omega} \cap \{N_{n,m,k} : m \in \omega\}$$

is infinite for all $t \in Fin(\omega_1 \setminus \beta_n, 2)$ with $x = x^n_k \in [t]$. Note that $s \setminus s_n \in Fin(\omega_1 \setminus \beta, 2) \subset Fin(\omega_1 \setminus \beta_n, 2)$ hence $[[s \setminus s_n]]^{<\omega} \cap \{N_{n,m,k} : m \in \omega\} \neq \emptyset$. Since $\{N_{n,m,k} : m \in \omega\} \subset$

$[[s_n]]^{<\omega}$ we have that $[[s]]^{<\omega} \cap \{N_{n,m,k} : m \in \omega\} \neq \emptyset$ as well. This completes our proof. □

Theorem 2.2.51 motivates the following

**Question 2.2.52.** Is every countable HFD space R-*nw*-selectivity?

# Chapter 3

# Recent studies on Topological Games

The first studied topological game was the Banach-Mazur game which appeared in the Scottish Book (1935-1941), a collection of notes recording the ideas of a group of mathematicians who gathered in the Scottish Café in Lwów (Poland) to discuss about various mathematical topics. This game, described in Problem 43 by Banach and Mazur, provided a new way to explore topological properties using game theory. It became fundamental in understanding concepts like Baire spaces and complete metric spaces.

Maurice René Fréchet and Gustave Choquet further developed the field by introducing the Choquet game, which focused on properties of open sets and bases in topological spaces, particularly in the context of topological vector spaces.

David Gale and Frank Stewart, in the 1950s, expanded the study of infinite games, linking these ideas to automata theory and formal languages. Their work provided a broader understanding of strategic games with infinite moves.

Menger and Hurewicz, in the 1920s and 1930s, explored games involving open covers, leading to the concepts of Menger and Hurewicz selection properties, which are closely related to compactness and paracompactness in topology.

Fritz Rothberger introduced the Rothberger game, focusing on the coverage and selection of dense sets, contributing to the understanding of real number sets.

In recent years, researchers like Scheepers have continued to extend the principles of selection and analyze infinite games in topology, applying these concepts to set theory, function spaces, and topology.

In conclusion, topological games are valuable tools for investigating the structural properties of topological spaces.

All uncited results in this chapter are either trivial remarks or can be found in [5].

## 3.1 Introduction to topological games related to selection principles

Scheepers [61, 62] introduced a systematical study on Topological Games associated to Selection Principles. Topological games are infinite games played by two different players, ALICE and BOB, on a topological space $X$. We assume that the length (number of innings) of the games is $\omega$, if it is not differently specified, and the two players pick in each inning some topological objects of a fixed space. The strategies of the two players are a priori defined, they are some functions that take care of the game history. At the end there is only one winner, so a draw is not allowed. This construction give rise to two properties on a topological space $X$, fixed a particular game $G$: "ALICE has a winning strategy in the game $G$ on $X$"; "BOB has a winning strategy in the game $G$ on $X$". Of course, since there is not draw, it is impossible for a space to have both these properties, but it can be that the negation of both of them holds. In this case we say that the game $G$ is indeterminate on the space $X$. Given two families of topological objects $\mathcal{A}$ and $\mathcal{B}$, the followings are two games associated to selection principles.

$G_1(\mathcal{A}, \mathcal{B})$ : is played according to the following rules.

- for every $n \in \omega$ ALICE chooses $A_n \in \mathcal{A}$;
- BOB answers picking $b_n \in A_n$ for each $n \in \omega$;
- the winner is BOB if $\{b_n : n \in \omega\} \in \mathcal{B}$, otherwise ALICE wins.

$G_{fin}(\mathcal{A}, \mathcal{B})$ : is played according to the following rules.

- for every $n \in \omega$ ALICE chooses $A_n \in \mathcal{A}$;
- BOB answers picking a finite subset $B_n \subseteq A_n$ for each $n \in \omega$;
- the winner is BOB if $\bigcup\{B_n : n \in \omega\} \in \mathcal{B}$, otherwise ALICE wins.

The game $G_1(\mathcal{O}, \mathcal{O})$ is strictly related to the property $S_1(\mathcal{O}, \mathcal{O})$ (the Rothberger property), in fact it is called Rothberger game. Similarly, the game $G_{fin}(\mathcal{O}, \mathcal{O})$ is strictly related to the property $S_{fin}(\mathcal{O}, \mathcal{O})$ (the Menger property), in fact it is called Menger game.

This two games were largely studied and some important characterizations of "ALICE does not have a winning strategy"and "BOB has a winning strategy"have been given. Despite this some questions are still open.

We denote by $Bob \uparrow G$ on $X$, the fact that "BOB has a winning strategy in the game $G$ on $X$"and by $Alice \not\uparrow G$ on $X$, the fact that "ALICE does not have a winning strategy in the game $G$ on $X$".

**Remark 3.1.1.** In general the following implications hold.

1. $Bob \uparrow G_1(\mathcal{A}, \mathcal{B}) \implies Bob \uparrow G_{fin}(\mathcal{A}, \mathcal{B})$;
2. $Alice \not\uparrow G_1(\mathcal{A}, \mathcal{B}) \implies Alice \not\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$;
3. $Bob \uparrow G_1(\mathcal{A}, \mathcal{B}) \implies Alice \not\uparrow G_1(\mathcal{A}, \mathcal{B}) \implies S_1(\mathcal{A}, \mathcal{B})$;
4. $Bob \uparrow G_{fin}(\mathcal{A}, \mathcal{B}) \implies Alice \not\uparrow G_{fin}(\mathcal{A}, \mathcal{B}) \implies S_{fin}(\mathcal{A}, \mathcal{B})$.

For some properties the last two implication of points 3 and 4 are, in fact, characterizations, that is, $Alice \not\uparrow G(\mathcal{A}, \mathcal{B}) \iff S(\mathcal{A}, \mathcal{B})$.

The following impications hold:

$$X \text{ is a countable space} \implies Bob \uparrow \text{Rothberger}(X) \implies$$
$$\implies Alice \not\uparrow \text{Rothberger}(X) \implies X \text{ is Rothberger.}$$

As we say in Remark 3.1.1, the viceversa of the third implication of this chain holds.

**Theorem 3.1.2.** [54] A space $X$ is Rothberger if, and only if, $Alice \not\uparrow \text{Rothberger}(X)$.

The reader can find the proof of this previous theorem in [68].

Actually, even the first implication of the chain can be reverse in some class of spaces.

**Theorem 3.1.3.** [70][36] Let $X$ be a space in which each point is a $G_\delta$ (equivalently, $\psi(X) \leq \omega$). Then $Bob \uparrow \text{Rothberger}(X)$ if, and only if, $X$ is countable.

A similar argument is valid for the Menger case. Indeed,

$$X \text{ is a } \sigma\text{-compact space} \implies \textit{Bob} \uparrow \text{Menger}(X) \implies \\ \implies \textit{Alice} \not\uparrow \text{Menger}(X) \implies X \text{ is Menger.}$$

**Theorem 3.1.4.** [61][67] A space $X$ is Menger if, and only if, $\textit{Alice} \not\uparrow \text{Menger}(X)$.

**Theorem 3.1.5.** [69] Let $X$ be a metrizable space. Then $\textit{Bob} \uparrow \text{Menger}(X)$ if, and only if, $X$ is $\sigma$-compact.

Two topological games $G$ and $H$ are called dual if both

"$\textit{Alice} \uparrow G \iff \textit{Bob} \uparrow H$"and

"$\textit{Alice} \uparrow H \iff \textit{Bob} \uparrow G$"

hold. Sometimes this dual vision could be useful to apply different techniques in demonstrations. It is known that the following game is the dual of the Rothberger game for every space. We will see that in some cases it is useful to consider this game rather than the Rothberger one.

**Definition 3.1.6.** The point-open game on $X$ (point-open$(X)$) is played according to the following rules.

- for every $n \in \omega$ ALICE chooses $x_n \in X$;
- BOB answers picking an open subset $U_n$ such that $x_n \in U_n$ for each $n \in \omega$;
- the winner is ALICE if $\{U_n : n \in \omega\}$ is a cover of the space, otherwise BOB wins.

Replacing "point"with "compact"one obtains the compact-open game.

