A BLOW-UP RESULT FOR A NONLINEAR WAVE EQUATION ON MANIFOLDS: THE CRITICAL CASE

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ABSTRACT. We consider a inhomogeneous semilinear wave equation on a noncompact complete Riemannian manifold $(\mathbb{M}, g)$ of dimension $N \geq 3$, without boundary. The reaction exhibits the combined effects of a forcing term with critical exponent, and of a nonnegative perturbation term. Using a rescaled test function argument together with appropriate estimates, we show that the problem (i.e., left open by Zhang in his interesting paper [11]) admits no global solution. Moreover, in the special case when $\mathbb{M} = \mathbb{R}^3$, our result improves that of this author. Namely, our main result is valid without assuming that the initial values are compactly supported.

1. INTRODUCTION

In the general field of partial differential equations, considerable efforts have been taken to understanding the blow-up structural mechanism for solutions of evolution equations. For applications, we refer to reaction-diffusion processes of fluid mechanics and turbolence flows, which are often represented by wave equations. This research topic continues to attract the interest of both pure mathematicians and scientistists in applied theories. Clearly, pure mathematicians are not motivated by the applications, but their challenge is to depict the behavior of solutions and to carry out an asymptotic analysis of solutions near any kinds of singularities. To understand the meaning of these sentences, we recall briefly what is referred as blow-up. Indeed, the problem to establish sufficient conditions for the existence and non-existence of solutions is crucial. Moreover, in the time analysis of the evolution equations, we speak about global and local results, meaning that the study focuses on the positive real axis $(0, T)$, with $T \in \mathbb{R}_+ \cup \{+\infty\}$ or just on a finite interval $[a, b]$ of the same positive real axis (with $a, b > 0$). Now, by a blow-up we mean that a solution function is unbounded at some point of the spatial domain (that is, a loss of regularity). This is the key behavior towards deserving the absence of global solutions to a given problem (namely, global non-existence results). In addition, we speak about finite blow-up whenever we consider also the case where the loss of regularity is dictated

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by unboundedness of certain derivative of the solution functions (that is, again a loss of regularity). As we will show later, the nonlinear reaction term of the particular wave equation plays a crucial role in the blow-up structural mechanism of solutions. Strictly controlling the growth of any nonlinearities, then we can give precise informations on the asymptotic behavior of solutions (mainly near blow-ups). On this basis, the initial prototype over which we construct this paper is given by the the classical wave equation

$$u_{tt} - \Delta u = |u|^p \quad \text{in } (0,\infty)\times\mathbb{R}^N, \quad p>1. \tag{1.1}$$

The large-time behavior of solutions to (1.1) has been studied extensively since four decades, focusing on the role of a "number" named critical exponent, and denoted by $p_c(\cdot)$. This number plays the role of a bifurcation parameter in distinguishing between existence and non-existence of solutions, but also in determing their asymptotic behavior. In view of the preamble, we summarize saying that it is critical with regard to the occurrence of any blow-ups. The critical exponent may depend on the geometry of the problem and in particular by the boundary constraints and additional parameters in the leading equation; basically it depends on the dimension of the domain space. Here, we consider the dimension N of $\mathbb{R}^N$ as the main argument of $p_c(\cdot)$ (that is, we refer to $p_c(N)$). Now, we give a short survey of the literature closely related to our finding. Inspired by the seminal work of John [5] in the three dimensional case, Strauss [9] conjectured that for all $N\geq 2$, there exists a critical exponent $p_c(N)$ for the global existence question to (1.1) with compactly supported initial values, and it should be the positive root of the polynomial

$$(N-1)p^2-(N+1)p-2=0.$$

After twenty-five years of efforts, this conjecture is finally showed to be true for every $N\geq 2$, see for instance [2–4, 6–8, 10, 12] and the references therein. Namely, for $N\geq 2$, let

$$p_c(N)=\frac{N+1+\sqrt{N^2+10N-7}}{2(N-1)}.$$

As a typical result, looking at the literature, we can encounter the following bifurcation statement:

(a) If the exponent p lies in the interval $(1,p_c(N)]$, then for any $(u,u_t)|_{t=0}$ compactly supported with positive average, the solution to (1.1) blows up in a finite time.
(b) If the exponent p is greater than the critical exponent $p_c(N)$, we can find positive compactly supported initial conditions such that the solution to (1.1) exists globally in time.

