Revista Matemática Complutense (2026) 39:117–133
https://doi.org/10.1007/s13163-025-00529-5

Nonlinear singular problems with an indefinite perturbation

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Received: 11 May 2023 / Accepted: 12 May 2025 / Published online: 23 July 2025

Abstract

We consider a nonlinear Dirichlet problem driven by a nonhomogeneous differential operator and with a reaction which exhibits the competing effects of a parametric singular term and of a superlinear perturbation. Using variational methods together with truncation and comparison techniques, we show that for all small values of the parameter, the problem has at least two positive smooth solutions.

Keywords Nonhomogeneous differential operator · Regularity theory · Nonlinear maximum principle · Upper and lower solutions · Truncations · Strong comparisons · Singular term

Mathematics Subject Classification Primary 35J60 · 35J66 · 35J75

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear singular problem

$$\begin{aligned} &-\operatorname{div} a(\nabla u) = \lambda\beta(z)u^{-\eta} + f(z,u) \text{ in } \Omega,\\ &u\big|_{\partial\Omega} = 0,\ \lambda > 0,\ 0 < \eta < 1,\ u > 0. \end{aligned} \tag{P_λ}$$

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In this problem the map $a : \mathbb{R}^N \to \mathbb{R}^N$ involved in the differential operator is continuous, strictly monotone (thus maximal monotone too) and satisfies certain other growth and regularity conditions listed in hypotheses H_1 (see Section 2). These hypotheses provide a general framework in which we can fit many differential operators of interest such as the p-Laplacian and the (p, q)-Laplacian. In the reaction we have the competing effects of two terms of different nature. One is the parametric singular term $\lambda\beta(z)u^{-\eta}$ and the other is a Carathéodory perturbation $f(z, x)$ (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous), which is $(p - 1)$-superlinear as $x \to +\infty$ and also changes sign. Using variational tools together with truncation and comparison techniques, we prove a multiplicity theorem when the parameter $\lambda > 0$ is small.

Problem (P_λ) was investigated by Papageorgiou-Rădulescu-Repovš [12], under the hypothesis that the perturbation $f(z, x)$ is positive. This is a convenient condition which simplifies the analysis of the problem, since the unique solution of the purely singular problem (no perturbation present), is a lower solution of (P_λ) and we can use it to bypass the singularity and deal with C^1-functionals on which we can use the minimax theorems of the critical point theory. In [12] the authors prove an existence and multiplicity theorem which is global in the parameter $\lambda > 0$ (a bifurcation-type theorem, see Theorem 21 of [12]). The positivity of the perturbation is present in most works on singular problems. We mention the works of Coclite-Palmieri [1], Haitao [5], Sun-Wu-Long [20] (semilinear problems), Giacomoni-Schindler-Takač [3] (p-Laplacian equations) and Papageorgiou-Rădulescu-Zhang [13], Papageorgiou-Vetro-Vetro [15] (anisotropic (p, q)-equations). In a recent paper, Papageorgiou-Vetro-Vetro [16] considered a nonparametric singular (p, q)-equation, with a negative perturbation. Hence, the energy functional of the problem is coercive. They show that the problem has a unique positive solution. When dealing with singular problems, the energy functional of the problem is not C^1 and so we can not use the results of the critical point theory. Therefore, we need to find ways to bypass the singularity and deal with C^1-functionals. When the perturbation of the singular term is nonnegative (as is the case in most works in the literature, see [1, 3, 5, 12, 20]), the solution of the purely singular problem can serve as a lower solution for the problem and can be used as a tool to bypass the singularity and deal with C^1-functionals. This is no longer possible for our problem because of hypothesis $H_2\,(iv)$ which forces the perturbation to be negative near the origin. To overcome this difficulty we need to come up with a new approach, suitable for our situation. So, we propose to consider two new auxiliary problems, the first one, problem (Q_λ) is motivated by hypothesis $H_2\,(iv)$. The second one is defined on the smallest open ball B_ρ containing the domain Ω and exploits the properties of the solution of (Q_λ). This way we are able to overcome the indefiniteness of the perturbation function and generate an ordered pair of upper and lower solutions, which permits us to proceed and eventually prove a multiplicity theorem for problem (P_λ). Summarizing we can state the following multiplicity theorem.

Theorem 1 *If hypotheses H_0, H_1, H_2 hold, then for all $\lambda > 0$ small problem (P_λ) has at least two positive solutions $u_0, \widehat{u} \in \operatorname{int} C_+$.*

As a direction for further work, it will be interesting to examine if we can improve our result and prove a multiplicity theorem which is global in the parameter $\lambda > 0$.

2 Mathematical background - hypotheses

The two main spaces in the analysis of problem (P_λ) are the Sobolev space $W_0^{1,p}(\Omega)$ $(1 < p < \infty)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\}$. On account of the Poincaré inequality, on $W_0^{1,p}(\Omega)$ we consider the norm $\|\cdot\|$ defined by

$$\|u\| = \|\nabla u\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

We know that

$$W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega) \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right).$$

The space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : 0 \le u(z) \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$ and $\frac{\partial u}{\partial n} = (\nabla u, n)_{\mathbb{R}^N}$.

If $u : \Omega \to \mathbb{R}$ is a measurable function, then we define

$$u^{\pm}(z) = \max\{\pm u(z), 0\} \text{ for all } z \in \Omega.$$

We have $u = u^+ - u^-$, $|u| = u^+ + u^-$. Moreover, if $u \in W_0^{1,p}(\Omega)$, then $u^{\pm} \in W_0^{1,p}(\Omega)$.