**Theorem 3.1.7.** [36] The point-open game and the Rothberger game are dual on every space.

**Remark 3.1.8.** By using Theorem 3.1.7, it is straightforward to see that the unit interval of the real line $\mathbb{I} = [0, 1]$ is a Menger not Rothberger space. Since $\mathbb{I}$ is compact, it is Menger and it is not Rothberger. Indeed, it is easy to see that $\textit{Bob} \uparrow \text{point-open}(\mathbb{I})$ (BOB just need to pick open subsets whose sum of the measures of them is strictly less than 1), hence $\textit{Alice} \uparrow \text{Rothberger}(\mathbb{I})$. Therefore, by Theorem 3.1.2, $\mathbb{I}$ is not Rothberger.

The natural observations that one can make is to consider, as dual of the Menger game, a game in which a finite subset of the space replace the single point in the point-open game. It is proved that such game (finite-open game) is equivalent to the point-open then it cannot be the dual of the Menger game. However, the following theorem shows that the compact-open game could be a good candidate for this goal.

**Theorem 3.1.9.** [69] The following implications hold on every space $X$.

1. $\textit{Alice} \uparrow \text{compact-open}(X) \implies \textit{Bob} \uparrow \text{Menger}(X)$;
2. $\textit{Alice} \uparrow \text{Menger}(X) \implies \textit{Bob} \uparrow \text{compact-open}(X)$;
3. if $X$ is regular, then $\textit{Bob} \uparrow \text{Menger}(X) \implies \textit{Alice} \uparrow \text{compact-open}(X)$.

Theorem 3.1.9 allows to prove the following result.

**Theorem 3.1.10.** [70] Let $X$ be a regular space in which each point is $G_\delta$ (or equivalently, $\psi(X) \leq \omega$). If *Bob* $\uparrow G_{fin}(\mathcal{O}, \mathcal{O})$ on $X$, then $X$ is $\sigma$-compact.

This theorem is a generalization of Theorem 3.1.5, since every metrizable space is perfectly normal (hence a regular space in which each compact subset is $G_\delta$).

**Question 3.1.11.** Are the Menger game and the compact-open game dual on every regular space?

In [6] the authors provided, under CH, a regular example (the Sierpiński set, i.e., an uncountable subset of the real line such that every intersection with a zero-set is countable) in which the compact-open game and the Menger game are not dual, since *Bob* $\uparrow$ compact-open$(X)$, but *Alice* $\not\uparrow$ Menger$(X)$.
Then the following question is still open.

**Question 3.1.12.** Is there a ZFC-consistent model in which the compact-open game and the Menger game are dual on every regular space?

Another important aim in the topological games theory is to study the determinancy of games.
Firstly we ask if the Rothberger game is undeterminate for some subset of the real line $\mathbb{R}$. By Theorems 3.1.2 and 3.1.3, it is clear that one has to find an uncountable Rothberger space.

**Theorem 3.1.13.** [50][49] (CH) There exists a Luzin set, i.e., an uncountable subset $X \subseteq \mathbb{R}$ that has countable intersection with every nowhere dense subset of $\mathbb{R}$.

*Proof.* $\mathbb{R}$ is second countable. Therefore, the cardinality of the topology $\tau$ is precisely the cardinality of the power set of the basis, i.e., $2^\omega$. Assuming CH, we can index the set of all the nowhere dense closed subsets of $\mathbb{R}$ in $\omega_1$. Let $\{F_\alpha : \alpha \in \omega_1\}$ be this set. Now, for each $\alpha \in \omega_1$, we choose a point in a set defined recursively as $x_\alpha \in \mathbb{R} \setminus (\{x_\beta : \beta < \alpha\} \cup \bigcup_{\beta<\alpha} F_\beta)$, such a choice is possible because $\mathbb{R}$ is a Baire space (i.e., the union of countably many nowhere dense subsets is co-dense). Thus, $\{x_\alpha : \alpha \in \omega_1\}$ is the desired Luzin set. Indeed, fix a nowhere dense $G$, then $\overline{G} = F_\alpha$ for some $\alpha$. □

**Theorem 3.1.14.** [56] Every Luzin set is Rothberger.

*Proof.* Let $X \subseteq \mathbb{R}$ be a Luzin set, and let $D = \{d_n : n \in \omega\}$ be a countable dense subset of $X$. It is always possible to choose a countable dense subset of $X$ since $\mathbb{R}$ is second countable, and this property is hereditary. Therefore, $X$ is also second countable, hence separable. Fix any sequence $\{\mathcal{U}_n : n \in \omega\}$ of covers of $X$ made by open sets of $\mathbb{R}$. For each $n \in \omega$, choose $U_{2n} \in \mathcal{U}_{2n}$ such that $d_n \in U_{2n}$. Consider the set $A = X \setminus \bigcup_{n\in\omega} U_{2n}$. We will show that $A$ is nowhere dense in $\mathbb{R}$. Fix an open subset $U$ of $\mathbb{R}$ such that $U \cap X \neq \emptyset$ then $U \cap X$ is open in $X$ and then some $d_n \in U \cap X$, then $\emptyset \neq U \cap U_{2n} \subseteq U \cap (\mathbb{R} \setminus A)$. Therefore $\mathbb{R} \setminus A$ is co-dense in $\mathbb{R}$. Since $X$ is Luzin, it follows that the set $X \cap A$ is countable, then it can be trivially covered by certain $U_{n+1} \in \mathcal{U}_{n+1}$. Thus, we have established that $X \subseteq \bigcup_{n\in\omega} U_n$, therefore $X$ is Rothberger. □

**Corollary 3.1.15.** [6] (CH) There exists a subset $X \subseteq \mathbb{R}$ such that the Rothberger game on $X$ is undetermined.

Another study on the determinacy of the Rothberger game can be carried out under the Borel Conjecture, whose definition follows.
Reacall that a subset $X \subseteq \mathbb{R}$ is said to have strong measure zero if, for every sequence $\{\varepsilon_n\}_{n\in\omega}$ of positive real numbers, there exists a sequence $\{I_n\}_{n\in\omega}$ of intervals in $\mathbb{R}$ such that $X \subseteq \bigcup_{n\in\omega} I_n$, and for every $n \in \omega$, the measure of $I_n$ is precisely $\varepsilon_n$.

**Definition 3.1.16.** [28] The Borel Conjecture states:
"every subset of $\mathbb{R}$ with strong measure zero is countable".

Laver proved the following Theorem.

**Theorem 3.1.17.** [48] The Borel Conjecture is consistent with ZFC.

**Theorem 3.1.18.** [57] The following statements are equivalent.

1. Every Rothberger subset of $\mathbb{R}$ is countable.
2. The Borel Conjecture holds.

**Corollary 3.1.19.** [6] The following statements are equivalent.

1. The Rothberger game on $X$ is determined for every $X \subseteq \mathbb{R}$.
2. The Borel Conjecture holds.

**Remark 3.1.20.** From the results seen in this paragraph, we deduce a significant fact in Set Theory: in ZFC, we cannot assume the coexistence of the Continuum Hypothesis and the Borel Conjecture. In fact, if we assume both to be true, we would have that "all Rothberger spaces would be countable" (by the Borel Conjecture), while we know that the Continuum Hypothesis implies the existence of the Luzin set (a Rothberger set more than countable).

About the study of the determinacy of the Menger game Fremlin and Miller construct a subset of the real line which is Menger not $\sigma$-compact.

Given a space $X$ is possible to introduce the games $G_1(\mathcal{D},\mathcal{D})$, that we call R-separable game on $X$ (denoted by R-separable$(X)$) and $G_{fin}(\mathcal{D},\mathcal{D})$ that we call M-separable game on $X$ (denoted by M-separable$(X)$). Clearly, they are related to the R-separability ($S_1(\mathcal{D},\mathcal{D})$) and to the M-separability ($S_{fin}(\mathcal{D},\mathcal{D})$) in the sense of Remark 3.1.1.

The following facts are easy to see.

$$\pi w(X) \leq \omega \implies Bob \uparrow \text{R-separable}(X) \implies$$
$$\implies Alice \not\uparrow \text{R-separable}(X) \implies X \text{ is R-separable.}$$

Berner and Juhász [18] introduced the Point-picking game.

**Definition 3.1.21.** Let $\alpha$ be an ordinal and $\mathcal{P}$ be a class of spaces satisfying a certain property. The Point-picking game on $X$ (denoted by $G^{\mathcal{P}}_{\alpha}(X)$) is played according to the following rules.

- for every $\beta < \alpha$ ALICE chooses an open set $U_\beta$ in $X$;
- BOB answers picking a point $x_\beta \in U_\beta$ for each $\beta < \alpha$;

- the winner is ALICE if the collection of points $\{x_\beta : \beta < \alpha\}$ belongs to $\mathcal{P}$, otherwise BOB wins.