Remark 1.1. *The range interval of exponents p in the statement (a) leads to the conclusion that the critical exponent $p_c(N)$ belongs to the blow-up situation.*

In [11], Zhang considered for the first time the problem for inhomogeneous semilinear wave equation of the form

$$\begin{cases} u_{tt} - \Delta u & = & |u|^p + w(x) \quad \text{in } (0,\infty)\times\mathbb{M}, \\ (u(0,x), u_t(0,x)) & = & (u_0(x), u_1(x)) \quad \text{in } \mathbb{M}. \end{cases} \tag{1.2}$$

In the equation, by Δ we denote the Laplace Beltrami operator on $\mathbb{M}$, and by $(\mathbb{M}, g)$ we denote a noncompact complete Riemannian manifold of dimension $N \geq 3$, with Riemannian metric g (also known as metric tensor, which defines a appropriate inner product; see Section 2). In the right hand side, the positive forcing term $|u|^p$ $(p > 1)$ is perturbed by an additional nonnegative term w with some regularities (that is, the function $w = w(x) \geq 0$ is in $L^1_{loc}(\mathbb{M})$). Moreover, the initial values condition is a way to fix the geometry of the problem (recall the above discussion on the arguments of $p_c(\cdot)$).

Let $B(x,\rho)$ be the geodesic ball centered at x with radius ρ, given by

$$B(x,\rho) = \{y \in \mathbb{M} :\ d_{\mathbb{M}}(x,y) < \rho\},$$

where $d_{\mathbb{M}}$ is the induced distance metric on $\mathbb{M}$, and $|B(x,\rho)|$ denotes the volume of $B(x,\rho)$. Thus, problem (1.2) was investigated under the following set of hypotheses (see [11]):

$(A1)$ $u_i \in C(\mathbb{M})$, $i = 0, 1$. The possible solutions to (1.2) are classical in the sense of belonging to $C^2((0,T)\times\mathbb{M}) \cap C([0,T)\times\mathbb{M})$.

$(A2)$ $\mathbb{M}$ has nonnegative Ricci curvature. We can find $x_0 \in \mathbb{M}$ with $C_{x_0} = \emptyset$, where C_{x_0} means the cut locus of x_0. If r denotes the distance from x_0 and $\sqrt{A}$ denotes the volume density, we impose the restriction

$$\left| \frac{\partial\left(\ln\sqrt{A}\right)}{\partial r} \right| \leq \frac{C}{r} \quad \text{for some } C > 0.$$

$(A3)$ f is a nonnegative function satisfying:

$$Cf(\rho) \geq |B(x,\rho)| \geq C^{-1}f(\rho), \quad \text{for all } x \in \mathbb{M}, \text{ all } \rho > 0, \text{ some } C > 0,$$
$$f(\rho) \sim \rho^\alpha \text{ for some } \alpha > 2 \text{ as } \rho \to \infty,$$
$$f(\rho) \sim \rho^N \quad \text{as } \rho \to 0^+.$$

Referring to the above bifurcation statement (see (a) and (b)), Zhang [11] established a similar result as follows.

Theorem 1.1. *If* $(A1)$–$(A3)$ *hold, then we distinguish the following situations:*

(i) *If the exponent p lies in the interval $(1, \frac{\alpha}{\alpha-2})$ and $w \not\equiv 0$, then problem (1.2) admits no global solution for all u_0, u_1.*

(ii) *When $\mathbb{M} = \mathbb{R}^3$ (hence $p = \frac{N}{N-2} = 3$) and $w \not\equiv 0$, then problem (1.2) admits no global solution, provided that $u_0, u_1 \in C(\mathbb{M})$ are compactly supported.*

(*iii*) *If the exponent* p *is greater than* $\frac{\alpha}{\alpha-2}$*, then problem* (1.2) *admits global solutions for some* $u_0, u_1 \in C(\mathbb{M})$ *and some* $w \in L^1_{loc}(\mathbb{M})$.