Given two measurable functions $v, w : \Omega \to \mathbb{R}$ such that $v(z) \le w(z)$ for a.a. $z \in \Omega$, we define

$$\begin{aligned}
&[v, w] = \{h \in W_0^{1,p}(\Omega) \ :\ v(z) \le h(z) \le w(z) \text{ for a.a. } z \in \Omega\},\\
&[v) = \{h \in W_0^{1,p}(\Omega) \ :\ v(z) \le h(z) \text{ for a.a. } z \in \Omega\},\\
&\operatorname{int}_{C_0^1(\overline{\Omega})}[v, w] = \text{ the interior in } C_0^1(\overline{\Omega}) \text{ of } [v, w] \cap C_0^1(\overline{\Omega}).
\end{aligned}$$

Our hypotheses on the data of problem (P_λ) are the following:

H_0: $\beta \in L^\infty(\Omega)$, $\beta(z) > 0$ for a.a. $z \in \Omega$, $0 < \eta < 1$.

Let $k \in C^1(0, \infty)$ with $k(t) > 0$ for all $t > 0$ and assume that

$$\begin{aligned}
&0 < \widehat{c}_0 \le \frac{k'(t)t}{k(t)} \le \widehat{c},\\
&c_0 t^{p-1} \le k(t) \le c_1(t^{s-1} + t^{p-1})\\
&\text{for some } c_0, c_1 > 0, \text{ all } t > 0,\ \ 1 < s < p < N.
\end{aligned} \tag{1}$$

The hypotheses on the map $a(\cdot)$ are the following:

H_1: $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$, with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0, \infty)$, $t \to a_0(t)t$ is strictly increasing on $(0, \infty)$, $a_0(t)t \to 0$ as $t \to 0^+$ and $-1 < \lim\limits_{t \to 0^+} \dfrac{a_0'(t)t}{a_0(t)}$;

(ii) there exists $c_3 > 0$ such that

$$|\nabla a(y)| \leq c_3 \frac{k(|y|)}{|y|} \text{ for all } y \in \mathbb{R}^N \backslash \{0\};$$

(iii) $\dfrac{k(|y|)}{|y|}|\xi|^2 \leq (\nabla a(y)\xi, \xi)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N \backslash \{0\}$, all $\xi \in \mathbb{R}^N$;

(iv) if $G_0(t) = \int_0^t a_0(s)s ds$, then $0 \leq pG_0(t) - a_0(t)t^2$ for all $t > 0$ and there exists $1 < q \leq p$ such that

$$\limsup_{t \to 0^+} \frac{qG_0(t)}{t^q} \leq c^*.$$

Remark 1 Conditions H_1 (i), (ii), (iii) are from Lieberman [10] and from Pucci-Serrin [18]. In fact hypothesis H_1 (i) is stronger than that in Lieberman [10], because in addition to global regularity, we want to use the nonlinear Hopf maximum principle of Pucci-Serrin [18] (p. 120). They lead to a global regularity theory (that is, up to the boundary, see Lieberman [10]) and to nonlinear versions of the Hopf maximum principle (see again [18]). Hypothesis H_1 (iv) is particular for the needs of our problem, but it is mild and it is satisfied in all cases of interest (see the Examples below). Same conditions on the map $a(\cdot)$ were used in [12].

Evidently $G_0(\cdot)$ is strictly convex and strictly increasing. We set

$$G(y) = G_0(|y|) \text{ for all } y \in \mathbb{R}^N.$$

We see that $G \in C^1(\mathbb{R}^N)$ is convex and $G(0) = 0$. We have

$$\begin{aligned} \nabla G(y) &= G_0'(|y|)\frac{y}{|y|} \quad \text{(by the chain rule)} \\ &= a_0(|y|)|y|\frac{y}{|y|} = a_0(|y|)y = a(y), \\ \nabla G(0) &= 0. \end{aligned}$$

Therefore, $G(\cdot)$ is the primitive of $a(\cdot)$ and so the convexity of $G(\cdot)$ implies that

$$G(y) \leq (a(y), y)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N. \tag{2}$$

Using hypotheses H_1 (i), (ii), (iii) and (1), we deduce the following properties for the map $a(\cdot)$.

Lemma 1 *If hypotheses* H_1 (i), (ii), (iii) *hold, then*

(a) $y \to a(y)$ *is continuous and strictly monotone* (*thus maximal monotone too*)*;*

(b) $|a(y)| \leq c_4(|y|^{s-1} + |y|^{p-1})$ *for all* $y \in \mathbb{R}^N$*, some* $c_4 > 0$*;*

(c) $\frac{c_0}{p-1}|y|^p \leq (a(y), y)_{\mathbb{R}^N}$ *for all* $y \in \mathbb{R}^N$.

Combining these properties with (2), we deduce the following bilateral growth properties for the primitive $G(\cdot)$.

Corollary 1 *If hypotheses* H_1 (i), (ii), (iii) *hold, then*

$$\frac{c_0}{p(p-1)}|y|^p \leq G(y) \leq c_5(1 + |y|^p) \text{ for all } y \in \mathbb{R}^N, \text{ some } c_5 > 0.$$

Remark 2 Therefore $G(\cdot)$ exhibits balanced growth and this leads to a global regularity theory (see Lieberman [10]).

Examples Let

(a) $a(y) = |y|^{p-2}y$, $1 < p < \infty$.
This map corresponds to the p-Laplace differential operator.

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$, $1 < q < p < \infty$.
This map corresponds to the (p, q)-Laplacian (the sum of a p-Laplacian and of a q-Laplacian).

(c) $a(y) = (1 + |y|^2)^{\frac{p-2}{2}}y$, $1 < p < \infty$.
The resulting operator is the p-mean curvature operator.

(d) $a(y) = |y|^{p-2}y\left(1 + \frac{1}{1+|y|^p}\right)$, $1 < p < \infty$.
The resulting differential operator is used in plasticity theory (see Roubíček [19]).

The hypotheses on the perturbation $f(z, x)$ are the following.

