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Calculus Artefacts in Chinese Textbooks: Variational Approaches with Prospective Primary Teachers

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The first part of the paper presents some theoretical reflections and some methodological notes about a Professional Development (PD) path worked out during the last two years by Italian researchers for prospective primary teachers. The theoretical construct of Cultural Transposition defines the framework of the PD path's activities and the related research. It was used to define an interesting cultural lens to delineate possible new approaches for effective pre-service teacher education programs in particular for the primary level. The defined methodology was based on the possibility to reflect on the he decentralization of didactic practices based on a specific cultural context through one or more contacts with other "realities" coming out from different selected cultural contexts. In the second part of the paper, we particularly focus on one of the stimuli proposed in the PD's path: the Chinese "variation" as a systematic method adopted in Chinese textbooks. In this context, we concentrate our analysis on the use of three counting art artifacts as the Abacus, the Straws, and the Chinese abacus, the Suanpan (算盘).

***Keywords*:** teacher education, cultural transposition, Chinese mathematics, problems with variation, pre-service teacher training program

Introduction[1]

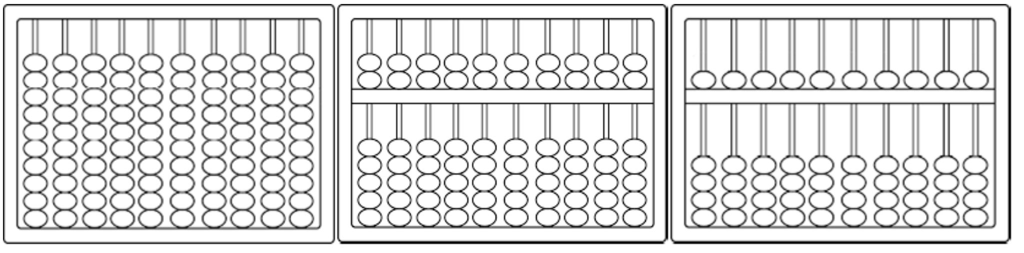
[1] In the following we will denote the Western classical abacus with the wording Abacus (note the capital letter), the Chinese abacus as Suanpan, the Japanese abacus as Soroban (Figure 0). The expression abacus (note the lowercase) will be used to denote all of them. Straws are the mathematical tool, a straw is an object, which use leads to Straws. In the text the expression *x book, y volume* means the y-th book of the x-th year. For each year, so for each book, in China there are only two volumes, one for the first period, another for the second part of the year. Through the exposition we quoted some pages of these books; we built a path independent by the order of presentation of these topics into the books because we aimed to give a whole idea of the use of artefacts, throughout the primary education in China. The number of each figure is connected to the order of our exposition, and to when the topic connected is deeply analysed.

Looking at new scenarios that School is living in recent years, the classroom realities that teachers and students observe are nowadays changing, enriched by new modes, new stimuli, new routines, new didactical processes that come from inside and outside social and cultural classroom contexts. These scenarios are in many cases very complex and difficult to study for the Mathematics Education Communities. Bishop was one of the first researchers who highlighted the importance of recognizing mathematics practises as social phenomena that are embedded in those cultures and those societies that generated them (Bishop, 1988). D'Ambrosio's research underlines that taking care of historical, cultural and social issues in mathematical practises contributes to the understanding of cultures and the mathematics itself (2006). Nowadays this awareness is well known by Mathematics Education Communities. In the literature there are many works that, through qualitative and/or quantitative approaches (An et al., 2004; An et al., 2006; Bartolini Bussi et al., 2014; Bartolini Bussi et al., 2017; Di Paola, 2016; Mellone & Ramploud, 2015), pointed out how mathematics education researchers, coming into contact with educational practises adopted in other cultural contexts, are able to *deconstruct* (Derrida, 1967) them, reconsidering the themes of educational intentionality defined as background of their educational practices(Mellone et al., 2019).