Comparing Theorem 1.1(i)-(ii) with (a), we note that Zhang works with the critical exponent (that is, the value $p = \frac{\alpha}{\alpha-2}$) only in the case $\mathbb{M} := \mathbb{R}^3$. So there is a lack of precise information on the behavior of solutions to (1.2) as $p = \frac{\alpha}{\alpha-2}$ in the case $\mathbb{M} \neq \mathbb{R}^3$. Moreover, Zhang posed this problem in [11, Remark 4]. The aim of this paper is to solve this situation. Namely, we are concerned with the critical wave equation

$$\left\{ \begin{array}{lll} u_{tt} - \Delta u & = & |u|^{p^*} + w(x) \quad \text{in } (0,\infty)\times\mathbb{M}, \\ (u(0,x), u_t(0,x)) & = & (u_0(x), u_1(x)) \quad \text{in } \mathbb{M}, \end{array} \right. \tag{1.3}$$

where $p^* := \frac{\alpha}{\alpha-2}$, again under the set of hypotheses ($A1$)–($A3$). Using a rescaled test function argument together with appropriate estimates, we show that the critical exponent belongs to the blow-up case. Our main result is given by the following theorem.

Theorem 1.2. *If* ($A1$)–($A3$) *hold and* $w \not\equiv 0$*, then problem* (1.3) *admits no global solution.*

No additional hypotheses on the initial values u_i, $i = 0, 1$, are assumed in Theorem 1.2. Consequently, in the case $\mathbb{M} = \mathbb{R}^3$, Theorem 1.2 improves the corresponding result in Theorem 1.1(ii).

In Section 2 we prove a technical lemma and hence we give the proof of our main result.

2. Technical Lemma and Proof of Theorem 1.2

Let $\mathbb{M}$ be a N-dimensional smooth manifold (that is, C^∞-differentiable manifold), $T\mathbb{M}$ be the tangent bundle of $\mathbb{M}$, $X, Y : \mathbb{M} \to T\mathbb{M}$ be two vector fields, and $\Gamma(T\mathbb{M})$ be the vector space of vector fields. According to the classical theory of Riemannian manifolds, we adopt the following notation

$$\langle X, Y\rangle_g := g(X,Y) \quad \text{and} \quad |X|_g := \sqrt{\langle X, X\rangle}, \quad \text{for all } X, Y \in \Gamma(T\mathbb{M}),$$

where g is the involved Riemannian metric on $\mathbb{M}$.

Throughout this section, C will denote a positive constant which may change from line to line. Also, dV will be the standard Riemmanian volume element (that is, the volume density), dS being the standard Riemmanian surface element.

The following lemma will be used later in the proof of Theorem 1.2.

Lemma 2.1. *If* ($A3$) *holds, then the estimate*

$$\int_{B(x_0,R)\setminus B(x_0,\sqrt{R})} d_{\mathbb{M}}(x_0,x)^{-\alpha}\, dV \leq C \ln R \tag{2.1}$$

is true, for sufficiently large R.

Proof. Hypothesis $(A3)$ allows us to find $\rho_0 > 0$ satisfying

$$|B(x,\rho)| \leq C\rho^{\alpha}, \quad \text{for all } x \in \mathbb{M},\ \rho \geq \rho_0. \tag{2.2}$$

Now, we can find $r_0 \in \left[\max\left\{1, \frac{\rho_0}{2}\right\}, \rho_0 + 1\right]$ and consider the set $\mathcal{R}$ of real-numbers $R := 2^{\ell} r_0$, for some $\ell \in \mathbb{N}$, satisfying the condition $R > (\rho_0 + 1)^2$. Next, for every $R \in \mathcal{R}$ we have

$$\begin{aligned}
\int_{B(x_0,R)\setminus B(x_0,\sqrt{R})} d_{\mathbb{M}}(x_0,x)^{-\alpha}\, dV &\leq \int_{B(x_0,R)\setminus B(x_0,r_0)} d_{\mathbb{M}}(x_0,x)^{-\alpha}\, dV \\
&= \sum_{i=1}^{\ell} \int_{B(x_0,2^i r_0)\setminus B(x_0,2^{i-1}r_0)} d_{\mathbb{M}}(x_0,x)^{-\alpha}\, dV \\
&\leq \sum_{i=1}^{\ell} (2^{i-1} r_0)^{-\alpha} \int_{B(x_0,2^i r_0)} dV.
\end{aligned}$$