H_2: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leq \widehat{a}(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geq 0$, with $\widehat{a} \in L^\infty(\Omega)$ and $p < r < p^* = \frac{Np}{N-p}$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\lim_{x\to+\infty} \frac{F(z, x)}{x^p} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) there exists $\tau \in \left((r-p)\frac{N}{p}, p^*\right)$ such that

$$0 < \vartheta_0 \leq \liminf_{x\to+\infty} \frac{f(z, x)x - pF(z, x)}{x^\tau} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) there exist $\delta \in (0, 1)$ and $k_s > 0$ such that

$$-\beta(z) \leq f(z, x) \leq -k_s < 0 \text{ for a.a. } z \in \Omega, \text{ all } 0 < s \leq x \leq \delta,$$

and for every $\rho > 0$ there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function $x \to f(z, x) + \widehat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0, \rho]$.

Remark 3 Since we look for positive solutions and the above hypotheses on $f(z, \cdot)$ concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may assume that $f(z, x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$. Hypotheses H_2 (ii), (iii) imply that

$$\lim_{x \to +\infty} \frac{f(z, x)}{x^{p-1}} = +\infty \text{ uniformly for a.a. } z \in \Omega.$$

So, the perturbation of the singular term is $(p-1)$-superlinear. However, we do not use the Ambrosetti-Rabinowitz condition (see, for example, Willem [21], p. 46), which is common in the literature, when we consider superlinear problems. Our hypotheses are less restrictive and incorporate in our framework superlinear nonlinearities with slower growth as $x \to +\infty$, which fail to satisfy the Ambrosetti-Rabinowitz condition. Consider the following function

$$f(z, x) = \begin{cases} -\beta(z)x^+ & \text{if } x \leq 1, \\ x^{p-1} \ln x - \beta(z)x^{s-1} & \text{if } 1 < x, \end{cases} \text{ with } 1 < s \leq p, \ z \in \Omega.$$

This function satisfies hypotheses H_2 above, but fails to satisfy the Ambrosetti-Rabinowitz condition.

Let $V : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega)$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ be the nonlinear map defined by

$$\langle V(u), h \rangle = \int_\Omega (a(\nabla u), \nabla h)_{\mathbb{R}^N} dz \ \text{ for all } u, h \in W_0^{1,p}(\Omega).$$

This operator has the following properties (see Gasiński-Papageorgiou [2], p. 279).

Proposition 1 *If hypotheses H_1 (i), (ii), (iii) hold, then the operator $V(\cdot)$ is bounded (maps bounded sets to bounded ones), continuous, strictly monotone (hence maximal monotone too) and it is $(S)_+$, that is, $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to\infty} \langle V(u_n), u_n - u \rangle \leq 0$ imply that $u_n \to u$ in $W_0^{1,p}(\Omega)$.*

3 An auxiliary problem

In this section we consider an auxiliary Dirichlet problem, the unique solution of which will serve as a lower solution for problem (P_λ) and this way we will be able to bypass the singularity. We will also generate an ordered upper solution.

Let $\mu > \frac{p-1}{\eta}$ and consider the following auxiliary Dirichlet problem

$$\begin{gathered} -\operatorname{div} a(\nabla u) = \lambda^\mu \beta(z) \text{ in } \Omega, \\ u\big|_{\partial\Omega} = 0, \ \lambda > 0, \ u > 0. \end{gathered} \tag{Q_λ}$$

Proposition 2 *If hypotheses H_0, H_1 hold and $\lambda > 0$, then problem (Q_λ) has a unique solution $\underline{u}_\lambda \in \operatorname{int} C_+$ and $\underline{u}_\lambda \to 0$ in $C_0^1(\overline{\Omega})$ as $\lambda \to 0^+$.*

Proof From Proposition 1 we know that the operator $V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is maximal monotone and strictly monotone. We have

$$\begin{aligned} &\langle V(u), u\rangle \geq \frac{c_0}{p(p-1)}\|\nabla u\|_p^p \quad \text{(see Lemma 1)},\\ &\Rightarrow V(\cdot) \text{ is coercive.}\end{aligned}$$

But a maximal monotone coercive operator is surjective (see [11], p. 135). So, we can find $\underline{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$V(\underline{u}_\lambda) = \lambda^\mu \beta \text{ in } W^{-1,p'}(\Omega).$$

We act with $-\underline{u}_\lambda^- \in W_0^{1,p}(\Omega)$ and since $\beta(\cdot) > 0$ (see hypotheses H_0), we infer that $\underline{u}_\lambda \geq 0$, $\underline{u}_\lambda \neq 0$. Moreover, the strict monotonicity of $V(\cdot)$ implies that this solution $\underline{u}_\lambda$ is unique. From Ladyzhenskaya-Uraltseva [8] (p. 286), we have that $\underline{u}_\lambda \in L^\infty(\Omega)$. If $m > \frac{N}{p}$, using the Moser iteration process, we have

$$\|\underline{u}_\lambda\|_\infty \leq \lambda^{\frac{\mu}{p-1}}\|\beta\|_m^{\frac{1}{p-1}}$$

(see also Guedda-Veron [4], Proposition 1.3). Then, letting $m \to \infty$ and using Problem 1.9 of Gasiński-Papageorgiou [2], we obtain

$$\|\underline{u}_\lambda\|_\infty \leq \lambda^{\frac{\mu}{p-1}}\|\beta\|_\infty^{\frac{1}{p-1}}. \tag{3}$$

From Lieberman [10], we know that we can find $\alpha \in (0,1)$ and $c_6 > 0$ such that

$$\underline{u}_\lambda \in C_0^{1,\alpha}(\overline{\Omega}),\ \ \|\underline{u}_\lambda\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c_6 \text{ for all } \lambda \in (0,1].$$