If this awareness is, in this sense, well known in research, we can't say the same for many School contexts about the related use by teachers of cultural diversity approaches in their own mathematics teaching (Di Paola, 2016). In recent years, considering the important of these aspects for the classroom teaching practice (An, 2008, 2009; Marton, 1997, 2008; Sun, 2011b) and with the aim to reduce this gap, some Professional Development (PD) paths for teachers were designed and realized in many countries (e.g., Bartolini Bussi & Martignone, 2013; Chen, 2017; Mellone et al., 2019; Ramploud & Di Paola, 2013). This paper presents some theoretical and methodological notes about a PD's path implemented during the last two years in Palermo for prospective primary teachers (around 250 selected on a voluntary basis). We argue about one of the proposed stimuli related to maybe the most significant mathematics tools for Chinese primary classroom practises: the Chinese *variation* approach (Sun, 2011b). In the following paragraphs, we concentrate our analysis on the use of three counting artefacts such as Abacus, Straws, Suanpan（算盘）and their "variational" use in Chinese mathematics school textbooks (e.g., Vv, Aa., 2012). These stimuli were proposed to PD's pre-service teachers due to the hypothesis that contact with other "realities", coming out from different selected cultural contexts and, specifically, to the possibility to investigate these artefacts, studied through a cultural perspective, can favour in pre-service teachers a possible key of reflection their own epistemology, their beliefs, their silent assumption in teaching/learning mathematics (e.g., Bartolini Bussi & Mariotti, 2008).

In particular, in this paper, we refer to the use of variation cultural transposition aimed at a possible early approach to the algebraic relationship in primary level, not strongly proposed in many Italian classrooms.

Figure 0
The Design of Various Kind of Abacus Involved



the Abacus,
the classic Western abacus
(in this example with ten beads)

the Suanpan,
the Chinese abacus

the Soroban,
the Japanese abacus

Theoretical Framework

Mathematics Teachers Professional Learning

To define the notion of teacher professional learning it is not simple: Some recent studies tried to do it by looking it through different perspectives (e.g., Hodgen & Askew, 2011; Potari, 2013; Skott, 2013). Goldsmith, Doerr, and Lewis (2014) identified 10 characteristics of professional learning, one of which was changes in teachers' practice. Simon and Tzur (1999) defined these as:

> Not only everything teachers do that contributes to their teaching (planning, assessing, interacting with students) but also everything teachers think about, know, and believe about what they do. In addition, teachers' intuitions, skills, values, and feelings about what they do are part of their practice. Thus, we see a teacher's practice as a conglomerate that cannot be understood by looking at parts split off from the whole. (pp. 253-254)

According to this view, the importance of looking to professional learning where teachers' practises are analysed from the perspective of the researcher appears evident. To favour pre-service teachers in defining reflection on their own epistemology, their beliefs, their didactical assumption in teaching/learning mathematics is complex. All these aspects are not cultural-free. In recent papers we discussed the necessity in referring to the Derrida (1967) *deconstruction* as a process that arises as an attitude that serves to continually deconstruct a culture, that is, to put in place a radical critique (e.g., Di Paola, 2021; Mellone et al., 2019; Mellone et al., 2021; Ramploud & Di Paola, 2013). According to this view, which differs from Bishop's approach engaged in the search of equivalence among cultures, we are more concerned with the investigation of differences among mathematics teaching/learning practises and their use in school classrooms.

The Cultural Transposition

The *Cultural Transposition* construct focused on the idea to set the "condition for decentralising the didactic practice of a specific cultural context through contact with the didactic practises of different cultural contexts" (Mellone et al., 2019, p. 199). This construct is inspired by the Skovsmose's (1994) approach according to which mathematics is seen as an imperceptible "construction" that plays an important role in the subtended connected societies and in general in the human condition. In this perspective, offering the possibility to prospective primary teachers to "get in touch" with different educational practises, coming out from different cultural contexts can help them not only to become more aware of their social and cultural paradigm as regards classroom teaching practises, but also to deconstruct their thought. As we just wrote, of course, this "changing process" is very complex and needs more and more opportunities of reflection and *contaminations* (Bartolini Bussi et al., 2014). We are also convinced that use of the *Cultural Transposition* construct in a teacher's training is useful to promote future more effective educational choices related to classroom practises (Di Paola, 2016; Mellone et al., 2019). With this awareness, the PD path was designed with the aim to favour in all involved prospective teachers a passage, a transition, from a previous own condition to another one, more complex, but also more rich, critical, conscious and stable.