By construction, $2^i r_0 \geq \rho_0$ for all $i = 1, 2, \cdots, \ell$ and, since (2.2) holds, also we conclude the estimate (2.1). Indeed, we have

$$\begin{aligned}
\int_{B(x_0,R)\setminus B(x_0,\sqrt{R})} d_{\mathbb{M}}(x_0,x)^{-\alpha}\, dV &\leq C \sum_{i=1}^{\ell} (2^{i-1} r_0)^{-\alpha} (2^i r_0)^{\alpha} \\
&\leq C\ell \\
&= \frac{C}{\ln 2} (\ln R - \ln r_0) \\
&\leq C \ln R \quad \text{(recall that } r_0 \geq 1).
\end{aligned}$$

□

We now develop the proof of Theorem 1.2, whose statement is given at the end of Section 1. We recall that we will use a rescaled test function argument, and hence before proving our result we introduce some auxiliary functions:

- Let $\eta \in C^{\infty}(\mathbb{R})$ be such that
$$\eta \geq 0, \quad \eta \not\equiv 0, \quad \operatorname{supp}(\eta) \subset (0,1).$$
- Let $F : \mathbb{R} \to [0,1]$ be a smooth function satisfying
$$F(s) = \begin{cases} 1 & \text{if} \quad s \leq 0, \\ 0 & \text{if} \quad s \geq 1. \end{cases}$$
- For sufficiently large R, let
$$\eta_R(t) = \eta\left(\frac{t}{R^2}\right), \quad \text{for all } t > 0,$$

$$G_R(r) = F\left(\frac{\ln\left(\frac{r}{\sqrt{R}}\right)}{\ln\sqrt{R}}\right), \quad \text{for all } r > 0,$$

$$F_R(x) = G_R(d_{\mathbb{M}}(x_0, x)), \quad \text{for all } x \in \mathbb{M} \setminus \{x_0\}.$$

Combining appropriately these functions, for α given in $(A3)$ we define the key test function

$$\varphi_R(t,x) = \eta_R^\alpha(t) F_R^\alpha(x), \quad \text{for all } (t,x) \in (0,\infty) \times \mathbb{M} \ (R \text{ large enough}), \tag{2.3}$$

and involve it in the following proof.

Proof of Theorem 1.2. Fixed R sufficiently large, we set

$$Q_R := (0, R^2) \times B(x_0, R).$$

Now, we prove by contradiction the non-existence of global solution. So, we start with an eventually global solution to problem (1.3), which we denote by u. We multiply both the sides of the inhomogeneous semilinear wave equation in (1.3) by φ_R (see (2.3)). We integrate over the set Q_R, thus obtaining

$$\int_{Q_R} (u_{tt} - \Delta u)\, \varphi_R \, dV\, dt = \int_{Q_R} |u|^{p^*} \varphi_R \, dV\, dt + \int_{Q_R} w(x) \varphi_R \, dV\, dt,$$

Integrating by parts, we get

$$\begin{aligned}
&\int_{Q_R} u(\varphi_R)_{tt}\, dV\, dt - \int_{Q_R} u \Delta\varphi_R \, dV\, dt + \int_{(0,R^2)\times\partial B(x_0,R)} \langle \nu, \operatorname{grad} \varphi_R \rangle_g \, u\, dS\, dt \\
&- \int_{(0,R^2)\times\partial B(x_0,R)} \langle \nu, \operatorname{grad} u \rangle_g \, \varphi_R \, dS\, dt \\
&= \int_{Q_R} |u|^{p^*} \varphi_R \, dV\, dt + \int_{Q_R} w(x) \varphi_R \, dV\, dt,
\end{aligned}$$

where by $\nu \in \Gamma(T\partial B(x_0, R))$, we mean the outward-pointing normal unit vector field on $\partial B(x_0, R)$.