We know that $C_0^{1,\alpha}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ compactly and from (3) we have $\underline{u}_\lambda \to 0$ in $L^\infty(\Omega)$ as $\lambda \to 0^+$. It follows that

$$\underline{u}_\lambda \to 0 \text{ in } C_0^1(\overline{\Omega}) \text{ as } \lambda \to 0^+.$$

Finally, from the nonlinear Hopf maximum principle of Pucci-Serrin [18] (p. 120), we have $\underline{u}_\lambda \in \operatorname{int} C_+$. □

On account of Proposition 2, we can find $\lambda^* > 0$ such that

$$0 \leq \underline{u}_\lambda(z) \leq \delta \text{ for all } z \in \overline{\Omega}, \text{ all } 0 < \lambda \leq \lambda^*.$$

Then, hypothesis H_2 (iv) says that

$$-\beta(z) \leq f(z, \underline{u}_\lambda(z)) \text{ for a.a. } z \in \Omega, \text{ all } 0 < \lambda \leq \lambda^*. \tag{4}$$

For $\lambda \in (0, \lambda^*]$, we have

$$\begin{aligned}
&-\operatorname{div} a(\nabla \underline{u}_\lambda) - \lambda\beta(z)\underline{u}_\lambda^{-\eta} - f(z, \underline{u}_\lambda) \\
&\leq \lambda^\mu \beta(z) - \frac{\lambda\beta(z)}{\|\underline{u}_\lambda\|_\infty^\eta} + \beta(z) \quad (\text{see } (4)) \\
&\leq \beta(z)\left[\lambda^\mu - \frac{\lambda}{\lambda^{\frac{\mu\eta}{p-1}}\|\beta\|_\infty^{\frac{\eta}{p-1}}} + 1\right] \quad (\text{see } (3)).
\end{aligned}$$

Recall $\mu > \frac{p-1}{\eta}$, hence $\frac{\mu\eta}{p-1} > 1$ and so we can find $\lambda_1^* \leq \lambda^*$ such that

$$-\operatorname{div} a(\nabla \underline{u}_\lambda) - \lambda\beta(z)\underline{u}_\lambda^{-\eta} - f(z, \underline{u}_\lambda) \leq 0 \text{ in } \Omega, \text{ for all } 0 < \lambda \leq \lambda_1^*. \tag{5}$$

Next let $\rho > 0$ be large so that $\overline{\Omega} \subseteq B_\rho = \{z \in \mathbb{R}^N : |z| < \rho\}$. Then, given $\vartheta > 0$ we consider the following Dirichlet problem

$$-\operatorname{div} a(\nabla u) = \vartheta \text{ in } B_\rho, \ u\big|_{\partial B_\rho} = 0.$$

From Proposition 8 of Papageorgiou-Rădulescu-Repovš [12], we know that this problem has a unique solution $\overline{u}_\vartheta \in \operatorname{int} C_+(\overline{B}_\rho)$ and $\overline{u}_\vartheta \to 0$ in $C_0^1(\overline{B}_\rho)$ as $\vartheta \to 0^+$. So, we can find $\vartheta_0 > 0$ such that

$$0 \leq \overline{u}_\vartheta(z) \leq \delta \text{ for all } z \in \overline{B}_\rho, \text{ all } \vartheta \in (0, \vartheta_0]. \tag{6}$$

Since $\overline{u}_\vartheta \in \operatorname{int} C_+(\overline{B}_\rho)$ and $\overline{\Omega} \subseteq B_\rho$, we have $0 < m_0 = \min_{\overline{\Omega}} \overline{u}_{\vartheta_0}$. Choosing B_ρ to be the smallest ball containing $\overline{\Omega}$, we will have $m_0 \in (0, 1]$. Let $\lambda_2^* = \min\left\{\lambda_1^*, \frac{\vartheta_0 m_0^\eta}{\|\beta\|_\infty}\right\}$. Then for $0 < \lambda \leq \lambda_2^*$, we have

$$\begin{aligned}
&-\operatorname{div} a(\nabla \underline{u}_\lambda) = \lambda\beta(z) < \vartheta_0 = -\operatorname{div} a(\nabla \overline{u}_{\vartheta_0}) \text{ in } \Omega, \\
&\Rightarrow \ \langle V(\underline{u}_\lambda), h\rangle \leq \langle V(\overline{u}_{\vartheta_0}), h\rangle \text{ for all } h \in W_0^{1,p}(\Omega), \ h \geq 0.
\end{aligned}$$

Choosing $h = (\underline{u}_\lambda - \overline{u}_{\vartheta_0})^+ \in W_0^{1,p}(\Omega)$, we obtain

$$\int_{\{\overline{u}_{\vartheta_0} < \underline{u}_\lambda\}} (a(\nabla \underline{u}_\lambda) - a(\nabla \overline{u}_{\vartheta_0}), \nabla \underline{u}_\lambda - \nabla \overline{u}_{\vartheta_0})_{\mathbb{R}^N} dz \leq 0.$$

The strict monotonicity of $a(\cdot)$ implies that

$$|\{\overline{u}_{\vartheta_0} < \underline{u}_\lambda\}|_N = 0$$

with $|\cdot|_N$ denoting the Lebesgue measure on $\mathbb{R}^N$. Therefore

$$\underline{u}_\lambda \leq \overline{u}_{\vartheta_0} \text{ for all } \lambda \in (0, \lambda_2^*]. \tag{7}$$

We have

$$\begin{aligned}
&-\operatorname{div} a(\nabla \overline{u}_{\vartheta_0}) - \lambda\beta(z)\overline{u}_{\vartheta_0}^{-\eta} - f(z, \overline{u}_{\vartheta_0}) \\
&\geq \vartheta_0 - \lambda \frac{\|\beta\|_\infty}{m_0^\eta} \quad \text{(see (6) and hypothesis } H_2\ (iv)) \\
&\geq 0 \quad \text{(recall the choice of } \lambda_2^* \text{ and that } 0 < \lambda \leq \lambda_2^*).
\end{aligned} \tag{8}$$