The *Variation* Approach

With the aim to present and discuss the implicit epistemological landscape behind the didactics of mathematics in China and so in the textbooks, in this section we briefly describe the Chinese "variation" as a systematic method adopted in the Chinese textbooks and in their teaching/learning practice (e.g., Bartolini Bussi et al., 2014; Caie & Nie, 2007; Di Paola, 2016; Mellone et al., 2019; Mellone et al., 2021; Ramploud & Di Paola, 2013; Sun, 2011b). In the last twenty years, many researchers underlined the importance of *variation* approach as necessary conditions for deep learning (e.g., Marton & Booth, 1997; Sun, 2011b) and in particular for mathematics learning. It is difficult to characterize cultural and textual features of variation tradition: *Variation* is typically expressed by Rowland (2008) as a practice "in structured exercises varies considerably from country to country and from text to text." According to Fan et al. (2004), one of the main theoretical tools in the Chinese mathematics is the use of *variation* approach to problems, Bianshi（变式）, that means "changing form." There are two fundamental directions of variations: The first is called *conceptual variation* (CV). When teachers or students try to draw the meaning of the definition by putting one next to other examples and counterexamples, when they try to classify problems and specify the connotation through essential and non-essential variation of instances, they are using *conceptual variation*. This method is quite abstract however in this way it's possible to train critical thought and analysing ability, working on strategic examples as a part of a more complex problem to face. The second way to vary problems is called *procedural variation* (PV). Following the historical development of a concept or analysing every step that carries to a definition, pupils build their knowledge using *procedural variation*. This way of learning is more dynamic and naturally connected with problem solving/posing (Di Paola & Spagnolo, 2009; Spagnolo & Di Paola, 2010). Sun (2011a) underlined the use of *variation* as tool "to discern and compare the invariant features of the relationship among concepts, solutions and contexts, and provide opportunities for making connections, since comparison is considered the pre-condition to perceive the structures, dependencies, and relationships that may lead to mathematical abstraction" (p.107).

Yakes and Star (2009) looked to *variation* as a critical means for comparing and develop flexibility for learning algebra just since first school years. The issue of variations, in this sense, perfectly reproduces one of Chinese proverb: □□□□□□□□□ - "*no clarification, without comparison*", and puts in evidence the crucial "in contrast" to the assumption used in many western textbooks around the world: "*to consolidate one topic, or skill, before moving on to another*" (Rowland, 2008). In Chinese textbooks there are no repetitions: there are only little shifts, little *variations*, that make us pass forward. But on the other hand, students cannot forget some themes: So there are hard problems where a student must use all his/her knowledge to solve them, to recap what he/she remembers. According to Sun (2011b), a slow but permanent flow moves us in a smooth way from one topic, studied with some tools, to the same topic faced with new tools (*One Problem, Multiple Solutions or Multiple Changes*) or to another topic studied with the same instruments (*Multiple Problems, One Solution*), applied in an original way. Even if exercises are quite similar in the different books, along the vertical curriculum there are little, important, significant, changes useful to an early, informal, approach to abstraction in terms of pre-algebraic thought (Di Paola et al., 2016; Wu, 2017). This is not common in Western primary education. It is clear that the study of this framework, coming far from the Western context, could offer to "contaminated" mathematics researchers and teachers the possibility to re-think many teaching design activities and textbook structures (concepts, pictures, concrete objects, diagrams, etc.) often too much Eurocentric.

The PD's Teaching Path: Methodological Notes

According to the declared aim and what literature discusses on PD path, based on the same research subject (e.g., Bartolini Bussi et al., 2017;

Mellone et al., 2019; Rowland, 2008; Sun, 2011b, 2019; Yakes & Star, 2011) the PD' training activities was designed following this frame:

1. Pre/Post-questionnaire: Teachers' consciousness about classroom practises and cultural resources;
2. Theoretical contribution on the Cultural Transposition paradigm, analysis on different school systems and typical classroom practises (e.g., Italian, Chinese, Singaporean, Arabic, African);
3. Focus group: *Why/When/Where/What/How culturally transpose in your classroom*?;
4. Observation (by video of classroom activities, textbooks analysis, etc.) and critical discussion about teaching classroom practises coming from different cultural context;
5. Teachers' implementation of classroom activities designed according to the Cultural Transposition;
6. Written teaching report and group discussion: *What happens in your classroom*?