Hypothesis $(A2)$, recall that $C_{x_0} = \emptyset$, gives us the smoothness of

$$r : x \mapsto d_{\mathbb{M}}(x_0, x)$$

on $\mathbb{M} \setminus \{x_0\}$, thus obtaining that the test function φ_R is smooth too. On the other hand, by the definition of the function φ_R (see (2.3)), we deduce that $F_R|_{\partial B(x_0,R)} \equiv 0$ implies the following two relations on $(0, R^2) \times \partial B(x_0, R)$:

$$\langle \nu, \operatorname{grad} \varphi_R \rangle_g = \eta_R^\alpha \langle \nu, \operatorname{grad} F_R^\alpha \rangle_g = \eta_R^\alpha \langle \nu, \alpha F_R^{\alpha-1} \operatorname{grad} F_R \rangle_g = 0$$

and

$$\langle \nu, \operatorname{grad} u \rangle_g \, \varphi_R = \eta_R^\alpha \, \langle \nu, \operatorname{grad} u \rangle_g \, F_R^\alpha = 0.$$

Taking these relations into account, we obtain the following identity

$$\int_{Q_R} |u|^{p^*}\varphi_R\, dV\, dt + \int_{Q_R} w(x)\varphi_R\, dV\, dt = \int_{Q_R} u(\varphi_R)_{tt}\, dV\, dt - \int_{Q_R} u\Delta\varphi_R\, dV\, dt,$$

which yields to the inequality

$$\int_{Q_R} |u|^{p^*}\varphi_R\, dV\, dt+\int_{Q_R} w(x)\varphi_R\, dV\, dt \leq \int_{Q_R} |u||(\varphi_R)_{tt}|\, dV\, dt+\int_{Q_R} |u||\Delta\varphi_R|\, dV\, dt. \tag{2.4}$$

By appealing to the Young inequality, we provide the estimates to the two addends in the right hand side of (2.4) as follows:

$$\int_{Q_R} |u||(\varphi_R)_{tt}|\, dV\, dt \leq \frac{1}{2}\int_{Q_R} |u|^{p^*}\varphi_R\, dV\, dt + C\int_{Q_R} \varphi_R^{\frac{-1}{p^*-1}}|(\varphi_R)_{tt}|^{\frac{p^*}{p^*-1}}\, dV\, dt \tag{2.5}$$

and

$$\int_{Q_R} |u||\Delta\varphi_R|\, dV\, dt \leq \frac{1}{2}\int_{Q_R} |u|^{p^*}\varphi_R\, dV\, dt + C\int_{Q_R} \varphi_R^{\frac{-1}{p^*-1}}|\Delta\varphi_R|^{\frac{p^*}{p^*-1}}\, dV\, dt. \tag{2.6}$$

We combine (2.4), (2.5) and (2.6), thus obtaining

$$\int_{Q_R} w(x)\varphi_R\, dV\, dt \leq C\left(I_1(\varphi_R) + I_2(\varphi_R)\right), \tag{2.7}$$

where we use the notations

$$I_1(\varphi_R) := \int_{Q_R} \varphi_R^{\frac{-1}{p^*-1}}|(\varphi_R)_{tt}|^{\frac{p^*}{p^*-1}}\, dV\, dt \tag{2.8}$$

and

$$I_2(\varphi_R) := \int_{Q_R} \varphi_R^{\frac{-1}{p^*-1}}|\Delta\varphi_R|^{\frac{p^*}{p^*-1}}\, dV\, dt. \tag{2.9}$$

Taking (2.3) into account, we deduce that

$$\varphi_R^{\frac{-1}{p^*-1}}|(\varphi_R)_{tt}|^{\frac{p^*}{p^*-1}} = F_R^{\alpha}(x)\eta_R^{\frac{-\alpha}{p^*-1}}(t)\left|(\eta_R^{\alpha})''(t)\right|^{\frac{p^*}{p^*-1}}, \quad \text{for all } (t,x)\in Q_R.$$

This fact, together with (2.8), leads to the identity

$$I_1(\varphi_R) = \left(\int_0^{R^2} \eta_R^{\frac{-\alpha}{p^*-1}}(t)\left|(\eta_R^{\alpha})''(t)\right|^{\frac{p^*}{p^*-1}}\, dt\right)\left(\int_{B(x_0,R)} F_R^{\alpha}(x)\, dV\right). \tag{2.10}$$