Finally note that if $\widetilde{u} \in \operatorname{int} C_+$, then we can find $c_7 > 0$ such that

$$c_7 \widehat{d} \leq \widetilde{u} \tag{9}$$

with $\widehat{d}(\cdot)$ being the distance function from $\partial\Omega$, that is, $\widehat{d}(z) = d(z, \partial\Omega)$ for all $z \in \overline{\Omega}$. Then for every $h \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
\int_\Omega \left(\frac{|h|}{\widetilde{u}^\eta}\right)^p dz &\leq c_8 \int_\Omega \left(\frac{|h|}{\widehat{d}^\eta}\right)^p dz \text{ for some } c_8 > 0 \text{ (see (9))} \\
&= c_8 \int_\Omega (\widehat{d}^{1-\eta})^p \left(\frac{|h|}{\widehat{d}}\right)^p dz \\
&\leq c_9 \int_\Omega \left(\frac{|h|}{\widehat{d}}\right)^p dz \text{ for some } c_9 > 0 \\
&\leq c_{10} \|\nabla h\|_p^p \text{ for some } c_{10} > 0 \\
&\quad \text{(using Hardy's inequality, see [11], p. 66).}
\end{aligned}$$

Moreover, from (9) and the Lemma (see also its proof) of Lazer-McKenna [9], we have for $\lambda \in (0, \lambda_2^*]$

$$\underline{u}_\lambda^{-\eta} \in L^1(\Omega) \quad (\text{hence } \overline{u}_{\vartheta_0}^{-\eta} \in L^1(\Omega), \text{ see (7)}).$$

4 Positive solutions

In this section we will use the solutions $\underline{u}_\lambda, \overline{u}_{\vartheta_0}$ from the previous section, together with variational tools and truncation and comparison techniques, in order to prove a multiplicity theorem for problem (P_λ) when $\lambda > 0$ is small.

Proposition 3 *If hypotheses H_0, H_1, H_2 hold and $\lambda \in (0, \lambda_2^*]$ is small, then problem (P_λ) has a positive solution $u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\underline{u}_\lambda, \overline{u}_{\vartheta_0}]$.*

Proof From (7) we see that we can introduce the following truncation of the reaction of the problem

$$\widehat{\gamma}_\lambda(z,x)=\begin{cases}\lambda\beta(z)\underline{u}_\lambda(z)^{-\eta}+f(z,\underline{u}_\lambda(z)) & \text{if } x<\underline{u}_\lambda(z),\\ \lambda\beta(z)x^{-\eta}+f(z,x) & \text{if } \underline{u}_\lambda(z)\le x\le \overline{u}_{\vartheta_0}(z),\\ \lambda\beta(z)\overline{u}_{\vartheta_0}(z)^{-\eta}+f(z,\overline{u}_{\vartheta_0}(z)) & \text{if } \overline{u}_{\vartheta_0}(z)<x.\end{cases} \tag{10}$$

This is a Carathéodory function. We set $\widehat{\Gamma}_\lambda(z,x)=\int_0^x\widehat{\gamma}_\lambda(z,s)ds$ and introduce the functional $\widehat{\varphi}_\lambda: W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u)=\int_\Omega G(\nabla u)dz-\int_\Omega\widehat{\Gamma}_\lambda(z,u)dz \text{ for all } u\in W_0^{1,p}(\Omega).$$

We know that $\widehat{\varphi}_\lambda\in C^1(W_0^{1,p}(\Omega))$ (see Papageorgiou-Smyrlis [14], Proposition 3). From Corollary 1 and (10) we see that

$$\widehat{\varphi}_\lambda(\cdot) \text{ is coercive.}$$

Also, using the Sobolev embedding theorem, we see that

$$\widehat{\varphi}_\lambda(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

So, by the Weierstrass-Tonelli theorem, we can find $u_0\in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}&\widehat{\varphi}_\lambda(u_0)=\inf\left\{\widehat{\varphi}_\lambda(u): u\in W_0^{1,p}(\Omega)\right\},\\ &\Rightarrow \langle\widehat{\varphi}_\lambda'(u_0),h\rangle=0 \text{ for all } h\in W_0^{1,p}(\Omega),\\ &\Rightarrow \langle V(u_0),h\rangle=\int_\Omega\widehat{\gamma}_\lambda(z,u_0)hdz \text{ for all } h\in W_0^{1,p}(\Omega).\end{aligned} \tag{11}$$

In (11) first we choose $h=(u_0-\overline{u}_{\vartheta_0})^+\in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}&\langle V(u_0),(u_0-\overline{u}_{\vartheta_0})^+\rangle\\ &=\int_\Omega[\lambda\beta(z)\overline{u}_{\vartheta_0}^{-\eta}+f(z,\overline{u}_{\vartheta_0})](u_0-\overline{u}_{\vartheta_0})^+dz\\ &\le\langle V(\overline{u}_{\vartheta_0}),(u_0-\overline{u}_{\vartheta_0})^+\rangle \quad \text{(see (8))},\\ &\Rightarrow\ u_0\le\overline{u}_{\vartheta_0} \quad \text{(see Proposition 1).}\end{aligned}$$