In the same paper [18], the authors prove some connection between the fact that the space has a certain $\pi$-weight and some point-picking game.
In what follows we consider only this game in the case $\alpha = \omega$, the class $\mathcal{P}$ is always the collection $\mathcal{D}$ of all dense subsets of a given space $X$ and we continue call it point-picking game on $X$ (denoted by $G^{\mathcal{D}}(X)$).

In [62] the author considered the point-picking game and proved the following result.

**Theorem 3.1.22.** [62, Theorems 7 and 8] The R-separable game and the point-picking game are dual on every topological space.

Moreover the author showed the following result.

**Theorem 3.1.23.** [62, Theorem 3] The following are equivalent for every space $X$.

1. BOB $\uparrow$ R-separable$(X)$;
2. $\pi w(X) \leq \omega$.

In Chapter 2, we have seen that the M-and R-nw-selective properties are strictly stronger than M- and R-separability, respectively. The following example shows that not even the stronger condition "BOB $\uparrow$ R-separable game"implies that the space is R- or M-nw-selective.

**Example 3.1.24.** A countable space $X$ such that BOB $\uparrow$ R-separable$(X)$ (hence BOB $\uparrow$ M-separable$(X)$), which is not M-nw-selective space (hence not R-nw-selective).

The Fréchet-Urysohn fan space $S_\omega$ is a typical example of a countable space with only one non-isolated point the fan-tightness of which is not countable. Then, by Proposition 2.1.10, $S_\omega$ is not M-nw-selective. Since, obviously, $S_\omega$ has countable $\pi$-weight, then BOB $\uparrow$ R-separable$(X)$. $\triangle$

## 3.2 On some topological games involving networks

In this section we introduce some games related to the selection principles R-nw-selective and M-nw-selective and we study the behaviour of them. In particular, we investigate characterizations of the facts that the players have a winning strategy, we compare the games to each other, and we try to construct a possible dual game for the "R-"case.

### 3.2.1 The R-nw-selective game

**Definition 3.2.1.** Let $X$ be a space with $nw(X) = \omega$. The R-nw-selective game, denoted by R-nw-selective$(X)$, is played according to the following rules. ALICE chooses a countable network $\mathcal{N}_0$ and BOB answers picking an element $N_0 \in \mathcal{N}_0$. Then ALICE chooses another countable network $\mathcal{N}_1$ and BOB answers in the same way and so on for countably many innings. At the end BOB wins if the set $\{N_n : n \in \omega\}$ of his selections is a network.

Simultaneously we consider the possible dual version of the R-nw-selective game.

**Definition 3.2.2.** The (Point, Open)-Set game on a space $X$, denoted by PO-set$(X)$, is played according to the following rules. ALICE chooses a point $x_0$ and an open set $U_0$ containing $x_0$. Then BOB picks $N_0$ a subset of $X$ such that $x_0 \in N_0 \subseteq U_0$. The game goes ahead in this way for every $n \in \omega$ and ALICE wins if the set $\{N_n : n \in \omega\}$ of BOB's choices is a network.

**Theorem 3.2.3.** Let $X$ be a space. BOB $\uparrow$ R-nw-selective$(X)$ if, and only if, the space $X$ is countable and second countable.

*Proof.* Clearly, if $X$ is a countable second contable space then it is easy to construct a winning strategy for BOB in the R-nw-selective game on $X$.
Let $\mathbb{M}$ be the collection of all countable networks of $X$. Let $\sigma$ be a winning strategy for Bob.
First we prove that the space is countable.
Claim 1. $|\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})}| \leq 1$.
Indeed, suppose that two distinct points points, say $x$ and $y$, belong to all the clousure of the possible answers to $\mathcal{N}$, for any $\mathcal{N} \in \mathbb{M}$. Fix any countable network $\mathcal{N}$ and observe that $\mathcal{N}' = \{N \in \mathcal{N} : \{x,y\} \cap \overline{N} = \emptyset\} \cup \{\{x\},\{y\}\}$ is also a network in $X$ such that no element of $\mathcal{N}'$ contains the set $\{x,y\}$ in its closure. This gives a contradiction.
Claim 2. There exists a countable $\mathbb{M}' \subset \mathbb{M}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}'} \overline{\sigma(\mathcal{N})} = \bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})}$.
Indeed, if $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})} = \{x\}$ (it is the same if $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})} = \emptyset$), the complements of all the closures form an open cover of $X \setminus \{x\}$ (or $X$) and then, since having countable network implies hereditary Lindelöfness, we can obtain a countable subcover of $X \setminus \{x\}$ (or of the all space $X$).
Claim 1. and Claim 2. hold for any inning $n \in \omega$, that is
Claim 1(n). $|\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N}_0, ..., \mathcal{N}_{n-2}, \mathcal{N})}| \leq 1$.
Claim 2(n). There exists a countable $\mathbb{M}' \subset \mathbb{M}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}'} \overline{\sigma(\mathcal{N}_0, ..., \mathcal{N}_{n-2}, \mathcal{N})} = \bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N}_0, ..., \mathcal{N}_{n-2}, \mathcal{N})}$.
The proof of Claim 1(n). and Claim 2(n). is analogous to the one of Claim 1. and Claim 2., respectively.
Consider the following tree of possible evolution of the R-nw-selective game on $X$. By claim 2. there exists $(\mathcal{N}^n_\emptyset)_{n\in\omega}$, that is countably many possible choices of ALICE in the first inning $\mathcal{N}^0_\emptyset, \mathcal{N}^1_\emptyset, \mathcal{N}^2_\emptyset, ...$, such that $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})} = \bigcap_{n\in\omega} \overline{\sigma(\mathcal{N}^n_\emptyset)}$.
Fix, for example, the branch with $\mathcal{N}^0_\emptyset$ then there exists a sequence $(\mathcal{N}^n_{<0>})_{n\in\omega}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N}^0_\emptyset, \mathcal{N})} = \bigcap_{n\in\omega} \overline{\sigma(\mathcal{N}^0_\emptyset, \mathcal{N}^n_{<0>})}$.
Again consider, for example, $\mathcal{N}^1_{<0>}$, then there exists a sequence $(\mathcal{N}^n_{<0,1>})_{n\in\omega}$ such that $\bigcap_{n\in\omega} \overline{\sigma(\mathcal{N}^0_\emptyset, \mathcal{N}^1_{<0>}, \mathcal{N}^n_{<0,1>})} = \bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N}^0_\emptyset, \mathcal{N}^1_{<0>}, \mathcal{N})}$. By Claim 1, each intersection is empty or contains only one element. If the intersection $\bigcap_{n\in\omega} \overline{\sigma(\mathcal{N}^n_\emptyset)}$ is non-empty we call this element $x^\emptyset$, otherwise we go on; if $\bigcap_{n\in\omega} \overline{\sigma(\mathcal{N}^0_\emptyset, \mathcal{N}^n_{<0>})}$ is not empty we call this element $x^{<0>}$; if the intersection $\bigcap_{n\in\omega} \overline{\sigma(\mathcal{N}^0_\emptyset, \mathcal{N}^1_{<0>}, \mathcal{N}^n_{<0,1>})}$ is not empty we call this element $x^{<0,1>}$, and so on. We obtain a subset $X_0 = \{x^s : s \in \omega^{<\omega}\}$ and now we want to prove that $X_0 = X$. By contradiction, assume there exists $y \in X \setminus X_0$. Then $y \notin \bigcap_{n\in\omega} \overline{\sigma(\mathcal{N}^n_\emptyset)}$; hence there exists an element of the sequence $\{\sigma(\mathcal{N}^n_\emptyset) : n \in \omega\}$, say $\sigma(\mathcal{N}^{k_0}_\emptyset)$, such that $y$ does not belong to it. By hypotesis, $y \notin \bigcap_{n\in\omega} \overline{\sigma(\mathcal{N}^{k_0}_\emptyset, \mathcal{N}^n_{<k_0>})}$; hence there exists an element of $\{\sigma(\mathcal{N}^n_{<k_0>}) : n \in \omega\}$, say $\sigma(\mathcal{N}^{k_1}_{<k_0>})$, such that $y$ does not belong to it. Again, $y \notin \bigcap_{n\in\omega} \overline{\sigma(\mathcal{N}^{k_0}_\emptyset, \mathcal{N}^{k_1}_{<k_0>}, \mathcal{N}^n_{<k_0,k_1>})}$, there exists an element of $\{\sigma(\mathcal{N}^n_{<k_0,k_1>}) : n \in \omega\}$, say $\sigma(\mathcal{N}^{k_2}_{<k_0,k_1>})$, such that $y$ does not belong to it. Proceeding in this way we obtain