A simple calculation is carried out which gives us the inequality

$$\left|(\eta_R^{\alpha})''(t)\right| \leq CR^{-4}\eta_R^{\alpha-2}(t), \quad \text{for all } t\in(0,R^2),$$

thus obtaining

$$\begin{aligned}
& \int_0^{R^2} \eta_R^{\frac{-\alpha}{p^*-1}}(t)\left|(\eta_R^\alpha)''(t)\right|^{\frac{p^*}{p^*-1}}\,dt \\
\leq\ & CR^{\frac{-4p^*}{p^*-1}} \int_0^{R^2} \eta_R^{\alpha-\frac{2p^*}{p^*-1}}(t)\,dt \\
=\ & CR^{2(1-\alpha)} \int_0^1 \eta^{\alpha-\frac{2p^*}{p^*-1}}(t)\,dt \quad \text{(by a change of variables)} \\
=\ & CR^{2(1-\alpha)}. \qquad (2.11)
\end{aligned}$$

Hypothesis $(A3)$, together with the definition of the function F_R, implies that

$$\begin{aligned}
\int_{B(x_0,R)} F_R^\alpha(x)\,dV &\leq \int_{B(x_0,R)} dV \\
&= |B(x_0,R)| \\
&\leq CR^\alpha. \qquad (2.12)
\end{aligned}$$

Merging (2.10), (2.11), and (2.12), we obtain the estimate

$$I_1(\varphi_R) \leq CR^{2-\alpha}. \tag{2.13}$$

Similarly it is possible to estimate $I_2(\varphi_R)$. Again, the definition of the function φ_R leads to

$$\varphi_R^{\frac{-1}{p^*-1}}|\Delta\varphi_R|^{\frac{p^*}{p^*-1}} = \eta_R^\alpha(t)F_R^{\frac{-\alpha}{p^*-1}}(x)\,|\Delta(F_R^\alpha)(x)|^{\frac{p^*}{p^*-1}}, \quad \text{for all } (t,x)\in Q_R.$$

This time, using (2.9) we deduce that

$$I_2(\varphi_R) = \left(\int_0^{R^2} \eta_R^\alpha(t)\,dt\right)\left(\int_{B(x_0,R)} F_R^{\frac{-\alpha}{p^*-1}}(x)\,|\Delta(F_R^\alpha)(x)|^{\frac{p^*}{p^*-1}}\,dV\right). \tag{2.14}$$

To provide more information about the first integral in the right hand side of (2.14), we get

$$\begin{aligned}
\int_0^{R^2} \eta_R^\alpha(t)\,dt &= \int_0^{R^2} \eta^\alpha\left(\frac{t}{R^2}\right)dt \\
&= R^2\int_0^1 \eta^\alpha(s)\,ds \quad \text{(by a change of variables),}
\end{aligned}$$

that is,

$$\int_0^{R^2} \eta_R^\alpha(t)\,dt = CR^2. \tag{2.15}$$

Next, starting from the identity

$$\Delta(F_R^\alpha) = \alpha F_R^{\alpha-1}\Delta F_R + \alpha(\alpha-1)F_R^{\alpha-2}|\nabla F_R|_g^2,$$

we deduce that

$$\begin{aligned} |\Delta(F_R^\alpha)|^{\frac{p^*}{p^*-1}} &= |\Delta(F_R^\alpha)|^{\frac{\alpha}{2}} \\ &\leq C\left(F_R^{\frac{(\alpha-1)\alpha}{2}}|\Delta F_R|^{\frac{\alpha}{2}} + F_R^{\frac{(\alpha-2)\alpha}{2}}|\nabla F_R|_g^\alpha\right), \end{aligned}$$

which leads to the inequality

$$\begin{aligned} F_R^{\frac{-\alpha}{p^*-1}}|\Delta(F_R^\alpha)|^{\frac{p^*}{p^*-1}} &= F_R^{\frac{-\alpha(\alpha-2)}{2}}|\Delta(F_R^\alpha)|^{\frac{\alpha}{2}} \\ &\leq C\left(F_R^{\frac{\alpha}{2}}|\Delta F_R|^{\frac{\alpha}{2}} + |\nabla F_R|_g^\alpha\right) \\ &\leq C\left(|\Delta F_R|^{\frac{\alpha}{2}} + |\nabla F_R|_g^\alpha\right). \end{aligned}$$

Taking the definition of the function F_R into account, we obtain

$$\int_{B(x_0,R)} F_R^{\frac{-\alpha}{p^*-1}}|\Delta(F_R^\alpha)|^{\frac{p^*}{p^*-1}}\,dV \leq C\int_{B(x_0,R)\setminus B(x_0,\sqrt{R})}\left(|\Delta F_R|^{\frac{\alpha}{2}} + |\nabla F_R|_g^\alpha\right)dV. \tag{2.16}$$