Authors of this paper were involved in all phases of the PD' path, as plenary speakers and group leaders of the focus group; two teachers/researchers in mathematics education supported their work and participated also to analyse the collected data. The pre-service teachers were engaged for 220 hours. During all PD's phases the data were collected by video and audio recording. These were opportunely analysed by the authors of this paper and the two teachers/researchers and were used to redefine step by step the designed PD design. Pre-service teachers were excited to participate and to analyse for the first-time mathematics classroom practises, coming out from different cultural contexts. They never did it before during their teacher training. According to what we proposed, they noticed and actively discussed analogies and differences in classroom setting, teachers and students' behaviours, classroom practises, tasks design and use with students, etc.

In the next section we discuss one of the inputs proposed during the fourth PD path's phase based on the *variation* approach (Bartolini Bussi et al., 2014; Sun, 2011b, 2019). Others PD's stimuli were discussed in Buttitta & Di Paola (2021) and Di Paola (2021). With this aim we focus our attention to some examples discussed in Chinese textbooks, connected with possible classroom or workshop didactical practises using calculus artefacts (Bartolini Bussi & Mariotti, 2008).

We proceed to next papers, the qualitative and quantitative analysis of the data related to the questionnaires, the focus group and the written report that we collected.

Examples of *Variation* Approach on Calculus Artefacts in Chinese Textbooks

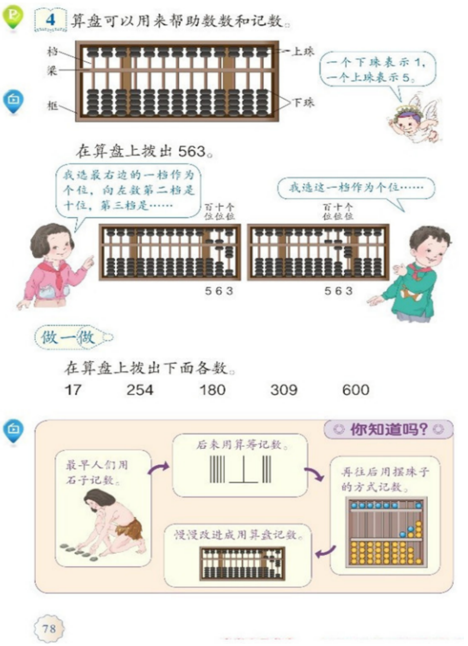
As we said before and as we showed during the PD course, the other-culture way of teaching mathematics can be analysed through the mathematics textbooks; this should be seen just as a first step toward a more aware way to do and teach mathematics, as a first step in the analysis of a whole educational system, entirely unknown. We focus on numbers, and how in the textbooks are figured out the main properties of the elementary operations (e.g., the symmetry between addition and subtraction) through the use of the Abacus, the straws and the Chinese abacus, the Suanpan (算盘). To be clearer in the pages below we discuss firstly about each artefact by itself even if in the Chinese textbooks these are always used one next to the other, at the same time, and this is one of the main gaps between Chinese and western approaches (Di Paola & Spagnolo, 2010; Spagnolo & Di Paola, 2010).

The Suanpan and the Positional Numeral System

The layout of the books in China is well balanced: for each mathematical topic, there is a strict connection between figures (from pictures of objects to graphs), formalisation (from numbers to letters) and language. The use of the same artefacts drives students all along the first years of primary school to more abstract concepts (Figure 4). It is not the same in Italian textbooks (Mellone et al, 2019).

Figure 1

A First Sight of the Suanpan (Second Book, Second Volume; page 78) (Vv.Aa., 2012)



One of the most relevant calculus artefacts through human history is the Abacus. Many times, the Abacus is seen in a static way, we do counts and then we move the beads to represent the result, while it can be helpful: The Abacus can count for us! The Chinese abacus is by far more powerful, but it’s also quite difficult to use at a first moment. This is the reason why the Suanpan is introduced at the second year (at the middle of the second book, second volume, Figure 1) into an historical frame while the Abacus is used since the first year to compare numbers (at the middle of the first book, second volume, Figure 2; but also, until the first book of the fourth year to compare five-digit numbers). Into the text there is an explanation of each piece of the Suanpan, also from a mathematical viewpoint: each bead of the group of five on the rod at the bottom of the deck counts as one, each of the group of two on the rod at the top of the deck counts as five. At this time the student already knows, as we will see discussing the Abacus, the principle that moving to the left, on columns, every unit becomes ten times bigger. Implicitly the reader assume that the Suanpan is “better”/smarter than other abacus because, as it is presented (Figure 1**)**, the Suanpan is the final evolution from the use of stones (*calculi*) through Chinese counting-rod form (vertical and horizontal lines) numbers and a schematic ancient Chinese abacus, actually a Japanese one, less “expressive” because with fewer beads, but with the same tenet of the concrete Chinese abacus.