Next in (11) we use the test function $h=(\underline{u}_\lambda-u_0)^+\in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}&\langle V(u_0),(\underline{u}_\lambda-u_0)^+\rangle\\ &=\int_\Omega[\lambda\beta(z)\underline{u}_\lambda^{-\eta}+f(z,\underline{u}_\lambda)](\underline{u}_\lambda-u_0)^+dz \quad \text{(see (10))}\end{aligned}$$

$$
\begin{aligned}
&\geq \langle V(\underline{u}_\lambda), (\underline{u}_\lambda - u_0)^+\rangle \quad \text{(see (5))},\\
&\Rightarrow \ \underline{u}_\lambda \leq u_0 \quad \text{(see Proposition 1)}.
\end{aligned}
$$

So, we have proved that

$$u_0 \in [\underline{u}_\lambda, \overline{u}_{\vartheta_0}]. \tag{12}$$

From (12), (10) and (11), we infer that u_0 is a positive solution of problem (P_λ). Moreover, from Theorem B1 of Giacomoni-Schindler-Takač [3], we have that

$$u_0 \in [\underline{u}_\lambda, \overline{u}_{\vartheta_0}] \cap C_+.$$

Let $\rho = \|\overline{u}_{\vartheta_0}\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H_2\,(iv)$. Then

$$
\begin{aligned}
&-\operatorname{div} a(\nabla u_0) - \lambda\beta(z)u_0^{-\eta} + \widehat{\xi}_\rho u_0^{p-1}\\
&= f(z, u_0) + \widehat{\xi}_\rho u_0^{p-1}\\
&\leq f(z, \overline{u}_{\vartheta_0}) + \widehat{\xi}_\rho \overline{u}_{\vartheta_0}^{p-1} \quad \text{(see (12) and hypothesis } H_2\,(iv))\\
&\leq -\operatorname{div} a(\nabla \overline{u}_{\vartheta_0}) - \lambda\beta(z)\overline{u}_{\vartheta_0}^{-\eta} + \widehat{\xi}_\rho \overline{u}_{\vartheta_0}^{p-1} \quad \text{(see (8))}\\
&\Rightarrow \ u_0(z) < \overline{u}_{\vartheta_0}(z) \text{ for all } z \in \overline{\Omega}\\
&\quad \text{(see hypothesis } H_2\,(iv) \text{ and [12], Proposition 6).}
\end{aligned} \tag{13}
$$

Also we have

$$
\begin{aligned}
&-\operatorname{div} a(\nabla \underline{u}_\lambda) - \lambda\beta(z)\underline{u}_\lambda^{-\eta} + \widehat{\xi}_\rho \underline{u}_\lambda^{p-1}\\
&\leq f(z, \underline{u}_\lambda) + \widehat{\xi}_\rho \underline{u}_\lambda^{p-1} \quad \text{(see (5))}\\
&\leq f(z, u_0) + \widehat{\xi}_\rho u_0^{p-1} \quad \text{(see (12) and hypothesis } H_2\,(iv))\\
&= -\operatorname{div} a(\nabla u_0) - \lambda\beta(z)u_0^{-\eta} + \widehat{\xi}_\rho u_0^{p-1} \text{ in } \Omega.
\end{aligned} \tag{14}
$$

Note that

$$
\begin{aligned}
&-\operatorname{div} a(\nabla \underline{u}_\lambda) - \lambda\beta(z)\underline{u}_\lambda^{-\eta} + \widehat{\xi}_\rho \underline{u}_\lambda^{p-1}\\
&= \lambda\beta(z) - \lambda\beta(z)\underline{u}_\lambda^{-\eta} + \widehat{\xi}_\rho \underline{u}_\lambda^{p-1}\\
&= \lambda\beta(z)\left(1 - \frac{1}{\underline{u}_\lambda^{\eta}}\right) + \widehat{\xi}_\rho \underline{u}_\lambda^{p-1}.
\end{aligned}
$$

From Proposition 2 we know that $\underline{u}_\lambda \to 0$ in $C_0^1(\overline{\Omega})$. Since

$$\lambda\beta(z)\left(1 - \frac{1}{\underline{u}_\lambda^{\eta}}\right) \leq \lambda\beta(z)\left(1 - \frac{1}{\|\underline{u}_\lambda\|_\infty^{\eta}}\right),$$

we see that we can find $\lambda_3^* \in (0, \lambda_2^*]$ such that

$$\lambda\beta(z)\left(1 - \frac{1}{\underline{u}_\lambda^\eta}\right) + \widehat{\xi}_\rho \underline{u}_\lambda^{p-1} < 0 \leq f(z, u_0) + \widehat{\xi}_\rho u_0^{p-1}$$

for a.a. $z \in \Omega$, all $\lambda \in (0, \lambda_3^*]$. Therefore from (14) and Proposition 2.3 of Papageorgiou-Winkert [17], we have

$$u_0 - \underline{u}_\lambda \in \operatorname{int} C_+. \tag{15}$$

From (13) and (15) we conclude that for $\lambda \in (0, \lambda_3^*]$ we have

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\underline{u}_\lambda, \overline{u}_{\vartheta_0}].$$

□

Let $\gamma_\lambda(z, x)$ be the Carathéodory function defined by

$$\gamma_\lambda(z, x) = \begin{cases} \lambda\beta(z)\underline{u}_\lambda(z)^{-\eta} + f(z, \underline{u}_\lambda(z)) & \text{if } x \leq \underline{u}_\lambda(z), \\ \lambda\beta(z)x^{-\eta} + f(z, x) & \text{if } \underline{u}_\lambda(z) < x. \end{cases} \tag{16}$$

We set $\Gamma_\lambda(z, x) = \int_0^x \gamma_\lambda(z, s)ds$ and consider the C^1-functional $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \int_\Omega G(\nabla u)dz - \int_\Omega \Gamma_\lambda(z, u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

From (10) and (16) we see that

$$\varphi_\lambda\Big|_{[\underline{u}_\lambda, \overline{u}_{\vartheta_0}]} = \widehat{\varphi}_\lambda\Big|_{[\underline{u}_\lambda, \overline{u}_{\vartheta_0}]}. \tag{17}$$

In the next proposition, the "C-condition" refers to the Cerami condition. We mention that a similar result can be found in [12] (see the proof of Proposition 18). However, there the assumptions are different. The proof in [12] uses a quasimonotonicity condition on

$$e_\lambda(z, x) = \lambda\left(1 - \frac{p}{1-\eta}\right)x^{1-\eta} + f(z, x)x - pF(z, x)$$

(see hypothesis $H(f)$ (iii)). Here instead we use hypothesis H_2 (iii) which does not involve the singular term.