Recalling that F_R is radial, we deduce that

$$\Delta F_R(x) = G_R''(r) + \frac{N-1}{r}G_R'(r) + G_R'(r)\frac{\partial\left(\ln\sqrt{A}\right)}{\partial r},$$

for all $x\in B(x_0,R)\setminus B(x_0,\sqrt{R})$, $r=d_{\mathbb{M}}(x_0,x)$. Next, we use hypothesis $(A2)$ to conclude that

$$|\Delta F_R(x)| \leq C\left(|G_R''(r)| + \frac{1}{r}|G_R'(r)|\right).$$

Simple and elementary calculations allow us to obtain the estimates

$$|G_R''(r)| \leq \frac{C}{r^2\ln\sqrt{R}} \quad \text{and} \quad |G_R'(r)| \leq \frac{C}{r\ln\sqrt{R}}.$$

It follows that, for all $x\in B(x_0,R)\setminus B(x_0,\sqrt{R})$, we have

$$|\Delta F_R|^{\frac{\alpha}{2}} + |\nabla F_R|_g^\alpha \leq \frac{C}{r^\alpha(\ln\sqrt{R})^{\frac{\alpha}{2}}}. \tag{2.17}$$

Using (2.16) together with (2.17), we get

$$\int_{B(x_0,R)} F_R^{\frac{-\alpha}{p^*-1}}|\Delta(F_R^\alpha)|^{\frac{p^*}{p^*-1}}\,dV \leq \frac{C}{(\ln\sqrt{R})^{\frac{\alpha}{2}}}\int_{B(x_0,R)\setminus B(x_0,\sqrt{R})} d_{\mathbb{M}}(x_0,x)^{-\alpha}\,dV.$$

At this point we use Lemma 2.1, thus obtaining

$$\int_{B(x_0,R)} F_R^{\frac{-\alpha}{p^*-1}}|\Delta(F_R^\alpha)|^{\frac{p^*}{p^*-1}}\,dV \leq C(\ln R)^{1-\frac{\alpha}{2}}. \tag{2.18}$$

Returning to (2.14), and using (2.15), and (2.18), we find

$$I_2(\varphi_R) \leq CR^2(\ln R)^{1-\frac{\alpha}{2}}. \tag{2.19}$$

Next we work with the perturbation term w. The choice of the test function φ_R (see (2.3)) together with (2.15), lead to the following estimate

$$\begin{aligned}\int_{Q_R} w(x)\varphi_R\, dV\, dt &= \left(\int_0^{R^2} \eta_R^\alpha(t)\, dt\right)\left(\int_{B(x_0,R)} w(x)F_R^\alpha(x)\, dV\right)\\ &\geq CR^2 \int_{B(x_0,\sqrt{R})} w(x)\, dV\\ &\geq CR^2. \end{aligned} \tag{2.20}$$

In order to obtain this result, we take into account the assumption that w is nonnegative and it is positive somewhere (recall $w \not\equiv 0$). Combining all the obtained estimates (that is merging (2.7), (2.13), (2.19) and (2.20)), we arrive at the following inequality

$$0 < C \leq R^{-\alpha} + (\ln R)^{1-\frac{\alpha}{2}}. \tag{2.21}$$

In the limit as R goes to infinity, (2.21) leads to contradiction. It follows that there isn't any global solution u, and hence problem (1.3) admits no global solution. □

3. Conclusions

The qualitative analysis of different forms of wave equations on (compact, non-compact, complete, connected) Riemannian manifolds is less carried out than the analogous analysis on Euclidean spaces. The interest for manifolds, originates from the fact that the manifold can contain key information and details about wave processes arising in many different contexts of structural and fluid mechanics. Of course, it is well-known that the points of a manifold can be mapped to a subset of Euclidean space via local charts (that is, set of continuous functions with continuous inverses). However, working on the geometry of the manifold with and without boundary conditions, it is possible to develop sharp conditions useful in observing, controlling, and hence stabilizing system dynamics (see, for example, the work of Bardos-Lebeau-Rauch [1], for the case of manifold with boundary). Here we established a non-existence result for a inhomogeneous semilinear wave equation in the critical exponent case, on a manifold without additional boundary condition.

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