Figure 2

A First Comparison Between Numbers, Using Abacus (First book, Second Volume; page 42) (Vv.Aa., 2012)

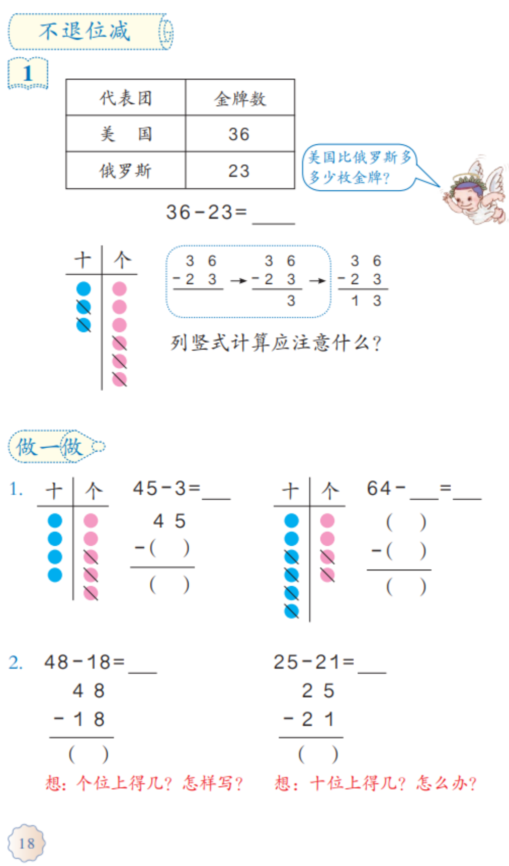

5
42里面有4个十。
37里面有3个十。
42 > 37
百位 十位 个位
百位 十位 个位
23和25十位上的数相同。
这样的数怎样比较大小呢？
23 < 25
做一做
1. 在◯里填上 “>” “<” 或 “=”。
百位 十位 个位
百位 十位 个位
29 ◯ 30
百位 十位 个位
百位 十位 个位
81 ◯ 18
2. 在◯里填上 “>” “<” 或 “=”。
41 ◯ 45
68 ◯ 78
69 ◯ 69
57 ◯ 56
80 ◯ 90
98 ◯ 89
42

The Abacus and Its Mathematical Relationships

To understand how the Suanpan can be seen only in part as something new we have to do a step back, and follow the textbooks, because before the Suanpan, there are other artefacts, indeed the Abacus and the Straws, which the student gets used to from the beginning of the school. Going back, since the first book (first and second volume of the first year) there are many patterns to underline the connections between the Abacus, as the Straws, on one hand and addition and an early form of subtraction on the other (Figure 4). In the middle of the first book, second volume the Abacus is used for easy operations (sum and subtraction) and to focus attention on the differences between adding/subtracting a unit or a ten. As a matter of fact, the link between sum and subtraction might look trivial, but not at the first year of school! When a new one-digit number is introduced, there are partially empty schemes to fill to get all the combinations of smaller numbers which added together give the new bigger number.

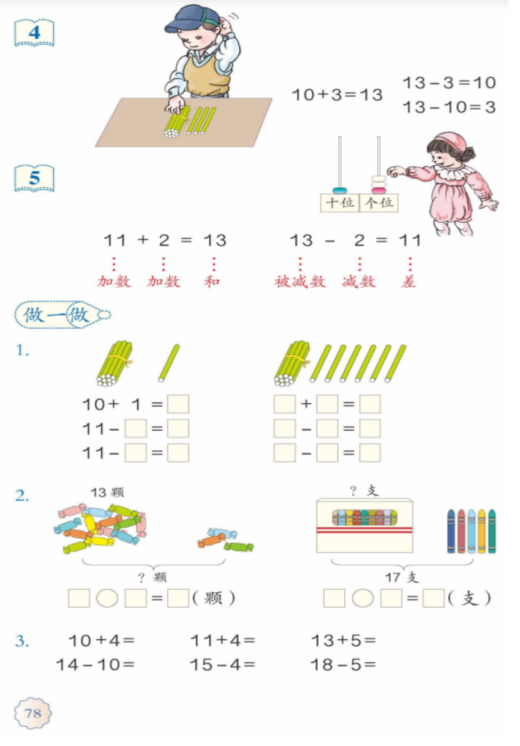
Figure 3

From Practice to Formalisation: A Smooth, Parallel, Path (Second Book, First Volume; page 18) (Vv.Aa., 2012)