a branch consisting of elements that do not contain $y$; a contradiction, because such a branch is a network due to the fact that $\sigma$ is a winning strategy for BOB. Then $X$ is countable.
Now we prove that $X$ is second countable.
Claim 3. If $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})} = \{x\}$, there exists an open set $V$ such that $x \in V \subset \bigcup_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})}$.
Indeed, assume by contradiction that for every open set $V$ such that $x \in V$ there exists $y_V \in V \setminus \overline{\sigma(\mathcal{N})}$, for every $\mathcal{N} \in \mathbb{M}$. Let $\mathcal{N}$ be a countable network and consider the family $\mathcal{N}' = (\mathcal{N} \setminus \mathcal{N}_x) \cup \{\{x, y_V\} : V \in \tau_x\}$, where $\tau_x$ denotes the family of all open sets containing $x$ and $\mathcal{N}_x = \{N \in \mathcal{N} : x \in \overline{N}\}$. Since $X$ is countable, $\mathcal{N}'$ is countable. Now we prove that $\mathcal{N}'$ is a network. Clearly, for construction $\mathcal{N}'$ is a network at $x$. Let $y \in X$, $y \neq x$, and let $A$ be an open set such that $y \in A$. Since $X$ is $T_2$, there exists an open set $B$ such that $y \in B$ and $x \notin \overline{B}$. Then there exists $N \in \mathcal{N}$ such that $y \in N \subset A \cap B$. Therefore $N \in \mathcal{N} \setminus \mathcal{N}_x$.
Claim 4. If $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})} = \{x\}$, there exists a countable $\mathbb{M}' \subset \mathbb{M}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}'} \overline{\sigma(\mathcal{N})} = \{x\}$ and also such that $\bigcup_{\mathcal{N}\in\mathbb{M}'} \overline{\sigma(\mathcal{N})} = \bigcup_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})}$.
Recall that, by Claim 2 there exists a countable subset $\mathbb{M}^* \subset \mathbb{M}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})} = \bigcap_{\mathcal{N}\in\mathbb{M}^*} \overline{\sigma(\mathcal{N})}$; further, since $X$ is countable, $\bigcup_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})}$ is countable and then there exists a countable subset $\mathbb{M}^{**} \subset \mathbb{M}$ such that $\bigcup_{\mathcal{N}\in\mathbb{M}^{**}} \overline{\sigma(\mathcal{N})} = \bigcup_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})}$. Then $\mathbb{M}' = \mathbb{M}^* \cup \mathbb{M}^{**}$.
Even Claim 3. and Claim 4. can be given for any inning $n \in \omega$, that is
Claim 3(n). If $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N}_0, ..., \mathcal{N}_{n-2}, \mathcal{N}))} = \{x\}$, there exists an open set $V$ such that $x \in V \subset \bigcup_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N}_0, ..., \mathcal{N}_{n-2}, \mathcal{N}))}$.
Claim 4(n). If $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N}_0, ..., \mathcal{N}_{n-2}, \mathcal{N}))} = \{x\}$, there exists a countable $\mathbb{M}' \subset \mathbb{M}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}'} \overline{\sigma(\mathcal{N}_0, ..., \mathcal{N}_{n-2}, \mathcal{N}))} = \{x\}$ and also such that $\bigcup_{\mathcal{N}\in\mathbb{M}'} \overline{\sigma(\mathcal{N}_0, ..., \mathcal{N}_{n-2}, \mathcal{N}))} = \bigcup_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N}_0, ..., \mathcal{N}_{n-2}, \mathcal{N}))}$.
The proof of Claim 3(n). and Claim 4(n). is analogous to the one of Claim 3. and Claim 4., respectively.
Consider the construction of the tree in the previous part of the proof. We know that $|\bigcap_{n\in\omega} \overline{\sigma(\mathcal{N}^n_\emptyset)}| \leq 1$. If $\bigcap_{n\in\omega} \overline{\sigma(\mathcal{N}^n_\emptyset)} \neq \emptyset$, fix $V_\emptyset$ as in Claim 3. If $\bigcap_{k\in\omega} \overline{\sigma(\mathcal{N}^n_\emptyset, \mathcal{N}^k_{<n>})} \neq \emptyset$, fix $V_{<n>}$ as in Claim 3 and so on. Now we prove that $\{V_s : s \in \omega^{<\omega}\}$ is a base. If it is not true, then there exist $x \in X$ and an open set $A$ with $x \in A$ such that for every $s \in \omega^{<\omega}$ such that $x \in V_s$, $V_s$ is not contained in $A$. In the first inning, we have a family $\mathbb{M}'$ of countably many networks obtained as in Claim 4. Consider the intersection $\bigcap_{N\in\mathbb{M}'} \overline{\sigma(\mathcal{N})}$. If $\bigcap_{N\in\mathbb{M}'} \overline{\sigma(\mathcal{N})} = \emptyset$, we can pick an $\mathcal{N} \in \mathbb{M}'$, such that $x \notin \sigma(\mathcal{N})$. If $\bigcap_{N\in\mathbb{M}'} \overline{\sigma(\mathcal{N})} = \{y\}$ we have two cases: if $y \neq x$, we can pick an $\mathcal{N} \in \mathbb{M}'$, such that $x \notin \sigma(\mathcal{N})$; if $y = x$, then we can pick, if there exists an $\mathcal{N} \in \mathbb{M}'$, such that $x \notin \sigma(\mathcal{N})$, otherwise by hypothesis and Claim 3 we can pick an $\mathcal{N} \in \mathbb{M}'$, such that $\overline{\sigma(\mathcal{N})}$ is not contained in $A$. Then, proceeding in this way for each inning, we find a branch of the tree, i.e., our construction provides a winning strategy for ALICE in the R-nw-selective game on $X$ which is a contradiction. □

The following proposition shows that the (Point, Open)-set game is a good candidate to be the dual of the R-nw-selective game.

**Proposition 3.2.4.** Let $X$ be a space. The following implications hold.

1. ALICE $\uparrow$ PO-set$(X) \implies$ BOB $\uparrow$ R-nw-selective$(X)$.
2. ALICE $\uparrow$ R-nw-selective$(X) \implies$ BOB $\uparrow$ PO-set$(X)$.
3. BOB $\uparrow$ R-nw-selective$(X) \implies$ ALICE $\uparrow$ PO-set$(X)$.

*Proof.* The proof of Items 1. and 2. is trivial and Item 3. is an easy consequence of Theorem 3.2.3. □

The following question is still open.

**Question 3.2.5.** Does BOB $\uparrow$ PO-set$(X)$ imply ALICE $\uparrow$ R-nw-selective$(X)$?

Now we study the determinacy of the R-nw-selective game.

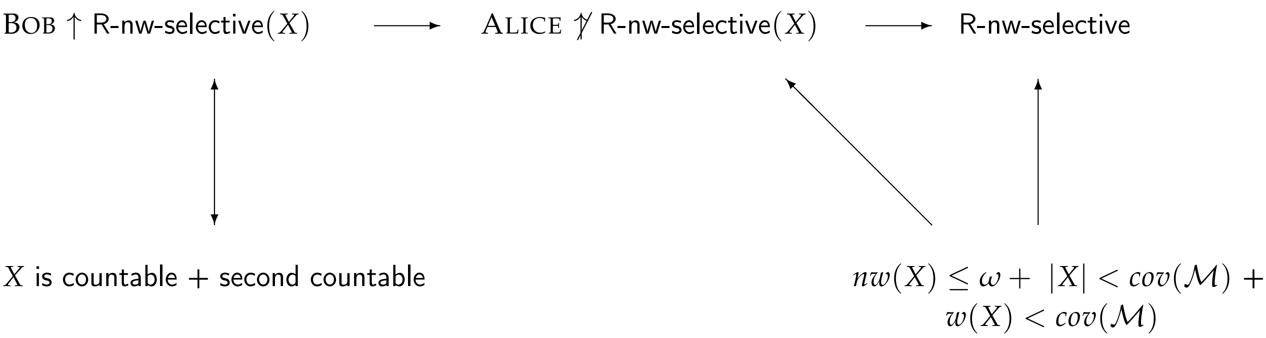
**Proposition 3.2.6.** Let $X$ be a space with $nw(X) = \omega$. If $|X| < cov(\mathcal{M})$ and $w(X) < cov(\mathcal{M})$, then ALICE $\not\uparrow$ R-nw-selective$(X)$.