We point out that Proposition 4 will be used to apply the Mountain pass Theorem, while Proposition 5 that follows is need in order to guarantee that the mountain pass critical point is a solution of (P_λ).

Proposition 4 *If hypotheses H_0, H_1, H_2 hold and $\lambda \in (0, \lambda_2^*]$ is small, then the functional $\varphi_\lambda(\cdot)$ satisfies the C-condition.*

Proof Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi_\lambda(u_n)| \le c_8 \text{ for some } c_8 > 0, \text{ all } n \in \mathbb{N}, \tag{18}$$
$$(1 + \|u_n\|)\varphi'_\lambda(u_n) \to 0 \text{ in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \tag{19}$$

From (19) we have

$$\left|\langle V(u_n), h\rangle - \int_\Omega \gamma_\lambda(z, u_n)h dz\right| \le \frac{\varepsilon_n \|h\|}{1 + \|u_n\|}$$
$$\text{for all } h \in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+. \tag{20}$$

In (20) first we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\frac{c_0}{p-1}\|\nabla u_n^-\|_p^p &\le \int_\Omega f(z, \underline{u}_\lambda)(-u_n^-)dz \\ &\le c_9\|u_n^-\| \text{ for some } c_9 > 0, \text{ all } n \in \mathbb{N}, \\ \Rightarrow \{u_n^-\}_{n\in\mathbb{N}} &\subseteq W_0^{1,p}(\Omega) \text{ is bounded.}\end{aligned} \tag{21}$$

From (18) and (21) we have

$$\int_\Omega pG(\nabla u_n^+)dz - \int_\Omega p\Gamma_\lambda(z, u_n^+)dz \le c_{10}$$
$$\text{for some } c_{10} > 0, \text{ all } n \in \mathbb{N} \text{ (see hypothesis } H_2\ (i)). \tag{22}$$

In (20), we use the test function $h = u_n^+ \in W_0^{1,p}(\Omega)$. Then

$$-\int_\Omega (a(\nabla u_n^+), \nabla u_n^+)_{\mathbb{R}^N} dz + \int_\Omega \gamma_\lambda(z, u_n^+)u_n^+ dz \le c_{11}\|u_n^+\|_\tau$$
$$\text{for some } c_{11} > 0, \text{ all } n \in \mathbb{N}. \tag{23}$$

We add (22) and (23) and obtain

$$\begin{aligned}&\int_\Omega \left[pG(\nabla u_n^+) - (a(\nabla u_n^+), \nabla u_n^+)_{\mathbb{R}^N}\right] dz + \int_\Omega \left[\gamma_\lambda(z, u_n^+)u_n^+ - p\Gamma_\lambda(z, u_n^+)\right] dz \\ &\le c_{12}(1 + \|u_n^+\|_\tau) \text{ for some } c_{12} > 0, \text{ all } n \in \mathbb{N}, \\ &\Rightarrow \int_\Omega \left[\gamma_\lambda(z, u_n^+)u_n^+ - p\Gamma_\lambda(z, u_n^+)\right] dz \le c_{12}(1 + \|u_n^+\|_\tau) \text{ for all } n \in \mathbb{N}, \\ &\quad \text{(see hypothesis } H_1\ (iv)) \\ &\Rightarrow \int_\Omega \left[f(z, u_n^+)u_n^+ - pF(z, u_n^+)\right] dz \le \frac{\lambda\eta}{1-\eta}\int_\Omega (u_n^+)^{1-\eta}dz + c_{12}(1 + \|u_n^+\|_\tau) \\ &\le c_{13}(1 + \|u_n^+\|_\tau) \text{ for some } c_{13} > 0, \text{ all } n \in \mathbb{N}.\end{aligned} \tag{24}$$

Here we have used (16) and Theorem 13.17, p. 196, of Hewitt-Stromberg [6]. Hypotheses H_2 (i), (iii) imply that

$$\begin{aligned}&\widehat{\beta}_0 x^\tau - c_{14} \leq f(z,x)x - pF(z,x)\\&\text{for a.a. } z \in \Omega, \text{ all } x \geq 0, \text{ some } \widehat{\beta}_0 \in (0,\beta_0),\ \ c_{14} > 0.\end{aligned} \tag{25}$$

Using (25) in (24), we obtain

$$\begin{aligned}&\|u_n^+\|_\tau^\tau \leq c_{15}(1+\|u_n^+\|_\tau) \text{ for some } c_{15} > 0, \text{ all } n \in \mathbb{N},\\&\Rightarrow \{u_n^+\}_{n\in\mathbb{N}} \subseteq L^\tau(\Omega) \text{ is bounded (recall } \tau > 1).\end{aligned} \tag{26}$$

From hypothesis H_2 (iii) we see that we can assume that

$$\tau < r < p^*.$$

Let $t \in (0,1)$ be such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^*}. \tag{27}$$

From the interpolation inequality (see Hu-Papageorgiou [7], p. 82), we have

$$\begin{aligned}&\|u_n^+\|_r \leq \|u_n^+\|_\tau^{1-t}\|u_n^+\|_{p^*}^t\\&\Rightarrow \|u_n^+\|_r^r \leq c_{16}\|u_n^+\|^{tr} \text{ for some } c_{16} > 0, \text{ all } n \in \mathbb{N}\\&\text{(use (26) and the Sobolev embedding theorem).}\end{aligned} \tag{28}$$