The same happens with two-digit numbers (Figure 3 as Figure 4): the first items are already filled to clear the idea, then some are empty, finally the student has to write all by himself/herself. So, the Abacus is just one of many strategies adopted. It's used to connect topics and to make some parts meaningful. For example, the Abacus can be the start (Figure 3).

Figure 4
Where We Want to Arrive, It's Their Beginning (First Book, First Volume; page 78) (Vv.Aa., 2012)



On this page at the beginning of the first volume of the second year there is a bridge and not a gap: readers are going to achieve a well-built, step-by-step, mathematical representation of numbers and their operations. The new formalisation (column method) reminds us of the pattern of the Abacus: working on each column, every dot is like a bead, and then (like a) unit. So, numbers have different shapes: at the first step you can touch them (beads), then you can see them (dots), finally you should abstract and work with them using only the mathematical alphabet. As usual in Chinese textbooks, it is common to find *graphics operations*, due to a schematization of an "instrument", in this case, the Abacus, used to improve a deep visualisation of a more abstract topiche contemporary use of these artefacts (Abacus as Suanpan) in Chinese textbooks guides students to a concrete and clear vision, then general and abstract understanding, of the base-10 positional notation.

The use of Abacus helps pupils to uncover the decimal meaning embodied into the positional representation system; the approach with Suanpan in Chinese textbooks is delayed because ten in this artefact is the sum of the values of the two five-valued beads, so, even if this idea is only a bit more abstract, it's also quite indirect. So, the Abacus is a useful instance to investigate the concept of radix-base (ten in this case) and the meaning of positional notation as conventional but coherent and in such a way necessary (some system must exist).

Studying Chinese school textbooks and their use in classrooms, highlight how for several Chinese primary teachers the Abacus (in all its shapes) is one of the best ways to improve counting ability in a fast, easy and concrete way, without being captive by the artefact: the instrument is "inside us", it strengthens our representation of numbers and operations. In the proposed *variation* approach between/through Abacus as well as Suanpan, the stress is not only on the algorithmic confidence or ability on counting of our pupils (one final goal) but firstly on making them "feel" the *structure* of the positional numeral system. This approach underlines relationships, regularities, invariants etc. and makes students discover a possible pre-algebraic thought working from the first school beginning on topics easier than the algebraic unknown quantities calculation (Di Paola et al., 2016; Wu, 2017). As we just said before, this is one of the main gaps between Chinese and western approaches (Di Paola, 2021; Mellone et al., 2019; Spagnolo & Di Paola, 2010)

Counting with Straws

The same process, the heart of the base-10 positional numeral system, it's the focus of the use of the straws assembled in groups of tens. Ten straws united together means ten units, so a unit of the upper order; these are like a new unit, of another order; but if necessary, we can untie this group to get again ten units and then add or subtract them with other units of the same order. This approach for operations is shown systematically in Chinese textbooks since the beginning of the first volume, the second year where it is possible to see operations in columns between two-digit numbers and the analogous operations with Straws, as happens using the Abacus. In a more informal way, Straws can be seen since the middle of the first book, the second volume where for the first-time students learn to tie and untie to do sums and subtractions on one-digit numbers. Going further back the idea of adding a unit (indeed a straw) to a number to get the following number can be seen since the middle of the first book, first volume where the Abacus and straws are used at the same time to represent numbers, in a static way, and are used to make easier to achieve the concept of the positional representation (comparing numbers with the same digits in different positions) and the idea of being a "bigger number" through a meaningful sight of every unit, divided by order (in columns or groups of straws) (Figure 4 and 2).