*Proof.* Suppose, by contradiction, that $\sigma$ is a winning strategy for ALICE in the R-nw-selective$(X)$ and fix a base $\mathcal{B}$ of cardinality $w(X)$. Construct a countable tree using the strategy $\sigma$ in such a way that $\sigma(\langle\rangle) = \mathcal{N}_0$; for each $N_0 \in \mathcal{N}_0$ apply the strategy and so on. Look at this tree as the poset of all finite branches ordered with the inverse natural extension. The nodes in this tree are the countable networks that are images under the function $\sigma$. Fix $x \in X$ and $B \in \mathcal{B}$ containing $x$. The set $D_{(x,B)}$ of all the finite sequences of the tree such that there exists an element of the sequence that is a $\sigma(\langle ..., N\rangle)$ with $x \in N \subset B$, is dense in the poset. Since the cardinality of the family $\{D_{(x,B)} : x \in X \text{ and } B \in \mathcal{B}\}$ is less than $cov(\mathcal{M})$ there exists a generic filter whose union is a branch of the tree that intersects all the dense sets of the family. This gives us a contradiction because this branch witnesses that there is a play in the R-nw-selective game on $X$ in which ALICE applies her strategy but BOB wins. □

**Example 3.2.7.** ($\omega_1 < cov(\mathcal{M})$) Consider a subspace $X \subset \mathbb{R}$ of cardinality $\omega_1$. By Theorem 3.2.3 and Proposition 3.2.6 the R-nw-selective game on $X$ is indeterminate.

**Question 3.2.8.** Is there any ZFC example of a space in which the R-nw-selective game turns out to be indeterminate?

The following diagram shows all the relations found above.




**Question 3.2.9.** Does R-nw-selectivity of a space $X$ imply ALICE $\not\uparrow$ R-nw-selective$(X)$?

Using Corollary 2.2.11 it is possible to give a partial answer to Question 3.2.9.

**Proposition 3.2.10.** Let $X$ be a space with $nw(X) = \omega$ and $w(X) < cov(\mathcal{M})$. Then the following are equivalent.

1. $|X| < cov(\mathcal{M})$;
2. ALICE $\not\uparrow$ R-nw-selective$(X)$;
3. $X$ is R-nw-selective.

Now we will show that if BOB is forced to select a fixed number of elements from each network, then the respective game is equivalent to the R-nw-selective game for BOB. Let $Nw$ denote the class of all countable networks of a fixed space $X$. Let $k \in \omega$ and $G_k(Nw, Nw)$ on $X$ be the game played in the following way: ALICE chooses a countable network $\mathcal{N}_0$ and BOB answers picking a subset $\mathcal{F}_0 \subset \mathcal{N}_0$ such that $|\mathcal{F}_0| = k$. Then ALICE chooses another countable network $\mathcal{N}_1$ and BOB answers picking a subset $\mathcal{F}_1 \subset \mathcal{N}_1$ such that $|\mathcal{F}_1| = k$ and so on for countably many innings. At the end BOB wins if the set $\bigcup\{\mathcal{F}_n : n \in \omega\}$ of his selections is a network.

**Proposition 3.2.11.** BOB $\uparrow$ R-nw-selective($X$) if, and only if, BOB $\uparrow$ $G_k(Nw, Nw)$ on $X$.

*Proof.* It suffices to prove that BOB $\uparrow$ $G_k(Nw, Nw)$ on $X$ implies that the space $X$ is countable and second countable. In fact the proof is similar to the one of Theorem 3.2.3. Let $\sigma$ be a winning strategy for BOB in the $G_k(Nw, Nw)$ on $X$ and let $\mathbb{M}$ be the collection of all countable networks of the space $X$. We just need to prove the following claims.
Claim 1. $|\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}| \leq k$.
Assume that $x_0, ..., x_k$ are $k+1$ distinct points of $X$. Take any countable network $\mathcal{N}$ in the space $X$ and observe that the family $\mathcal{N}' = \{N \in \mathcal{N} : x_i \notin \overline{N} \text{for every} i = 0, ..., k\} \cup \{\{x_0\}, ..., \{x_k\}\}$ is still a network in $X$ and no element of $\mathcal{N}'$ contains more than one point of the set $\{x_1, ..., x_k\}$ in its closure. Now our claim easily follows.
Claim 2. There exists $\mathbb{M}' \subset \mathbb{M}$ countable such that $\bigcap_{\mathcal{N}\in\mathbb{M}'} \overline{\bigcup \sigma(\mathcal{N})} = \bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$.
Claim 3. If $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})} = \{x\}$, there exists an open set $V$ such that $x \in V \subset \bigcup_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$.
Claim 4. If $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})} = \{x\}$, there exists $\mathbb{M}' \subset \mathbb{M}$ countable such that $\bigcap_{\mathcal{N}\in\mathbb{M}'} \overline{\bigcup \sigma(\mathcal{N})} = \{x\}$ and also such that $\bigcup_{\mathcal{N}\in\mathbb{M}'} \overline{\bigcup \sigma(\mathcal{N})} = \bigcup_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$.
The proof of Claims 2., 3. and 4. are similar to the ones in Theorem 3.2.3. □

Also, it is straightfoward to prove the following result.

**Proposition 3.2.12.** ALICE $\uparrow$ $G_k(Nw, Nw)$ on $X$ implies that ALICE $\uparrow$ R-nw-selective($X$).

**Question 3.2.13.** Is it true that if ALICE $\uparrow$ R-nw-selective($X$) then ALICE $\uparrow$ $G_k(Nw, Nw)$ on $X$?

### 3.2.2 The M-nw-selective game

**Definition 3.2.14.** Let $X$ be a space with $nw(X) = \omega$. The M-nw-selective game, denoted by M-nw-selective$(X)$, is played according to then following rules. ALICE chooses a countable network $\mathcal{N}_0$ and BOB answers picking a finite subset $\mathcal{F}_0 \subset \mathcal{N}_0$. Then ALICE chooses another countable network $\mathcal{N}_1$ and BOB answers in the same way and so on for countably many innings. At the end BOB wins if the set $\bigcup\{\mathcal{F}_n : n \in \omega\}$ of his selections is a network.

**Proposition 3.2.15.** $(MA[\mathfrak{d}])$ Let $X$ be a space with $nw(X) = \omega$. If $|X| < \mathfrak{d}$ and $w(X) < \mathfrak{d}$, then ALICE $\not\uparrow$ M-nw-selective$(X)$.

*Proof.* Similar to the proof of Proposition 3.2.6. □

Using Corollary 2.2.8 it is possible to give the following equivalences.

**Proposition 3.2.16.** ($MA[\mathfrak{d}]$) Let $X$ be a space with $nw(X) = \omega$ and $w(X) < \mathfrak{d}$. The following are equivalent:

1. $|X| < \mathfrak{d}$;
2. ALICE $\not\uparrow$ M-nw-selective$(X)$;
3. $X$ is M-nw-selective.

However, it is worthwhile to pose the following question.