From (20) with $h = u_n^+ \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}&\int_\Omega (a(\nabla u_n^+), \nabla u_n^+)_{\mathbb{R}^N} dz \leq c_{17}\|u_n^+\| + \int_\Omega f(z,u_n^+)u_n^+ dz\\&\text{for some } c_{17} > 0, \text{ all } n \in \mathbb{N} \text{ (see (16))},\\&\Rightarrow \frac{c_0}{p-1}\|u_n^+\|^p \leq c_{18}(\|u_n^+\| + \|u_n^+\|^{tr} + 1)\\&\text{for some } c_{18} > 0, \text{ all } n \in \mathbb{N} \text{ (see (28) and } H_2\ (i)).\end{aligned} \tag{29}$$

From (27), we have

$$\begin{aligned}&tr = \frac{p^*(r-\tau)}{p^*-\tau},\\&\Rightarrow tr < p \text{ (see hypothesis } H_2\ (iii)).\end{aligned}$$

Therefore from (29), we infer that

$$\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{30}$$

From (21) and (30) it follows that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega),\ \ u_n \to u \text{ in } L^r(\Omega). \tag{31}$$

In (20) we choose $h = (u_n - u) \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (31). Then

$$\begin{aligned}&\lim_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0,\\ &\Rightarrow u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 1).}\end{aligned}$$

Therefore $\varphi_\lambda(\cdot)$ satisfies the C-condition. □

In what follows by K_{φ_λ} we denote the critical set of $\varphi_\lambda(\cdot)$, that is,

$$K_{\varphi_\lambda} = \left\{u \in W_0^{1,p}(\Omega) : \varphi_\lambda'(u) = 0\right\}.$$

The next proposition localizes K_{φ_λ}.

Proposition 5 *If hypotheses H_0, H_1, H_2 hold and $\lambda \in (0, \lambda_2^*]$ is small, then $K_{\varphi_\lambda} \subseteq [\underline{u}_\lambda) \cap \operatorname{int} C_+$.*

Proof Let $u \in K_{\varphi_\lambda}$. Then

$$\langle V(u), h\rangle = \int_\Omega \gamma_\lambda(z, u) h dz \text{ for all } h \in W_0^{1,p}(\Omega).$$

We choose $h = (\underline{u}_\lambda - u)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\langle V(u), (\underline{u}_\lambda - u)^+\rangle &= \int_\Omega [\lambda\beta(z)\underline{u}_\lambda^{-\eta} + f(z, \underline{u}_\lambda)](\underline{u}_\lambda - u)^+ dz \quad \text{(see (16))}\\ &\geq \langle V(\underline{u}_\lambda), (\underline{u}_\lambda - u)^+\rangle \quad \text{(see (5))},\\ &\Rightarrow\ \underline{u}_\lambda \leq u.\end{aligned}$$

As before from the regularity theory of Giacomoni-Schindler-Takač [3], we have

$$K_{\varphi_\lambda} \subseteq [\underline{u}_\lambda) \cap \operatorname{int} C_+.$$

□

Now we are ready to produce a second positive solution for problem (P_λ) with $\lambda \in (0, \lambda_2^*]$ small.

Proposition 6 *If hypotheses* H_0, H_1, H_2 *hold and* $\lambda \in (0, \lambda_2^*]$ *is small, then problem* (P_λ) *has a second positive solution* $\widehat{u} \in \operatorname{int} C_+$.

Proof From Proposition 3 and its proof, we know that we already have a positive solution

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\underline{u}_\lambda, \overline{u}_{\vartheta_0}], \tag{32}$$

which is a global minimizer of $\widehat{\varphi}_\lambda(\cdot)$. From (17) and (32) it follows that

$$\begin{aligned} & u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda(\cdot), \\ & \Rightarrow u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda(\cdot) \text{ (see [13])}. \end{aligned} \tag{33}$$

We assume that K_{φ_λ} is finite or otherwise on account of Proposition 5, we already have an infinity of positive smooth solutions for (P_λ) and so we are done. Then from (33) and Theorem 5.7.6, p. 449, of [11], we can find $\rho \in (0, 1)$ small such that

$$\varphi_\lambda(u_0) < \inf\{\varphi_\lambda(u) : \|u - u_0\| = \rho\} = m_\lambda \text{ (see (36))}. \tag{34}$$

If $u \in \operatorname{int} C_+$, then by hypothesis H_2 (ii), we have

$$\varphi_\lambda(tu) \to -\infty \text{ as } t \to \infty. \tag{35}$$

From Proposition 4, we know that

$$\varphi_\lambda(\cdot) \text{ satisfies the } C\text{-condition}. \tag{36}$$

Then (34), (35) and (36) permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} & \widehat{u} \in K_{\varphi_\lambda} \subseteq [\underline{u}_\lambda) \cap \operatorname{int} C_+, \ m_\lambda \le \varphi_\lambda(\widehat{u}), \\ & \Rightarrow \widehat{u} \ne u_0 \text{ and } \widehat{u} \text{ is a second positive solution of } (P_\lambda). \end{aligned}$$

□

Combining Proposition 3 and Proposition 6, we get the existence of two positive solutions for problem (P_λ), as stated in Theorem 1.

Acknowledgements The authors wish to thank the two anonymous referees for their constructive remarks. The second author is supported by the research fund of University of Palermo: ”FFR 2025 Calogero Vetro”.

Funding Open access funding provided by Università degli Studi di Palermo within the CRUI-CARE Agreement. The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Data Availability This paper has no associated data and material.

Declarations

Competing interests The authors read and approved the final manuscript. The authors have no relevant financial or non-financial interests to disclose.

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