A more dynamic use of Straws appears at the same time (one page after, in the first book, second volume) where is introduced more dynamic use of the Abacus, until that moment used, quite often with the Straws, only to represent and compare numbers, and not for operations, as said before. Also, in this case, these two very powerful artefacts are used together, in complementarity, to build a strong feeling for numbers and operations since the first year and, in a deeper way, during the second year. As we just said before, comparisons between artefacts give students a significant opportunity for early approach to abstracting, according to meaningful experiences. This kind of *variation* approach is not so common in western school textbooks' educational context (Spagnolo & Di Paola, 2010). For example, the carry-borrow operation, showed following the CT framework to the PD preservice teachers appear as the same process in the case of Straws and the Abacus: even if something changes, from a more abstract point of view, putting one next to the other these material artefacts yield the mathematical pattern to stay and shine, all over the differences.

Changing to Improve: *Variation* in Counting Artefacts

We focused more on the first four Chinese books of the first two grades because until that moment the use of artefacts, especially the Abacus, is stronger, after that point the use of these instruments is less systematic, also because the new operations, starting by multiplication, become, even if feasible, a bit tricky using an Abacus. In the first volume of the fourth book the Abacus returns, and it is the excuse to introduce new and more powerful instruments (like calculators and computers). Across history and reality situations it is possible to reach the calculators, which will be used starting by this moment to make counts faster, because pupils are already aware of the meaning of counting (Figure 5**).**

Along this analysis, we saw how in China using different registers and artefacts Abacus, Straws, different mathematical languages) as different attitudes (concrete, operative, symbolic) make easier to face some mathematical topics, building a parallelism between near concepts like addition and subtraction. Each of these ways of representing things can help students to overcome a different difficulty and allow them to deeply understand something hard and new, being alongside pupils like something familiar also in new occasions, especially during abstraction and generalisation.

In our opinion, the "static/algorithmic" training in the Italian school is a massive problem: in many cases, in Italian textbooks, we find a huge number of exercises, of all the same type. Many times, the focus is on theory, and exercises are used, not chosen. At the same time, since the first steps in Chinese textbooks there is no division between knowledge, everything varies! As we just wrote before, of course, this *variation* approach favors handling different tools at the same time, to choose which of them is the best for the problem, and this is synonymous with being well trained, because students can be free, without limits, and while he/she is able to face a problem, he/she is also able to solve that problem in more than one way (Fan et al., 2004). Analogues subjects are submitted into the same pages because these topics are connected: the way of teaching (didactics) follows the matter (Mathematics), in a smart way, without putting together really hard definitions but making students feel the affinities, first of all in an *informal way*. Then growing up they will see again, but from a higher point of view some of these patterns and topics, which they had already met and connected, because presented simultaneously. This is something that holds for a long time in us, as former students, and in our cultural heritage, as human beings: both history and mathematics witness that.

Figure 5

From History and Daily Life to the Future (Fourth Book, First Volume; page 23-24) (Vv.Aa., 2012)

Some Conclusion … What Can We bring to Our Classrooms?

As seen, the *variation* approach is a very broad and flexible tool, which can help us to redesign many mathematical topics, using comparison, problem-solving strategies, and different registers. An original suggestion, inspired by the topics we talked about, melted with something new can be as follows: the use in workshop situations of different artefacts, e.g., the Abacus, to inquire the differences and similarities between addition and subtraction. The main artefacts could be the compared use of the Abacus and its Chinese (Suanpan) and Japanese (Soroban) versions; the theoretical framework should be the idea of *variation* (*procedural variation* especially); the first goal can be to let students understand the deep connection between addition and subtraction and how works our numerical system, the second goal to teach students the way of using Abacus to improve their counting abilities. To be clearer we will use the deeper classification of variation problems by Sun (2011a, 2011b).

On one hand teachers can propose to use *one artefact at a time* to face problems like a simple addition (2+3=_), a modified form of addition (from one number we want to get another bigger one adding something we don't know) (2+_=5), and finally a subtraction (5-3=_ and 5-_=2); so this pattern is the Multiple Problems, One Solution (and one instrument) (MPOS, □□□□, varying presentations). This suggestion can work well as a group workshop lesson: defining a possible *semiotic mediation* based on different artefacts (Bartolini Bussi & Mariotti, 2008) each group can use just a type of abacus, meanwhile further groups are using other kind of abacus. Along this direction we use *procedural variation*: we are building step by step the idea of subtraction, starting from something known (addition).