**Question 3.2.17.** Does the M-nw-selectivity of a space $X$ imply that ALICE $\not\uparrow$ M-nw-selective$(X)$?

**Theorem 3.2.18.** Let $X$ be a regular space such that BOB $\uparrow$ M-nw-selective(X). Then $X$ is $\sigma$-compact.

*Proof.* Let $\mathbb{M}$ be the collection of all countable networks of $X$ and $\sigma$ a winning strategy for BOB in M-nw-selective(X).
Claim 1: $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$ is compact.
Indeed, put $K = \bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$, let $\mathcal{U}$ be a cover made by open sets of $X$ and $\mathcal{N} \in \mathbb{M}$. Consider the network $\mathcal{N}' = \{N \in \mathcal{N} : \overline{N} \subset U \text{ for some } U \in \mathcal{U}\} \cup \{N \in \mathcal{N} : \overline{N} \cap K = \emptyset\}$. Then $K \subset \sigma(\langle \mathcal{N}' \rangle)$ and considering the corresponding open sets we extract from $\mathcal{U}$ a finite subcover of $K$.
Claim 2: There exists a countable subset $\mathbb{M}' \subset \mathbb{M}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}'} \overline{\bigcup \sigma(\mathcal{N})} = \bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$.
The proof is similar to the one of Claim 2. in Theorem 3.2.3 and, as in there, these claims are true also for all the other innings.
There exists $(\mathcal{N}^n_\emptyset)_{n\in\omega}$, that is countably many possible first innings $\mathcal{N}^0_\emptyset, \mathcal{N}^1_\emptyset, \mathcal{N}^2_\emptyset, \ldots$, such that $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})} = \bigcap_{n\in\omega} \overline{\bigcup \sigma(\mathcal{N}^n_\emptyset)}$.
If Alice chooses $\mathcal{N}^0_\emptyset$, we can find $(\mathcal{N}^n_{<0>})_{n\in\omega}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N}^0_\emptyset, \mathcal{N})} = \bigcap_{n\in\omega} \overline{\bigcup \sigma(\mathcal{N}^0_\emptyset, \mathcal{N}^n_{<0>})}$.
If then Alice chooses $\mathcal{N}^1_{<0>}$, we can find $(\mathcal{N}^n_{<0,1>})_{n\in\omega}$ such that $\bigcap_{n\in\omega} \overline{\bigcup \sigma(\mathcal{N}^0_\emptyset, \mathcal{N}^1_{<0>}, \mathcal{N}^n_{<0,1>})} = \bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N}^0_\emptyset, \mathcal{N}^1_{<0>}, \mathcal{N})}$. By Claims 1 and 2, each intersection, if it is not empty, is a compact subset. If the intersection is empty, we do not do anything and if $\bigcap_{n\in\omega} \overline{\bigcup \sigma(\mathcal{N}^n_\emptyset)}$ is a compact, we call this subset $K^\emptyset$. If $\bigcap_{n\in\omega} \overline{\bigcup \sigma(\mathcal{N}^0_\emptyset, \mathcal{N}^n_{<0>})}$ is a compact subset, we call it $K^{<0>}$. If $\bigcap_{n\in\omega} \overline{\bigcup \sigma(\mathcal{N}^0_\emptyset, \mathcal{N}^1_{<0>}, \mathcal{N}^n_{<0,1>})}$ is a compact subset, we call this element $K^{<0,1>}$, and so on. Consider the set $X_0 = \bigcup\{K^s : s \in \omega^{<\omega}\}$. Now we prove that $X_0 = X$. By contradiction, assume there exists $y \in X \setminus X_0$. Then $y \notin \bigcap_{n\in\omega} \overline{\bigcup \sigma(\mathcal{N}^n_\emptyset)}$; hence there exists $n_0 \in \omega$ such that $y \notin \overline{\bigcup \sigma(\mathcal{N}^{n_0}_\emptyset)}$. Again, $y \notin \bigcap_{n\in\omega} \overline{\bigcup \sigma(\mathcal{N}^{n_0}_\emptyset, \mathcal{N}^n_{\langle n_0\rangle})}$; hence there exists $n_1 \in \omega$ such that $y \notin \overline{\bigcup \sigma(\mathcal{N}^{n_0}_\emptyset, \mathcal{N}^{n_1}_{\langle n_0\rangle})}$. Proceeding in this way we obtain a branch (or an evolution of the M-nw-selective$(X)$) in which BOB does not win, a contradiction, because $\sigma$ is a winning strategy. Then $X$ is $\sigma$-compact. ☐

Recall that a space is called $\sigma$-(metrizable compact) if it is union of countably many metrizable compact spaces. Then it is possible to obtain the following corollary.

**Corollary 3.2.19.** Let $X$ be a regular space in which BOB $\uparrow$ M-nw-selective$(X)$. Then $X$ is $\sigma$-(metrizable compact).

*Proof.* By the previous theorem, $X$ is $\sigma$-compact. Put $X = \bigcup_{n\in\omega} X_n$, where each $X_n$ is compact. Since the space $X$ has countable netweight, then each $nw(X_n) = \omega$ for every $n \in \omega$. By compactness of every $X_n$, each $X_n$ is second countable. Therefore each $X_n$ is metrizable. □

The following is a consistent example showing that the M-nw-selective game can be indeterminate.

**Example 3.2.20.** $(MA[\mathfrak{d}] + \omega_1 < \mathfrak{d})$ Consider a subset $X$ of the irrational numbers having cardinality $\omega_1$. By Proposition 3.2.15, ALICE $\not\uparrow$ M-nw-selective$(X)$. Since $X$ is not $\sigma$-compact, by Theorem 3.2.18 we have that BOB $\not\uparrow$ M-nw-selective$(X)$.

We prove the following result.

**Proposition 3.2.21.** If $X$ is a countable space in which BOB $\uparrow$ M-nw-selective$(X)$. Then $X$ is second countable.

*Proof.* Similar to the the proof of Theorem 3.2.3 replacing Claims 3. and 4. with the following.
Claim $3'$. If $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})} = \{x\}$, there exists an open set $V$ such that $x \in V \subset \bigcup_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$.
Claim $4'$. If $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})} = \{x\}$, there exists $\mathbb{M}' \subset \mathbb{M}$ countable such that $\bigcap_{\mathcal{N}\in\mathbb{M}'} \overline{\bigcup \sigma(\mathcal{N})} = \{x\}$ and also such that $\bigcup_{\mathcal{N}\in\mathbb{M}'} \overline{\bigcup \sigma(\mathcal{N})} = \bigcup_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$. □

The next result uses the previous proposition to state that in the class of countable spaces the M-nw-selective and the R-nw-selective games are equivalent for BOB.

**Corollary 3.2.22.** Let $X$ be a countable space. The following are equivalent.

1. BOB $\uparrow$ R-nw-selective$(X)$;
2. BOB $\uparrow$ M-nw-selective$(X)$;
3. $X$ is second countable.

# Appendix A

# Introduction to set-theoretic topology

The modern study of Set Theory started with a single paper in 1874 by Georg Cantor: “On a Property of the Collection of All Real Algebraic Numbers”, for this reason Cantor is commonly considered the founder of this theory.
The non-formalized systems investigated during this early stage go under the name of “naive set theory”. After the discovery of paradoxes within naive set theory (such as Russell’s paradox, Cantor’s paradox and the Burali-Forti paradox), various axiomatic systems were proposed in the early twentieth century, of which Zermelo–Fraenkel set theory (with or without the axiom of choice) is still the best-known and most studied.

Many mathematicians had struggled with the concept of infinity, for instance Zeno of Elea in the West and Indian mathematicians in the East. Modern understanding of infinity began in 1870–1874, and was motivated by Cantor’s work in real analysis. In Set Theory two important cardinals associated with $\omega$ are $\omega_1$ and $\mathfrak{c}$, respectively the successor of $\omega$ and the cardinality of $\mathcal{P}(\omega)$ or, equivalently, of ${}^{\omega}\omega$. Cantor proved that $\omega_1 \leq \mathfrak{c}$, Cohen and Gödel showed that both $\omega_1 < \mathfrak{c}$ and $\omega_1 = \mathfrak{c}$ are consistent in ZFC. One can define other cardinals associated with $\omega$ and each one lies between $\omega_1$ and $\mathfrak{c}$. Of course the interest of these cardinals is purely set-theoretical and it is surprising the relations that they have with topological properties.
Consider the Baire space ${}^{\omega}\omega$ and introduce the order relations “$\leq^*$" and “$\leq$" defined as follows. For $f, g \in {}^{\omega}\omega$, denote with $f \leq^* g$ the fact that $f(n) \leq g(n)$ for all but finitely many $n$ and $f \leq g$ means $f(n) \leq g(n)$ for all $n \in \omega$.
A subset $B \subseteq {}^{\omega}\omega$ is called *bounded* if there is $g \in {}^{\omega}\omega$ such that $f \leq^* g$ for every $f \in B$.
A subset $D \subseteq {}^{\omega}\omega$ is called *dominating* or *cofinal* if for each $g \in {}^{\omega}\omega$ there is $f \in D$ such that $g \leq^* f$.
Now one can introduce two cardinal numbers: the bounding number $\mathfrak{b}$ and the dominating number $\mathfrak{d}$, based on the previous definitions. $\mathfrak{b}$ is the minimal cardinality of an unbounded subset of ${}^{\omega}\omega$ and $\mathfrak{d}$ is the minimal cardinality of a dominating subset of ${}^{\omega}\omega$. It is proved that the value of $\mathfrak{d}$ does not change if one considers the relation $\leq$ instead of $\leq^*$ [32, Theorem 3.6].
Let’s denote by $\mathcal{M}$ the family of all meager subsets of $\mathbb{R}$. $cov(\mathcal{M})$ is the minimum of the cardinalities of subfamilies $\mathcal{U} \subseteq \mathcal{M}$ such that $\bigcup \mathcal{U} = \mathbb{R}$. However, another description of the cardinal $cov(\mathcal{M})$ is the following one.

**Theorem A.0.1.** ([8] and [9, Theorem 2.4.1]) $cov(\mathcal{M})$ is the minimum cardinality of a subset $F \subset {}^{\omega}\omega$ such that for every $g \in {}^{\omega}\omega$ there is $f \in F$ such that $f(n) \neq g(n)$ for all but finitely many $n \in \omega$.

Thus if $F \subset {}^{\omega}\omega$ and $|F| < cov(\mathcal{M})$, then there is $g \in {}^{\omega}\omega$ such that for every $f \in F$, $f(n) = g(n)$ for infinitely many $n \in \omega$.
For a topological property $\mathcal{K}$, let's denote with non($\mathcal{K}$) the minimum cardinality of a subspace of $\mathbb{R}$ that does not have property $\mathcal{K}$. It is well known that non(Menger) $= \mathfrak{d}$, non(Hurewicz) $= \mathfrak{b}$, non(Rothberger) $= cov(\mathcal{M})$ (see [35, 44, 60]).

**Proposition A.0.2.** If $X$ is a Lindelöf space of cardinality strictly less than $\mathfrak{d}$, then $X$ is Menger.

*Proof.* Suppose by contradiction that $X$ is not Menger. Let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers witnessing the fact that $X$ is not Menger. Since $X$ is Lindelöf one can suppose that each cover is countable, then $\mathcal{U}_n = \{U^n_m : m \in \omega\}$. For every $x \in X$ and $n \in \omega$ define $f_x(n) = \min\{m \in \omega : x \in U^n_m\}$. Put $D = \{f_x : x \in X\}$ and prove that it is dominating. Fix $g \in {}^{\omega}\omega$, for each $n \in \omega$, $g(n) = m_n$. One have that $\bigcup_{n\in\omega} \bigcup_{i=0}^{m_n} U^n_i \subsetneq X$, then there exists $x \in X$ such that $x \notin \bigcup_{n\in\omega} \bigcup_{i=0}^{m_n} U^n_i$. Therefore for every $n \in \omega$ and $i = 0, ..., m_n$, $x \notin U^n_i$. Hence $f_x(n) > m_n = g(n)$ for every $n \in \omega$. Since $\phi : X \to {}^{\omega}\omega$, defined by $\phi(x) = f_x$, is a function and the image of $\phi$ is $D$, then $|X| \geq |D| \geq \mathfrak{d}$, a contradiction. □

**Proposition A.0.3.** If $X$ is a Lindelöf space of cardinality strictly less than $\mathfrak{b}$, then $X$ is Hurewicz.

*Proof.* Suppose by contradiction that $X$ is not Hurewicz. Let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers witnessing the fact that $X$ is not Hurewicz. Since $X$ is Lindelöf one can suppose that each cover is countable, then $\mathcal{U}_n = \{U^n_m : m \in \omega\}$. For every $x \in X$ and $n \in \omega$ define $f_x(n) = \min\{m \in \omega : x \in U^n_m\}$. Put $B = \{f_x : x \in X\}$ and prove that it is unbounded. Fix $g \in {}^{\omega}\omega$, for each $n \in \omega$, $g(n) = m_n$. For every $n \in \omega$ put $\mathcal{V}_n = \{U^n_m : m \leq m_n\}$, then there exists $x \in X$ such that $x \in \bigcup \mathcal{V}_n$ for finitely many $n \in \omega$. Then $f_x(n) > g(n)$ for every but finitely many $n \in \omega$. Hence $g \leq^* f_x$. Since $\phi : X \to {}^{\omega}\omega$, defined by $\phi(x) = f_x$, is a function and the image of $\phi$ is $B$, then $|X| \geq |B| \geq \mathfrak{b}$, a contradiction. □

**Proposition A.0.4.** If $X$ is a Lindelöf space of cardinality strictly less than $cov(\mathcal{M})$, then $X$ is Rothberger.

*Proof.* Suppose by contradiction that $X$ is not Rothberger. Let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers witnessing the fact that $X$ is not Rothberger. Since $X$ is Lindelöf one can suppose that each cover is countable, then $\mathcal{U}_n = \{U^n_m : m \in \omega\}$. For every $x \in X$ and $n \in \omega$ define $f_x(n) = \min\{m \in \omega : x \in U^n_m\}$. Put $F = \{f_x : x \in X\}$ and prove that $|F| \geq cov(\mathcal{M})$, by using Theorem A.0.1. Fix $g \in {}^{\omega}\omega$, for each $n \in \omega$, $g(n) = m_n$. One have that $\bigcup_{n\in\omega} U^n_{m_n} \subsetneq X$, then there exists $x \in X$ such that $x \notin \bigcup_{n\in\omega} U^n_{m_n}$. Therefore for every $n \in \omega$, $x \notin U^n_{m_n}$. Hence $f_x(n) \neq m_n = g(n)$ for every $n \in \omega$. Since $\phi : X \to {}^{\omega}\omega$, defined by $\phi(x) = f_x$, is a function and the image of $\phi$ is $F$, then $|X| \geq |F| \geq cov(\mathcal{M})$, a contradiction. □

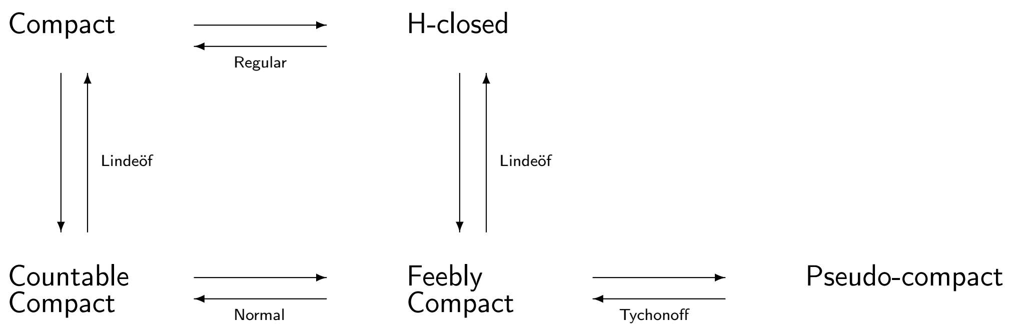
# Appendix B

# Variations of compacteness

For a subset $A$ of a topological space $X$ we will denote by $[A]^{<\lambda}$ ($[A]^{\lambda}$) the family of all subsets of $A$ of cardinality $< \lambda$ ($= \lambda$). A space is called H-closed if it is Hausdorff and it is closed in every Hausdorff space in which it is embedded. Moreover, a more operative characterization of H-closedness of a space $X$ is given by the following statement: "for every open cover $\mathcal{U}$ of $X$ there exists a finite subfamily $\mathcal{V} \in [\mathcal{U}]^{<\omega}$ such that $\overline{\bigcup \mathcal{V}} = X$". This defininition makes clear why this property is a generalization of compactness.
Two other characterizations of H-closed spaces are that each open filter (open ultrafilter) on the space has a nonempty adherence (convergers, respectively). Clearly, a space is H-closed iff $wL(X) < \omega$ (consult [55] for more details on H-closed spaces).

Recall that a space is feebly compact if every locally finite family of open subsets is finite. Equivalently, if every countable cover has a finite subfamily whose union is dense. It is know that in the class of Tychonoff spaces feebly compactness and pseudo-compactness are equivalent properties. Actually, in [34] the definition of pseudo-compactness includes the Tychonoffness of the space, so that pseudo-compactness and feebly compactness are exatly the same properties. However, some authors simply call pseudo-compact a space whose image through any continuous real valued function is bounded.

The following diagram sums up the connections between these properties.



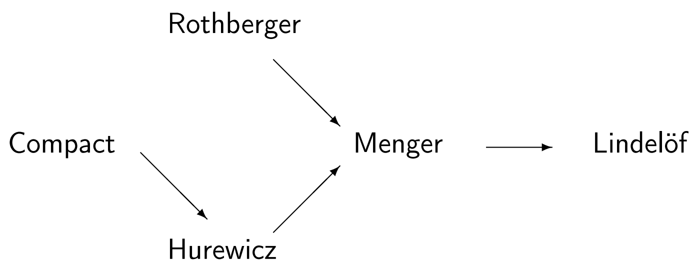

In general covering properties represent a class widely studied in topology, in fact they are particularly used to characterize other properties and also there are amazing connections and relations between them and some well know "small"cardinal numbers in Set Theory. Popular among such properties are the Menger, Hurewich and Rothberger ones.