On the other hand, teachers can fix the same problem and then let "use" (in the sense of Bartolini Bussi & Mariotti, 2008) different artefacts at the same time; in this case the pattern is One Problem, Multiple Changes (and multiple instruments) (OPMC, □□□□, varying conditions and conclusions) as One Problem, Multiple Solutions (and multiple approaches) (OPMS, □□□□, varying solutions). In this case there is a strong comparison between different instances of the same pattern; the aim is to find the common mathematical structure, so we are in such a way into the *conceptual variation* frame.

According to the stimuli that we proposed, during the fifth PD' phase teachers designed similar teaching variation activities, testing these at schools, following what they "learn by Chinese teachers". The results were impressive. Following what Giovanna, a prospective teacher wrote in her post-questionnaire - "Thanks to Cultural transposition approach, used for this interesting course, I rethought my knowledge, I rethought me, as future mathematics teacher. The problems with variation are really powerful to find mathematics relation in a word-problem and can help students to approach Algebra in an informal way. I will treasure this in my future teaching" - approaching "variation" permitted them to be "contaminated" and culturally rethink their own typical mathematical classroom practises aimed to new teaching designing.

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Appendix: English summary of the writings in the figures

Figure 1: The title of the section says, "abacus can help you counting". There is a short description of the parts of Chinese abacus: cross-rod and upper and lower beads. It is said that lower bead means one, upper bead means five. Into the two cartoons is explained the way of counting using an abacus: the right column is *a ones-column, the second, starting from right, is a tens-column, and so on with the third. In the next section is asked to dial assigned numbers using Chinese abacus. At the bottom there is a brief history of counting tools (see Figure 4): stones (lculi*), rods, beads (into a Japanese abacus), (Chinese) abacus. Note that the colours of beads in the Japanese abacus are used to underline the different or the same value of the beads: yellow for ones, blue for fives.

Figure 2: In the cartoon the two children say "there are 4 tens in 42" and "there are 3 tens in 37". This is visualized using Straws. So, 23 and 25 have the same number of tens or the same digit in the tens-position. The female child asks then how to compare these kinds of numbers. In the next section is asked to fill using ">", "<" or "=" the empty space between the two numbers compared. Note that the Abacus is used to compare just the first set of numbers. Note also that in the Abacus each column, starting from right, is labelled as units-position (个位), tens-position (十位), hundreds-position (百位).

Figure 3: Into the table there are data about numbers of gold medal won by USA and Russia. Then is asked how many medals America have more than Russia. Then is asked what you have to care about during the so called *vertical* (-writing subtraction) (竖式). Colour (pink and blue) of dots relates to one-value (个) and ten-value (十). In the following part there is a comparison between Abacus-like operations (*vertical* subtraction) and column operations. The last two red questions focus on ones and tens: in such cases subtracting, ones and tens become empty.

Figure 4: The boy on the left is deeply understanding the difference between tens and ones through the subtraction. On the Abacus on which the girl is working on, on the first column (starting from right) is written units-position (个位), on the second is written tens-position (十位). Then is shown the symmetry (the root of *variation* approach) between addition and subtraction: 11 (addend) + 2 (addend) = 13 (sum), 13 (minuend) – 2 (subtrahend) = 11 (difference). The following section (called “do it”) is to fill.

Figure 5 (first one): The title is “knowledge of computational tools”; in each illustration (starting from the left top) there is a fundamental stage in the history of counts: rods (used like *calculi*) and Chinese abacus from China (between 2000 and 1000 years ago) and slide rule and calculator from West (around XVII century). In the XX century there are electronic calculators, around 40’s the electronic computer. At the bottom of the page there is a brief path of the development of technologies of computers: from PC to laptop and tablet. The computational power of fastest computer (the book is composed before 2012) it is said to be around 10^{20} elementary operations per second.

Figure 5 (second one): Despite more powerful tools, abacus is sometimes used nowadays (around 10’s) in China, in traditional shops for examples. The young girl asks in the cartoon how much counts one of the upper beads and how much counts one of the lower beads; it is useful to remind even if this information is already known by the reader as shown before. Note that the two abaci next to her are a Suanpan and a Soroban. In the bottom part is asked if the reader can write down the numbers performed on Suanpan. On each column of the Chinese abacus are written the growing powers of ten, starting from 1 to 10^9.

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