**Definition B.0.1.** A space $X$ is Menger if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of $X$ one can select finite $\mathcal{F}_n \subset \mathcal{U}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} \mathcal{F}_n$ covers $X$.

**Definition B.0.2.** A space $X$ is Rothberger if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of $X$ one can select $F_n \in \mathcal{U}_n$, $n \in \omega$, such that $\{F_n : n \in \omega\}$ covers $X$.

**Definition B.0.3.** A space $X$ is Hurewicz if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of $X$ one can select finite $\mathcal{F}_n \subset \mathcal{U}_n$, $n \in \omega$, such that for every $x \in X$, $x \in \bigcup \mathcal{F}_n$ for all but finitely many $n$.

It is well known that these properties lie between two famous and basic covering properties: Compactness and Lindelöfness.



# Appendix C

# A gentle introduction to Forcing

Let $(\mathbb{P}, \leq, 1)$ be a poset, i.e. a partially ordered set. The element 1 is the maximum with respect to the relation $\leq$. Sometimes, when there is no reason to specify we omitt $\leq$ or 1.

- $D \subseteq \mathbb{P}$ is a dense subset (in the sense of posets) if for every $p \in \mathbb{P}$ there exists $d \in D$ such that $d \leq p$.
- $p, q \in \mathbb{P}$ are called compatible ($p \not\perp q$) if there exists $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$, otherwise they are called incompatible ($p \perp q$).
- A poset $(\mathbb{P}, \leq, 1)$ is said to have the countable chain condition (ccc is the sense of posets) if every antichain (subset of $\mathbb{P}$ having pairwise incompatible elements) is countable.
- A subset $G \subseteq \mathbb{P}$ is called a filter if
  (a) the maximum 1 belongs to $G$;
  (b) for every $p, q \in G$ there exists $r \in G$ such that $r \leq p$ and $r \leq q$;
  (c) for each $p, q \in \mathbb{P}$, if $p \in G$ and $p \leq q$, then $q \in G$.
- Let $\langle \mathbb{P}, \leq \rangle$ be a poset and let $\mathcal{D}$ be a family of dense subsets of $\mathbb{P}$. We say that a filter $G$ in $\mathbb{P}$ is $\mathcal{D}$-generic if $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

The partial orders used in the context of $\mathcal{D}$-generic filters will often be called *forcings*. Also, if $\langle \mathbb{P}, \leq \rangle$ is a forcing then elements of $\mathbb{P}$ are called *conditions*. For conditions $p, q \in \mathbb{P}$ we say that $p$ is stronger than $q$ provided $p \leq q$.

**Theorem C.0.1.** *(Rasiowa-Sikorski lemma)*Let $\langle \mathbb{P}, \leq \rangle$ be a poset and $p \in \mathbb{P}$. If $\mathcal{D}$ is a countable family of dense subsets of $\mathbb{P}$ then there exists a $\mathcal{D}$-generic filter $G$ in $\mathbb{P}$ such that $p \in G$.

*Proof.* Let $\mathcal{D} = \{D_n : n < \omega\}$. We define a sequence $\langle p_n : n < \omega \rangle$ by induction on $n < \omega$. We start by picking $p_0 \in D_0$ such that $p_0 \leq p$. We continue by choosing $p_{n+1} \in D_{n+1}$ such that $p_{n+1} \leq p_n$. Now let $E = \{p_n : n < \omega\}$ and put $G = \{p \in \mathbb{P} : \exists q \in E(q \leq p)\}$. Then $G$ is a filter in $\mathbb{P}$ intersecting every $D \in \mathcal{D}$. □

The Rasiowa-Sikorski lemma is one of the most fundamental facts in the theory of forcing. Its importance, however, does not come from its complexity. It rather gives us the language of forcing.
More precisely, a poset $\mathbb{P}$ used to construct an object will usually be built on the basis of an attempted inductive construction of the object. That is, conditions will be chosen as a "description of the current stage of induction". The inductive steps will be related to the dense subsets of $\mathbb{P}$ in the sense that the density of a particular

set $D_x = \{p \in \mathbb{P} : \varphi(p,x)\}$ will be equivalent to the fact that at an arbitrary stage $q$ of the inductive construction we can make the next inductive step by extending the condition $q$ to $p$ having the property $\varphi(p,x)$. In particular, the family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ will always represent the set of all inductive conditions of which we have to take care, and a $\mathcal{D}$-generic filter in $\mathbb{P}$ will be an "oracle"that "takes care of all our problems", and from which we will recover the desired object. Evidently, if the number $|\mathcal{D}|$ of conditions we have to take care of is not more than the number of steps in our induction, then usually the (transfinite) induction will be powerful enough to construct the object, and the language of forcing will be redundant. Consider a transitive set $M$. For every formula $\psi$ of the language of set theory we define a formula $\psi^M$, called its relativization to $M$. It is obtained by replacing in $\psi$ each unbounded quantifier $\forall x$ or $\exists x$ with its bounded counterpart $\forall x \in M$ or $\exists x \in M$. For example, if $\psi_0$ is a sentence from the axiom of extensionality

$$\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y]$$

then $\psi_0^M$ stands for

$$\forall x \in M \forall y \in M [\forall z \in M (z \in x \leftrightarrow z \in y) \rightarrow x = y].$$

In particular, if $\psi(x_1,\ldots,x_n)$ is a formula with free variables $x_1,\ldots,x_n$ and $t_1,\ldots,t_n \in M$ then $\psi^M(t_1,\ldots,t_n)$ says that $\psi(t_1,\ldots,t_n)$ is true under the interpretation that all variables under quantifiers are bound to $M$. In other words, $\psi^M(t_1,\ldots,t_n)$ represents the formula $\psi(t_1,\ldots,t_n)$ as seen by a "person living inside $M$".

For a transitive set $M$ and a formula $\psi$ (with possible parameters from $M$) we say that "$\psi$ is true in $M$" and write $M \models \psi$ if $\psi^M$ is true.

Forcing consistency proofs will be based on the following fundamental principle.

**Forcing principle**: In order to prove that the consistency of ZFC implies the consistency of ZFC + "$\psi$" it is enough to show (in ZFC) that every countable transitive model $M$ of ZFC can be extended to a countable transitive model $N$ of ZFC + "$\psi$".

**Cohen Reals**

A *Cohen real* is a type of real number added to a model $V$ of set theory through *Cohen forcing*. The Cohen forcing is defined on the partial order $\mathbb{P}$ of finite subsets of $2^{<\omega}$, the set of finite functions from $\omega$ to 2. Formally, $\mathbb{P}$ is the set of partial functions $p : \omega \to 2$ with finite domain, ordered by reverse inclusion:

$$p \leq q \iff p \supseteq q.$$

A *Cohen generic filter* $G \subseteq \mathbb{P}$ is a set of conditions such that for every dense set $D \subseteq \mathbb{P}$ in $V$, $G \cap D \neq \emptyset$. The addition of a Cohen real corresponds to adding a new object $x : \omega \to 2$ such that $x = \bigcup G$ for $G$ a generic filter. This new real $x$ satisfies that no function $f \in V$ with domain $\omega$ is such that $f = x$, ensuring that the real $x$ does not belong to the original model $V$.

**Random Reals**

A *random real* is a real number added via *random forcing*. The random forcing is constructed using the space $\mathbb{B}$, the Borel algebra of subsets of $[0,1]$ with positive Lebesgue measure, along with the inclusion relation:

$$p \leq q \iff \mu(p) \leq \mu(q).$$

Here, $\mu$ denotes the Lebesgue measure. A generic filter $G \subseteq \mathbb{B}$ is such that for every dense set $D \subseteq \mathbb{B}$, $G \cap D \neq \emptyset$. The element $r$ generated, $r = \bigcap G$ for a generic filter $G$, is a real number such that for every null set in the ground model $V$, $r$ does not belong to that set.

Cohen forcing uses the set $\mathbb{P}$ of finite functions from $\omega$ to 2, while random forcing uses the Borel algebra $\mathbb{B}$ on subsets of $[0, 1]$. Cohen reals avoid countable dense sets in the ground model, whereas random reals have statistical properties similar to a random sequence with respect to the Lebesgue measure. A Cohen real does not belong to any countable dense subset of the ground model $V$, while a random real is constructed to be "random" with respect to any preexisting measure in $V$.

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